Application of Optimal Transport to Evolutionary PDEs

5 -Applications

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Outline

1. Thin film equation as the gradient flow of the Dirichlet functional
   - in collaboration with U. Gianazza, G. Toscani, D. Matthes, R. McCann
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Starting point: a family of 4th order equations in $\mathbb{R}^d$

We look for non-negative solutions to the nonlinear 4th order evolution PDEs

$$\partial_t u + \text{div} \left( u \ D(u^{\alpha-1}\Delta u^\alpha) \right) = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^d, \quad \alpha \in [1/2, 1],$$

with the initial condition

$$0 \leq u(0, \cdot) = u_0 \in L^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} |x|^2 u_0 \, dx < +\infty.$$
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$\alpha = 1$: thin film

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$$\partial_t u + \text{div} \left( u \ D(\nabla u) \right) = 0$$

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$$\partial_t u + \text{div} \left( u \frac{\Delta \sqrt{u}}{\sqrt{u}} \right) = 0$$

Here we focus on the thin film case $\alpha = 1$ with mobility/diffusion coefficient $u$. The more general equation

$$\partial_t u + \text{div}(m(u) \, D(\Delta u)) = 0,$$

where, e.g. $m(u) = u^m$

has been studied (mainly in dimension $d = 1, 2, 3$) by many authors:

[Bernis-Friedman ’90, Bertsch-Dal Passo-Garcke-Grüen ’98–’04; review: Becker-Grün ’05.; asymptotic behaviour: Carrillo-Toscani ’02, Carlen-Ulusoy ’07]
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Structure of the equation

In the **thin film case**

\[ \partial_t u + \text{div} \left( u D(\Delta u) \right) = 0 \]
Structure of the equation

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$$\partial_t u + \text{div} \left( u \, D(\Delta u) \right) = 0$$

Continuity equation + nonlinear condition

$$\partial_t u + \text{div} \left( u \, v \right) = 0, \quad v = -D \left( \frac{\delta \Phi}{\delta u} \right)$$

where

$$\frac{\delta \Phi}{\delta u} = -\Delta u$$
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The generating functional is

$$\Phi(u) := \frac{1}{2} \int_{\mathbb{R}^d} |Du|^2 \, dx$$
The “Wasserstein gradient” of the Dirichlet functional

Standard technique: choose a vector field $\xi \in C_\infty_c(\mathbb{R}^d; \mathbb{R}^d)$ and the flow $X$

$$\frac{d}{dt} X_t(x) = \xi(X_t(x)), \quad X_0(x) = x; \quad M_\varepsilon := (X_\varepsilon)_\# M; \quad \sim \frac{d}{d\varepsilon} \Phi(M_\varepsilon)_{|\varepsilon=0}.$$ 

Wasserstein gradient $g = -v$:

$$\int_{\mathbb{R}^d} \langle g, \xi \rangle \, dM = \frac{d}{d\varepsilon} \Phi(M_\varepsilon)_{|\varepsilon=0}. $$

As usual $M \leftrightarrow u$, $M_\varepsilon \leftrightarrow u_\varepsilon$. In view of the continuity equation, we choose directly $\xi = \nabla \zeta:$
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\frac{d}{d\varepsilon} \Phi(M_\varepsilon)_{\varepsilon=0} = \frac{1}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta u^2 - 2D^2 \zeta D u \cdot D u - \Delta \zeta |D u|^2 \, dx
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$$\left. \frac{d}{d\varepsilon} \Phi(M_\varepsilon) \right|_{\varepsilon=0^+} = \frac{1}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta \, u^2 - 2D^2 \zeta \nabla D u \cdot D u - \Delta \zeta \, |D u|^2 \, dx$$

Equation for the velocity: $v = -g$,

$$\int_{\mathbb{R}^d} \text{div}(u v) \zeta \, dx = -\int_{\mathbb{R}^d} \langle v, \nabla \zeta \rangle u \, dx = \frac{1}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta \, u^2 - 2D^2 \zeta \nabla D u \cdot D u - \Delta \zeta \, |D u|^2 \, dx$$
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\frac{d}{d\varepsilon} \Phi(M_\varepsilon) \big|_{\varepsilon=0} = \frac{1}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta \left( \frac{u^2}{2} - 2D^2 \zeta D u \cdot D u - \Delta \zeta |D u|^2 \right) \, dx
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\]

It corresponds to the weak formulation of the thin film equation

\[
\partial_t u + \frac{1}{2} \Delta^2 (u^2) - \partial_{x_i} x_j (\partial_{x_i} u \partial_{x_j} u) - \frac{1}{2} \Delta |D u|^2 = 0 \quad \Leftrightarrow \quad \partial_t u + \text{div} (u D \Delta u) = 0
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Discrete equation: \( M^n_\tau \leftrightarrow U^n_\tau \)

\[
\int_{\mathbb{R}^d} \zeta (U^n_\tau - U^{n-1}_\tau) \, dx + \frac{\tau}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta (U^n_\tau)^2 - 2D^2 \zeta D U^n_\tau \cdot D U^n_\tau - \Delta \zeta |D U^n_\tau|^2 \, dx = o(\tau)
\]
Main problem

Discrete equation:

\[ \int_{\mathbb{R}^d} \zeta \left( U^n_T - U^{n-1}_T \right) \, dx + \frac{\tau}{2} \int_{\mathbb{R}^d} \Delta^2 \zeta (U^n_T)^2 - 2D^2 \zeta \, \text{D} U^n_T \cdot \text{D} U^n_T - \Delta \zeta \, |\text{D} U^n_T|^2 \, dx = o(\tau) \]

Strong compactness in $W^{1,2}$ in order to pass to the limit in the quadratic term

\[ \int_{\mathbb{R}^d} 2D^2 \zeta \, \text{D} U^n_T \cdot \text{D} U^n_T \, dx \]
First variation along auxiliary flows

**MAIN IDEA:** take the first variation of the minimum problem

\[ U^n \in \arg\min_V \frac{W^2(V, U^{n-1})}{2\tau} + \Phi(V) \]

along the (Wasserstein) gradient flow \( S^\Psi \) generated by other "good" auxiliary functionals \( \Psi \).
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**HEURISTICS:** in an euclidean space $S^\Phi, S^\Psi$ corresponds to

$$u_t := S_t^\Phi(u_0) \text{ solves } \frac{d}{dt} u = -\nabla \Phi(u), \quad w_t := S_t^\Psi(w_0) \text{ solves } \frac{d}{dt} w = -\nabla \Psi(w)$$
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$$\mathbf{U} \in \arg\min_{\mathbf{V}} \left\{ \frac{W^2(\mathbf{V}, \mathbf{U}_{n-1})}{2\tau} + \Phi(\mathbf{V}) \right\}$$

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If $u_0 = w_0$ then we have the "commutation" identity

$$\frac{d}{d\varepsilon} \Phi(w_{\varepsilon})\big|_{\varepsilon=0^+} = \frac{d}{d\varepsilon} \Psi(u_{\varepsilon})\big|_{\varepsilon=0^+}$$
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\]

**RECIPE:** if the derivative of the \textit{(main) functional} \( \Phi \) along the \textit{(auxiliary) flow} \( S^\Psi \) is negative

then \( \Psi \) is a \textbf{Lyapunov functional} for the \textbf{main flow} \( S^\Phi \)
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\[ U^n \tau \in \arg \min_V W^2(V, U^{n-1}) + \Phi(V) \]

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**RECIPE:** if the derivative of the (main) functional \( \Phi \) along the (auxiliary) flow \( S^\Psi \) is negative

then \( \Psi \) is a Lyapunov functional for the main flow \( S^\Phi \)

Look for good flows \( S^\Psi \) having \( \Phi \) as Lyapunov functional
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\[ U^n_\tau \in \arg\min_V \frac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V) \]

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If \( u_0 = w_0 \) then we have the \textit{“commutation”} identity

\[ \frac{d}{d\varepsilon} \Phi(w_\varepsilon) \bigg|_{\varepsilon=0^+} = \frac{d}{d\varepsilon} \Psi(u_\varepsilon) \bigg|_{\varepsilon=0^+} \quad \left( = -\langle \nabla \Phi(w_0), \nabla \Psi(u_0) \rangle \right) \]

**RECIPE:** if the derivative of the \textit{(main) functional} \( \Phi \) along the \textit{(auxiliary) flow} \( S^\Psi \) is negative (up to lower order terms)

then \( \Psi \) is a \textbf{Lyapunov functional} for the \textbf{main flow} \( S^\Phi \) (up to lower order terms).

Look for good flows \( S^\Psi \) having \( \Phi \) as Lyapunov functional.
A Lyapunov-type estimate at the discrete level in the Wasserstein space

Suppose that \( \Psi \) generates a good flow \( w_t = S_t^\Psi(w) \) satisfying the EVI:

\[
\frac{d}{dt} \frac{1}{2} W^2(S_t^\Psi(w), z) \leq \Psi(z) - \Psi(S_t^\Psi(w))
\]  

(EVI)
A Lyapunov-type estimate at the discrete level in the Wasserstein space

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$$\frac{d}{dt} \frac{1}{2} W^2(S_t^{\Psi}(w), z) \leq \Psi(z) - \Psi(S_t^{\Psi}(w)) \quad \text{(EVI)}$$

We call $\mathcal{D}$ the dissipation of $\Phi$ along $S^{\Psi}$

$$\mathcal{D}(w) := \left[ - \frac{d}{d\varepsilon} \Phi(S_{\varepsilon}^{\Psi}(w)) \right]_{\varepsilon=0^+} = \limsup_{\varepsilon \downarrow 0} \frac{\Phi(w) - \Phi(S_{\varepsilon}^{\Psi}(w))}{\varepsilon}$$
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$$\mathcal{D}(w) := \limsup_{\varepsilon \downarrow 0} \frac{\Phi(w) - \Phi(S^\Psi_\varepsilon(w))}{\varepsilon}$$

Theorem (Discrete flow-interchange estimate)

If $U^n_\tau$ is a minimizer of $V \mapsto \frac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)$ then

$$\Psi(U^n_\tau) + \tau \mathcal{D}(U^n_\tau) \leq \Psi(U^{n-1}_\tau)$$
A Lyapunov-type estimate at the discrete level in the Wasserstein space

Suppose that $\Psi$ generates a good flow $w_t = S^\Psi_t (w)$ satisfying the EVI:

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We call $\mathcal{D}$ the dissipation of $\Phi$ along $S^\Psi$

$$\mathcal{D} (w) := - \limsup_{\varepsilon \downarrow 0} \frac{\Phi (w) - \Phi (S^\Psi_{\varepsilon} (w))}{\varepsilon}$$

**Theorem (Discrete flow-interchange estimate)**

If $U^n_\tau$ is a minimizer of $V \mapsto \frac{W^2 (V, U^{n-1}_\tau)}{2\tau} + \Phi (V)$ then

$$\Psi (U^n_\tau) + \tau \mathcal{D} (U^n_\tau) \leq \Psi (U^{n-1}_\tau)$$

**PROOF:**

$$0 \leq \frac{d}{d\varepsilon} \left. W^2 (S^\Psi_{\varepsilon} (U^n_\tau), U^{n-1}_\tau) \right|_{\varepsilon = 0^+} + \Phi (S^\Psi_{\varepsilon} (U^n_\tau)) \quad \text{(by the minimality of } U^n_\tau)$$
A Lyapunov-type estimate at the discrete level in the Wasserstein space

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\[
\frac{d}{dt} \frac{1}{2} W^2(w_t, z) \leq \Psi(z) - \Psi(S_t^\Psi(w))
\]

(EVI)

We call \( D \) the dissipation of \( \Phi \) along \( S^\Psi \)

\[
D(w) := \limsup_{\varepsilon \downarrow 0} \frac{\Phi(w) - \Phi(S^\Psi_\varepsilon(w))}{\varepsilon}
\]

Theorem (Discrete flow-interchange estimate)

If \( U^*_n \tau \) is a minimizer of \( V \mapsto \frac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V) \) then

\[
\Psi(U^*_n) + \tau D(U^*_n) \leq \Psi(U^{n-1}_\tau)
\]

PROOF:

\[
0 \leq \frac{d}{d\varepsilon} \left[ \frac{W^2(S^\Psi_\varepsilon(U^n_\tau), U^{n-1}_\tau)}{2\tau} + \Phi(S^\Psi_\varepsilon(U^n_\tau)) \right]_{\varepsilon=0^+} \tag{by the minimality of \( U^n_\tau \)}
\]

\[
\leq \frac{\Psi(U^{n-1}_\tau) - \Psi(U^n_\tau)}{\tau} - D(U^n_\tau) \tag{by the EVI, with \( z = U^{n-1}_\tau, \ w = U^n_\tau \)}
\]
A Lyapunov-type estimate at the discrete level in the Wasserstein space

Suppose that $\Psi$ generates a good flow $w_t = S^\Psi_t(w)$ satisfying the EVI:

$$\frac{d}{dt} \frac{1}{2} W^2(S^\Psi_t(w), z) \leq \Psi(z) - \Psi(S^\Psi_t(w)) - \frac{\kappa}{2} W^2(w_t, z) \quad \text{(EVI)}$$

We call $D$ the dissipation of $\Phi$ along $S^\Psi$

$$D(w) := \left[ - \frac{d}{d\varepsilon} \Phi(S^\Psi_\varepsilon(w)) \right]_{\varepsilon=0^+} = \limsup_{\varepsilon \downarrow 0} \frac{\Phi(w) - \Phi(S^\Psi_\varepsilon(w))}{\varepsilon}$$

**Theorem (Discrete flow-interchange estimate)**

If $U^n_\tau$ is a minimizer of $V \mapsto \frac{W^2(V, U^{n-1}_\tau)}{2\tau} + \Phi(V)$ then

$$\Psi(U^n_\tau) + \tau D(U^n_\tau) \leq \Psi(U^{n-1}_\tau) - \frac{\kappa}{2} W^2(U^n_\tau, U^{n-1}_\tau).$$

**PROOF:**

$$0 \leq \frac{d}{d\varepsilon} \frac{W^2(S^\Psi_\varepsilon(U^n_\tau), U^{n-1}_\tau)}{2\tau} + \Phi(S^\Psi_\varepsilon(U^n_\tau))) \bigg|_{\varepsilon=0^+} \quad \text{(by the minimality of } U^n_\tau)$$

$$\leq \frac{\Psi(U^{n-1}_\tau) - \Psi(U^n_\tau)}{\tau} - D(U^n_\tau) \quad \text{(by the EVI, with } z = U^{n-1}_\tau, w = U^n_\tau)$$
Auxiliary flows for the thin film equation (II)

\[ \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^d} |Du|^2 \, dx \] decays on the heat flow

\[ \partial_t w - \Delta w = 0 \]

with

\[ D(w) = -\frac{d}{d\varepsilon} \Phi(S^\varepsilon(w)) \bigg|_{\varepsilon=0} = \int_{\mathbb{R}^d} |\Delta w|^2 \, dx = \int_{\mathbb{R}^d} |D^2 w|^2 \, dx \]
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\[ \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^d} |Du|^2 \, dx \] decays on the heat flow

\[ \partial_t w - \Delta w = 0 \quad \iff \quad \partial_t w - \text{div} \left( w \, D \log w \right) = 0 \]

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\[ D(w) = -\frac{d}{d\varepsilon} \Phi(S^\varepsilon(w)) \bigg|_{\varepsilon=0} = \int_{\mathbb{R}^d} |\Delta w|^2 \, dx = \int_{\mathbb{R}^d} |D^2 w|^2 \, dx \]

The heat equation is the Wasserstein gradient flow of the relative entropy functional \( \mathcal{H}(w) := \int_{\mathbb{R}^d} w \log w \, dx \).
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The discrete flow-interchange estimates shows that \( \mathcal{H} \) is a Lyapunov functional and satisfies

\[ \mathcal{H}(U^n_\tau) + \tau \int_{\mathbb{R}^d} |D^2 U^n_\tau|^2 \, dx \leq \mathcal{H}(U^{n-1}_\tau). \]
Auxiliary flows for the thin film equation (II)

\[ \Phi(u) = \frac{1}{2} \int_{\mathbb{R}^d} |Du|^2 \, dx \] decays on the heat flow

\[ \partial_t w - \Delta w = 0 \quad \Leftrightarrow \quad \partial_t w - \text{div} \left( w \, D \log w \right) = 0 \]

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In term of \( U_\tau \) it corresponds to

\[ \int_0^T \int_{\mathbb{R}^d} |D^2 U_\tau|^2 \, dx \, dt \leq C. \]
Main result

Assume that the non-negative initial condition $u_0 \in L^1(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} |x|^2 u_0(x) \, dx < +\infty, \quad \mathcal{H}(u_0) = \int_{\mathbb{R}^d} u_0 \log u_0 \, dx < +\infty.$$
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**Theorem**

There exists an infinitesimal subsequence of time steps $\tau_k \downarrow 0$ such that

$$U_{\tau_k} \to u \text{ pointwise in } L^1(\mathbb{R}^d) \text{ and in } L^2(0,T;W^{1,2}(\mathbb{R}^d)) \text{ as } k \uparrow \infty.$$
Main result

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**Theorem**

There exists an infinitesimal subsequence of time steps $\tau_k \downarrow 0$ such that

$$
U_{\tau_k} \rightarrow u \quad \text{pointwise in } L^1(\mathbb{R}^d) \text{ and in } L^2(0,T; W^{1,2}(\mathbb{R}^d)) \quad \text{as } k \uparrow \infty
$$

$u \in C^0([0, +\infty); L^1(\mathbb{R}^d)) \cap L^2_{\text{loc}}([0, +\infty); W^{2,2}(\mathbb{R}^d))$ is a non-negative global solution of the weak formulation of thin film equation

$$
\partial_t u + \frac{1}{2} \Delta^2(u^2) - \partial_{x_i x_j}(\partial_{x_i} u \partial_{x_j} u) - \frac{1}{2} \Delta |D u|^2 = 0
$$
Outline

1. Thin film equation as the gradient flow of the Dirichlet functional
   - in collaboration with U. Gianazza, G. Toscani, D. Matthes, R. McCann

2. The $L^2$-gradient flow of the simplest polyconvex functional
   - in collaboration with L. Ambrosio, S. Lisini

3. The sticky particle system
   - in collaboration with L. Natile
Polyconvex functionals

\[ \mathcal{F}(u) = \int_{\Omega} F(Du) \, dx \]

where

\[ F(A) = \Phi(A, M_2(A), \cdots, M_{d-1}(A), \det A), \quad \text{and } \Phi \text{ is convex}; \]

\[ M_2(A), \cdots, M_{d-1}(A), M_d(A) = \det A \quad \text{are the minors of } A. \]
Polyconvex functionals

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If \( \Phi \) is superlinear then the functional \( \mathcal{F} \) is lower semicontinuous in \( L^2(\Omega; \mathbb{R}^d) \) [J. Ball].

**Well posedness of the variational problems**

\[ \min_U \frac{1}{2\tau} \int_{\Omega} |U - U^{n-1}_\tau|^2 \, dx + \mathcal{F}(U) \]
Polyconvex functionals

\[ \mathcal{F}(u) = \int_{\Omega} F(Du) \, dx \]

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If \( \Phi \) is superlinear then the functional \( \mathcal{F} \) is lower semicontinuous in \( L^2(\Omega; \mathbb{R}^d) \) [J. Ball].

Well posedness of the variational problems

\[ \min_U \frac{1}{2\tau} \int_{\Omega} |U - U_{n-1}^\tau|^2 \, dx + \mathcal{F}(U) \]

Nevertheless, no general results are known for gradient flows of polyconvex functionals and for their variational approximation.
The “simplest” polyconvex functional

\[ F(A) := \Phi(\det A), \quad \mathcal{F}(u) := \int_{\Omega} \Phi(\det D u(x)) \, dx \]

under the additional constraint that

- \( u \) is a \textbf{diffeomorphism} between \( \Omega \) and \( u(\Omega) \), \( \det D u(x) > 0 \),
- \( u(\Omega) \) is contained in a target open set \( \mathcal{U} \).
The “simplest” polyconvex functional

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- \( u(\Omega) \) is contained in a target open set \( U \).
The “simplest” polyconvex functional

\[ \mathcal{F}(u) := \int_{\Omega} \Phi(\det D\mathbf{u}(x)) \, dx \]

under the additional constraint that
\[ \mathbf{u} \] is a diffeomorphism between \( \Omega \) and \( \mathbf{u}(\Omega) \), \( \det D\mathbf{u}(x) > 0 \),
\( \mathbf{u}(\Omega) \) is contained in a target open set \( \mathcal{U} \).

Difficulties (besides polyconvexity):

- lack of coercivity (\( \mathcal{F} \) controls only \( \det D\mathbf{u} \))
- lack of lower semicontinuity in \( L^2(\Omega; \mathcal{U}) \).
The form of the PDE

\[ F(A) = \Phi(\det A), \quad DF(A) = (\text{cof } A)^T \Phi'(\det A), \]

since

\[ \frac{\partial \det A}{\partial A^i_{\alpha}} = (\text{cof } A)_{\alpha}^i \quad \text{where} \quad \sum_{\alpha} A^i_{\alpha} (\text{cof } A)_{\alpha}^j = \det A \delta_{i,j} \quad \forall i, j. \]

\[ \delta \mathcal{F}(u, \xi) = \int_{\Omega} \Phi'(\det Du) \text{cof } Du \cdot D\xi \, dx \]
The form of the PDE

\[ F(A) = \Phi(\det A), \quad DF(A) = (\text{cof } A)^T \Phi'(\det A), \]

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\[ \delta \mathcal{F}(u, \xi) = \int_\Omega \Phi'(\det Du) \text{cof } Du \cdot D\xi \, dx \]

Gradient flow

\[ \partial_t u - \text{div} \left( \Phi'(\det Du) \text{cof } Du \right) = 0 \]
A differential approach [Evans, Gangbo, Savin]

Make the transformation

\[ y = u(x), \quad \rho(y) := \frac{1}{\det Du(x)} = \frac{1}{\det Du \circ u^{-1}(y)} = u#(L^d_{|\Omega}) \]
A differential approach [Evans, Gangbo, Savin]

Make the transformation

\[ y = u_t(x), \quad \rho_t(y) := \frac{1}{\det D u_t(x)} = \frac{1}{\det D u_t} \circ u_t^{-1}(y) = u_\#(L^d_{|\Omega}) \]
A differential approach [Evans, Gangbo, Savin]

Make the transformation

\[ y = u_t(x), \quad \rho_t(y) := \frac{1}{\det D u_t(x)} = \frac{1}{\det D u_t} \circ u^{-1}(y) = u_#(\mathcal{L}^d|\Omega) \]

\( \rho \) solves the nonlinear diffusion PDE

\[ \begin{aligned}
    \partial_t \rho - \text{div}(\rho D\phi'(\rho)) &= 0 \quad \text{in} \; \mathcal{U} \times (0, +\infty), \\
    \rho(x, 0) &= \rho_0(x) \quad \text{in} \; \mathcal{U}; \\
    \partial_n \rho &= 0 \quad \text{on} \; \partial \mathcal{U} \times (0, +\infty)
\end{aligned} \]

where \( \phi(\rho) := \rho \Phi(1/\rho) \)
Recovering \( u \)

Step 1: put

\[ \phi(\rho) := \rho \Phi(1/\rho) \]
Recovering $u$

Step 1: put

$$\phi(\rho) := \rho \Phi(1/\rho)$$

$$\begin{cases} 
\partial_t \rho - \text{div}(\rho \nabla \phi'(\rho)) = 0 \quad \text{in } \mathcal{U}, \\
\rho(\cdot, 0) = \rho_0, \quad \partial_n \rho = 0 \quad \text{on } \partial \mathcal{U} 
\end{cases}$$

Step 2: solve the PDE

Step 3: build the vector field

$$V(t, y) = -\nabla \phi'\left(\rho(t, y)\right)$$

Step 4: compute the flow

$$\dot{Y}(t, y) = V(t, Y(t, y))$$

$Y(0, y) = y$

Step 5: $u(t, x) = Y(t, u_0(x))$

Main problem: Prove that the $L^2$-Minimizing Movement scheme converges to this solution
Recovering $u$

Step 1: put

\[ \phi(\rho) := \rho \Phi(1/\rho) \]

\[ \begin{aligned} \partial_t \rho - \text{div}(\rho \nabla \phi'(\rho)) &= 0 \quad \text{in } \mathcal{U}, \\ \rho(\cdot, 0) &= \rho_0, \quad \partial_n \rho = 0 \quad \text{on } \partial \mathcal{U} \end{aligned} \]

Step 2: solve the PDE

Step 3: build the vector field

\[ V(t, y) = -\nabla \phi'(\rho_t(y)) \]

Step 4: compute the flow

\[ Y(t, y) = \phi'(\rho_t(y)) \]

Step 5: $u(t, x) = Y(t, u_0(x))$
Recovering $u$

Step 1: put

$$\phi(\rho) := \rho \Phi(1/\rho)$$

$$\frac{\partial \rho}{\partial t} - \text{div}(\rho \nabla \phi'(\rho)) = 0 \quad \text{in } \mathcal{U},$$

$$\rho(\cdot, 0) = \rho_0, \quad \partial_n \rho = 0 \quad \text{on } \partial \mathcal{U}$$

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$$V(t, y) = -\nabla \phi'(\rho_t(y))$$

Step 2: solve the PDE

$$\begin{cases} \dot{Y}(t, y) = V(t, Y(t, y)) \\ Y(0, y) = y \end{cases}$$

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Step 4: Compute the flow

Step 5
Recovering $u$

Step 1: put

$$\phi(\rho) := \rho \Phi(1/\rho)$$

$$\begin{cases}
\partial_t \rho - \text{div}(\rho \nabla \phi'(\rho)) = 0 & \text{in } \mathcal{U}, \\
\rho(\cdot, 0) = \rho_0, & \partial_n \rho = 0 & \text{on } \partial \mathcal{U}
\end{cases}$$

Step 2: solve the PDE

$$V(t, y) = -\nabla \phi'(\rho_t(y))$$

$$\begin{cases}
\dot{Y}(t, y) = V(t, Y(t, y)) \\
Y(0, y) = y
\end{cases}$$

$$u(t, x) = Y(t, u_0(x))$$

Step 3: build the vector field

Step 4: Compute the flow

Step 5

Main problem:

Prove that the $L^2$-Minimizing Movement scheme converges to this solution
Transporting the functional $\mathcal{F}$

$$
\mathcal{F}(u) = \int_\Omega \Phi(\det Du(x)) \, dx = \int_\mathcal{U} \Phi(\det D(u^{-1}(y))) \rho(y) \, dy
$$

$$
= \int_\mathcal{U} \Phi\left(\frac{1}{\rho(y)}\right) \rho(y) \, dy = \int_\mathcal{U} \phi(\rho(y)) \, dy = \mathcal{G}(\rho)
$$
Transporting the functional $\mathcal{F}$

$$
\mathcal{F}(u) = \int_{\Omega} \Phi(\det D\mathbf{u}(x)) \, dx = \int_{\mathcal{U}} \Phi(\det D\mathbf{u}(\mathbf{u}^{-1}(y))) \rho(y) \, dy
$$

$$
= \int_{\mathcal{U}} \Phi \left( \frac{1}{\rho(y)} \right) \rho(y) \, dy = \int_{\mathcal{U}} \phi(\rho(y)) \, dy = \mathcal{G}(\rho)
$$

$\Phi(s) = 1/s$

$\phi(\rho) = \rho^2$

$\partial_t \rho - \Delta \rho^2 = 0$  Porous media equation
Transporting the functional $\mathcal{F}$

$$\mathcal{F}(u) = \int_{\Omega} \Phi(\det Du(x)) \, dx = \int_{\mathcal{U}} \Phi(\det Du(u^{-1}(y))) \rho(y) \, dy$$

$$= \int_{\mathcal{U}} \Phi \left( \frac{1}{\rho(y)} \right) \rho(y) \, dy = \int_{\mathcal{U}} \phi(\rho(y)) \, dy = \mathcal{G}(\rho)$$

$\Phi(s) = -\log s$

$\phi(\rho) = \rho \log \rho$

$\partial \rho - \Delta \rho = 0$  Heat equation
Transporting the functional $\mathcal{F}$

$$\mathcal{F}(u) = \int_\Omega \Phi(\det Du(x)) \, dx = \int_\mathcal{U} \Phi(\det Du(u^{-1}(y))) \rho(y) \, dy$$

$$= \int_\mathcal{U} \Phi\left(\frac{1}{\rho(y)}\right) \rho(y) \, dy = \int_\mathcal{U} \phi(\rho(y)) \, dy = \mathcal{G}(\rho)$$

$\Phi(s) = s \log s$

$\phi(\rho) = -\log \rho$

$\partial_t \rho - \Delta \log \rho = 0$
Transporting the functional $\mathcal{F}$

\[
\mathcal{F}(u) = \int_{\Omega} \Phi(\det Du(x)) \, dx = \int_{\mathcal{U}} \Phi(\det Du(u^{-1}(y))) \rho(y) \, dy
\]

\[
= \int_{\mathcal{U}} \Phi\left(\frac{1}{\rho(y)}\right) \rho(y) \, dy = \int_{\mathcal{U}} \phi(\rho(y)) \, dy = \mathcal{G}(\rho)
\]

$\Phi(s) = 1/s + s^2$

$\phi(\rho) = 1/\rho + \rho^2$
Transporting the variational problem

\[ U \rightsquigarrow R = \frac{1}{\det D U} \circ U^{-1} , \quad \begin{cases} \mathcal{F}(U) = \int_{\Omega} \Phi(\det D U) \, dx = \\ \mathcal{G}(R) = \int_{\Omega} \phi(R) \, dy \end{cases} \]
Transporting the variational problem

\( U \mapsto R = \frac{1}{\det D\mathbf{U}} \circ \mathbf{U}^{-1}, \)

\[
\begin{aligned}
\mathcal{F}(\mathbf{U}) &= \int_{\Omega} \Phi(\det D\mathbf{U}) \, dx = \\
\mathcal{G}(R) &= \int_{\Omega} \phi(R) \, dy
\end{aligned}
\]

Given \( U_{\tau}^{n-1} \mapsto R_{\tau}^{n-1} \) find \( U^n \in \text{Diff}(\Omega; \mathcal{U}) \) solution of

\[
\min_{\mathbf{U}} \mathcal{F}(\mathbf{U}) + \frac{1}{2\tau} \| \mathbf{U} - U_{\tau}^{n-1} \|^2_{L^2(\Omega; \mathbb{R}^d)}
\]
Transporting the variational problem

\[
U \rightsquigarrow R = \frac{1}{\det DU} \circ U^{-1}, \quad \begin{cases}
F(U) &= \int_{\Omega} \Phi(\det DU) \, dx = \\
G(R) &= \int_{\Omega} \phi(R) \, dy
\end{cases}
\]

Given \( U^{n-1}_\tau \rightsquigarrow R^{n-1}_\tau \) find \( U^n \in \text{Diff}(\Omega; \mathcal{U}) \) solution of

\[
\min_{U} \mathcal{F}(U) + \frac{1}{2\tau} \left\| U - U^{n-1}_\tau \right\|_{L^2(\Omega; \mathbb{R}^d)}^2
\]

\[
\min_{R} \left( \min_{U} \mathcal{F}(U) + \frac{1}{2\tau} \left\| U - U^{n-1}_\tau \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right)
\]
Transporting the variational problem

\[ U \sim \Rightarrow R = \frac{1}{\det D(U)} \circ U^{-1}, \quad \begin{cases} \mathcal{F}(U) = \int_{\Omega} \Phi(\det D(U)) \, dx = \\ \mathcal{G}(R) = \int_{\Omega} \phi(R) \, dy \end{cases} \]

Given \( U^{n-1}_\tau \sim \Rightarrow R^{n-1}_\tau \) find \( U^n \in \text{Diff}(\Omega; \mathcal{U}) \) solution of

\[
\min_U \mathcal{F}(U) + \frac{1}{2\tau} \left\| U - U^{n-1}_\tau \right\|_{L^2(\Omega; \mathbb{R}^d)}^2
\]

\[
\min_R \left( \min_U \mathcal{F}(U) + \frac{1}{2\tau} \left\| U - U^{n-1}_\tau \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right)
\]

\[
\min_R \left( \mathcal{G}(R) + \min_U \frac{1}{2\tau} \left\| U - U^{n-1}_\tau \right\|_{L^2(\Omega; \mathbb{R}^d)}^2 \right)
\]
Transporting the variational problem

\[
U \leadsto R = \frac{1}{\det DU} \circ U^{-1}, \quad \left\{ \begin{array}{l}
\mathcal{F}(U) = \int_{\Omega} \Phi(\det DU) \, dx = \\
\mathcal{G}(R) = \int_{\Omega} \phi(R) \, dy
\end{array} \right.
\]

Given \( U^{n-1}_\tau \leadsto R^{n-1}_\tau \) find \( U^n \in \text{Diff}(\Omega; \mathcal{U}) \) solution of

\[
\min_U \mathcal{F}(U) + \frac{1}{2\tau} \| U - U^{n-1}_\tau \|^2_{L^2(\Omega; \mathbb{R}^d)}
\]

\[
\min_R \left( \min_U \mathcal{F}(U) + \frac{1}{2\tau} \| U - U^{n-1}_\tau \|^2_{L^2(\Omega; \mathbb{R}^d)} \right)
\]

\[
\min_R \left( \mathcal{G}(R) + \min_U \frac{1}{2\tau} \| U - U^{n-1}_\tau \|^2_{L^2(\Omega; \mathbb{R}^d)} \right)
\]

Problem: given a density \( R \) in \( \mathcal{U} \) and \( U^{n-1}_\tau \leadsto R^{n-1}_\tau \) solve

\[
\min_{U \leadsto R} \| U - U^{n-1}_\tau \|^2_{L^2(\Omega; \mathbb{R}^d)}
\]
Optimal transportation

Minimize \( \int_{\Omega} |U - U^{n-1}_\tau|^2 \, dx \) under the constraint \( U \leadsto R \).
Optimal transportation

Minimize $\int_\Omega |U - U_{\tau}^{n-1}|^2 \, dx$ under the constraint $U \sim R$.

Write $U = T \circ U_{\tau}^{n-1}$, $T : \mathcal{U} \to \mathcal{U}$, $T_\#(R_{\tau}^{n-1}) = R$. 
Optimal transportation

Minimize \( \int_{\Omega} |U - U_{\tau}^{n-1}|^2 \, dx \) under the constraint \( U \sim R \).

Write \( U = T \circ U_{\tau}^{n-1}, \ T : \mathcal{U} \to \mathcal{U}, \ T_{\#}(R_{\tau}^{n-1}) = R \)

\[
\int_{\Omega} |U - U_{\tau}^{n-1}|^2 \, dx = \int_{\Omega} |T(U_{\tau}^{n-1}) - U_{\tau}^{n-1}|^2 \, dx \\
= \int_{\mathcal{U}} |T(y) - y|^2 \, R_{\tau}^{n-1}(y) \, dy
\]
A Wasserstein gradient flow

The piecewise constant interpolant $R_τ$ of the discrete solution of the variational algorithm

$$\min_U \mathcal{F}(U) + \frac{1}{2τ} \|U - U_τ^{n-1}\|^2_{L^2(Ω;\mathbb{R}^d)} = \min_R \mathcal{G}(R) + \frac{1}{2τ} W^2(R, R_τ^{n-1})$$
A Wasserstein gradient flow

The piecewise constant interpolant $R_\tau$ of the discrete solution of the variational algorithm

$$
\min_U \mathcal{F}(U) + \frac{1}{2\tau} \| U - U^{n-1}_\tau \|_{L^2(\Omega;\mathbb{R}^d)}^2 = \min_R \mathcal{G}(R) + \frac{1}{2\tau} W^2(R, R^{n-1}_\tau)
$$

converge to the solution of the nonlinear PDE

$$
\begin{cases}
\partial_t \rho + \text{div}(\rho \mathbf{v}) = 0 & \text{in } \mathcal{U} \times (0, +\infty) \quad \text{(continuity equation)} \\
\mathbf{v} = - \nabla \phi'(\rho) & \text{(Nonlinear condition)} \\
\rho(y, 0) = \rho_0(y), \quad \partial_n \rho = 0 & \text{on } \partial\mathcal{U} \times (0, +\infty).
\end{cases}
$$
A Wasserstein gradient flow

The piecewise constant interpolant $R_\tau$ of the discrete solution of the variational algorithm

$$\min_U \mathcal{F}(U) + \frac{1}{2\tau} \|U - U_{\tau}^{n-1}\|_{L^2(\Omega; \mathbb{R}^d)}^2 = \min_R \mathcal{G}(R) + \frac{1}{2\tau} W^2(R, R_{\tau}^{n-1})$$

converge to the solution of the nonlinear PDE

$$\begin{cases}
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\mathbf{v} = -\nabla \phi' (\rho) & \text{(Nonlinear condition)} \\
\rho(y, 0) = \rho_0(y), \quad \partial_n \rho = 0 & \text{on } \partial \mathcal{U} \times (0, +\infty).
\end{cases}$$

Optimal error estimate:

$$\sup_t W^2(R_\tau(t), \rho(t)) \leq \tau \mathcal{G} (\rho_0)$$
Iterated optimal transport maps

\[ U^n = Y^n \circ U^0 \]

\[ \min_R \int_{\mathcal{U}} \phi(R) \, dy + \frac{1}{2\tau} W^2(R, R^{n-1}_\tau) \sim R^n_\tau \]
Iterated optimal transport maps

\[
\begin{align*}
U_n &= Y_n \circ U_0 \\
\min \int_R \phi(R) \, dy + \frac{1}{2\tau} W^2(R, R^{n-1}_\tau) &\Rightarrow R^n_	au \\
R^n_\tau, \ Y^n_\tau &\text{ solve the PDE.}
\end{align*}
\]

\[
\frac{Y^n_\tau - Y^{n-1}_\tau}{\tau} = V^n_\tau(Y^n_\tau), \quad V^n_\tau = -\nabla \phi'(R^n_\tau)
\]
Iterated optimal transport maps

\[
\begin{align*}
\min_{R} \int_{\mathcal{U}} \phi(R) \, dy + \frac{1}{2\tau} W^2(R, R^{n-1}_\tau) & \quad \sim \quad R^n_	au \\
U^0 \quad \Omega \quad U^n = Y^n \circ U^0
\end{align*}
\]

\(R^n_	au, Y^n_	au\) solve the PDE. How to pass to the limit?

\[
\frac{Y^n_	au - Y^{n-1}_\tau}{\tau} = V^n_	au (Y^n_	au), \quad V^n_	au = -\nabla \phi'(R^n_{\tau})
\]
Convergence of the iterated maps

Main problem:

\[
\frac{d}{dt} Y_\tau(t, y) = V_\tau(t, Y_\tau(t, y)), \quad V_\tau(t, y) = -\nabla \phi'(R_\tau(t, y))
\]

as \( \tau \to 0 \)

\[
\frac{d}{dt} Y(t, y) = V(t, Y(t, y)), \quad V(t, y) = -\nabla \phi'(\rho(t, y))
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Convergence of the iterated maps

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as \( \tau \to 0 \) \( \downarrow \) \( \downarrow \) \( \downarrow \) \( \downarrow \) \( \downarrow \) ?

\[
\frac{d}{dt} Y(t, y) = V(t, Y(t, y)), \quad V(t, y) = -\nabla \phi'(\rho(t, y))
\]

Difficulties:

- No regularity estimate for \( V_\tau \)
Convergence of the iterated maps

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as \( \tau \to 0 \) \( \downarrow \) \( \downarrow \) \( \downarrow \) \( \downarrow \) \( ? \)

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Difficulties:

- No regularity estimate for \( V_\tau \)
- No lower density bound for \( R_\tau \).
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as \( \tau \to 0 \) ↓ ↓ ↓ ↓ ?

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Difficulties:

- No regularity estimate for \( V_\tau \)
- No lower density bound for \( R_\tau \).
- Only weak convergence of \( V_\tau R_\tau \) to \( V\rho \) (DiPerna-Lions, Ambrosio-theory cannot be applied)
Convergence of the iterated maps

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\[ \frac{d}{dt} Y(t, y) = V(t, Y(t, y)), \quad V(t, y) = -\nabla \phi'(\rho(t, y)) \]

Difficulties:

- No regularity estimate for \( V_\tau \)
- No lower density bound for \( R_\tau \).
- Only weak convergence of \( V_\tau R_\tau \) to \( V \rho \) (DiPerna-Lions, Ambrosio-theory cannot be applied)
- convergence of the energy:

\[
\lim_{\tau \downarrow 0} \int_0^T \int_{\mathcal{U}} |V_\tau(t, y)|^2 \, R_\tau(t, y) \, dy \, dt = \int_0^T \int_{\mathcal{U}} |V(t, y)|^2 \, \rho(t, y) \, dy \, dt
\]
A first result: convergence of flows

Suppose that $V_\tau, Y_\tau, \mu_\tau = \rho_\tau \mathcal{L}^d$ are given with

$$\frac{d}{dt} Y_\tau(t, y) = V_\tau(t, Y_\tau(t, y)), \quad \mu_{\tau, t} = (Y_\tau(t, \cdot)) \# \mu_{\tau, 0}$$
A first result: convergence of flows

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\frac{d}{dt} Y_\tau(t, y) = V_\tau(t, Y_\tau(t, y)), \quad \mu_{\tau,t} = (Y_\tau(t, \cdot)) \# \mu_{\tau,0}
\]

- \( \mu_{\tau,t} \rightharpoonup \mu_t \) narrowly,

\[
\lim_{\tau \downarrow 0} \int_0^T \int_U \left( Y_\tau(t, y) - \frac{d}{dt} Y_\tau(t, y) \right)^2 \, d\mu_{\tau,t}(y) = 0.
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A first result: convergence of flows

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- $\mu_{\tau,t} \rightharpoonup \mu_t$ narrowly,
- $V_\tau \mu_\tau \rightharpoonup V \mu$ in the distribution sense
A first result: convergence of flows

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- $\mu_{\tau, t} \to \mu_t$ narrowly,
- $V_\tau \mu_\tau \to V \mu$ in the distribution sense

$$\lim_{\tau \downarrow 0} \int_0^T \int_{\mathcal{U}} |V_\tau(t, y)|^2 \, d\mu_{\tau, t}(y) \, dt = \int_0^T \int_{\mathcal{U}} |V(t, y)|^2 \, d\mu_t(y) \, dt$$
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\frac{d}{dt} Y_\tau(t, y) = V_\tau(t, Y_\tau(t, y)), \quad \mu_\tau, t = (Y_\tau(t, \cdot)) \# \mu_\tau, 0
\]

- \( \mu_\tau, t \to \mu_t \) narrowly,
- \( V_\tau \mu_\tau \to V \mu \) in the distribution sense

\[
\lim_{\tau \downarrow 0} \int_0^T \int_U |V_\tau(t, y)|^2 d\mu_\tau, t(y) dt = \int_0^T \int_U |V(t, y)|^2 d\mu_t(y) dt
\]

- \( V \) is a “tangent vector field”, i.e. \( V \in \{ \nabla \psi : \psi \in C^\infty_c(U) \} \)
A first result: convergence of flows

Suppose that $V_\tau, Y_\tau, \mu_\tau = \rho_\tau \mathcal{L}^d$ are given with

$$\frac{d}{dt} Y_\tau(t, y) = V_\tau(t, Y_\tau(t, y)), \quad \mu_{\tau,t} = (Y_\tau(t, \cdot)) \# \mu_{\tau,0}$$

- $\mu_{\tau,t} \rightharpoonup \mu_t$ narrowly,
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- $\lim_{\tau \downarrow 0} \int_0^T \int_{\mathcal{U}} \left| V_\tau(t, y) \right|^2 d\mu_{\tau,t}(y) dt = \int_0^T \int_{\mathcal{U}} \left| V(t, y) \right|^2 d\mu_t(y) dt$

- $V$ is a “tangent vector field”, i.e. $V \in \{\nabla \psi : \psi \in C^\infty(\mathcal{U})\}$
- The limit ODE admits a unique solution for $\mu_0$-a.e. $y \in \mathcal{U}$. 
A first result: convergence of flows

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- $V$ is a “tangent vector field”, i.e. $V \in \{ \nabla \psi : \psi \in C_\infty(U) \}$
- The limit ODE admits a unique solution for $\mu_0$-a.e. $y \in U$.

Then there exists a unique flow $Y$ solving

$$\dot{Y}(t, y) = V(t, Y(t, y)), \quad Y(0, y) = y$$

$$\lim_{\tau \downarrow 0} \int_0^T \max_t |Y_\tau(t, y) - Y(t, y)|^2 d\mu_0(y) = 0.$$
Reconstruction of the gradient flow of $\mathcal{F}$

Suppose that $\rho_0 \in C^\alpha(\mathcal{U})$, $\mathcal{J}(\rho_0) = \int_{\mathcal{U}} \phi(\rho_0) \, dy < +\infty$.

- The discrete transports $Y_\tau$ converge to $Y$ in the sense of $L^2(\mathcal{U}; L^\infty(0, T))$

$$\lim_{\tau \downarrow 0} \int_0^T \max_t \left| Y_\tau(t, y) - Y(t, y) \right|^2 \rho_0(y) \, dy = 0.$$ 

and the discrete solutions $U_\tau(t, x) = Y_\tau(t, u_0(x))$ converge to $u(t, x) = Y(t, u_0(x))$. 

Reconstruction of the gradient flow of $\mathcal{F}$

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- The limit flow $Y$ solves the ODE

$$\begin{cases}
    \dot{Y}(t, y) = V(t, Y(t, y)) \\
    Y(0, y) = y
\end{cases}$$

where $V(t, y) = -\nabla \phi'(\rho_t(y))$.
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Y(0, y) = y
\end{cases}$$

where $V(t, y) = -\nabla \phi'(\rho_t(y))$

- $\rho$ is the unique solution of the nonlinear diffusion equation

$$\begin{cases} 
\partial_t \rho - \text{div}(\rho D\phi' (\rho)) = 0 \quad \text{in } \mathcal{U}, \\
\rho(y, 0) = \rho_0(y), \quad \partial_n \rho = 0 \quad \text{on } \partial \mathcal{U}
\end{cases}$$
Outline

1 Thin film equation as the gradient flow of the Dirichlet functional
   in collaboration with U. Gianazza, G. Toscani, D. Matthes, R. McCann

2 The $L^2$-gradient flow of the simplest polyconvex functional
   in collaboration with L. Ambrosio, S. Lisini

3 The sticky particle system
   in collaboration with L. Natile
Starting point: motion of a finite number of particles.

Discrete particle model

$N$ particles $P_i := (m_i, x_i, v_i)$, $i = 1, \ldots, N$,
with positive mass $m_i$ satisfying $\sum_{i=1}^{N} m_i = 1$
ordered positions $x_1 < x_2 < \ldots < x_{N-1} < x_N$, 

\[ P_1 \quad P_2 \quad P_3 \quad P_4 \]
Starting point: motion of a finite number of particles.

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ordered positions $x_1 < x_2 < \ldots < x_{N-1} < x_N,$
and velocities $v_i$.

At the initial time $t = 0$ the particles are disjoint and start to move freely with constant velocity:

\[
x_i(t) := x_i(0) + v_i(0)t, \quad v_i(t) := v_i.
\]
**Starting point: motion of a finite number of particles.**

### Discrete particle model

- **$N$ particles** $P_i := (m_i, x_i, v_i), \quad i = 1, \ldots, N$, with positive mass $m_i$ satisfying $\sum_{i=1}^{N} m_i = 1$
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The first collision time $t = t^1$ correspond to

$$x_j(t^1) = x_{j+1}(t^1) = \ldots = x_k(t^1) \quad \text{for some indices } j < k.$$

The particles $P_j, P_{j+1}, \ldots, P_k$ collapse and stick in a new particle $P$
with mass $m := m_j + \ldots + m_k$ and

“barycentric” velocity $v := \frac{m_j v_j(t^1) + m_{j+1} v_{j+1}(t^1) + \ldots + m_k v_k(t^1)}{m}$
Starting point: motion of a finite number of particles.

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Measure-theoretic description

We thus have:

a (finite) sequence of collision times $0 < t^1 < t^2 < \ldots$
in each interval $[t^h, t^{h+1})$ a finite number $N^h$ of (suitably relabelled)particles $P_1(t), \ldots, P_{N^h}(t)$, $P_i(t) := (m_i, x_i(t), v_i(t))$. 
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We can introduce the measures

$$
\rho_t := \sum_{i=1}^{N^h} m_i \delta_{x_i(t)} \in \mathcal{P}(\mathbb{R}) \quad (\rho v)_t := \sum_{i=1}^{N^h} m_i v_i \delta_{x_i(t)} \in \mathcal{M}(\mathbb{R}) \quad \text{if } t \in [t^h, t^{h+1}).
$$
Measure-theoretic description

We thus have:

a **finite** sequence of collision times $0 < t^1 < t^2 < \ldots$

in each interval $[t^h, t^{h+1})$ a finite number $N^h$ of (suitably relabelled) particles $P_1(t), \ldots, P_{N^h}(t)$, $P_i(t) := (m_i, x_i(t), v_i(t))$.

We can introduce the measures

$$\rho_t := \sum_{i=1}^{N^h} m_i \delta_{x_i(t)} \in \mathcal{P}(\mathbb{R}) \quad (\rho v)_t := \sum_{i=1}^{N^h} m_i v_i \delta_{x_i(t)} \in \mathcal{M}(\mathbb{R}) \quad \text{if } t \in [t^h, t^{h+1})$$

They satisfy the **1-dimensional pressureless Euler system** in the sense of distributions

$$\begin{cases}
\partial_t \rho + \partial_x (\rho v) = 0, & \text{in } \mathbb{R} \times (0, +\infty); \quad \rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0, \\
\partial_t (\rho v) + \partial_x (\rho v^2) = 0, & \text{in } \mathbb{R} \times (0, +\infty).
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and the \textbf{OLEINIK entropy condition}

$$v_t(x_2) - v_t(x_1) \leq \frac{1}{t} (x_2 - x_1) \quad \text{for } \rho_t\text{-a.e. } x_1, x_2 \in \mathbb{R}, \ x_1 \leq x_2.$$
Main problem: continuous limit

Consider a sequence of discrete initial data $\mu^n_0 := (\rho^n_0, \rho^n_0 v^n_0)$ converging to $\mu_0 = (\rho_0, \rho_0 v_0)$ in a suitable measure-theoretic sense and let $\mu^n_t = (\rho^n_t, \rho^n_t v^n_t)$ be the (discrete) solution of SPS.
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- Prove that the limit $\mu_t = (\rho_t, \rho_t v_t)$ of the SPS $\mu^n_t = (\rho^n_t, \rho^n_t v^n_t)$ as $n \uparrow +\infty$ exists.
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- Find a suitable characterization of $\mu_t$
- Show that $(\rho_t, \rho_t v_t)$ solves the pressureless Euler system

\[
\begin{align*}
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\frac{\partial (\rho v)}{\partial t} + \frac{\partial}{\partial x} (\rho v^2) &= 0,
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and satisfy Oleinik entropy condition.
Main contributions

- Existence and convergence:
  - Grenier ’95, E-Rykov-Sinai ’96: first existence and convergence result.
  - Brenier-Grenier ’96: Characterization of the limit in terms of a suitable scalar conservation law, uniqueness.
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- Pioneering ideas which lies (more or less explicitly) at the core of the papers by E-Rykov-Sinaï and Brenier-Grenier have been introduced by Shnirelman ’86 and further clarified by Andrievsky-Gurbatov-Sobolevski˘ı’07 in a formal way.

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  - Bouchut-James ’95, Poupaud-Rascle ’97, Sobolevski˘ı’97, Boudin ’00: viscous regularization.
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Basic assumptions:

\[ \rho^n_0 \to \rho_0 \text{ in the } L^2\text{-Wasserstein distance}, \]
\[ v^n_0 = v_0 \text{ is given by a continuous function with (at most) linear growth.} \]

In particular the result cover the case when \( \rho^n_0, \rho_0 \) have a common compact support and \( \rho^n_0 \to \rho_0 \) weakly in the sense of distribution (or, equivalently, in the duality with continuous functions).

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The Brenier-Grenier formulation

For every probability measure $\rho \in \mathcal{P}(\mathbb{R})$ we introduce the cumulative distribution function

$$M_\rho(x) := \rho((\infty, x]), \quad x \in \mathbb{R},$$

so that $\rho = \partial_x M_\rho$ in $\mathcal{D}'(\mathbb{R})$. 

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**Theorem (Brenier-Grenier ’96)**

$M$ is the unique entropy solution of the scalar conservation law

$$\partial_t M + \partial_x A(M) = 0 \quad \text{in } \mathbb{R} \times (0, +\infty)$$

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A'(M_0(x)) = v_0(x).
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Monotone rearrangement

Point of view of 1-dimensional optimal transport: instead of using the cumulative distribution function \( M_\rho(x) = \rho((\infty, x]) \), we represent each probability measure \( \rho \) by its monotone rearrangement \( X_\rho : (0, 1) \rightarrow \mathbb{R} \)

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X_\rho(w) := \inf \left\{ x \in \mathbb{R} : M_\rho(x) > w \right\} \quad w \in (0, 1)
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The map $X_\rho$ is **nondecreasing and right-continuous** and it pushes the Lebesgue measure $\lambda := \mathcal{L}^1|_{(0,1)}$ on $(0, 1)$ onto $\rho$. 
Wasserstein distance and the $L^2$ isometry

The map $\rho \mapsto X_\rho$ is a **one-to-one correspondence** between

the space $\mathcal{P}_2(\mathbb{R})$ of probability measures with finite quadratic moment

$$m_2(\rho) = \int_{\mathbb{R}} |x|^2 \, d\rho(x) < +\infty$$

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$W_2(\rho^1, \rho^2)$ between $\rho^1, \rho^2 \in \mathcal{P}_2(\mathbb{R})$:

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In this way $\rho \leftrightarrow X_\rho$ is an isometry between $(\mathcal{P}_2(\mathbb{R}), W_2)$ and $(\mathcal{K}, \| \cdot \|_{L^2(0,1)})$. 
A metric space for the measure-momentum couples \((\rho, \rho v)\)

We consider the space of couples \((\rho, \rho v)\), with \(\rho \in \mathcal{P}_2(\mathbb{R})\) and \(v \in L^2_\rho(\mathbb{R})\):

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\mathcal{V}_2(\mathbb{R}) := \left\{ \mu = (\rho, \rho v) \subset \mathcal{P}_2(\mathbb{R}) \times \mathcal{M}(\mathbb{R}) : v \in L^2_\rho(\mathbb{R}) \right\}.
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thus \(\rho\) is a probability measure and \(\eta = \rho v\) is a finite signed measure in \(\mathcal{M}(\mathbb{R})\) with \(\int_{\mathbb{R}} |v(x)|^2 \, d\rho(x) < +\infty\).
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\(\mu_n = (\rho_n, \rho_n v_n)\) converges to \(\mu = (\rho, \rho v)\) in \(\mathcal{V}_2(\mathbb{R})\) if and only if

\[
W_2(\rho_n, \rho) \to 0, \quad \rho_n v_n \rightharpoonup \rho v \quad \text{weakly in} \ \mathcal{M}(\mathbb{R}), \quad \int_\mathbb{R} |v_n|^2 \, d\rho_n \to \int_\mathbb{R} |v|^2 \, d\rho.
\]
The fundamental estimate

Let $\mathcal{V}_{\text{discr}}(\mathbb{R})$ the collection of all the discrete measures in $\mathcal{V}_2(\mathbb{R})$ and let us denote by $\mathcal{S}_t : \mathcal{V}_{\text{discr}}(\mathbb{R}) \to \mathcal{V}_{\text{discr}}(\mathbb{R})$ the map associating to any discrete initial datum $(\rho_0, \rho_0 v_0) \in \mathcal{V}_{\text{discr}}$ the solution $(\rho_t, \rho_t v_t)$ of the (discrete) sticky-particle system. $\mathcal{S}_t$ is a semigroup in $\mathcal{V}_{\text{discr}}(\mathbb{R})$. 

Theorem (Stability with respect to the initial data)

Let $\mu^\cdot \in \mathcal{V}_2(\mathbb{R})$, $\mu^\cdot \in \mathcal{V}_2(\mathbb{R})$ be the solutions of the (discrete) sticky-particle system with initial data $\mu^0 \in \mathcal{V}_{\text{discr}}(\mathbb{R})$.

$\| \mu^1_t - \mu^2_t \|_2 \leq \| \mu^1_0 - \mu^2_0 \|_2 + \int_0^t U_x(\mu^1_r, \mu^2_r) \, dr \leq C(1 + t) \left[ \| \mu^1_0 \|_2 + \| \mu^2_0 \|_2 \right]$, for a suitable "universal" constant $C$ independent of $t$ and the data.
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For $\mu \in \mathcal{V}_2(\mathbb{R})$ we set

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**Theorem (Stability with respect to the initial data)**

Let $\mu^\ell_t = (\rho^\ell_t, \rho^\ell_t v^\ell_t) = \mathcal{S}_t[\mu^\ell_0]$, $\ell = 1, 2$, be the solutions of the (discrete) sticky-particle system with initial data $\mu^\ell_0 \in \mathcal{V}_{\text{discr}}(\mathbb{R})$.

$$W_2(\rho^1_t, \rho^2_t) \leq W_2(\rho^1_0, \rho^2_0) + tU_2(\mu^1_0, \mu^2_0),$$
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Let $\mu_\ell = (\rho_\ell, \rho_\ell v_\ell) = \mathcal{S}_t[\mu_0^\ell], \ell = 1, 2$, be the solutions of the (discrete) sticky-particle system with initial data $\mu_0^\ell \in \mathcal{V}_{\text{discr}}(\mathbb{R})$.

$$W_2(\rho_1^t, \rho_2^t) \leq W_2(\rho_1^0, \rho_2^0) + tU_2(\mu_0^1, \mu_0^2),$$

$$\int_0^t U_2^2(\mu_1^r, \mu_2^r) dr \leq C(1 + t) \left( [\mu_1]_2 + [\mu_2]_2 \right) \left( W_2(\rho_0^1, \rho_0^2) + U_2(\mu_0^1, \mu_0^2) \right),$$

for a suitable “universal” constant $C$ independent of $t$ and the data.
Evolution semigroup

Theorem (The evolution semigroup in $\mathcal{V}_2(\mathbb{R})$)

- The semigroup $\mathcal{I}_t$ can be uniquely extended by density to a right-continuous semigroup (still denoted $\mathcal{I}_t$) of strongly-weakly continuous transformations in $\mathcal{V}_2(\mathbb{R})$, thus satisfying

$$\mathcal{I}_{s+t}[\mu] = \mathcal{I}_s[\mathcal{I}_t[\mu]] \quad \forall s, t \geq 0, \quad \lim_{t \downarrow 0} D_2(\mathcal{I}_t[\mu], \mu) = 0.$$  

(2)

$\mathcal{I}_t$ complies with the same discrete stability estimates of the previous Theorem.
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- $(\rho_t, \rho_t v_t) = \mathcal{I}_t[\mu], \mu \in \mathcal{V}_2(\mathbb{R})$, is a distributional solution of Euler system satisfying Oleinik entropy condition.
A gradient flow formulation in $\mathcal{P}_2(\mathbb{R})$

The semigroup $\mathcal{S}_t$ can also be characterized by the (metric) gradient flow $\mathcal{G}_\tau$ of the $(-1)$-geodesically convex functional

$$\Phi(\rho) := -\frac{1}{2} W_2^2(\rho, \rho_0)$$

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**Theorem (The gradient flow of the opposite Wasserstein distance)**

If $\mu_t = (\rho_t, \rho_t v_t) = \mathcal{S}_t(\rho_0, \rho_0 v_0)$ is a solution of SPS then the rescaling $\tau = \log t$, $\hat{\mu}_\tau = \mu_t$, $\hat{\rho}_\tau = \rho_t$ satisfy

$$\hat{\rho}_{\tau+\delta} = \mathcal{G}_\delta(\hat{\rho}_\tau) \quad \text{or, equivalently} \quad \rho_{t e^{\delta}} = \mathcal{G}_\delta(\rho_t).$$
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The (rescaled) semigroup $\mathcal{G}$ provides a displacement extrapolation, i.e. a canonical way to extend Wasserstein geodesics after collisions.
A simple example

\[ \rho_0 \quad \rho_0 \quad \rho_0 \quad \rho_0 \]
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Extensions

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Adding a force induced by a potential $V$: 

$$ \partial_t \rho + \partial_x (\rho v) = 0, $$

$$ \partial_t (\rho v) + \partial_x (\rho v^2) = -\rho \partial_x V. $$

Adding a force induced by a smooth interaction potential $W$: 

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Adding a force induced by a non-smooth interaction potential, e.g. the Euler-Poisson system when $W(x) = \pm|x|$. 

Open problems: 

1. The SPS in the multidimensional case. 
2. The displacement-extrapolation problem.
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- (in collaboration with W. Gangbo and M. Westdickenberg) Adding a force induced by a potential $V$

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