The Semimartingale Property via Bounded Logarithmic Utility

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Abstract: This paper provides a new condition ensuring the preservation of semimartingality under enlargement of filtrations in the setting of continuous stochastic processes. Our main tool is the concept of utility maximization from mathematical finance. Normally, this approach would require an integration theory for non-semimartingales, however, by applying an approximation procedure we show how this obstacle can be circumvented. Finally, two examples are provided. The first one illustrates how our approach applies in situations not covered by the existing literature. The second one shows that our results do not carry over to processes with jumps.

Key Words: arbitrage, financial markets, logarithmic utility, semimartingales, stochastic processes, utility maximization

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1. INTRODUCTION AND SUMMARY

The question of whether an $\mathbb{F}$-semimartingale is also a $\mathbb{G}$-semimartingale - for filtrations $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,1]}$ with $\mathcal{F}_t \subseteq \mathcal{G}_t,\ t \in [0,1],$ - has been studied extensively in the literature and is known as the *problem of enlargement of filtrations*. This framework has been applied in mathematical finance as a model of inside trading. An $\mathbb{F}$-semimartingale is viewed as a model for the stock price, and different filtrations represent different levels of information different investors have access to. The larger amount of information the investor has at hand, the better he or she can predict future price movements, thereby increasing potential future wealth (compared to the less informed investor). The situation in which the better informed investor - let us call her *the insider* - is able to perfectly predict the future price movements and is able to take advantage of those predictions, is known as *arbitrage for the insider*. It is well-known that this property is closely related to the question of whether or not the stock price evolves as a $\mathbb{G}$-semimartingale, see e.g. Rogers (1997) and Björk and Hult (2005) and the discussions therein. In particular, Delbaen and Schachermayer (1994) guarantee $\mathbb{G}$-semimartingality under a condition related to the notion of no free lunch with vanishing risk (see Theorem 2.5 for the precise statement).

The study of the important special case where $\mathcal{G}_t \triangleq \mathcal{F}_t \vee \sigma(L)$, for some random variable $L \in \mathcal{F}_1$, was initiated in the seminal work of Itô (1978). Such enlargements are usually referred to as *initial enlargements* and have been investigated in great generality. Probably the most notable result is due to Jacod and is known as *Jacod's Criterion* (Theorem 10, p. 363 in Protter (2004)). It grants $\mathbb{G}$-semimartingality when the conditional distribution of L given $\mathcal{F}_t$ is well behaved. Initial enlargements were first studied in the context of mathematical finance in Karatzas and Pikovsky (1996), where the authors quantified the excess utility that an insider can generate by trading in a financial market. They show that this quantity is determined by the *information drift*, i.e. the knowledge of L adds to the drift of the stock price S, modeled by a continuous $\mathbb{F}$ - semimartingale. The results of Karatzas and Pikovsky (1996) have since been extended in various directions: Amendinger, Imkeller, and Schweizer (1998) relate the extra utility an insider can generate to the entropy of L, Elliott, Geman, and Korkie (1997) consider a more general continuous setting allowing multiple sources of uncertainty, whereas Elliott and Jeanblanc

(1999) study the discontinuous case. It is well-known that Jacod's Criterion is only sufficient for preservation of $\mathbb{G}$-semimartingality; several relaxations are provided in Imkeller, Pontier, and Weisz (2001) and Imkeller (2003). Finally, Corcuera, Imkeller, Kohatsu-Higa, and Nualart (2004) consider a model where L is only partially observed, but the variance of the signal noise reduces over time, thereby enabling the insider to make better and better forecasts of future price movements.

Another important class of filtration enlargements are so-called *progressive enlargements*. Here τ is a non-negative random variable and $\mathbb{G}$ is the minimal extension of $\mathbb{F}$ making τ a $\mathbb{G}$-stopping time. The study of progressive enlargements has been initiated by Barlow and subsequently extended by Jeulin and Yor in a series of papers, see e.g. the references in Protter (2004). This theory has found its way into utility maximization through the work of Imkeller (2002).

In the present paper we study the general problem of filtration enlargement and do not explicitly require any special relationship between $\mathbb{F}$ and $\mathbb{G}$ - in particular, our approach covers both initial and progressive enlargements. We are therefore automatically placed in a situation without an integration theory which would help us define the insider's wealth dynamics. In other words, there is no general theory of stochastic integration for $\mathbb{F}$-semimartingales with respect to $\mathbb{G}$-adapted integrands. One possible outlet is to introduce the theory of forward integration allowing integrands which are not necessarily $\mathbb{F}$-predictable and integrators in the class of $\mathbb{F}$-martingales. The theory of forward integration has been applied to study filtration enlargements in Biagini and Øksendal (2005) in the Brownian setting, and in Di Nunno, Meyer-Brandis, Øksendal, and Proske (2003) in the more general Lévy setting. In these papers Malliavin calculus and Wick analysis are used to grant the $\mathbb{G}$-semimartingale decomposition under the assumption that an optimal portfolio for the insider exists in the class of forward-integrable processes. In general, it can be difficult to ensure that a $\mathbb{G}$-optimizer exists since the standard duality results à la Kramkov and Schachermayer (1999) cannot be applied.

Our approach differs from the existing literature in that we restrict the investor to choose her strategy from the class of portfolios whose proportion of the wealth invested in the risky asset lies in the family of simple $\mathbb{G}$-adapted stochastic processes. The observation that the canonical definition of the stochastic exponential is possible as soon as the integrator has a pathwise quadratic variation and the integrand is simple, circumvents the lack of integration theory for (a priory) non-semimartingales. In the setting of continuous $\mathbb{F}$-semimartingales, our main result states that if the $\mathbb{G}$-investor has a uniform upper bound on the expected logarithmic utility arising from simple portfolios, then the stock price process remains a semimartingale with respect to the filtration $\mathbb{G}$. An explicit example show that the domain of applicability of our result differs from that of the already mentioned celebrated result of Delbaen and Schachermayer (1994). Our results differ from those of Biagini and Øksendal (2005) in several ways. First, we do not need the machinery of Malliavin calculus as the proof of our main theorem relies on a simple Hilbert-space argument. Second, we do not require the existence of an optimal trading strategy - we only need a uniform finite upper bound on the expected utilities of all simple portfolios. Finally, our approach is not necessarily restricted to the study of a Brownian motion - it can deal with any continuous process.

The transition to the class of $\mathbb{F}$-semimartingales with jumps presents us with an unexpected twist. We construct an example of an $\mathbb{F}$-semimartingale $\{S_t\}_{t\in[0,1]}$ with the property that it 1) does not remain a semimartingale under $\mathbb{G}$, and 2) admits a finite upper bound on the expected logarithmic utility from simple portfolios. In other words, our main result does not extend to the case of general discontinuous processes. From the modeling point of view, it shows that there exists a class of non-semimartingales which, if used as models for traded financial assets, would not be in infinite demand by risk-averse agents. Even though these processes will allow for arbitrage in the wide sense (there will be free lunch with vanishing risk), no risk-averse agent will choose to hold large positions in such strategies. From another point fo view, this example sheds some light on the use of general stochastic integrals as a mathematical idealization of the actually implementable integrals of simple, piecewise constant integrands. On one hand, such idealization brings a better understanding of the underlying phenomena and a host of explicitly computable quantities, as witnessed by the rapid expansion of the field of mathematical finance. On the other hand, the properties of the process S point to the existence of a wide gap between simple and general integrands when we focus our attention on the risk-averse agents.

The paper is organized as follows. After the Introduction, Section 2. focuses on the continuous processes and contains the proof of our main result, as well as an example illustrating its scope. Section 3. is devoted to an example showing the existence of a discontinuous non-semimartingale, trading in which can produce only bounded expected logarithmic utility.

2. CONTINUOUS PROCESSES

2.1. Notation

We start with the case where the stock-price process is modeled by a continuous $\mathbb{F}$-semimartingale S on the unit time horizon $[0,1]$, where $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,1]}$ is a complete and right-continuous filtration on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For any complete and right-continuous filtration $\mathbb{G} = \{\mathcal{G}_t\}_{t\in[0,1]}$, satisfying $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in [0,1]$, we let $\mathcal{H}^{simp}(\mathbb{G})$ denote the set of all stochastic processes $\{\pi_t\}_{t\in[0,1]}$ of the form:

$$\pi_t = \sum_{i=1}^{n} K_i \mathbf{1}_{(T_{i-1},T_i]}(t), \tag{2.1}$$

where $n \in \mathbb{N}$, $0 = T_0 \leq T_1 \leq \ldots, \leq T_n = 1$ are $\mathbb{G}$-stopping times and $K_i \in \mathbb{L}^\infty(\mathcal{G}_{T_{i-1}})$, $i = 1, \ldots, n$. Further,

$$\mathcal{H}^2(\mathbb{G}) \triangleq \{\pi : \pi \text{ is } \mathbb{G}\text{-predictable and } ||\pi||_{\mathcal{H}^2} < \infty\},$$

where $||\pi||^2_{\mathcal{H}^2} \triangleq \mathbb{E}\int_0^1 \pi_u^2\, d[S]_u \in [0,\infty]$. We shall also have use for the classes $\mathcal{H}^{simp}(\mathbb{F})$ and $\mathcal{H}^2(\mathbb{F})$, defined analogously, as well as $L(\mathbb{F})$, where

$$L(\mathbb{F}) \triangleq \left\{\pi \,:\, \pi \text{ is } \mathbb{F}\text{-predictable and } \int_0^1 \pi_u^2\, d[S]_u < \infty \text{ a.s.}\right\},$$

is the set of all $\mathbb{F}$-predictable S-integrable processes.

2.2. Canonical definition of stochastic exponentials

Even though we have no integration theory for the the general $\mathbb{G}$-adapted integrands with respect to S, we can still define the stochastic integral for $\mathcal{H}^{simp}(\mathbb{G})$ in the familiar way

$$(\pi \cdot S)_t = \int_0^t \pi_u\, dS_u \triangleq \sum_{i=1}^{n} K_i (S_{T_i \wedge t} - S_{T_{i-1}\wedge t}), \tag{2.2}$$

where the process $\pi \in \mathcal{H}^{simp}(\mathbb{G})$ is of the form (2.1). More importantly for our results, stochastic exponentials can be defined canonically as well. As usual, for

$\pi \in L(\mathbb{F})$, let $\mathcal{E}(\pi \cdot S)$ denote the unique solution Z to the Doléans-Dade stochastic differential equation

$$Z_t = Z_0 + \int_0^t Z_u \pi_u \, dS_u, \ t \in [0,1]. \tag{2.3}$$

We will show shortly that for $\pi \in \mathcal{H}^{simp}(\mathbb{G})$, the equation (2.3) still admits a pathwise solution $\mathcal{E}(\pi \cdot S)$ given by

$$\mathcal{E}(\pi \cdot S)_t \triangleq \exp(\int_0^t \pi_u \, dS_u - \frac{1}{2} \int_0^t \pi_u^2 \, d[S]_u). \tag{2.4}$$

where the dS-integral inside the exponential function is defined by (2.2). Of course, for $Z = \mathcal{E}(\pi \cdot S)$, the integrand $Z_u \pi_u$ appearing in (2.3) is not necessarily in $\mathcal{H}^{simp}(\mathbb{G})$. Nevertheless, the integral $\int_0^t Z_u \pi_u \, dS_u$ exists a.s. as a limit of Riemann sums and equals $Z_t - Z_0$. Indeed, by the semimartingale property of S, there exist an event $\Omega' \subset \Omega$ of full measure and a sequence $0 = \tau_0^n \leq \tau_1^n \leq \cdots \leq \tau_{m_n}^n = 1$, $n \in \mathbb{N}$ of stopping times subdividing the interval $[0,1]$ so that

$$\lim_n \sum_{k=1}^{m_n} (S_{\tau_k^n \wedge t} - S_{\tau_{k-1}^n \wedge t})^2 = [S]_t, \text{ and } \lim_n \sup_{1 \leq k \leq m_n} \Delta \tau_k^n = 0 \text{ on } \Omega'. \tag{2.5}$$

where $\Delta \tau_k^n \triangleq \tau_k^n - \tau_{k-1}^n$. Without loss of generality we can assume that the stopping times $T_0, T_1, \ldots, T_n$ in the definition (2.1) of π already belong to the subdivision $(\tau_k^n)_{n,k}$ for all n. This amounts to the assumption that π is constant on each interval $(\tau_{k-1}^n, \tau_k^n]$. Therefore, we have the following chain of equalities where we use the shorthand $\Delta S_k^n = S_{\tau_k^n} - S_{\tau_{k-1}^n}$

$$\begin{aligned} Z_1 - Z_0 &= \sum_{k=1}^{m_n} (Z_{\tau_k^n} - Z_{\tau_{k-1}^n}) \\ &= \sum_{k=1}^{m_n} Z_{\tau_{k-1}^n} \left(\exp(\int_{\tau_{k-1}^n}^{\tau_k^n} \pi_u \, dS_u - \frac{1}{2} \int_{\tau_{k-1}^n}^{\tau_k^n} \pi_u^2 \, d[S]_u) - 1 \right) \\ &= I_1^n + I_2^n, \end{aligned}$$

where $I_1^n = \sum_{k=1}^{m_n} Z_{\tau_{k-1}^n} \left(\exp(\pi_{\tau_k^n} \Delta S_k^n - \frac{1}{2}(\pi_{\tau_k^n} \Delta S_k^n)^2) - 1 - \pi_{\tau_k^n} \Delta S_k^n \right)$, and $I_2^n = \sum_{k=1}^{m_n} Z_{\tau_{k-1}^n} \pi_{\tau_k^n} \Delta S_k^n$. Thanks to the simple estimate $\left| e^{x - \frac{1}{2}x^2} - 1 - x \right| \leq e^{|x|} |x|^3$ we have $|I_1^n| \leq A_n B_n C_n D_n$, where

$$\begin{aligned} A_n &= \sup_{t \in [0,1]} Z_t, & B_n &= \sup_{1 \leq k \leq m_n} e^{|\pi_{\tau_k^n} \Delta S_k^n|}, \\ C_n &= \sum_{k=1}^{m_n} |\pi_{\tau_k^n}|^3 (\Delta S_k^n)^2, \text{ and} & D_n &= \sup_{1 \leq k \leq m_n} |\Delta S_k^n| \end{aligned}$$

Due to the (uniform) continuity of trajectories of processes Z and S, boundedness of π and the equation (2.5), we can easily conclude that $\limsup_n A_n B_n C_n < \infty$ and that $\lim_n D_n = 0$. Therefore $\lim I_1^n = 0$, and it follows that

$$\int_0^1 \pi_u Z_u \, dS_u \triangleq \lim_n \sum_{k=1}^{m_n} Z_{\tau_{k-1}^n} \pi_{\tau_k^n} \Delta S_k^n = Z_1 - Z_0, \text{ a.s.}$$

The same analysis applies to the case in which we replace Z_1 by Z_t for any $t \in [0,1]$.

2.3. The wealth process and the expected growth rates

Thanks to the above discussion, the process

$$W_t^\pi \triangleq W_0^\pi \mathcal{E}(\pi \cdot S)_t \tag{2.6}$$

can be interpreted as the wealth of a financial agent who invests in the asset S and uses π as a portfolio process More precisely π is the quantity-denominated proportion of the wealth invested in S, i.e. the number of shares of the asset S held in the agent's portfolio at time t is given by $\pi_t W_t^\pi$. As usual, W_0^π denotes the agent's initial wealth, from now on normalized to be equal to one dollar, i.e. $W_0^\pi \triangleq 1$. We always assume that $\pi \in \mathcal{H}^{simp}(\mathbb{G})$, so that the process $\mathcal{E}(\pi \cdot S)$ in (2.6) is well-defined. In mathematical finance it is customary to evaluate the performance of a trading strategy by computing the so-called *expected utility of terminal wealth*

$$\pi \mapsto \mathbb{E}[U(W_1^\pi)],$$

where U is a *utility function* - a concave and increasing function defined on the positive real semi-axis. The increase of U captures the fact that more wealth is better than less, but its concavity penalizes for any risk exposure. Different shapes of utility functions correspond to different "risk vs. return" preferences. In the present paper we deal exclusively with the case of $U(x) \triangleq \log(x)$. This is, arguably, the single most important utility function - certainly the most studied one. The expected logarithmic utility

$$L(\pi) \triangleq \mathbb{E}[\log(W_1^\pi)]$$

can also be interpreted as the expected average growth-rate of wealth over the interval $[0,1]$.

REMARK 2.1. *The portfolio which achieves the maximal growth rate (the growth optimal portfolio) has a long history originating from gambling considerations, see Breiman (1961) and Kelly (1956). More recently, the wealth process corresponding to this portfolio has been suggested as a pricing device by Platen (2002). The idea of using the growth optimal portfolio as a pricing tool stems from the fact that the concept of no arbitrage is often not strong enough to guarantee existence of a pricing measure in the traditional sense, but the growth optimal portfolio can still be formed. It is well-known that the wealth generated by any admissible portfolio becomes a supermartingale when measured in terms of the wealth process of the growth optimal portfolio, and consequently, no arbitrage pricing can be preformed without the existence of a pricing measure. The numèraire properties of the growth optimal portfolio and the corresponding pricing theory have been studied by e.g. Long (1990) and Becherer (2001).*

2.4. Excess growth for an insider

When the stock market exhibits no growth (i.e. when S is a martingale) it is intuitively obvious that the investor with access only to public information can expect no excess return in the market no matter what his/her risk exposure is. Formally,

$$\mathbb{E}[\log(W_1^\pi)] = \mathbb{E}[\int_0^1 \pi_u \, dS_u - \frac{1}{2}\int_0^1 \pi_u^2 \, d[S]_u] \leq 0, \text{ for } \pi \in \mathcal{H}^2(\mathbb{F}), \tag{2.7}$$

since the first term under the expectation operator is the terminal value of a uniformly integrable martingale (vanishing at $t = 0$), and the second term is a.s. negative. Moreover, the inequality in (2.7) becomes an equality if and only if $\pi_u(\omega) = 0$, $d[S]$-a.e, where $d[S]$ denotes the random measure on the predictable sets generated by the process $[S]$. This formal argument aligns perfectly with the intuitive one stated above.

What happens when the agent has access to additional (insider) information? In this paper we shall be concerned with the following variant of this question:

QUESTION 2.2. *What can be said about the semimartingale property of the process S w.r.t. filtration $\mathbb{G}$, if the excess growth (logarithmic utility) of the insider's portfolios admits a uniform upper bound?*

One of our main results is the following answer to Question 2.2 when the process S is continuous.

THEOREM 2.3. *Let $\{S_t\}_{t\in[0,1]}$ be a continuous $\mathbb{F}$-adapted stochastic process. Suppose that the excess growth of the insider's portfolios is (uniformly) bounded from above, i.e. suppose that*

$$\sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathbb{E}[\log(W_1^\pi)] < \infty. \tag{2.8}$$

Then S is a $\mathbb{G}$-semimartingale with the decomposition $S_t = \hat{S}_t + \int_0^t \alpha_u \, d[S]_u$, $t \in [0,1]$, where $\hat{S}_t$ is a $\mathbb{G}$-martingale, and $\{\alpha_t\}_{t\in[0,1]}$ is a $\mathbb{G}$-predictable process satisfying

$$\mathbb{E}[\int_0^1 \alpha_u^2 \, d[S]_u] < \infty. \tag{2.9}$$

Conversely, suppose that the process $\{S_t\}_{t\in[0,1]}$ is a $\mathbb{G}$-semimartingale with the decomposition $S_t = M_t + A_t$, $t \in [0,1]$, where $\{M_t\}_{t\in[0,1]}$ is a local martingale, and $\{A_t\}_{t\in[0,1]}$ is a process of finite variation. Then

$$\sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathbb{E}[\log(W_1^\pi)] = \frac{1}{2}\mathbb{E}[\int_0^1 \alpha_u^2 \, d[S]_u],$$

if M is a martingale, and the process A is of the form $A_\cdot = \int_0^\cdot \alpha_u \, d[S]_u$, for some process $\alpha \in \mathcal{H}^2(\mathbb{G})$. Otherwise, $\sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathbb{E}[\log(W_1^\pi)] = +\infty$.

Proof. ($\Rightarrow$) Let $\mathcal{L}(\pi) = \mathbb{E}[\log(W_1^\pi)]$, so that for $\pi \in \mathcal{H}^{simp}(\mathbb{G})$

$$\mathbb{E}[\int_0^1 \pi_u \, dS_u] - \frac{1}{2}\mathbb{E}[\int_0^1 \pi_u^2 \, d[S]_u] = \mathcal{L}(\pi) \le C,$$

where $C \triangleq \sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathcal{L}(\pi) \ge 0$. Therefore, the functional $\Lambda : \mathcal{H}^{simp}(\mathbb{G}) \to \mathbb{R} \cup \{\pm\infty\}$ given by $\Lambda(\pi) \triangleq \mathbb{E}[\int_0^t \pi_u \, dS_u]$, is linear and finite-valued and admits the following bound

$$\Lambda(\pi) \le C + \frac{1}{2}||\pi||_{\mathcal{H}^2}^2,$$

where $||\pi||_{\mathcal{H}^2}^2 = \mathbb{E}[\int_0^1 \pi_u^2 \, d[S]_u]$. This bound can be strengthened if we note that for each $\gamma > 0$ we have

$$\Lambda(\pi) = \frac{1}{\gamma}\Lambda(\gamma\pi) \le \frac{C}{\gamma} + \frac{\gamma}{2}||\pi||_{\mathcal{H}^2}^2. \tag{2.10}$$

A minimization of the right-most part of (2.10) with respect to γ yields that

$$\Lambda(\pi) \le \sqrt{2C}||\pi||_{\mathcal{H}^2},$$

from which we can easily conclude that Λ is a continuous linear functional defined on the linear subspace $\mathcal{H}^{simp}(\mathbb{G})$ of $\mathcal{H}^2(\mathbb{G})$. It is well known that $\mathcal{H}^{simp}(\mathbb{G})$ is dense in $\mathcal{H}^2(\mathbb{G})$ with respect to the topology induced by the norm $||\cdot||_{\mathcal{H}^2}$ (Proposition 2.8, p. 137 in Karatzas and Shreve (1991)). Therefore, the linear functional Λ admits a unique linear and continuous extension to $\mathcal{H}^2(\mathbb{G})$. The Riesz representation theorem guarantees the existence of a process $\{\alpha_t\}_{t\in[0,1]} \in \mathcal{H}^2(\mathbb{G})$ such that

$$\mathbb{E}\int_0^1 \pi_u \, dS_u = \Lambda(\pi) = \mathbb{E}\int_0^1 \pi_u \alpha_u \, d[S]_u, \tag{2.11}$$

for all $\pi \in \mathcal{H}^{simp}(\mathbb{G})$. For a $\mathbb{G}$-stopping time τ and $\pi \in \mathcal{H}^{simp}(\mathbb{G})$ defined by $\pi_t = \mathbf{1}_{\{t\leq\tau\}}$, the equation (2.11) becomes

$$\mathbb{E}[S_\tau] = \mathbb{E}\int_0^\tau \alpha_u \, d[S]_u, \text{ or } \mathbb{E}[\hat{S}_\tau] = 0, \tag{2.12}$$

where $\hat{S}_t = S_t - \int_0^t \alpha_u \, d[S]_u$. Since (2.12) holds for all $\mathbb{G}$-stopping times τ it follows that $\{\hat{S}_t\}_{t\in[0,1]}$ is a martingale. Therefore, the process $\{S_t\}_{t\in[0,1]}$ is a $\mathbb{G}$-semimartingale with the decomposition

$$S_t = \hat{S}_t + \int_0^t \alpha_u \, d[S]_u.$$

($\Leftarrow$) Suppose that $\{S_t\}_{t\in[0,1]}$ is a continuous $\mathbb{G}$-semimartingale with the decomposition $S_t = M_t + A_t$, where M is a local martingale and A is a process of finite variation. When $\sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathbb{E}[\log(W_1^\pi)] < \infty$, the ($\Rightarrow$) part of the proof tells us that M is a martingale, and that A must be of the form $A_t = \int_0^t \alpha_u \, d[S]_u$ for some predictable process $\{\alpha_t\}_{t\in[0,1]}$ with $\mathbb{E}[\int_0^1 \alpha_u^2 \, d[S]_u] < \infty$. For any $\pi \in \mathcal{H}^{simp}(\mathbb{G})$ we then have

$$\begin{aligned}\mathcal{L}(\pi) &= \mathbb{E}\int_0^1 \pi_u \, dS_u - \frac{1}{2}\,\mathbb{E}\int_0^1 \pi_u^2 \, d[S]_u \\ &= \mathbb{E}\int_0^1 (\pi_u\alpha_u - \frac{1}{2}\pi_u^2) \, d[S]_u \leq \frac{1}{2}\,\mathbb{E}\int_0^1 \alpha_u^2 \, d[S]_u.\end{aligned} \tag{2.13}$$

Since $\mathcal{H}^{simp}(\mathbb{G})$ is dense in $\mathcal{H}^2(\mathbb{G})$, the process $\{\alpha_t\}_{t\in[0,1]}$ can be approximated by elements of $\mathcal{H}^{simp}(\mathbb{G})$, and so the inequality in (2.13) is sharp, i.e.

$$\sup_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathcal{L}(\pi) = \frac{1}{2}\,\mathbb{E}\int_0^1 \alpha_u^2 \, d[S]_u.$$

∎

Remark 2.4. *Theorem 2.3 can be understood as a test of semimartingality in a context wider than that of the enlargement of filtrations. We do not necessarily have to start with a process that is an $\mathbb{F}$-semimartingale. In fact, it will apply to any continuous stochastic process $\{S_t\}_{t\in[0,1]}$ for which the pathwise existence of the quadratic variation $[S]$ can be ascertained.*

2.5. An example

To finish the discussion of the continuous case, in this subsection we compare the domain of applicability of our main theorem to an existing similar result. More precisely, we contrast Theorem 2.3 with a result of Delbaen and Schachermayer (Delbaen and Schachermayer (1994), Theorem 7.2, p 504) in which the authors relate the semimartingale property of a stochastic process to the $\mathbb{L}^\infty$-closure of the set of outcomes of simple stochastic integrands with respect to it. The following is a restatement of that result in our framework - rewritten to conform with our notation and augmented by a useful strengthening (Delbaen and Schachermayer (1994), Theorem 7.6, p. 509) when $\{S_t\}_{t\in[0,1]}$ is a continuous process.

Theorem 2.5 (Delbaen and Schachermayer). *Let $\{S_t\}_{t\in[0,1]}$ be a locally bounded $\mathbb{F}$-adapted càdlàg stochastic process with the property*

$$\overline{\mathcal{C}^{simple}} \cap \mathbb{L}^\infty_+ = \{0\}\,, \tag{2.14}$$

where $\overline{(\cdot)}$ denotes the norm-closure in $\mathbb{L}^\infty$ and

$$\mathcal{C}^{simple} \triangleq \left\{ \int_0^1 \pi_u \, dS_u - \xi \,:\, \pi \in \mathcal{H}^{simp}(\mathcal{F}), \pi \text{ is admissible and } \xi \in \mathbb{L}^\infty_+ \right\}.$$

Then $\{S_t\}_{t\in[0,1]}$ is a semimartingale. Moreover, if $\{S_t\}_{t\in[0,1]}$ is a continuous process, then there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, such that $\{S_t\}_{t\in[0,1]}$ is a $\mathbb{Q}$-local martingale.

In order to show that our result is not a special case of Theorem 2.5 we exhibit the following example.

Example 2.6. *In the spirit of Remark 2.4, in this example we deal with a single augmented filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,1]}$, generated by a Brownian motion B. Let $\{\alpha_t\}_{t\in[0,T]}$ be an $\mathbb{F}$-progressively measurable process with the following two properties:*

1. $\mathbb{E}[\int_0^1 \alpha_u^2\, du] < \infty$, *and*

2. *The exponential process* $\{Z_t\}_{t\in[0,1]}$, $Z_t = \mathcal{E}(\int_0^1 \alpha_u\, dB_u) = \exp(\int_0^1 \alpha_u\, dB_u - \int_0^1 \alpha_u^2\, du)$ *is a local martingale, but not a true martingale.*

Let the stock-price process $\{S_t\}_{t\in[0,1]}$ *be given by* $S_t = B_t + \int_0^t \alpha_u\, du$. *It is clear that (by the second part of Theorem 2.3 itself) there is an upper bound on the expected logarithmic utility, i.e.*

$$\sup_{\pi\in\mathcal{H}^{simp}(\mathcal{F})} \mathbb{E}[\log(W_1^\pi)] = \frac{1}{2}\mathbb{E}[\int_0^1 \alpha_u^2\, du] < \infty.$$

Therefore, Theorem 2.3 can be applied to conclude that S *is a semimartingale. On the other hand. suppose that the semimartingality of* S *can be obtained as a conclusion of Theorem 2.5. Then, there would exist a probability measure* $\mathbb{Q}$, *equivalent to* $\mathbb{P}$ *with the property that* S *is a* $\mathbb{Q}$*-local martingale. By the Martingale Representation Theorem, the only candidate for the density martingale* $\{\mathbb{E}[\frac{d\mathbb{Q}}{d\mathbb{P}}|\mathcal{F}_t]\}_{t\in[0,1]}$ *of the Radon-Nikodym derivative* $\frac{d\mathbb{Q}}{d\mathbb{P}}$ *is the exponential process* $Z_t = \mathcal{E}(\int_0^1 \alpha_u\, dB_u)$. *By the assumption (2) above, the process* Z *is not a martingale, and therefore we can conclude that there is no such measure* $\mathbb{Q}$ *and Theorem 2.5 cannot be applied in this case.*

REMARK 2.7. *The existence of a process* $\{\alpha_t\}_{t\in[0,T]}$ *satisfying (1) and (2) above can be deduced, for example, from the book of Liptser and Shiryaev (2001), Example 6. p. 235. Here is another simple construction of such a process.*
Let $(B_t^1, B_t^2, B_t^3)_{t\in[0,\infty)}$ *be a 3-dimensional Brownian motion and let* $\{R_t\}_{t\in[0,1]}$ *be the corresponding 3-dimensional Bessel process*

$$R_t = \sqrt{1 + (B_t^1)^2 + (B_t^2)^2 + (B_t^3)^2}$$

started at $R_0 = 1$. *We will show that its inverse* $\alpha_t = 1/R_t$ *satisfies (1) and (2) above. It is well known that its inverse* $\alpha_t = 1/R_t$ *is an example of a strictly positive local martingale which is not a martingale (it follows easily by computation of the expectation function* $t \mapsto \mathbb{E}[\alpha_t]$, *or Exercise 3.36, p. 168 in Karatzas and Shreve (1991)). Moreover, there exists a Brownian motion* B, *such that the process* α *satisfies the following stochastic differential equation*

$$d\alpha_t = -\alpha_t^2 dB_t,$$

and thus $\alpha = \mathcal{E}(\int_0^{\cdot} -\alpha_u \, dB_u)$ satisfies property (2). Property (1) follows imediately from the fact (easily verifiable via direct computation) that the function $t \mapsto \mathbb{E}[\alpha_t^2]$ is finite and decreasing.

3. PROCESSES WITH JUMPS

3.1. Definition of the Wealth Process

In this section we investigate whether it is possible to extend the results of Theorem 2.3 to the case when the stochastic process $\{S_t\}_{t\in[0,1]}$ admits jumps. We are facing the same problem as in the previous section, i.e. the non-existence of the canonical theory of stochastic integration for non-semimartingales. In order to deal with it, we recall the definition of the set $\mathcal{H}^{simp}(\mathbb{G})$ of simple integrands: it contains all stochastic processes $\{\pi_t\}_{t\in[0,1]}$ of the form

$$\pi_t = \sum_{i=1}^{n} K_i \mathbf{1}_{(T_{i-1},T_i]}(t),$$

where $0 = T_0 \leq T_1 \leq T_2 \leq \cdots \leq T_n = 1$ are $\mathbb{G}$-stopping times and $K_i \in \mathbb{L}^\infty(\mathcal{G}_{T_{i-1}})$. With the motivation from subsection 2.2, we can give a canonical definition of a stochastic exponential $\mathcal{E}(\pi \cdot M)$ of a $\mathcal{H}^{simp}(\mathbb{G})$-stochastic integral $\pi \cdot M$ with respect to an $\mathbb{F}$-local martingale M:

$$\mathcal{E}(\pi \cdot M)_t = \exp\left(\pi \cdot M_t - \frac{1}{2}\int_0^t \pi_u^2 \, d[M]_u^c\right) \prod_{s\leq t}(1 + \pi_s \Delta M_s) \exp(-\pi_s \Delta M_s),$$

remembering the well-known expression

$$\mathcal{E}(X)_t = \exp(X_t - \frac{1}{2}[X]_t^c) \prod_{s\leq t}(1 + \Delta X_s) \exp(-\Delta X_s) \tag{3.1}$$

for the (semimartingale) Doléans-Dade exponential of a semimartingale X. In general, let $\{S_t\}_{t\in[0,1]}$ be an $\mathbb{F}$-local martingale, and let $\pi \in \mathcal{H}^{simp}(\mathbb{G})$ be a simple $\mathbb{G}$-adapted integrand, interpreted as a portfolio-proportion process. It is natural to define the wealth process $\{W_t^\pi\}_{t\in[0,1]}$ of an agent using π as an investment strategy by

$$\begin{aligned} W_t^\pi &= \mathcal{E}(\pi \cdot S)_t \\ &= \exp\left((\pi \cdot S)_t - \frac{1}{2}\int_0^t \pi_s^2 d[S]_t^c\right) \prod_{s\leq t}(1 + \pi_s \Delta S_s) \exp(-\pi_s \Delta S_s). \end{aligned} \tag{3.2}$$

3.2. A counterexample

A question analogous to Question 2.2 can be posed in this setting as well:

QUESTION 3.1. *Can we guarantee that S is a $\mathbb{G}$-semimartingale, if we know that the expected logarithmic utility of the insider's portfolios admits a uniform upper bound?*

Surprisingly, the answer is negative! In order to substantiate this claim, we exhibit an example of an $\mathbb{F}$-semimartingale $\{S_t\}_{t\in[0,1]}$ with the following properties:

$$\left.\begin{array}{ll} \text{(NS)} & S \text{ is } \textit{not} \text{ a } \mathbb{G}\text{-semimartingale, but} \\ \text{(FL)} & \sup\limits_{\pi\in\mathcal{H}^{simp}(\mathbb{G})} \mathbb{E}[\log(W_1^\pi)] < \infty. \end{array}\right\} \tag{3.3}$$

Before giving the details of our construction let us pause and try to explain the intuition behind the example. The central idea is that the introduction of jumps into the dynamics of the stock price can lead to a drastic restriction of the set of portfolios at the disposal of a logarithmic utility maximizer. Simply, any portfolio leading to a negative terminal wealth with positive probability will yield an expected utility of negative infinity, and is therefore clearly inferior to the constant portfolio $\pi \equiv 0$. Suppose that the process S jumps in an unpredictable fashion, while its continuous part fails the semimartingale property just barely. In that case we might be able to envision the situation in which the non-semimartingality of S cannot be exploited for unbounded gains in logarithmic utility due to the previously mentioned scarcity of useful portfolio strategies. In other words, any strategy that might lead to large wealth suffers from the risk of finishing negative with positive probability.

Our construction of the process S utilizes the following ingredients:

1. $\{B_t\}_{t\in[0,1]}$ ia a Brownian motion and $\mathbb{F}^B = \{\mathcal{F}_t^B\}_{t\in[0,1]}$ is the (right-continuous and complete) augmentation of the filtration generated by B.

2. $\{M_t\}_{t\in[0,1]}$ is the Gaussian martingale given by $M_t \triangleq \int_0^t \sigma(u)\,dB_u$, where

$$\sigma(t) = \frac{|\log(1-t)|^{-2/3}}{\sqrt{1-t}}\mathbf{1}_{\{t>\frac{1}{2}\}}.$$

3. $\{N_t^1\}_{t\in[0,1]}$ and $\{N_t^2\}_{t\in[0,1]}$ are two Poisson processes independent of $\mathcal{F}_1^B$ and of each other.

4. $\{N_t\}_{t\in[0,1]}$, is defined by $N_t \triangleq N_t^1 - N_t^2$ and $\mathbb{F}^N$ is the filtration generated by the process N (or, equivalently, by N^1 and N^2).

The process $\{S_t\}_{t\in[0,1]}$ is defined by

$$S_t \triangleq M_t + \int_0^t \frac{1}{1-u}\, dN_u,\ t \in [0,1].$$

S is clearly an $\mathbb{F}$-semimartingale, where $\mathbb{F}$ is the filtration generated by B and N, i.e. $\mathbb{F} \triangleq \mathbb{F}^B \vee \mathbb{F}^N$. Let the enlarged filtration $\mathbb{G}$ be defined by adding the information about the terminal value B_1 of the Brownian motion B to $\mathbb{F}$, i.e. $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(B_1)$. The properties (NS) and (FL) in (3.3), are now established through several lemmas:

LEMMA 3.2. *Property (NS) in (3.3) holds true, i.e.* $\{S_t\}_{t\in[0,1]}$ *is* not *a* $\mathbb{G}$-*semimartingale.*

Proof. It is enough to show that $\{M_t\}_{t\in[0,1]}$ is not a $\mathbb{G}$-semimartingale. This is, however, exactly the content of Theorem IV.7 in Protter (2004) and the example following it. ∎

LEMMA 3.3. *Let* $\pi \in \mathcal{H}^{simp}(\mathbb{G})$ *be a simple integrand and let* W^π *be the corresponding wealth process, as defined in (3.2). If* $\mathbb{P}[W_1^\pi > 0] = 1$ *then*

$$\pi_t \in (-(1-t), 1-t),\ \lambda \otimes \mathbb{P} - a.e,$$

where λ *denotes the Lebesgue measure on* $[0,1]$.

Before we prove Lemma 3.3, we need the following simple result:

LEMMA 3.4. *Let* $\{N_t\}_{t\in[0,1]}$ *be a difference of two independent* $\mathbb{G}$-*Poisson processes, and let* $\{\beta_t\}_{t\in[0,1]}$ *be a* $\mathbb{G}$-*predictable process taking values in the set* $\{-1,1\}$*. Then the process* $\{\tilde{N}_t\}_{t\in[0,1]}$*, defined by the integral* $\tilde{N}_t = \int_0^t \beta_s\, dN_s$ *can be decomposed into a difference of two independent* $\mathbb{G}$-*Poisson processes.*

Proof. Let $N_t = N_t^+ - N_t^-$ be the decomposition of N into two independent Poisson processes, and let $\beta_t^+ = \max(\beta_t, 0)$ and $\beta_t^- = \max(-\beta_t, 0)$ so that $\beta_t = \beta_t^+ - \beta_t^-$ and $\beta_t^+ + \beta_t^- = 1$, for all $t \in [0,1]$, a.s. The processes $\tilde{N}^+$ and $\tilde{N}^-$ defined by

$$\tilde{N}_t^+ \triangleq \int_0^t \beta_s^+\, dN_s^+ + \int_0^t \beta_s^-\, dN_s^-, \text{ and}$$
$$\tilde{N}_t^- \triangleq \int_0^t \beta_s^-\, dN_s^+ + \int_0^t \beta_s^+\, dN_s^-.$$

have the following properties

1. $\tilde{N}^+$ and $\tilde{N}^-$ are non-decreasing processes and increase only by jumps of magnitude 1.

2. $\tilde{N}^+_t - (\beta^+_t + \beta^-_t)t = \tilde{N}^+_t - t$ and $\tilde{N}^-_t - (\beta^+_t + \beta^-_t)t = \tilde{N}^-_t - t$ are martingales.

3. The intersection of the sets of jump-times for $\tilde{N}^+$ and $\tilde{N}_-$ is empty, almost surely.

Items (1) and (2) imply that $\tilde{N}^+$ and $\tilde{N}^-$ are $\mathbb{G}$-Poisson processes and (3) is enough to conclude that they are independent (see Brémaud (1981)). Therefore, $\tilde{N} = \tilde{N}^+ - \tilde{N}^-$ is a difference of two Poisson processes. ∎

Proof. (Of Lemma 3.3) Let the process $\hat{\pi}$ be defined as $\hat{\pi}_t \triangleq \pi_t/(1-t)1_{\{t<1\}}$, and suppose that the predictable set $A \triangleq \{(t,\omega) \in [0,1] \times \Omega \,:\, |\hat{\pi}_t(\omega)| \geq 1\}$ satisfies $(\lambda \otimes \mathbb{P})[A] > 0$. The expression (3.2) for the wealth W^π_1 can be split into two factors, one of which is an exponential and the other is the product of the following form

$$Y \triangleq \prod_{s\leq 1}(1+\pi_s \frac{1}{1-s}\Delta N_s) = \prod_{s\leq 1}(1+\hat{\pi}_s \Delta N_s).$$

The sign of W^π_1 is equal to the sign of Y, so in order to reach a contradiction, it will be enough to prove that $\mathbb{P}[Y \leq 0] > 0$.

Define the process $\tilde{N}_t \triangleq \int_0^t \operatorname{sgn}(\hat{\pi}_s)\, dN_s$, where $\operatorname{sgn}(x) = 1$ for $x \geq 0$ and $\operatorname{sgn}(x) = -1$, otherwise. By Lemma 3.4, there exist two independent Poisson processes $\tilde{N}^+$ and $\tilde{N}^-$ such that $\tilde{N} = \tilde{N}^+ - \tilde{N}^-$, and

$$Y = \prod_{s\leq 1}(1+|\hat{\pi}_s|\,\Delta\tilde{N}_s).$$

Let J be the event in which $\tilde{N}^-$ jumps exactly once on the set A, i.e. $J \triangleq \left\{\int_0^1 \mathbf{1}_A(s)\, d\tilde{N}^- = 1\right\}$. Since $|\hat{\pi}_s| \geq 1$ on A, it is easy to see that $\mathbb{P}[Y \leq 0] \geq \mathbb{P}[J]$. In order to show that $\mathbb{P}[J] > 0$, we first define $J' \triangleq \left\{\int_0^1 \mathbf{1}_A(s)\, d\tilde{N}^- \geq 1\right\} \supseteq J$. The martingale property of the process $X_t = \int_0^t \mathbf{1}_A(s)\, d\tilde{N}^-_s - \int_0^t \mathbf{1}_A(s)\, ds$ implies that

$$\mathbb{E}[\int_0^1 \mathbf{1}_A(s)\, d\tilde{N}^-_s] = \mathbb{E}[\int_0^1 \mathbf{1}_A(s)\, ds] = (\lambda \otimes \mathbb{P})[A] > 0,$$

showing that the $\mathbb{N} \cup \{0\}$-valued random variable $\int_0^t \mathbf{1}_A(s)\, d\tilde{N}^-_s$ has a strictly positive expectation, and thus $\mathbb{P}[J'] > 0$. Define τ^1 to be the first jump time of the process $\tilde{N}^-$. By the $\mathbb{G}$-Lévy property of the Poisson process $\tilde{N}$, the process $\hat{N}_t = \tilde{N}^-_{\tau_1+t} - \tilde{N}^-_{\tau_1}$ is a Poisson process, independent of $\mathcal{G}_{\tau_1}$. The probability that $\hat{N}$ will stay constant for one unit of time is strictly positive, and, consequently, so is the probability that $\tilde{N}^-$ will jump exactly once on A. This implies that $\mathbb{P}[Y\leq 0] > 0$ - a contradiction. ∎

Proposition 3.5. *There exists a constant $C < \infty$ such that*

$$\mathbb{E}[\log(W_1^\pi)] \leq C, \text{ for all } \pi \in \mathcal{H}^{simp}(\mathbb{G}). \tag{3.4}$$

Proof. By Lemma 3.3, it is enough to show that (3.4) is true for all $\pi \in \mathcal{H}^{simp}(\mathbb{G})$, with the additional property that $|\pi_s| < (1-s)$, $\lambda \otimes \mathbb{P}$-a.e.

The expression for W_1^π given in (3.2) factorizes into an exponential and a product of transformed jumps, so that $\mathbb{E}[\log(W_1^\pi)] \leq C(\pi) + J(\pi)$, where

$$\begin{aligned} C(\pi) &\triangleq \mathbb{E}[\log(\mathcal{E}(\pi \cdot M))_1] \quad \text{and} \\ J(\pi) &\triangleq \mathbb{E}[\sum_{s\leq 1} \log(1 + \frac{\pi_s}{1-s}\Delta N_s)] \leq \mathbb{E}[\sum_{s\leq 1} \frac{\pi_s}{1-s}\Delta N_s] = 0. \end{aligned}$$

To obtain a bound on $C(\pi)$ we first apply Jensen's inequality and then Fatou's Lemma to obtain

$$C(\pi) \leq \log(\mathbb{E}[\mathcal{E}(\pi \cdot M)_1]) \leq \liminf_{t\to 1} \log(\mathbb{E}[\mathcal{E}(\pi \cdot M)_t]).$$

Now, all we need is a uniform bound (in π and t) on $\mathbb{E}[\mathcal{E}(\pi \cdot M)_t]$, for $t<1$. This is accomplished by noting that the process M is a $\mathbb{G}$-semimartingale on any interval $[0,u], u<1$, with the semimartingale decomposition $M = \hat{M} + (M - \hat{M})$, where the $\mathbb{G}$-martingale $\hat{M}$ is given by :

$$\hat{M}_t \triangleq \int_0^t \sigma(u) \left(dB_u - \frac{B_1 - B_u}{1-u} du \right).$$

This allows us to write

$$\begin{aligned} \mathcal{E}(\pi \cdot M)_t &= \mathcal{E}\left(\int_0^t \pi_u \, d\hat{M}_u + \int_0^t \pi_u \sigma(u) \left(\frac{B_1 - B_u}{1-u} \right) du \right) \\ &= \exp\left((\pi \cdot \hat{M})_t - (\pi^2 \cdot [\hat{M}])_t \right) \\ &\times \exp\left(\frac{1}{2} \int_0^t \pi_u^2 \sigma(u)^2 du + \int_0^t \pi_u \sigma(u) \left(\frac{B_1 - B_u}{1-u} \right) du \right). \end{aligned}$$

Hölder's inequality, combined with the observation that the square of the exponential $\exp\left((\pi \cdot \hat{M})_t - (\pi^2 \cdot [\hat{M}])_t\right)$ is a martingale, yields:

$$\mathbb{E}[\mathcal{E}(\pi \cdot M)_t]^2 \leq \mathbb{E}\left[\exp\left(\int_0^t \pi_u^2 \sigma(u)^2 du + 2\int_0^t \pi_u \sigma(u)\left(\frac{B_1 - B_u}{1-u}\right) du\right)\right].$$

To see that this expectation can be bounded away from ∞, independently of t and π, we can use the bound $|\pi_t| \leq 1-t$, the explicit form of the function σ, and the fact that all exponential moments of the random variable $\sup_{t\in[0,1]} |B_t|$ are finite. ∎

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