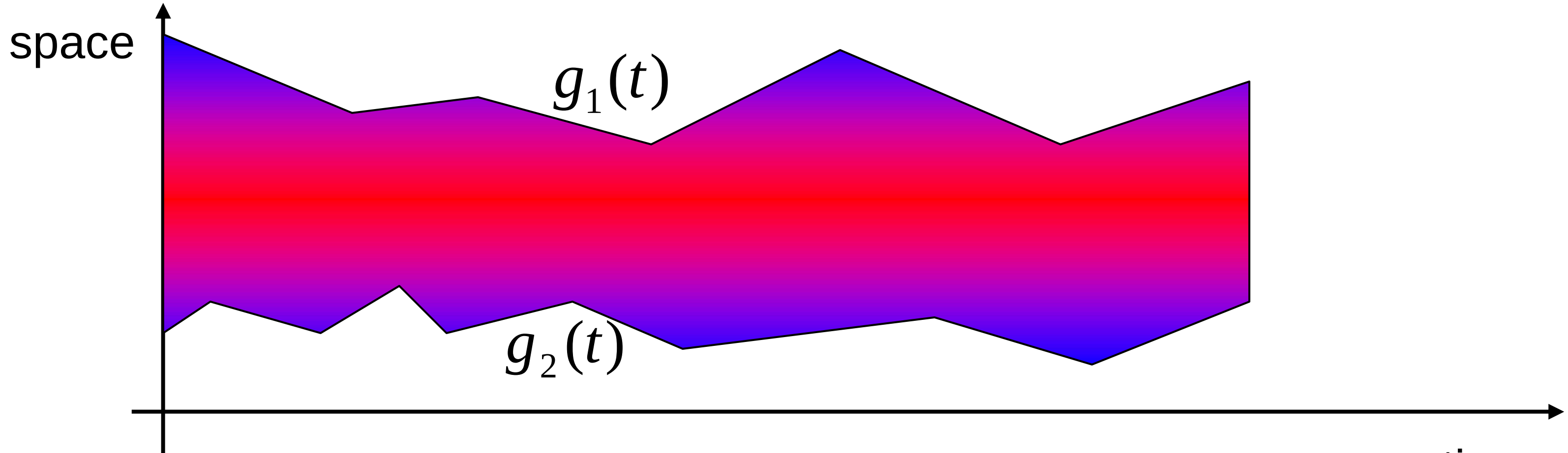
Essential Uses of Probability in Analysis

Part II. Domains with moving boundaries. The heat equation and reflected Brownian motion.

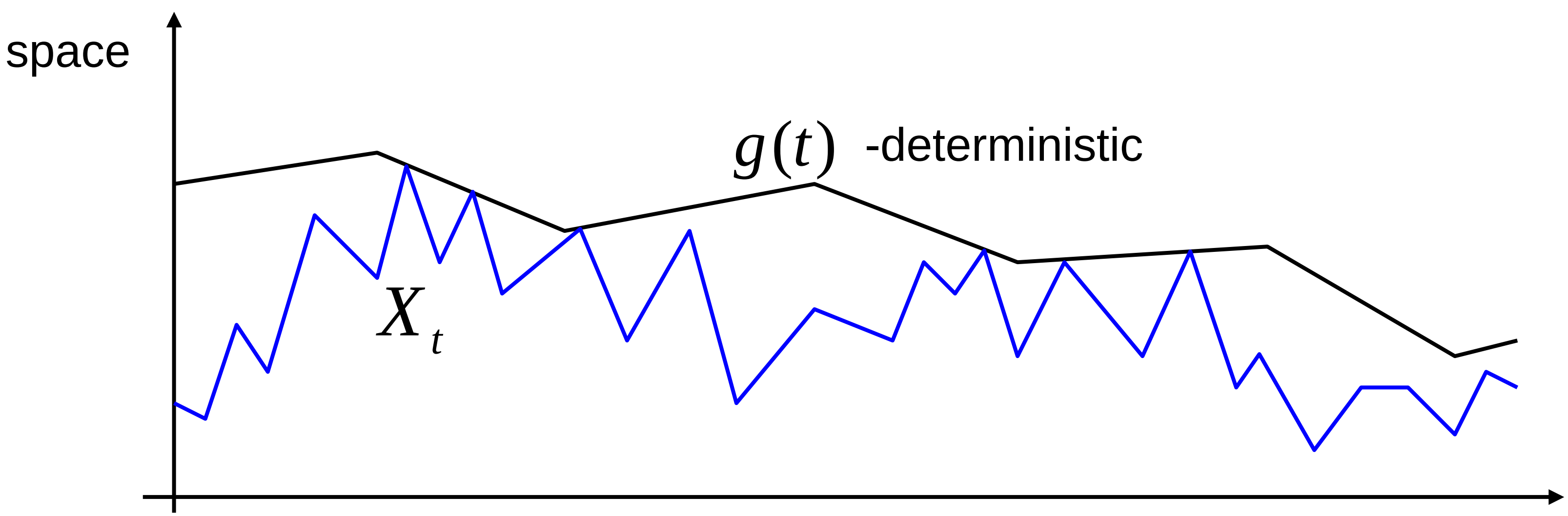
Krzysztof Burdzy

University of Washington

Time dependent domains



Heat equation, Neumann boundary conditions



Reflected Brownian motion

Reflected Brownian motion in time dependent domains

- **Cranston and Le Jan (1989)**
- **Knight (2001)**
- **Soucaliuc, Toth and Werner (2000)**
- **Zheng (1996)**
- **Bass and B (1999)**
- **Lewis and Murray (1995) – analysis, no probability**
- **Hofmann and Lewis (1996) – analysis, no probability**
- **Lepeltier and San Martin (2004)**
- **B, Chen and Sylvester (2003, 2004, 2004)**
- **B and Nualart (2002)**

Heat equation

$u(t,x)$ - temperature at time t at point x

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int_{-\infty}^{g(t)} u(t,x)\,dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \qquad (1)$$

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ u_x(t,x) = -g'(t)u(t,x), \quad x = g(t), \\ u(0,x) = u_0(x). \end{cases} \qquad (2)$$

Heat equation solutions – existence and uniqueness

Theorem. If $g(t)$ is C^3 then solutions to (1) and (2) exist, are unique and equal to each other.

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int_{-\infty}^{g(t)} u(t,x)dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ u_x(t,x) = -g'(t)u(t,x), \quad x = g(t), \\ u(0,x) = u_0(x). \end{cases} \tag{2}$$

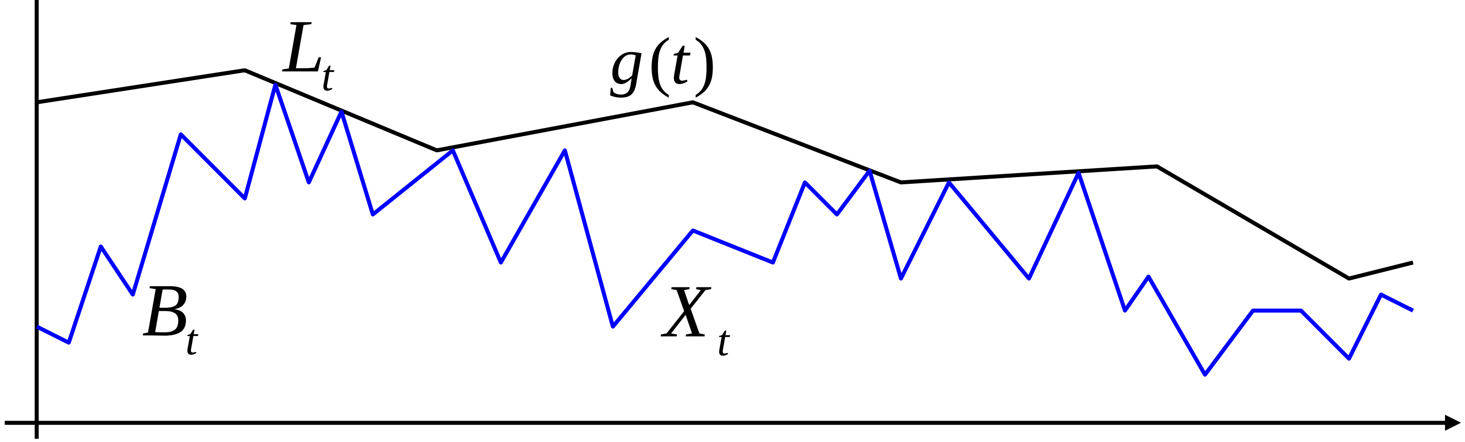
Lewis and Murray (1995), Hofmann and Lewis (1996)

Skorohod Lemma

$g(t), B_t$ - continuous functions

Lemma. There exists a unique continuous non-decreasing function L_t such that $X_t = B_t - L_t \le g(t)$ for every t and L_t does not increase when $X_t < g(t)$, i.e.,

$$\int_0^\infty 1_{(-\infty, g(t))}(X_s)\,dL_s = 0.$$



Heat equation solution via reflected Brownian motion

$g(t)$ - continuous function

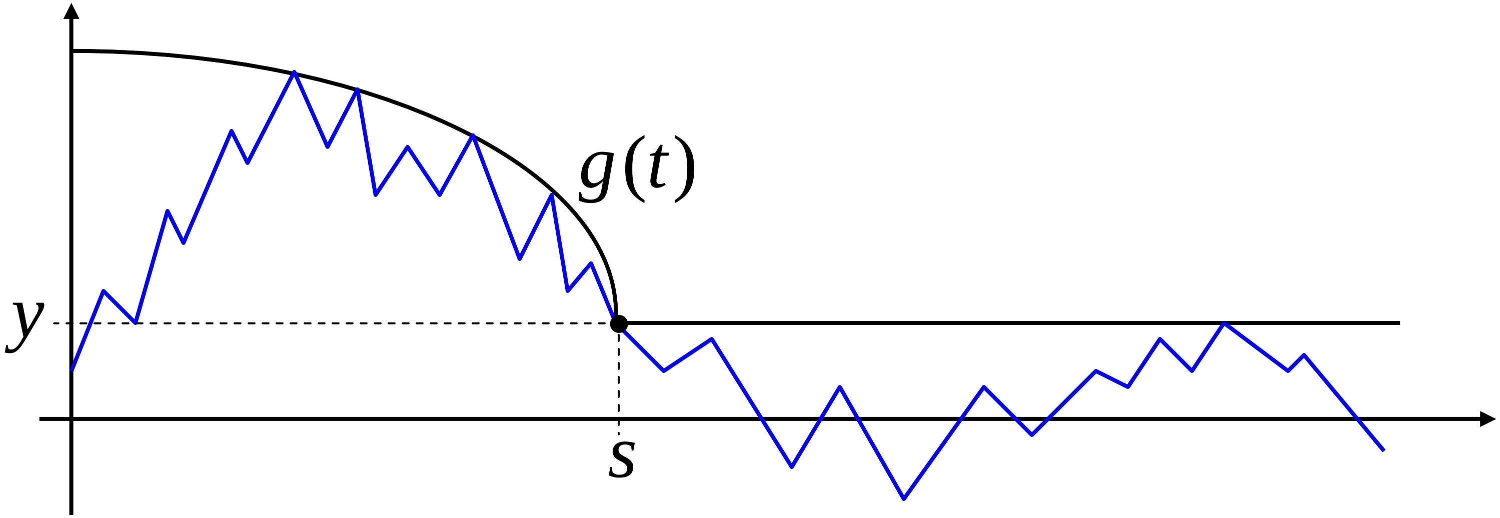
B_t - Brownian motion

$X_t = B_t - L_t$

Theorem. The function $u(t,x)dx = P(X_t \in dx)$ solves (1).

$$\begin{cases} \frac{1}{2}\Delta_x u(t,x) = u_t(t,x), \quad x < g(t),\ t > 0, \\ \int\limits_{(-\infty, g(t)]} u(t,x)dx = 1, \quad t \geq 0, \\ u(0,x) = u_0(x). \end{cases} \qquad (1)$$

Heat atoms

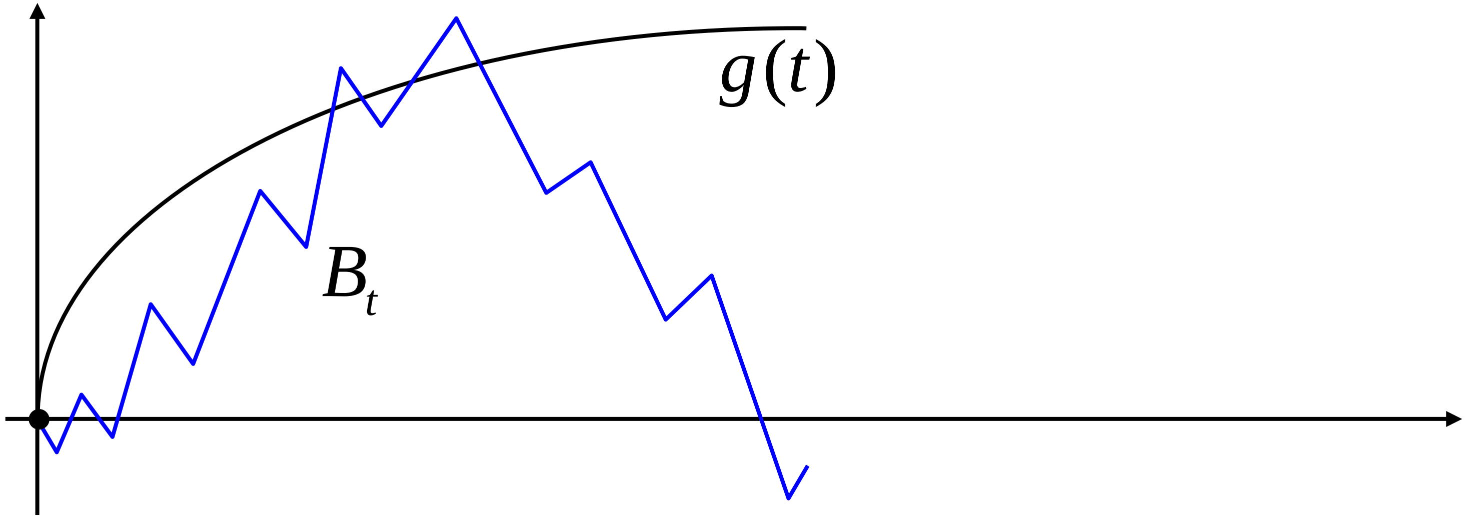


$$\int_{(-\infty,y)} u(s,x)dx < 1$$

$$P(X_s = y) > 0$$

Theorem. Heat atoms exist for some $g(t)$.

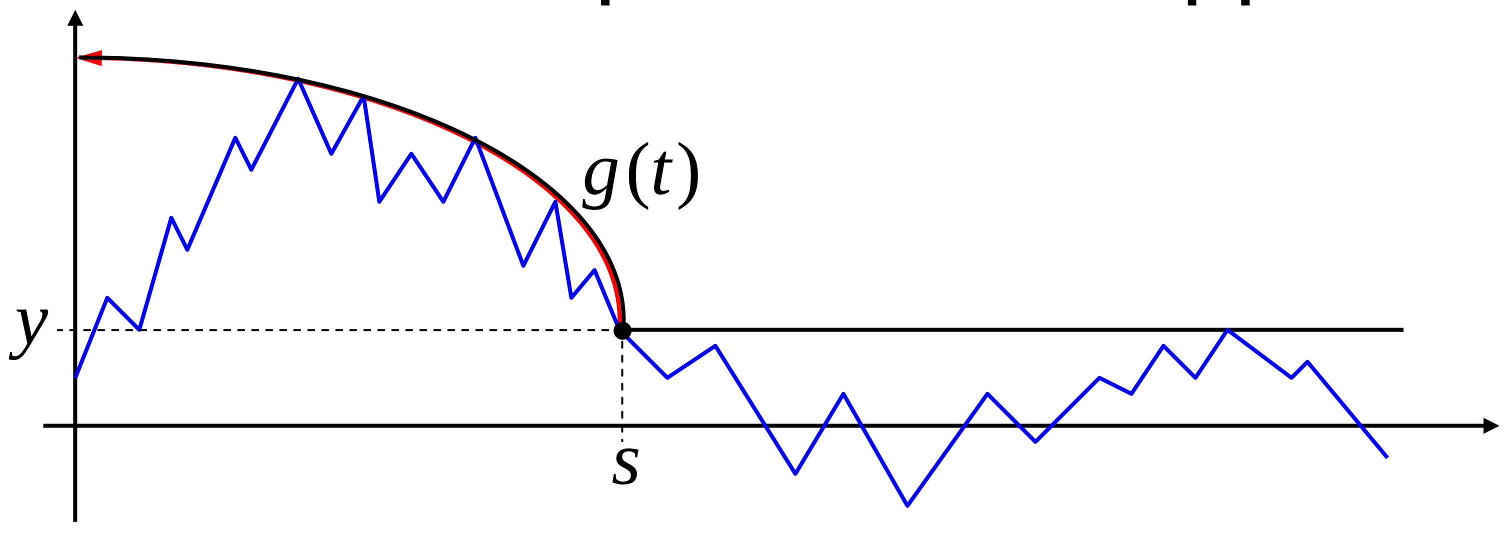
Upper functions for Brownian motion



$$P(\inf\{t > 0 : B_t = g(t)\} = 0) = 0$$

B_t - Brownian motion

Heat atoms – probabilistic approach



Theorem. $g(s)$ is a heat atom if an only if $f(t) = g(s-t) - g(s)$ is an upper function.

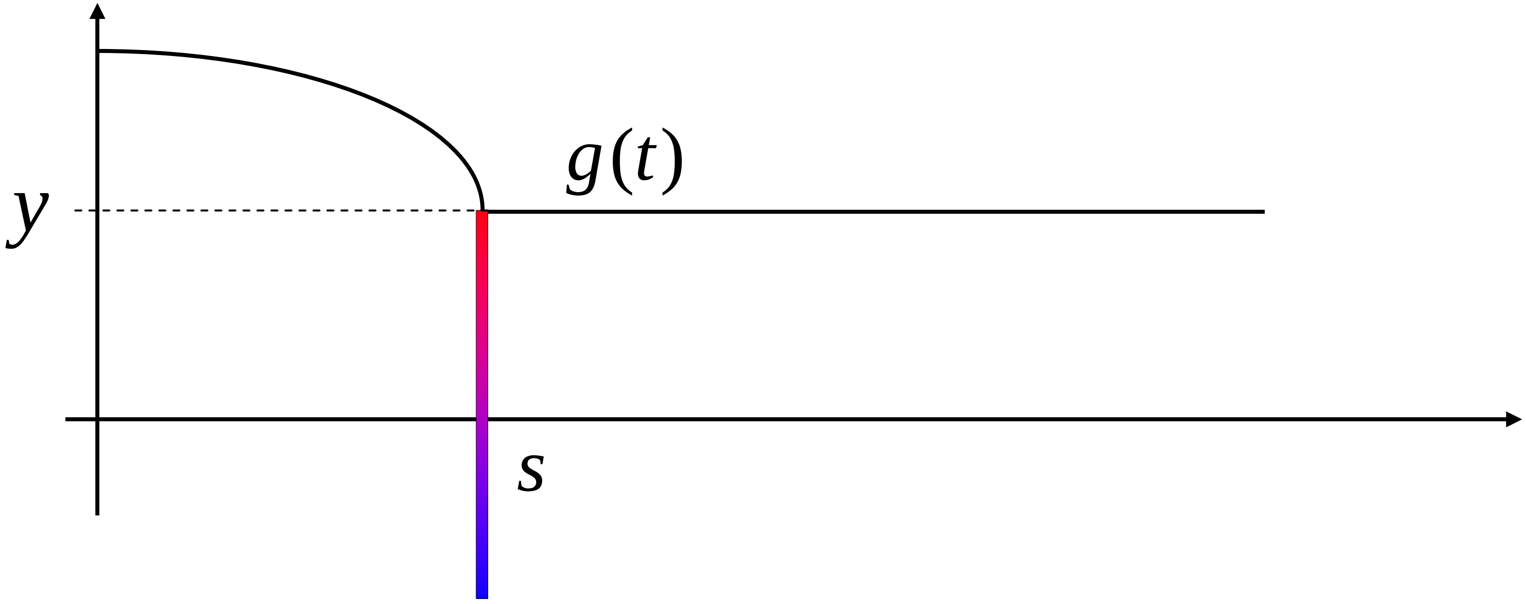
Kolmogorov's criterion: $f(t)$ is upper class if and only if

$$\int_0^1 t^{-3/2} f(t) \exp(-f^2(t)/(2t))\, dt < \infty$$

Example (LIL): $f(t) = (1+\varepsilon)\sqrt{2t \log|\log t|}$

$f(t)$ is upper class if and only if $\varepsilon > 0$

Singularities



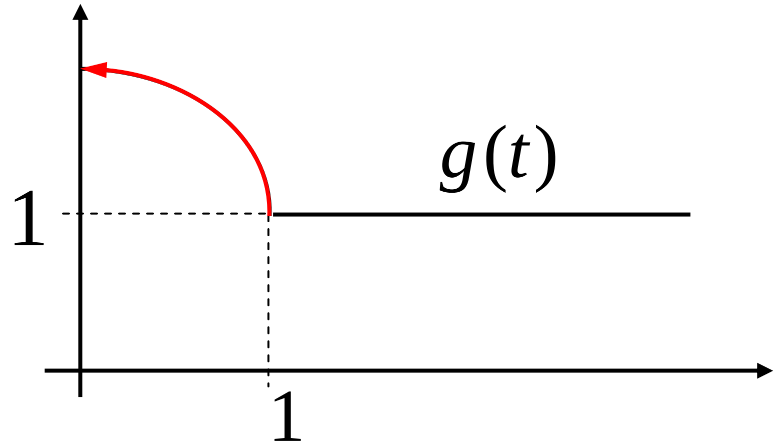
$$\limsup_{x \uparrow y} u(s,x) = \infty$$

Heat atoms and singularities

Theorem: There exist g_1, g_2, g_3, g_4 such that

	Singularity	Heat atom
g_1	No	No
g_2	Yes	No
g_3	Yes	Yes
g_4	No	Yes

Heat atoms and singularities - examples



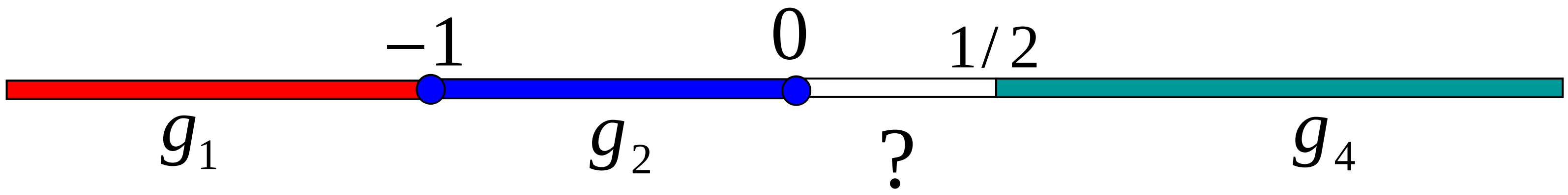
$$g(1-t) = 1 + \sqrt{t}\,|\log t|^{\beta}$$

$\beta \in (-\infty, -1) \Rightarrow g_1$

$\beta \in [-1, 0] \Rightarrow g_2$

??? $\Rightarrow g_3$

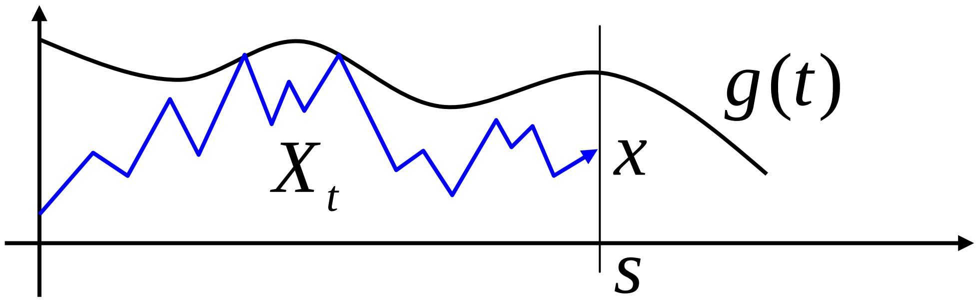
$\beta \in (1/2, \infty) \Rightarrow g_4$



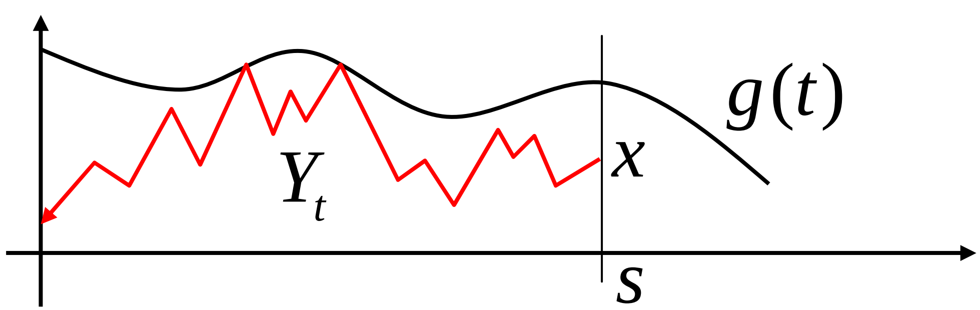
Conjecture: $\beta \in (0, 1/2] \Rightarrow g_4$

Probabilistic representations of heat equation solutions

$$u(s,x)dx = P(X_s \in dx)$$



$$u(s,x) = E^{0,x}\left[\exp\left(-\int_0^s 2g'(t)\,dL_t^Y\right)u(0,Y_s)\right]$$



Probabilistic representations of heat equation solutions (ctnd)

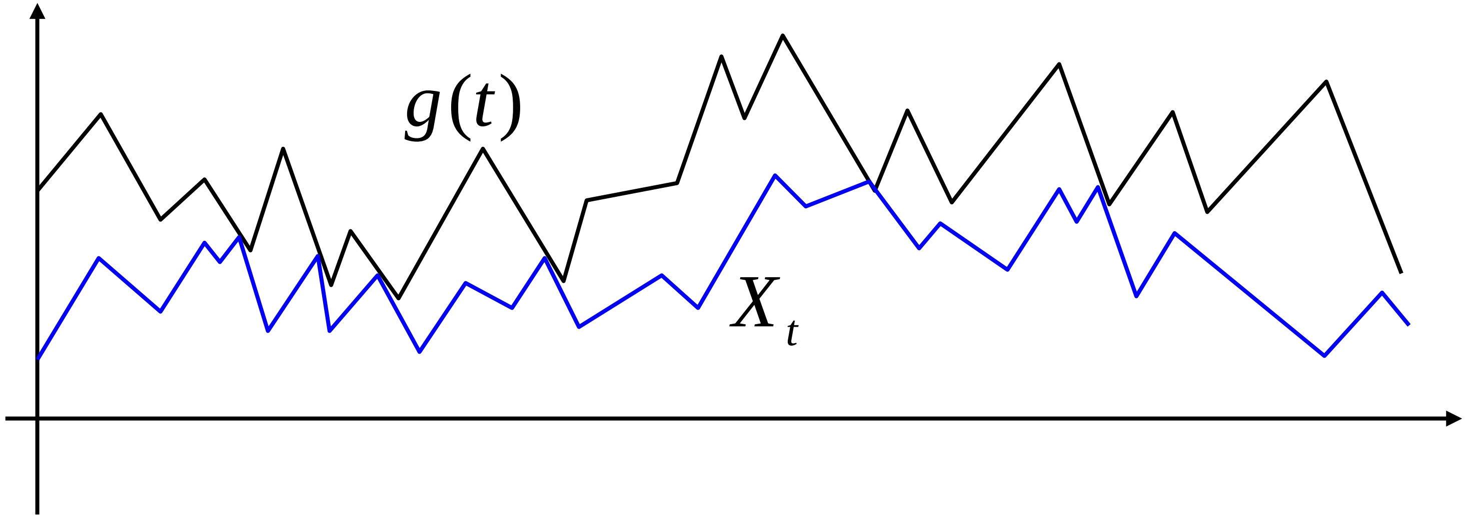
$$u(t,g(t)+x)=E\left[\exp\left(\int_0^t g'(t-s)dB_s-\tfrac{1}{2}\int_0^t (g'(t-s))^2ds-2\int_0^t g'(t-s)dL_s\right)\right]$$

B_t - standard Brownian motion

L_t - local time at 0 of reflected Brownian motion

$Y_t = x + B_t - L_t$ on $(-\infty, 0]$.

The set of heat atoms



$A(g) = \{t : g(t)$ is a heat atom $\}$

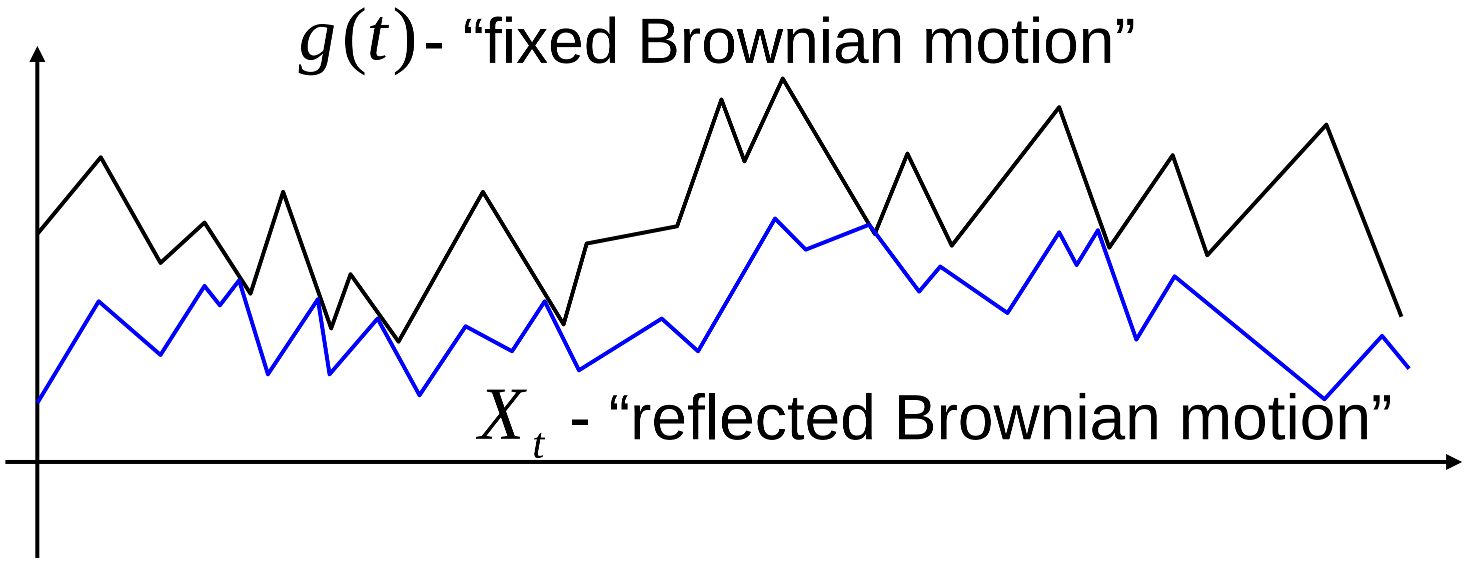
Theorem:

(i) $\forall g \quad \dim A(g) \leq 1/2$

(ii) $\exists g \quad \dim A(g) = 1/2$

Corollary: Lebesgue$(A(g)) = 0$

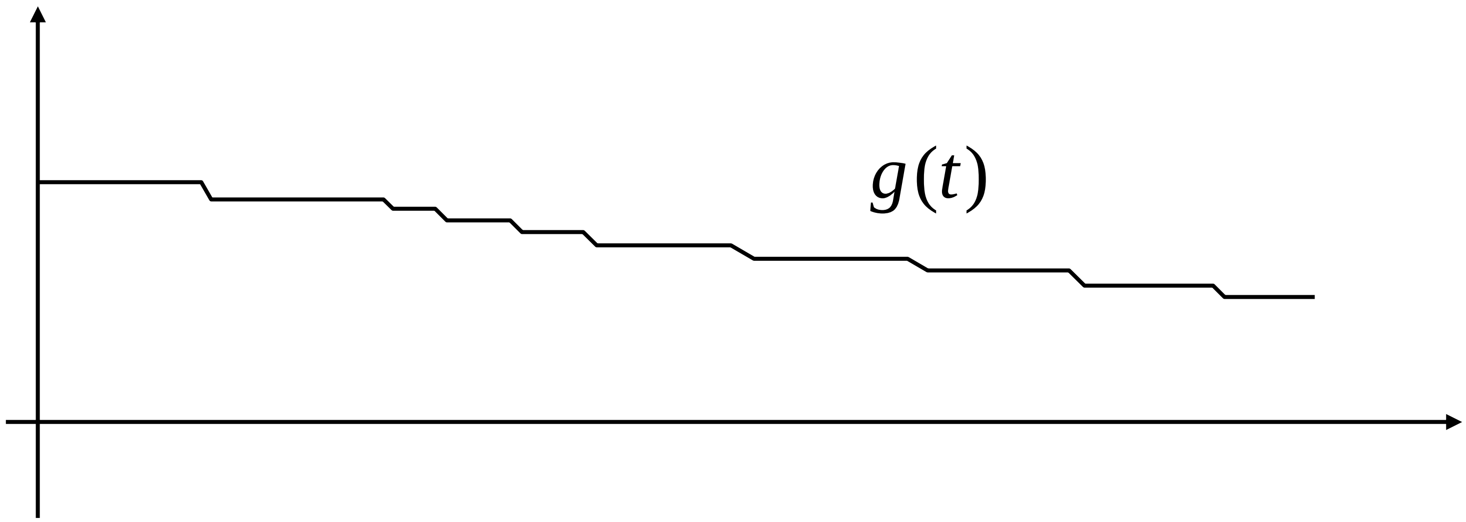
Brownian motion reflected on Brownian motion



Soucaliuc, Toth and Werner (2000)

Theorem: There are no heat atoms on Brownian path.

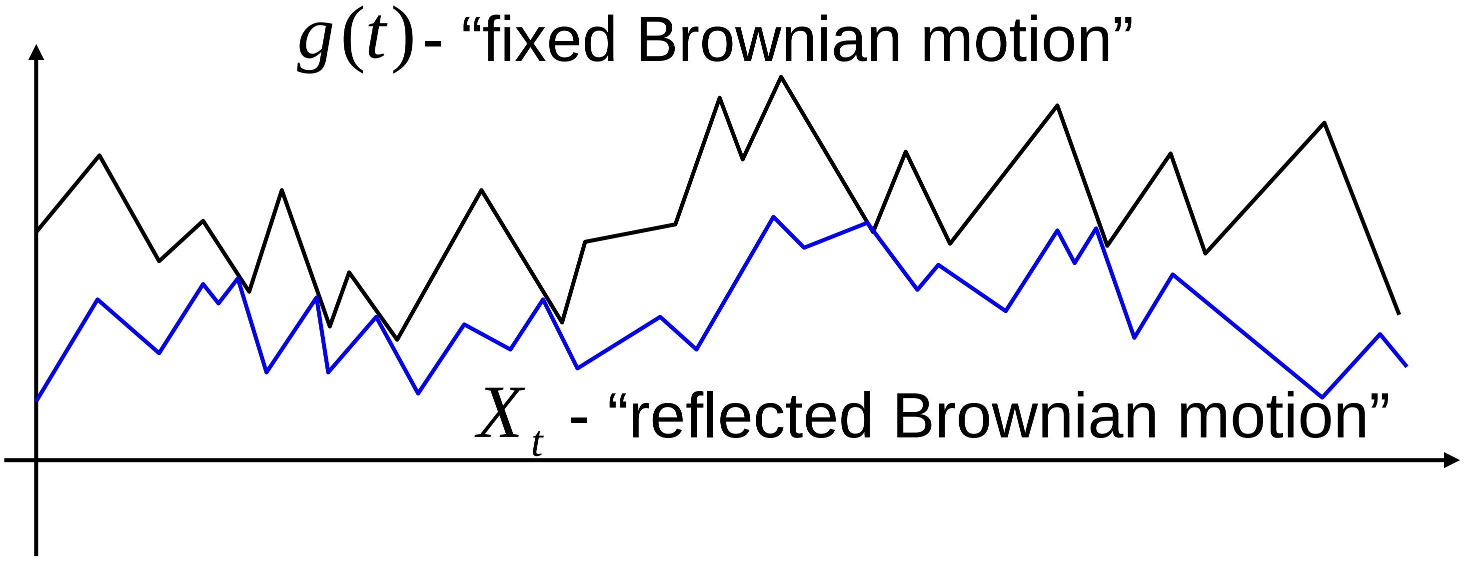
Stable boundary



$g(t)$ - inverse of a stable subordinator

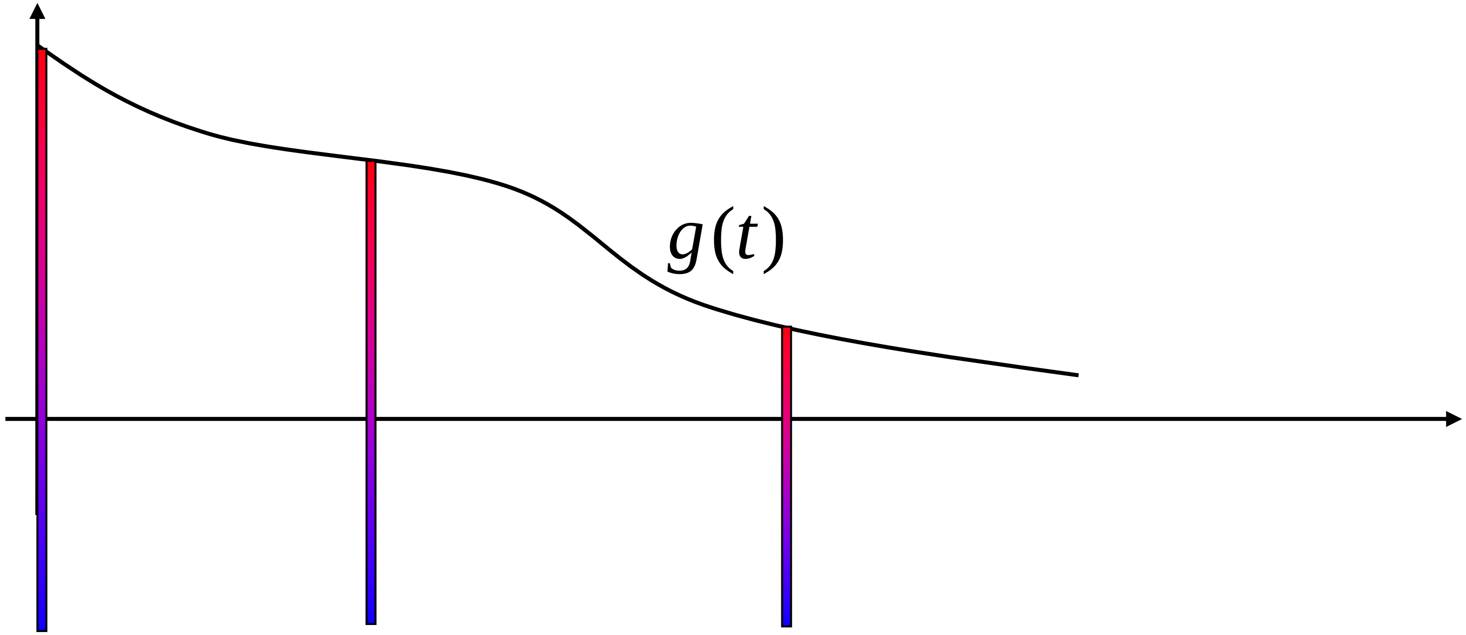
$$\dim A(g) = 1/2$$

Set of singularities



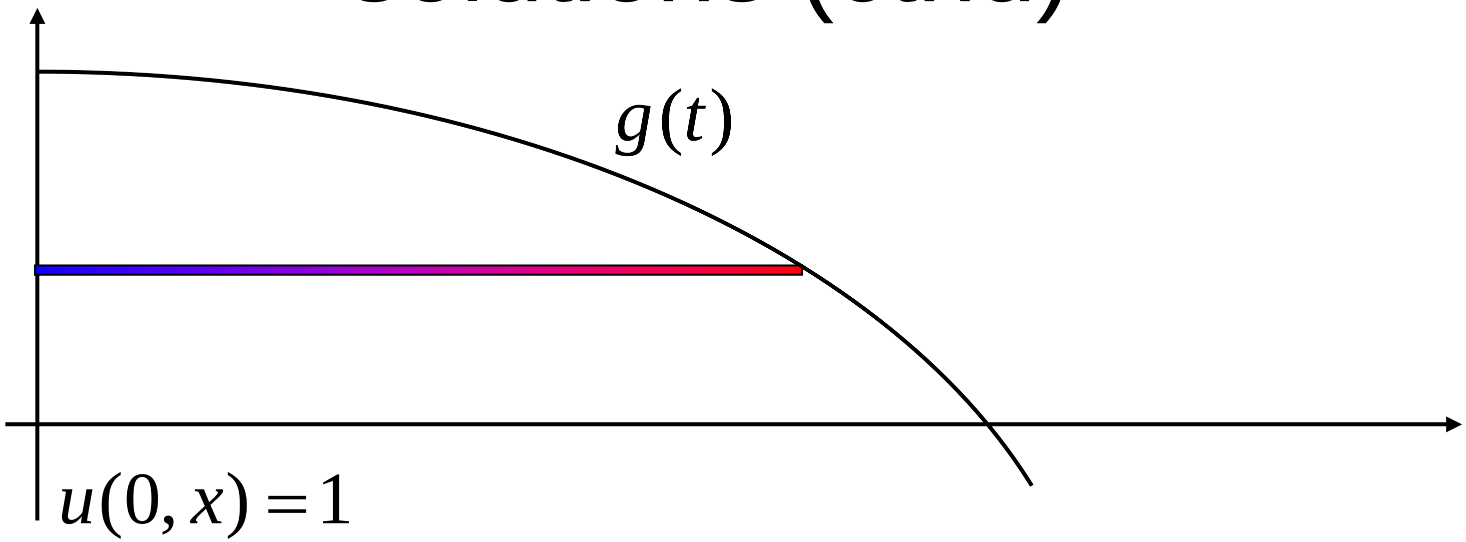
Theorem: Singularities are dense on a Brownian path.

Monotonicity of heat equation solutions



Theorem: If $t \to g(t)$ is decreasing and $x \to u(0,x)$ is increasing then for any $t > 0$, the function $x \to u(t,x)$ is increasing.

Monotonicity of heat equation solutions (ctnd)



Theorem: If $t \to g(t)$ is decreasing and concave and $u(0,x)=1$ then for any x, the function $t \to u(t,x)$ is increasing.

Monotonicity- probabilistic proof

$$u(s,x) = E^{0,x}\left[\exp\left(-\int_0^s 2g'(t)\,dL_t^Y\right)u(0,Y_s)\right]$$