Application of Optimal Transport to Evolutionary PDEs

1 - Gradient flows in linear spaces and their variational approximation

Giuseppe Savaré
http://www.imati.cnr.it/∼savare
Department of Mathematics, University of Pavia, Italy

2010 CNA Summer School
New Vistas in Image Processing and PDEs

Carnegie Mellon Center for Nonlinear Analysis, Pittsburgh, *June 7–12, 2010*

Outline

The basic ingredients of gradient flows

- **A functional** $\Phi : X \to \mathbb{R}$ function defined in some ambient space X (initially $X := \mathbb{R}^d$ for simplicity).
- **(metric) Velocity:** some norm $\|\cdot\|$ to measure the velocity (and the length/energy) of the curves $\boldsymbol{w} : t \in (a,b) \mapsto \boldsymbol{w}_t \in X$.

$$\textit{velocity} \quad \|\dot{\boldsymbol{w}}_t\|,$$

$$\textit{length} \quad \mathcal{L}[\boldsymbol{w}] := \int_a^b \|\dot{\boldsymbol{w}}_t\|\,\mathrm{d}t, \quad \textit{energy} \quad \mathcal{E}[\boldsymbol{w}] := \int_a^b \|\dot{\boldsymbol{w}}_t\|^2\,\mathrm{d}t$$

Typically $\|\cdot\|$ is the euclidean norm, but it could be a general one and **it could also depend on the point** (Riemannian/Finsler structure).
It is strictly related to a **distance** by the formulae

$$\textit{distance} \quad \mathsf{d}(\boldsymbol{w}_0, \boldsymbol{w}_1) := \inf\Big\{\mathcal{L}[\boldsymbol{w}] : \boldsymbol{w}(a) = \boldsymbol{w}_0,\ \boldsymbol{w}(b) = \boldsymbol{w}_1\Big\}$$

$$\textit{metric velocity} \quad |\dot{\boldsymbol{w}}_t| := \lim_{h\to 0} \frac{\mathsf{d}(\boldsymbol{w}_t, \boldsymbol{w}_{t+h})}{|h|} = \|\dot{\boldsymbol{w}}_t\|.$$

Heuristics: drection of maximal dissipation rate

Let $\mathrm{D}\Phi \in X^*$ denotes the differential of Φ.

Dissipation along a curve and chain rule: if $\boldsymbol{w} : t \in (a,b) \mapsto \boldsymbol{w}_t \in X$ is a smooth curve with time derivative $\dot{\boldsymbol{w}}_t := \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{w}_t$ then

$$\textit{Dissipation rate of } \Phi \textit{ along } \boldsymbol{w} := -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{w}_t) = -\langle \mathrm{D}\Phi(\boldsymbol{w}_t), \dot{\boldsymbol{w}}_t\rangle.$$

Basic rule: choose the direction of **maximal dissipation rate with respect to the given velocity** among all the curves trough a point $\boldsymbol{w}$:

$$\textit{Slope} \quad |\partial\Phi|(\boldsymbol{w}) := \sup\Big\{\frac{-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{w}_t)}{\|\dot{\boldsymbol{w}}_t\|} : \boldsymbol{w}_t = \boldsymbol{w},\ \dot{\boldsymbol{w}}_t \neq 0\Big\}$$

By the chain rule, the slope of Φ is the dual norm of its differential:

$$\textit{Slope} = |\partial\Phi|(\boldsymbol{w}) = \|-\mathrm{D}\Phi(\boldsymbol{w})\|_* = \limsup_{\boldsymbol{z}\to\boldsymbol{w}} \frac{\Phi(\boldsymbol{w}) - \Phi(\boldsymbol{z})}{\mathsf{d}(\boldsymbol{w},\boldsymbol{z})}.$$

A direction $\boldsymbol{v} = \dot{\boldsymbol{u}}_t$ is **of maximal slope** if it realizes the “sup“, i.e.

$$-\langle \mathrm{D}\Phi(\boldsymbol{u}), \boldsymbol{v}\rangle = \|\boldsymbol{v}\| \cdot \|-\mathrm{D}\Phi(\boldsymbol{u})\|_*.$$

By introducing the duality map $J := \mathrm{D}\big(\frac{1}{2}\|\cdot\|^2\big)$

$$\boxed{\boldsymbol{v} \quad \text{has the same direction of } J^{-1}(-\mathrm{D}\Phi(\boldsymbol{u})).}$$

When $\|\cdot\|$ is euclidean J is linear and $J^{-1}\mathrm{D}\Phi = \nabla\Phi$,

$$\boldsymbol{v} \quad \text{has the same direction of } -\nabla\Phi(\boldsymbol{u}).$$

In this case we usually identify X with its dual, and $\mathrm{D}\Phi$ with the gradient $\nabla\Phi$.

The choice of the speed

The velocity $\boldsymbol{v} = \dot{\boldsymbol{u}}$ *has the same direction of* $J^{-1}(-\mathrm{D}\Phi(\boldsymbol{u}))$.

To provide a complete description *the speed* (the norm of $\boldsymbol{v}$) has to be prescribed. In general, one can introduce an

$$\textit{increasing homeomorphism} \quad \beta : [0, +\infty) \to [0, +\infty)$$

and ask for

$$\boxed{\beta\big(\|\boldsymbol{v}\|\big) = \textit{slope} = |\partial\Phi|(\boldsymbol{u}) = \|\mathrm{D}\Phi(\boldsymbol{u})\|_*.}$$

Simplest (and typical) choice: $\boxed{\beta(r) = r,\ \textit{velocity=slope},\ \|\boldsymbol{v}\| = \| - \mathrm{D}\Phi|(\boldsymbol{u})\|_*.}$

More generally

$$\psi(r) := \int_0^r \beta(s)\,\mathrm{d}s, \quad \psi^*(r) := \int_0^r \beta^{-1}(s)\,\mathrm{d}s,$$

ψ^* is the (dual, conjugate) Legendre transform of ψ,

$$\beta\big(\|\boldsymbol{v}\|\big) = \| - \mathrm{D}\Phi|(\boldsymbol{u})\|_* \quad \Leftrightarrow \quad \|\boldsymbol{v}\|\,\| - \mathrm{D}\Phi(\boldsymbol{u})\|_* = \psi\big(\|\boldsymbol{v}\|\big) + \psi^*(\| - \mathrm{D}\Phi(\boldsymbol{u})\|_*).$$

The complete condition reads

$$\boxed{\mathrm{D}\Psi(\dot{\boldsymbol{u}}) = -\mathrm{D}\Phi(\boldsymbol{u})}$$

where

$$\Psi(\boldsymbol{v}) := \psi(\|\boldsymbol{v}\|) \quad \text{is the “dissipation potential”}$$

Doubly nonlinear evolution equation

$$\boxed{\mathrm{D}\Psi(\dot{\boldsymbol{u}}) = -\mathrm{D}\Phi(\boldsymbol{u})}$$

- Functional Φ.
- Velocity of a curve $\|\boldsymbol{v}_t\| = \|\dot{\boldsymbol{u}}_t\|$.
- Slope $\|-\mathrm{D}\Phi(\boldsymbol{u}_t)\|_* =$ "maximal dissipation rate" $= \sup\left\{\frac{-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{w}_t)}{\|\dot{\boldsymbol{w}}_t\|}\right\}$
- Speed function β, its primitive ψ, the dissipation potential $\Psi(\boldsymbol{v}) := \psi(\|\boldsymbol{v}\|)$

Problem

Find a curve $\boldsymbol{u}$ starting from $\boldsymbol{u}_0$ whose ***direction at each time realizes the maximal dissipation rate of*** *Φ and whose speed is linked to the slope by the equation $\beta(\|\dot{\boldsymbol{u}}_t\|) = \|-\mathrm{D}\Phi(\boldsymbol{u}_t)\|_*$.*

Along such a curve

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) &= \|\dot{\boldsymbol{u}}_t\|\,\|-\mathrm{D}\Phi(\boldsymbol{u}_t)\|_* = \psi(\|\dot{\boldsymbol{u}}_t\|) + \psi^*(\|-\mathrm{D}\Phi(\boldsymbol{u}_t)\|_*) \\ &= \Psi(\boldsymbol{u}_t) + \Psi^*(-\mathrm{D}\Phi(\boldsymbol{u}_t)). \end{aligned}$$

Along **any** cuve $\boldsymbol{w}$:

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{w}_t) &\le \|\dot{\boldsymbol{w}}_t\|\,\|-\mathrm{D}\Phi(\boldsymbol{w}_t)\|_* \le \psi(\|\dot{\boldsymbol{w}}_t\|) + \psi^*(\|-\mathrm{D}\Phi(\boldsymbol{w}_t)\|_*) \\ &= \Psi(\boldsymbol{w}_t) + \Psi^*(-\mathrm{D}\Phi(\boldsymbol{w}_t)). \end{aligned}$$

De Giorgi characterization of curves of maximal slope.

The "simplest" case: gradient flows

Norm velocity $\|\cdot\| \rightsquigarrow |\cdot|$ is euclidean like, J is a linear isometry $\|\cdot\|_* \rightsquigarrow |\cdot|$, $\nabla\Phi = J^{-1}(\mathrm{D}\Phi)$, $\beta(r) = r$, $\psi(r) = \psi^*(r) = \frac{1}{2}r^2$.

MAIN PROBLEM: find $\boldsymbol{u} : [0, +\infty) \to X$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_t = -\nabla\Phi(\boldsymbol{u}_t) \quad t \in [0, +\infty); \qquad \boldsymbol{u}_{|_{t=0}} = u_0 \tag{GF}$$

- **Starting level:** $X \approx \mathbb{R}^d$, finite dimensional euclidean space
  Φ is of class C^2, $\mathrm{D}^2\Phi \geq \boldsymbol{\lambda}\mathrm{I}$.
- **Slight variants:** $\Phi(\boldsymbol{u}) \rightsquigarrow \Phi_t(\boldsymbol{u}) := \Phi(\boldsymbol{u}) - \langle f_t, \boldsymbol{u}\rangle$, time dependent forcing term
  $X \approx \mathbb{M}^d$, smooth Riemannian manifold.
- **Applications to PDE's:** $X :=$ **Hilbert** (typically L^2-like),
  $\Phi : X \to (-\infty, +\infty]$ $\boldsymbol{\lambda}$-convex and just lower-semicontinuous
  $\nabla\Phi \rightsquigarrow \partial\Phi$, multivalued subdifferential of Φ, differential inclusions.
- **Relax $\boldsymbol{\lambda}$-convexity assumption**
- **Further step:** from linear to metric structures...

Main Estimate I: energy identity

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_t = -\nabla\Phi(\boldsymbol{u}_t)} \tag{GF}$$

Evaluating the dissipation rate of Φ

$$\boxed{-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) = |\dot{\boldsymbol{u}}_t|^2 = |\nabla\Phi(\boldsymbol{u}_t)|^2} \tag{I}$$

Integrating in time

$$\Phi(\boldsymbol{u}_t) + \int_0^t |\dot{\boldsymbol{u}}_s|^2 \,\mathrm{d}s = \Phi(\boldsymbol{u}_0)$$

"De Giorgi splitting": $|\dot{\boldsymbol{u}}|^2 = \frac{1}{2}|\dot{\boldsymbol{u}}|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{u})|^2$ (recall $\psi(r) = \psi^*(r) = \frac{1}{2}r^2$)

Curves of maximal slope

$$\Phi(\boldsymbol{u}_0) - \Phi(\boldsymbol{u}_t) = \int_0^t \Big(\frac{1}{2}|\dot{\boldsymbol{u}}_s|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{u}_s)|^2\Big)\,\mathrm{d}s \tag{I}$$

Along any other curve $\boldsymbol{w}$

$$\Phi(\boldsymbol{w}_0) - \Phi(\boldsymbol{w}_t) = \int_0^t -\nabla\Phi(\boldsymbol{w}_s)\cdot\dot{\boldsymbol{w}}_s \,\mathrm{d}s \le \int_0^t \Big(\frac{1}{2}|\dot{\boldsymbol{w}}_s|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{w}_s)|^2\Big)\,\mathrm{d}s$$

Energy identity and variational characterization of (GF)

Theorem

If a Lipschitz curve $\boldsymbol{u} : [0, +\infty) \to X$ *satisfies the differential inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) \le -\frac{1}{2}|\dot{\boldsymbol{u}}_t|^2 - \frac{1}{2}|\nabla\Phi(\boldsymbol{u}_t)|^2 \tag{1}$$

even in the weaker integrated form

$$\Phi(\boldsymbol{u}_t) + \int_0^t \Big(\frac{1}{2}|\dot{\boldsymbol{u}}_s|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{u}_s)|^2\Big)\,\mathrm{d}s \le \Phi(\boldsymbol{u}_0) \tag{2}$$

then $\boldsymbol{u}$ *is a solution of the Gradient Flow*

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_t = -\nabla\Phi(\boldsymbol{u}_t). \tag{GF}$$

Proof: Chain rule:

$$\Phi(\boldsymbol{u}_t) + \int_0^t \big\langle -\nabla\Phi(\boldsymbol{u}_s), \dot{\boldsymbol{u}}_s\big\rangle\,\mathrm{d}s = \Phi(\boldsymbol{u}_0) \tag{3}$$

Subtracting (3) to (2) we get

$$\int_0^t \Big(\frac{1}{2}|\dot{\boldsymbol{u}}_s|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{u}_s)|^2 - \big\langle -\nabla\Phi(\boldsymbol{u}_s), \dot{\boldsymbol{u}}_s\big\rangle\Big)\,\mathrm{d}s = \frac{1}{2}\int_0^t \Big|\dot{\boldsymbol{u}}_s + \nabla\Phi(\boldsymbol{u}_s)\Big|^2\,\mathrm{d}s \le 0$$

Exploting convexity

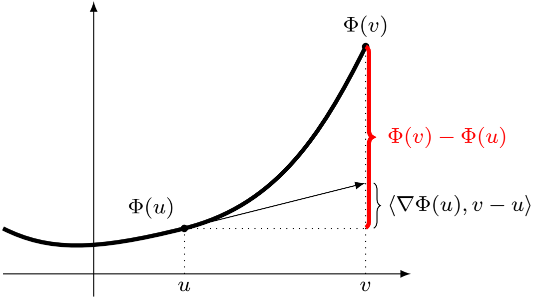
Convexity inequality: $\boldsymbol{w}_\theta = (1-\theta)\boldsymbol{w}_0 + \theta\boldsymbol{w}_1$,

$$\Phi(\boldsymbol{w}_\theta) \le (1-\theta)\Phi(\boldsymbol{w}_0) + \theta\Phi(\boldsymbol{w}_1) \quad \text{for every } \boldsymbol{w}_0, \boldsymbol{w}_1 \in X,\ \theta \in [0,1]$$

Hessian inequality: $\mathrm{D}^2\Phi \ge 0$

Subgradient property:

$$\langle \nabla\Phi(\boldsymbol{u}), v - \boldsymbol{u}\rangle \le \Phi(v) - \Phi(\boldsymbol{u})$$



Gradient monotonicity:

$$\big\langle \nabla\Phi(\boldsymbol{u}) - \nabla\Phi(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}\big\rangle \ge 0$$

$\boldsymbol{\lambda}$-convexity

Hessian inequality

$$\mathrm{D}^2\Phi(\boldsymbol{w}) \geq \boldsymbol{\lambda} I \quad \text{i.e.} \quad \langle \mathrm{D}^2\Phi(\boldsymbol{w})\xi, \xi\rangle \geq \boldsymbol{\lambda}|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^d.$$

$\boldsymbol{\lambda}$-convexity inequality: $\boldsymbol{w}_\theta := (1-\theta)\boldsymbol{w}_0 + \theta\boldsymbol{w}_1$, $\theta \in [0,1]$,

$$\Phi(\boldsymbol{w}_\theta) \leq (1-\theta)\Phi(\boldsymbol{w}_0) + \theta\Phi(\boldsymbol{w}_1) - \frac{\boldsymbol{\lambda}}{2}\theta(1-\theta)|\boldsymbol{w}_0 - \boldsymbol{w}_1|^2$$

Sub-gradient inequality

$$\langle \nabla\Phi(\boldsymbol{w}_1), \boldsymbol{w}_1 - \boldsymbol{w}_0\rangle - \frac{\boldsymbol{\lambda}}{2}|\boldsymbol{w}_1 - \boldsymbol{w}_0|^2 \geq \Phi(\boldsymbol{w}_1) - \Phi(\boldsymbol{w}_0) \geq \langle \nabla\Phi(\boldsymbol{w}_0), \boldsymbol{w}_1 - \boldsymbol{w}_0\rangle + \frac{\boldsymbol{\lambda}}{2}|\boldsymbol{w}_1 - \boldsymbol{w}_0|^2.$$

$\boldsymbol{\lambda}$-monotonicity of $\nabla\Phi$

$$\langle \nabla\Phi(\boldsymbol{w}_0) - \nabla\Phi(\boldsymbol{w}_1), \boldsymbol{w}_0 - \boldsymbol{w}_1\rangle \geq \boldsymbol{\lambda}|\boldsymbol{w}_0 - \boldsymbol{w}_1|^2.$$

Minimizer and Distance-Energy-Slope bounds for $\boldsymbol{\lambda} > 0$

When $\boldsymbol{\lambda} > 0$ then Φ has a unique minimizer $\boldsymbol{u}_{min}$ with $|\partial\Phi|(\boldsymbol{u}_{min}) = 0$ and

$$\frac{\boldsymbol{\lambda}}{2}|\boldsymbol{u} - \boldsymbol{u}_{min}|^2 \leq \Phi(\boldsymbol{u}) - \Phi(\boldsymbol{u}_{min}) \leq \frac{1}{2\boldsymbol{\lambda}}|\partial\Phi|^2(\boldsymbol{u})$$

Basic estimates (convex case, $\boldsymbol{\lambda} = 0$)

Contraction: if $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_t = -\nabla\Phi(\boldsymbol{u}_t)$, $\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{w}_t = -\nabla\Phi(\boldsymbol{w}_t)$ then

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_t - \boldsymbol{w}_t|^2 = \big\langle \dot{\boldsymbol{u}}_t - \dot{\boldsymbol{w}}_t, \boldsymbol{u}_t - \boldsymbol{w}_t\big\rangle = -\big\langle \nabla\Phi(\boldsymbol{u}_t) - \nabla\Phi(\boldsymbol{w}_t), \boldsymbol{u}_t - \boldsymbol{w}_t\big\rangle \le 0.$$

Contraction properties: uniqueness and stability

If $\boldsymbol{u}, \boldsymbol{w}$ are two trajectories with initial data $\boldsymbol{u}_0, \boldsymbol{w}_0$ then

$$|\boldsymbol{u}_t - \boldsymbol{w}_t| \le |\boldsymbol{u}_0 - \boldsymbol{w}_0|$$

Lyapunov functionals Take $\mathcal{F} : X \to \mathbb{R}$ and evaluate the derivative along GF:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(\boldsymbol{u}_t) = -\big\langle \nabla\mathcal{F}(\boldsymbol{u}_t), \nabla\Phi(\boldsymbol{u}_t)\big\rangle$$

Main choices:

$$\mathcal{F}(\boldsymbol{u}) := \Phi(\boldsymbol{u}) \quad \rightsquigarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) = -|\dot{\boldsymbol{u}}_t|^2 = -|\nabla\Phi(\boldsymbol{u}_t)|^2 \tag{I}$$

$$\mathcal{F}(\boldsymbol{u}) := \frac{1}{2}|\boldsymbol{u} - w|^2 \quad \rightsquigarrow \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_t - w|^2 = \big\langle \nabla\Phi(\boldsymbol{u}_t), v - \boldsymbol{u}_t\big\rangle \tag{II}$$

$$\mathcal{F}(\boldsymbol{u}) := \frac{1}{2}|\nabla\Phi(\boldsymbol{u})|^2 \quad \rightsquigarrow \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\Phi(\boldsymbol{u}_t)|^2 = -\big\langle \mathrm{D}^2\Phi(\boldsymbol{u}_t)\nabla\Phi(\boldsymbol{u}_t), \nabla\Phi(\boldsymbol{u}_t)\big\rangle \tag{III}$$

Estimate II: Evolution variational inequality for the distance

$$\mathcal{F}(\boldsymbol{u}) := \frac{1}{2}|\boldsymbol{u} - w|^2 \rightsquigarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_t - w|^2 = \big\langle \nabla\Phi(\boldsymbol{u}_t), w - \boldsymbol{u}_t \big\rangle \tag{I}$$

(by the subgradient inequality:) $\leq \Phi(w) - \Phi(\boldsymbol{u}_t)$

Evolution variational Inequality (EVI): differential form

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_t - w|^2 \leq \Phi(w) - \Phi(\boldsymbol{u}_t) \tag{EVI}$$

Integrating in time from 0 to t

$$\begin{aligned} \frac{1}{2}|\boldsymbol{u}_t - w|^2 - \frac{1}{2}|\boldsymbol{u}_0 - w|^2 &\leq \int_0^t \Big(\Phi(w) - \Phi(\boldsymbol{u}_s)\Big)\,\mathrm{d}s \\ &\leq t\Phi(w) - \int_0^t \Phi(\boldsymbol{u}_s)\,\mathrm{d}s \\ \text{(since } s \mapsto \Phi(\boldsymbol{u}_s) \text{ is nonincreasing)} \qquad &\leq t\Big(\Phi(w) - \Phi(\boldsymbol{u}_t)\Big) \end{aligned}$$

Evolution variational Inequality (EVI): integrated form

$$\frac{1}{2}|\boldsymbol{u}_t - w|^2 + (t-s)\Phi(\boldsymbol{u}_t) \leq \frac{1}{2}|\boldsymbol{u}_s - w|^2 + (t-s)\Phi(w) \quad \text{for every } 0 \leq s < t,\ w \in X.$$

Estimate III: decay of slope and velocity

$$\mathcal{F}(\boldsymbol{u}) := \frac{1}{2}|\nabla\Phi(\boldsymbol{u})|^2 \rightsquigarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\Phi(\boldsymbol{u}_t)|^2 = -\big\langle \mathrm{D}^2\Phi(\boldsymbol{u}_t)\nabla\Phi(\boldsymbol{u}_t), \nabla\Phi(\boldsymbol{u}_t)\big\rangle \qquad \text{(III)}$$

$$(\mathrm{D}^2\Phi \geq 0) \qquad \leq 0$$

Decay of the slope

$$\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\Phi(\boldsymbol{u}_t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}|\dot{\boldsymbol{u}}_t|^2 \leq 0 \qquad |\nabla\Phi(\boldsymbol{u}_t)| = |\dot{\boldsymbol{u}}_t| \leq |\nabla\Phi(\boldsymbol{u}_0)|, \qquad \text{(III)}$$

Regularizing estimates

Linear combination: $II + t \cdot I + \frac{1}{2}t^2 \cdot III \le 0$.

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}|\boldsymbol{u}_t - w|^2 + t\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big) + \frac{t^2}{2}|\nabla\Phi(\boldsymbol{u}_t)|^2\Big) \le 0}$$

Proof:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\boldsymbol{u}_t - w|^2 \le -\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(t\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big)\Big) \le \big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big) - t|\nabla\Phi(\boldsymbol{u}_t)|^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{t^2}{2}|\nabla\Phi(\boldsymbol{u}_t)|^2\Big) \le t|\nabla\Phi(\boldsymbol{u}_t)|^2.$$

Weighted estimate

$$\frac{1}{2}|\boldsymbol{u}_t - w|^2 + t\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big) + \frac{t^2}{2}|\nabla\Phi(\boldsymbol{u}_t)|^2 \le \frac{1}{2}|\boldsymbol{u}_0 - w|^2$$

Consequences: **istantaneous regularization.** For every $w \in X$

$$\boxed{\Phi(\boldsymbol{u}_t) \le \frac{1}{2t}|\boldsymbol{u}_0 - w|^2 + \Phi(w)} \qquad \boxed{|\nabla\Phi(\boldsymbol{u}_t)|^2 \le \frac{1}{t^2}|\boldsymbol{u}_0 - w|^2 + |\nabla\Phi(w)|^2}$$

If $w = \boldsymbol{u}_{min}$ is a **minimizer of Φ** then

$$\boxed{\Phi(\boldsymbol{u}_t) - \Phi(\boldsymbol{u}_{min}) \le C/t, \quad |\nabla\Phi(\boldsymbol{u}_t)| \le C/t^2.}$$

$\boldsymbol{\lambda}$-convexity ($\boldsymbol{\lambda} > 0$) and asymptotic behaviour

If Φ is $\boldsymbol{\lambda}$-convex with $\boldsymbol{\lambda} > 0$, we have a refined EVI:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\boldsymbol{u}_t - w|^2 + \frac{\boldsymbol{\lambda}}{2}|\boldsymbol{u}_t - w|^2 \le \Phi(w) - \Phi(\boldsymbol{u}_t).$$

and an exponential decay of the slope

$$\frac{\mathrm{d}}{\mathrm{d}t}|\partial\Phi|^2(\boldsymbol{u}_t) \le -2\boldsymbol{\lambda}|\partial\Phi|^2(\boldsymbol{u}_t) \quad \Rightarrow \quad \boxed{|\partial\Phi|^2(\boldsymbol{u}_t) \le \mathrm{e}^{-2\boldsymbol{\lambda}t}|\partial\Phi|^2(\boldsymbol{u}_0)}$$

On the other hand, Φ has a unique minimizer $\boldsymbol{u}_{min}$ and $\boldsymbol{\lambda}$-convexity yields

$$\frac{\boldsymbol{\lambda}}{2}|\boldsymbol{w} - \boldsymbol{u}_{\min}|^2 \le \Phi(\boldsymbol{w}) - \Phi(\boldsymbol{u}_{\min}) \le \frac{1}{2\boldsymbol{\lambda}}|\partial\Phi|^2(\boldsymbol{w})$$

Choosing $w := \boldsymbol{u}_{\min}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\boldsymbol{u}_t - \boldsymbol{u}_{\min}|^2 + \boldsymbol{\lambda}|\boldsymbol{u}_t - \boldsymbol{u}_{\min}|^2 \le 0 \quad \Rightarrow \quad \boxed{|\boldsymbol{u}_t - \boldsymbol{u}_{\min}|^2 \le \mathrm{e}^{-2\boldsymbol{\lambda}t}|\boldsymbol{u}_0 - \boldsymbol{u}_{\min}|^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi(\boldsymbol{u}_t) - \Phi(\boldsymbol{u}_{\min})\Big) = -|\partial\Phi|^2(\boldsymbol{u}_t) \le -2\boldsymbol{\lambda}\Big(\Phi(\boldsymbol{u}_t) - \Phi(\boldsymbol{u}_{\min})\Big)$$

so that

$$\boxed{\Phi(\boldsymbol{u}_t) - \Phi(\boldsymbol{u}_{\min}) \le \mathrm{e}^{-2\boldsymbol{\lambda}t}\Big(\Phi(\boldsymbol{u}_0) - \Phi(\boldsymbol{u}_{\min})\Big)}$$

Summary

Energy identity: $\Phi(\boldsymbol{u}_t) + \int_0^t \Big(\frac{1}{2}|\dot{\boldsymbol{u}}_s|^2 + \frac{1}{2}|\nabla\Phi(\boldsymbol{u}_s)|^2\Big)\,\mathrm{d}s = \Phi(\boldsymbol{u}_0)$

EVI: $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_t - v|^2 \le \Phi(v) - \Phi(\boldsymbol{u}_t)$

Slope decay: $t \mapsto |\nabla\Phi(\boldsymbol{u}_t)|^2 = |\dot{\boldsymbol{u}}_t|^2 = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t)$ is nonincreasing.

Contraction: $|\boldsymbol{u}_t - \boldsymbol{w}_t| \le |\boldsymbol{u}_0 - \boldsymbol{w}_0|$

Regularization: $\frac{1}{2}|\boldsymbol{u}_t - w|^2 + t\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big) + \frac{t^2}{2}|\nabla\Phi(\boldsymbol{u}_t)|^2 \le \frac{1}{2}|\boldsymbol{u}_0 - w|^2$

Asymptotic decay, $\boldsymbol{\lambda} > 0$

$$|\boldsymbol{u}_t - \boldsymbol{u}_{\min}|^2 \le \mathrm{e}^{-2\boldsymbol{\lambda} t}|\boldsymbol{u}_0 - \boldsymbol{u}_{\min}|^2$$

$$\Phi(\boldsymbol{u}_t) - \Phi(\boldsymbol{u}_{\min}) \le \mathrm{e}^{-2\boldsymbol{\lambda} t}\Big(\Phi(\boldsymbol{u}_0) - \Phi(\boldsymbol{u}_{\min})\Big)$$

$$|\partial\Phi|^2(\boldsymbol{u}_t) \le \mathrm{e}^{-2\boldsymbol{\lambda} t}|\partial\Phi|^2(\boldsymbol{u}_0)$$

Infinite dimension and non-smoothness

Applications in infinite dimensional Hilbert spaces X introduce new technical difficulties:

- Φ can take the value $+\infty$ and its *proper domain* $\mathsf{D}(\Phi) := \{\boldsymbol{x} \in X : \Phi(\boldsymbol{x}) < +\infty\}$ has empty interior.
- Φ is just lower semicontinuous and nowhere differentiable in the classical sense.
- The gradient $\nabla\Phi$ has to be replaced by the (Fréchet) subgradient $\partial\Phi$ whose domain $\mathsf{D}(\partial\Phi)$ is often a proper subset of $\mathsf{D}(\Phi)$.
- $\partial\Phi$ can be locally unbounded and multivalued.

Main example: integral functional in $X := L^2(\Omega)$, Ω being an open domain of $\mathbb{R}^m$, and

$$\Phi(\boldsymbol{u}) := \int_\Omega \varphi(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})\,\mathrm{d}x$$

where $\varphi : (x, u, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^m \mapsto \varphi(x, u, p) \in [0, +\infty)$ is a C^1-function, convex w.r.t. p. The first variation of Φ is

$$\frac{\delta\Phi}{\delta\boldsymbol{u}} := \partial_u\varphi(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u}) - \nabla\cdot\partial_p\varphi(x, \boldsymbol{u}, \mathrm{D}\boldsymbol{u})$$

and we want to solve the PDE

$$\partial_t\boldsymbol{u} + \frac{\delta\Phi}{\delta\boldsymbol{u}} = 0 \quad \text{in } [0, +\infty) \times \Omega$$

with some boundary conditions on $\partial\Omega$.

Gradient flows in L^2-spaces

The PDE

$$\partial_t \boldsymbol{u} + \frac{\delta \Phi}{\delta \boldsymbol{u}} = 0 \quad \text{in } \Omega \times [0, +\infty)$$

is the "formal" gradient flow of Φ in $L^2(\Omega)$.
Assuming e.g. 0 boundary condition and considering a smooth curve $t \in [0, T] \mapsto \boldsymbol{u}_t \in \mathrm{C}^1_0(\Omega)$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) &= \int_\Omega \Big(\partial_u \varphi(x, \boldsymbol{u}, \mathrm{D}_x \boldsymbol{u})\partial_t \boldsymbol{u} + \partial_p \varphi(x, \boldsymbol{u}, \mathrm{D}_x \boldsymbol{u})\mathrm{D}\partial_t \boldsymbol{u}\Big)\,\mathrm{d}x \\ &= \int_\Omega \Big(\partial_u \varphi(x, \boldsymbol{u}, \mathrm{D}_x \boldsymbol{u})\partial_t \boldsymbol{u} - \nabla \cdot \big(\partial_p \varphi(x, \boldsymbol{u}, \mathrm{D}_x \boldsymbol{u})\big)\partial_t \boldsymbol{u}\Big)\,\mathrm{d}x = \int_\Omega \frac{\delta \Phi}{\delta \boldsymbol{u}}\partial_t \boldsymbol{u}\,\mathrm{d}x\end{aligned}$$

If we choose the L^2-velocity $\|\partial_t \boldsymbol{u}\|_{L^2(\Omega)}$, then we can easily get the upper bound for the dissipation rate

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) \le \Big\| - \frac{\delta \Phi}{\delta \boldsymbol{u}}\Big\|_{L^2(\Omega)} \|\partial_t \boldsymbol{u}\|_{L^2(\Omega)}$$

with equality iff

$$\partial_t \boldsymbol{u} = -\frac{\delta \Phi}{\delta \boldsymbol{u}}, \quad |\partial \Phi|(\boldsymbol{u}) = \Big\|\frac{\delta \Phi}{\delta \boldsymbol{u}}\Big\|_{L^2(\Omega)}$$

The simplest example: Dirichlet integral in L^2

Consider $X := L^2(\mathbb{R}^m)$ and the integral functional

$$\mathfrak{D}(\boldsymbol{u}) := \frac{1}{2}\int_{\mathbb{R}^m} |\mathrm{D}\boldsymbol{u}|^2\,\mathrm{d}x \quad \text{if } \boldsymbol{u} \in W^{1,2}_0(\mathbb{R}^m); \quad \mathfrak{D}(\boldsymbol{u}) := +\infty \quad \text{otherwise.}$$

In this case $\mathfrak{D}$ is just lower semicontinuous (w.r.t. convergence in $L^2(\mathbb{R}^m)$) and its proper domain

$$\mathsf{D}(\mathfrak{D}) = W^{1,2}_0(\mathbb{R}^m) \overset{dense}{\subset} L^2(\mathbb{R}^m)$$

is the Sobolev space $W^{1,2}_0(\mathbb{R}^m)$ which is dense in $L^2(\Omega)$ but has empty interior. When $\boldsymbol{u} \in W^{2,2}(\mathbb{R}^m) \cap W^{1,2}_0(\mathbb{R}^m)$ its first variation is

$$\frac{\delta\mathfrak{D}}{\delta\boldsymbol{u}} := -\Delta\boldsymbol{u}$$

and the gradient flow of $\mathfrak{D}$ in $X = L^2(\Omega)$ should be a solution of the Heat equation

$$\partial_t\boldsymbol{u} - \Delta\boldsymbol{u} = 0 \quad \text{in } [0,+\infty)\times\mathbb{R}^m.$$

if we can identify $\dfrac{\delta\mathfrak{D}}{\delta\boldsymbol{u}}$ with "$\nabla\mathfrak{D}(\boldsymbol{u})$" or better with $\partial\mathfrak{D}(\boldsymbol{u})$, which is defined in the even smaller domain $\mathsf{D}(\partial\mathfrak{D}) = W^{2,2}(\mathbb{R}^m) \cap W^{1,2}_0(\mathbb{R}^m)$.

Subgradients and slope for convex functionals

Recall that in the smooth case

$$\boldsymbol{\xi} = \nabla\Phi(\boldsymbol{u}) \quad \Longleftrightarrow \quad \Phi(\boldsymbol{w}) = \Phi(\boldsymbol{u}) + \langle \boldsymbol{\xi}, \boldsymbol{w}-\boldsymbol{u}\rangle + o(\|\boldsymbol{w}-\boldsymbol{u}\|)$$

Suppose that $\Phi : X \to (-\infty, +\infty]$ is convex and lower semicontinuous with proper domain $\mathsf{D}(\Phi)$.

- The **subgradient of** Φ $\partial\Phi : X \rightrightarrows X$ is defined by

$$\boldsymbol{\xi} \in \partial\Phi(\boldsymbol{u}) \quad \Longleftrightarrow \quad \boldsymbol{u} \in \mathsf{D}(\Phi), \quad \Phi(\boldsymbol{w}) \ge \Phi(\boldsymbol{u}) + \langle \boldsymbol{\xi}, \boldsymbol{w}-\boldsymbol{u}\rangle + o(\|\boldsymbol{w}-\boldsymbol{u}\|)$$

  By convexity we also have

$$\boldsymbol{\xi} \in \partial\Phi(\boldsymbol{u}) \quad \Leftrightarrow \quad \boldsymbol{u} \in \mathsf{D}(\Phi), \quad \Phi(\boldsymbol{w}) \ge \Phi(\boldsymbol{u}) + \langle \boldsymbol{\xi}, \boldsymbol{w}-\boldsymbol{u}\rangle \quad \text{for every } \boldsymbol{w} \in X.$$

- The **proper domain of** $\partial\Phi$ is $\mathsf{D}(\partial\Phi) := \big\{\boldsymbol{u} \in \mathsf{D}(\Phi) : \partial\Phi(\boldsymbol{u}) \neq \emptyset\big\}$.
- The **minimal selection** $\partial^\circ\Phi$ is the element of minimal norm in $\partial\Phi(\boldsymbol{u})$.
- The **slope** of Φ is

$$|\partial\Phi|(\boldsymbol{u}) := \limsup_{\boldsymbol{w}\to\boldsymbol{u}} \frac{\big(\Phi(\boldsymbol{u}) - \Phi(\boldsymbol{w})\big)_+}{\|\boldsymbol{w}-\boldsymbol{u}\|} = \sup_{\boldsymbol{w}\neq\boldsymbol{u}} \frac{\big(\Phi(\boldsymbol{u}) - \Phi(\boldsymbol{w})\big)_+}{\|\boldsymbol{w}-\boldsymbol{u}\|}$$

Theorem

The slope $\boldsymbol{u} \mapsto |\partial\Phi|(\boldsymbol{u})$ is a lower semicontinuous functional ssatisfying

$$|\partial\Phi|(\boldsymbol{u}) = \begin{cases} \|\partial^\circ\Phi(\boldsymbol{u})\| & \text{if } \boldsymbol{u} \in \mathsf{D}(\partial\Phi) \\ +\infty & \text{otherwise} \end{cases}$$

Subgradient formulation of GF in the convex case

Definition

A locally absolutely continuous curve $\boldsymbol{u}:(0,+\infty)\to X$ is a Gradient Flow for Φ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{u}_t=\boldsymbol{v}_t,\quad \boldsymbol{v}_t\in-\partial\Phi(\boldsymbol{u}_t)\quad \text{for a.e. } t>0. \tag{GF}$$

Proposition *Let $\boldsymbol{u}:(0,+\infty)\to X$ be a locally absolutely continuous curve; GF is equivalent to the following equivalent properties:*

1. ***EVI (linear):***

$$\big\langle\dot{\boldsymbol{u}}_t,\boldsymbol{u}_t-w\big\rangle\le\Phi(w)-\Phi(\boldsymbol{u}_t)\quad \textit{a.e. in } (0,+\infty) \textit{ for every } w\in\mathsf{D}(\Phi)$$

2. ***EVI (Metric):*** $\mathsf{d}(\boldsymbol{u},w)=|\boldsymbol{u}-w|$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2(\boldsymbol{u}_t,w)\le\Phi(w)-\Phi(\boldsymbol{u}_t)\quad \textit{a.e. in } (0,+\infty), \textit{ for every } w\in\mathsf{D}(\Phi)$$

3. ***Maximal slope:*** $\boldsymbol{u}_t\in\mathsf{D}(\Phi)$ *for* $t>0$ *and*

$$\frac{1}{2}\int_0^t\Big(|\dot{\boldsymbol{u}}_r|^2+|\partial\Phi|^2(\boldsymbol{u}_r)\Big)\,\mathrm{d}r\le\Phi(\boldsymbol{u}_0)-\Phi(\boldsymbol{u}_t)$$

Main generation result: gradient flows of convex functionals in Hilbert spaces

Theorem (Komura, Kato, Crandall-Pazy, Brezis, etc...)

For every $\boldsymbol{u}_0 \in \overline{\mathsf{D}(\Phi)}$ *there exists a unique curve* $\boldsymbol{u}_t = \mathsf{S}_t[\boldsymbol{u}_0]$ *solution of* (GF) *such that* $\lim_{t\downarrow 0} \boldsymbol{u}_t = \boldsymbol{u}_0$.

- *The map* $t \mapsto \mathsf{S}_t[\cdot]$ *is a* ***continuous semigroup of contractions*** *in* $\overline{\mathsf{D}(\Phi)}$.
- $\boldsymbol{u}$ *is* ***locally Lipschitz*** *in* $(0,+\infty)$ *and for every* $t>0$ $\mathsf{S}_t[\boldsymbol{u}_0] \in \mathsf{D}(\partial\Phi) \subset \mathsf{D}(\Phi)$ *and satisfies the* ***regularization estimate***

$$\frac{1}{2}\mathsf{d}^2(\boldsymbol{u}_t, w) + t\big(\Phi(\boldsymbol{u}_t) - \Phi(w)\big) + \frac{t^2}{2}|\partial\Phi|(\boldsymbol{u}_t) \le \frac{1}{2}\mathsf{d}^2(\boldsymbol{u}_0, w) \quad \forall\, w \in D(\Phi)$$

- *The curves* $t \mapsto \boldsymbol{u}_t$ *and* $t \mapsto \Phi(\boldsymbol{u}_t)$ *are* ***right differentiable*** *at every* $t>0$ *and satisfies the* ***minimal selection principle***

$$\frac{\mathrm{d}}{\mathrm{d}t_+}\boldsymbol{u}_t = -\partial^\circ\Phi(\boldsymbol{u}_t),$$

$$t \mapsto -\frac{\mathrm{d}}{\mathrm{d}t_+}\Phi(\boldsymbol{u}_t) = |\dot{\boldsymbol{u}}_{t+}|^2 = |\partial\Phi|^2(\boldsymbol{u}_t) \quad \textit{is nonincreasing.}$$

A similar property holds for $t=0$ *if* $\boldsymbol{u}_0 \in \mathsf{D}(\partial\Phi)$ *and for the left derivative, except for an at most countable subset* $\mathcal{T} \subset (0,+\infty)$.

The Laplace operator as subgradient of $\mathfrak{D}$

Theorem

For every $\boldsymbol{u} \in \mathsf{D}(\mathfrak{D}) = W^{1,2}(\mathbb{R}^m)$ *the following properties are equivalent*

$$\boldsymbol{u} \in W^{2,2}(\mathbb{R}^m) \text{ and } \boldsymbol{\xi} := -\Delta \boldsymbol{u} \in X = L^2(\mathbb{R}^m) \tag{A}$$

$$\langle \boldsymbol{\xi}, \boldsymbol{w} - \boldsymbol{u} \rangle_{L^2(\mathbb{R}^m)} \le \mathfrak{D}(\boldsymbol{w}) - \mathfrak{D}(\boldsymbol{u}) \quad \text{for every } \boldsymbol{w} \in X \tag{B}$$

$$|\partial \mathfrak{D}|(\boldsymbol{u}) := \limsup_{\boldsymbol{w} \to \boldsymbol{u}} \frac{\mathfrak{D}(\boldsymbol{u}) - \mathfrak{D}(\boldsymbol{w})}{\|\boldsymbol{w} - \boldsymbol{u}\|_X} = \sup_{\boldsymbol{w} \neq \boldsymbol{u}} \frac{\mathfrak{D}(\boldsymbol{u}) - \mathfrak{D}(\boldsymbol{w})}{\|\boldsymbol{w} - \boldsymbol{u}\|_X} < +\infty. \tag{C}$$

In the case of (C) we also have $|\partial \mathfrak{D}|(\boldsymbol{u}) = \|\Delta \boldsymbol{u}\|_{L^2(\mathbb{R}^m)}$.

A simple variant: **Allen-Cahn equation.**
Choose a double-well potential $W : \mathbb{R} \to \mathbb{R}$ with

$$W''(r) \ge \boldsymbol{\lambda}, \qquad \lim_{r \to \pm\infty} r^{-1} W'(r) > 0.$$

The functional $\Phi(\boldsymbol{u}) := \frac{1}{2} \int_{\mathbb{R}^m} |\mathrm{D}\boldsymbol{u}|^2 \, \mathrm{d}x + \int_{\mathbb{R}^m} W(\boldsymbol{u}) \, \mathrm{d}x$ is $\boldsymbol{\lambda}$-convex

The L^2-gradient flow is $\boxed{\partial_t \boldsymbol{u} - \Delta \boldsymbol{u} + W'(\boldsymbol{u}) = 0}$

and applying the generation result one can find for any initial datum $\boldsymbol{u}_0 \in L^2(\mathbb{R}^m)$ a locally Lipschitz (in time) solution $\boldsymbol{u}$ with $\boldsymbol{u}_t \in W^{2,2}(\mathbb{R}^m)$ with $W'(\boldsymbol{u}_t) \in L^2(\mathbb{R}^m)$ for every $t > 0$.

Changing the dissipation

A different way to measure the velocity of an evolving family of functions $\boldsymbol{u} : \Omega \times [0, T] \to \mathbb{R}$:
represent $\boldsymbol{v} = \partial_t \boldsymbol{u}$ *as the Laplacian of a function* $\boldsymbol{z}$ *and take its* $W^{1,2}$*-seminorm:*

$$\|\boldsymbol{v}\| := \|\nabla \boldsymbol{z}\|_{L^2(\Omega;\mathbb{R}^m)} \quad \text{where} \quad -\Delta \boldsymbol{z} = \boldsymbol{v}, \quad \boldsymbol{z} \in W^{1,2}_0(\Omega).$$

We are considering the homogeneous $W^{-1,2}(\Omega)$-norm of $\partial_t \boldsymbol{u}$.
Choosing $\boldsymbol{z}_t := -\Delta^{-1} \boldsymbol{v}$ (with homogeneous B.C.) we have

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) &= -\int_\Omega \frac{\delta\Phi}{\delta\boldsymbol{u}} \partial_t \boldsymbol{u}_t \,\mathrm{d}x = \int_\Omega \frac{\delta\Phi}{\delta\boldsymbol{u}} \Delta \boldsymbol{z}_t \,\mathrm{d}x \\ &= -\int_\Omega \nabla \frac{\delta\Phi}{\delta\boldsymbol{u}} \nabla \boldsymbol{z}_t \,\mathrm{d}x \le \Big\| \nabla \frac{\delta\Phi}{\delta\boldsymbol{u}} \Big\|_{L^2(\Omega;\mathbb{R}^m)} \|\nabla \boldsymbol{z}\|_{L^2(\Omega;\mathbb{R}^m)} \end{aligned}$$

and we thus expect that

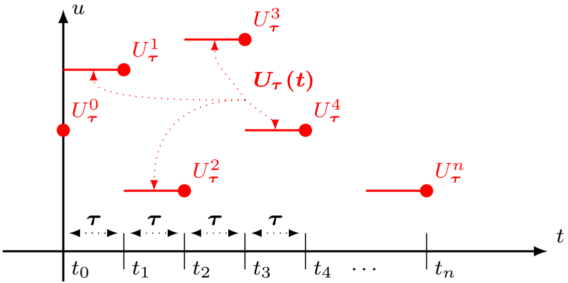
$$|\partial\Phi|(\boldsymbol{u}) := \Big\| \nabla \frac{\delta\Phi}{\delta\boldsymbol{u}} \Big\|_{L^2(\Omega;\mathbb{R}^m)}, \quad \text{with} \quad \frac{\delta\Phi}{\delta\boldsymbol{u}} \in W^{1,2}_0(\Omega)$$

$$\textit{Maximal dissipation rate} \quad \rightsquigarrow \quad \boldsymbol{z}_t := -\frac{\delta\Phi}{\delta\boldsymbol{u}}$$

$$\textit{Gradient flow:} \qquad \boxed{\partial_t \boldsymbol{u} = \Delta \frac{\delta\Phi}{\delta\boldsymbol{u}}}$$

Construction of the semigroup S: implicit Euler scheme

- Choose a partition of $(0, +\infty)$ of ***step size*** $\boldsymbol{\tau} > 0$



- Starting from $U_{\tau}^{0} := \boldsymbol{u}_0$ **find recursively** $U_{\tau}^{n}, \; n = 1, 2, \ldots,$

$$\frac{U_{\tau}^{n} - U_{\tau}^{n-1}}{\boldsymbol{\tau}} + \nabla\Phi(U_{\tau}^{n}) = 0 \quad \Leftrightarrow \quad U_{\tau}^{n} \in \operatorname*{argmin}_{V} \frac{\mathsf{d}^2(V, U_{\tau}^{n-1})}{2\tau} + \Phi(V)$$

- $\boldsymbol{U_{\tau}}$ is the **piecewise constant** interpolant of $\{U_{\tau}^{n}\}_n$.
- **Uniform Cauchy estimate in the convex case**

$$|\boldsymbol{U_{\eta}} - \boldsymbol{U_{\tau}}| \le (\sqrt{\boldsymbol{\tau}} + \sqrt{\boldsymbol{\eta}})(\Phi(\boldsymbol{u}_0) - \Phi_{\min}).$$

[Brezis, Crandall-Liggett,. . . , Baiocchi, Rulla, Nochetto-S.-Verdi]

Possible applications

Brezis, Crandall, Liggett, Bénilan, Pazy, J.L.Lions, Kato, Barbu, . . . ∼'70	*Contraction semigroups in Hilbert spaces, quasilinear parabolic P.D.E.'s, variational inequalities,. . .*
De Giorgi, Degiovanni, Marino, Saccon, Tosques, . . . ('80-'90)	*Abstract theory of minimizing movements and curves of maximal slope*
Luckhaus-Sturzenecker, Almgren-Taylor-Wang, . . . , Jost, Mayer,... ∼'90	*Geometric evolution problems, flows of harmonic maps, . . .*
Luckhaus, Visintin, Mielke-Theil-Levitas, Mielke, Rossi-S., Dal Maso, Serfaty, . . . '90∼'10	*Phase transitions, hystheresis, doubly nonlinear equations, Ginzburg-Landau, . . .*
Otto, Jordan, Kinderlehrer, Walkington, Agueh, Ghossoub, Carrillo-McCann-Villani, Ambrosio-Gigli-S., . . . '98∼'10	*Diffusion equations, Wasserstein distance*

In general only ***convergence results possibly up to subsequences*** are known...

Different directions...

1. **The "weakest" theory**: gradient flows are just

   limit (up to subsequences) of the Minimizing Movement Method.

   Applying lower semicontinuity and compactness arguments, the variational approximation is useful to construct a candidate solution, which is then studied by "ad hoc" meethods.

2. **Curves of maximal slope:** Extends to general metric space the differential identity

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\boldsymbol{u}_t) = \frac{1}{2}|\dot{\boldsymbol{u}}_t|^2 + \frac{1}{2}|\partial\Phi|^2(\boldsymbol{u}_t)$$

   and gives more insight on the solution, its stability, and its limit behaviour It has interesting results also in Hilbert/Banach spaces.

3. **The Hilbert-like theory**: it is modeled on the results for

   convex (or λ-convex) functionals in Euclidean/Hilbert spaces

   and gives the strongest results under restrictive assumptions on the
   - functional ϕ $\rightsquigarrow$ **"convexity"**
   - space $\rightsquigarrow$ **"Euclidean like"**

   Solution are characterized by the Evolution Variational Inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\mathsf{d}^2(\boldsymbol{u}_t, \boldsymbol{w}) \le \Phi(\boldsymbol{w}) - \Phi(\boldsymbol{u}_t)$$

Some general references on nonlinear semigroups in Hilbert/Banach spaces and applications

M. G. Crandall and T. M. Liggett.
Generation of semi-groups of nonlinear transformations on general Banach spaces.
Amer. J. Math., 93:265–298, 1971.

H. Brézis.
Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert.
North-Holland Publishing Co., Amsterdam, 1973.
North-Holland Mathematics Studies, No. 5. Notas de Matemática (50).

Viorel Barbu.
Nonlinear semigroups and differential equations in Banach spaces.
Editura Academiei Republicii Socialiste România, Bucharest, 1976.
Translated from the Romanian.

R. E. Showalter.
Monotone operators in Banach space and nonlinear partial differential equations.
American Mathematical Society, Providence, RI, 1997.

Some references to the metric theory of gradient flows and minimizing movements

Ennio De Giorgi, Antonio Marino, and Mario Tosques.
Problems of evolution in metric spaces and maximal decreasing curve.
Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8), 68(3):180–187, 1980.

Antonio Marino, Claudio Saccon, and Mario Tosques.
Curves of maximal slope and parabolic variational inequalities on nonconvex constraints.
Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 16(2):281–330, 1989.

Ennio De Giorgi.
New problems on minimizing movements.
In Claudio Baiocchi and Jacques Louis Lions, editors, *Boundary Value Problems for PDE and Applications*, pages 81–98. Masson, 1993.

Luigi Ambrosio.
Minimizing movements.
Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5), 19:191–246, 1995.

Luigi Ambrosio, Nicola Gigli, and Giuseppe Savaré.
Gradient flows in metric spaces and in the space of probability measures.
Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2005.

Sara Daneri, Giuseppe Savaré.
Lecture notes on gradient flows and optimal transport
"Optimal transportation: Theory and applications", Grenoble, June 2009.