Sensitivity analysis of utility based prices and risk-tolerance wealth processes

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Based on joint papers with Mihai Sirbu (Columbia University).
Model of a financial market

There are $d + 1$ traded or liquid assets:

1. a savings account. We assume that the interest rate is 0:

$$B = \text{const}$$

2. $d$ stocks. The price process $S$ of the stocks is a semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$, where $T$ is a maturity.

Assumption (No Arbitrage)

$$\mathcal{Q} \neq \emptyset$$

where $\mathcal{Q}$ is the family of equivalent local martingale measures for $S$. 
Economic agent or investor

\( x \): initial capital

\( U \): utility function for consumption at \( T \) such that

- \( U : (0, \infty) \rightarrow \mathbb{R} \)
- \( U \) is strictly increasing
- \( U \) is strictly concave
- the Inada conditions:
  \[ U'(0) = \infty \quad U'(-\infty) = 0 \]
Pricing problem

Consider a family of \( N \) non-traded or illiquid European contingent claims with

- maturity \( T \)
- payment functions \( f = (f_i)_{1 \leq i \leq N} \)

**Question** What is the (marginal) price \( p(x) \) of the contingent claims \( f \)?

Intuitive definition: \( p(x) \) is the threshold such that given the chance to buy or sell at \( p^{\text{trade}} \) the investor will

- buy at \( p^{\text{trade}} < p(x) \)
- sell at \( p^{\text{trade}} > p(x) \)
- do nothing at \( p^{\text{trade}} = p(x) \)
Denote

\[
   u(x, q) = \sup_{X_0 = x} \mathbb{E}[U(X_T + \langle q, f \rangle)]
\]

the maximal **expected utility** from the portfolio \((x, q)\), where

\(x\): total wealth invested into liquid securities

\(q = (q^i)_{1 \leq i \leq N}\): a number of non traded options

**Definition** A (marginal) utility based price is a vector \(p(x)\) such that

\[
   u(x) := u(x, 0) \geq u(x', q')
\]

for any portfolio \((x', q')\) such that

\[
   x = x' + \langle q', p(x) \rangle.
\]
Computation of $p(x) = p(x, 0)$

The idea belongs to Mark Davis. Define the conjugate function

$$V(y) = \max_{x>0} \left[ U(x) - xy \right], \quad y > 0.$$ 

and consider the following dual optimization problem:

$$v(y) = \inf_{Q \in \mathcal{Q}} \mathbb{E} \left[ V \left( y \left( \frac{dQ}{dP} \right) \right) \right], \quad y > 0$$

The measure $Q(y)$, where the lower bound is attained, is called the minimal martingale measure for $y$.

Mark Davis argued that if $x = -v'(y)$ then

$$p(x) = \mathbb{E}_{Q(y)}[f].$$
Theorem (Hugonnier, K., Schachermayer)

Let \( y > 0 \), \( x = -v'(y) \) and \( X \) be a non-negative wealth process. The following conditions are equivalent:

1. for any contingent claim \( f \) such that

\[
|f| \leq K(1 + X_T)
\]

for some \( K > 0 \) a utility based price \( p(x) \) is uniquely defined.

2. the minimal martingale measure \( Q(y) \) exists and \( X \) is a uniformly integrable martingale under \( Q(y) \).

Moreover, in this case

\[
p(x) = \mathbb{E}_{Q(y)}[f].
\]
“Trading” problem

**Question** What quantity \( q = q(p^{\text{trade}}) \) the investor should trade (buy or sell) at the price \( p^{\text{trade}} \)?

Qualitatively, there are two different cases:

1. If \( f \) is **replicable**, then
   
   \[
   q(p^{\text{trade}}) = \begin{cases} 
   \text{any} & \text{if } p^{\text{trade}} = p(x) \\
   \infty & \text{if } p^{\text{trade}} \neq p(x)
   \end{cases}
   \]

2. else if \( f \) is **not replicable** and \( p^{\text{trade}} \) is an arbitrage-free price, then

   \[
   q(p^{\text{trade}}) = \begin{cases} 
   0 & \text{if } p^{\text{trade}} = p(x) \\
   \text{finite} & \text{if } p^{\text{trade}} \neq p(x)
   \end{cases}
   \]
**Goal:** study the dependence of prices on quantities.

**Reservation price** (different values for buying or selling)

**Marginal utility based price** for the claims $f$ given the portfolio $(x, q)$ is a vector $p(x, q)$ such that

$$u(x, q) \geq u(x', q')$$

for any pair $(x', q')$ such that

$$x + \langle q, p(x, q) \rangle = x' + \langle q', p(x, q) \rangle.$$  

In other words, given the portfolio $(x, q)$ the investor **will not trade** the options at $p(x, q)$.

If $u = u(x, q)$ is differentiable at $(x, q)$ then

$$p(x, q) = \frac{u_q(x, q)}{u_x(x, q)}.$$
Using the marginal utility based prices $p(x, q)$ we can compute the optimal quantity

$$q = q(p^{\text{trade}})$$

from the “equilibrium” condition:

$$p^{\text{trade}} = p(x - qp^{\text{trade}}, q)$$

**Main difficulty:** $p(x, q)$ is hard to compute except for the case $q = 0$. 
Sensitivity analysis of utility based prices

We study linear approximation for $p(x, q)$:

$$p(x + \Delta x, q) \approx p(x) + p'(x)\Delta x + D(x)q,$$

where

$$p^i(x) = p^i(x, 0), \quad i = 1, \ldots, N$$

$p'(x)$ is the derivative of $p(x)$ and

$$D^{ij}(x) = \frac{\partial p^i}{\partial q^j}(x, 0), \quad i, j = 1, \ldots, N.$$ 

The vector $p'(x)$ and the matrix $D(x)$ measure the sensitivity of $p(x, q)$ with respect to $x$ and $q$ at $(x, 0)$.
Question (Quantitative) How to compute $p'(x)$ and $D(x)$?

Closely related publications:

**J. Kallsen (02)**: formula for $D(x)$ for general semimartingale model but in a different framework of local utility maximization.

**V. Henderson (02)**: formula for $D(x)$ in the case of a Black-Scholes type model with basis risk and for power utility functions.
Question (Qualitative) When the following (desirable) properties hold true for any family of contingent claims $f$?

1. The marginal utility based price $p(x) = p(x, 0)$ does not depend (locally) on $x$, that is,

$$p'(x) = 0$$

2. The sensitivity matrix $D(x)$ has full rank for non replicable contingent claims:

$$D(x)q = 0 \iff \langle q, f \rangle$$

3. The sensitivity matrix $D(x)$ is symmetric, that is

$$D^{ij}(x) = D^{ji}(x), \quad i, j = 1, \ldots, N.$$
4. The sensitivity matrix $D(x)$ is strictly negatively defined for non-replicable options:
\[
\langle q, D(x)q \rangle < 0 \quad \Leftrightarrow \quad \langle q, f \rangle \text{ is not replicable}
\]

5. **Stability** of the linear approximation: for any $p^{\text{trade}}$ the linear approximation to the “equilibrium” equation:
\[
p^{\text{trade}} = p(x - qp^{\text{trade}}, q)
\]
that is,
\[
p^{\text{trade}} \approx p(x) - p'(x)qp^{\text{trade}} + D(x)q
\]
has the “correct” solution.

**Remark** Using the representation
\[
p(x, q) = \frac{u_q}{u_x}(x, q).
\]
one can see that all these properties are, in fact, **equivalent**!
Risk-tolerance wealth process

Fix $x > 0$. Recall that $-U'(x)/U''(x)$ is called the risk-tolerance coefficient of $U$ at $x$.

Denote by $\widehat{X}(x)$ the optimal solution of

$$u(x) := u(x, 0) = \sup_{X_0=x} \mathbb{E}[U(X_T)].$$

**Definition (K., Sirbu)** A maximal wealth process $R(x)$ is called the risk-tolerance wealth process if

$$R_T(x) = -\frac{U'(\widehat{X}_T(x))}{U''(\widehat{X}_T(x))}.$$
Some properties of $\mathbf{R}(x)$ (if it exists):

1. Initial value:

$$R_0(x) = -\frac{u'(x)}{u''(x)}.$$

2. Derivative of optimal wealth strategy:

$$\frac{R(x)}{R_0(x)} = X'(x) := \lim_{\Delta x \to 0} \frac{\hat{X}(x + \Delta x) - \hat{X}(x)}{\Delta x}.$$
Theorem (K., Sirbu) The following assertions are equivalent:

1. The risk-tolerance wealth process $R(x)$ exists for all $x > 0$.

2. The minimal martingale measures $Q(y)$ for different $y > 0$ coincide:
   \[ Q(y) = \hat{Q}, \quad y > 0. \]

3. Any (equivalently, each) of the “qualitative” properties 1–5 holds true.
Extremal cases:

1. Power utility functions:

\[ U(x, \alpha) = \frac{x^{1-\alpha} - 1}{1 - \alpha}, \quad (\alpha > 0). \]

For \( \alpha = 1 \) we have Bernoulli utility:

\[ U(x; 0) = \log x. \]

2. Model is “essentially complete”, that is, it has the same minimal martingale measure for any utility function.
Computation of $D(x)$

We choose

$$R(x)/R_0(x) = X'(x)$$

as a **numeraire** and denote

$$f^R = f_{R_0}(x)/R(x) : \text{discounted contingent claims}$$

$$X^R = X_{R_0}(x)/R(x) : \text{discounted wealth processes}$$

$$\mathbb{Q}^R : \text{the martingale measure for } R(x)/R_0(x)-\text{discounted wealth processes:}$$

$$\frac{d\mathbb{Q}^R}{d\mathbb{Q}} = \frac{R_T(x)}{R_0(x)}$$
Consider the $\mathbb{Q}^R$-martingale

$$P_t^R = \mathbb{E}_{\mathbb{Q}^R} \left[ f^R | \mathcal{F}_t \right].$$

and let

$$P^R = M + N, \quad N_0 = 0,$$

be its Kunita-Watanabe decomposition, where

1. $M$ is $R(x/R_0(x))$-discounted wealth process. Interpretation: **hedging process**.

2. $N$ is a martingale under $\mathbb{Q}^R$ which is orthogonal to all $R(x)/R_0(x)$-discounted wealth processes. Interpretation: **risk process**.
Denote by \( a(x) \) the relative risk-aversion coefficient of

\[
    u(x) = \max_{X_0=x} \mathbb{E}[U(X_T)],
\]

that is

\[
    \alpha(x) = -\frac{x u''(x)}{u'(x)}.
\]

**Theorem (K., Sirbu)** Assume that the risk-tolerance wealth process \( R(x) \) exists. Then

\[
    p'(x) = 0
\]

\[
    D(x) = -\frac{a(x)}{x} \mathbb{E}_{Q|R} \left[ N_T N_T' \right]
\]
Question  How to compute $D(x)$ in practice?

Inputs:

1. $Q$.  *Already implemented!* 

2. $R(x)/R_0(x)$.  Recall that 

$$
\frac{R(x)}{R_0(x)} = \lim_{\Delta x \to 0} \frac{\hat{X}(x + \Delta x) - \hat{X}(x)}{\Delta x}.
$$

*Decide what to do with one penny!* 

3. Relative risk-aversion coefficient $\alpha(x)$.  *Deduce from mean-variance preferences.*  In any case, this is just a number!
Black and Scholes model with basis risk

Traded asset:

\[ dS_t = S_t (\mu dt + \sigma dW_t) \]

Non traded asset:

\[ d\tilde{S} = (\tilde{\mu} dt + \tilde{\sigma} d\tilde{W}_t), \]

Denote by

\[ \rho = \frac{d\tilde{W} dW}{dt} \]

the correlation coefficient between \( S \) and \( \tilde{S} \). In practice, we want to chose \( S \) so that

\[ \rho \approx 1. \]
Consider contingent claims

\[ f = f(\tilde{S}) \]

whose payoffs are determined by \( \tilde{S} \).

To compute \( D(x) \) assume (as an example) the following choices:

1. \( \mathbb{Q} \) is a martingale measure for \( \tilde{S} \).

2. \( R(x)/R_0(x) = 1 \)

Then

\[ D_{ij}(x) = -\frac{\alpha(x)}{x}(1 - \rho^2)\text{Covar}_\mathbb{Q}(f_i, f_j). \]
Smoothness of the indirect utility

Consider the problem of expected utility maximization:

\[ u(x) = \sup_{X_0=x} \mathbb{E}[U(X_T)] \]

**Question** When \( u''(x) \) exists and is strictly positive?

We need additional conditions on the utility function \( U \) and the price processes \( S \).

**Assumption** *There are strictly positive constants \( c_1 \) and \( c_2 \) such that*

\[ c_1 < -\frac{xU''(x)}{U'(x)} < c_2, \quad x > 0. \]
**Definition**  A semimartingale $R$ is called **sigma-bounded** if

$$R = \int H \, dS$$

and $S$ is (locally) bounded.

**Assumption**  *For any numeraire* $X$

$$S^X = \left( \frac{1}{X}, \frac{S}{X} \right)$$

*is sigma-bounded.*

**Examples:**

1. $S$ is continuous or, more generally, depends on a finite number of Poisson processes

2. complete financial model

**Remark**  These two examples will not satisfy if sigma-boundedness is replaced by local boundedness.
Theorem (K., Sirbu) Let both assumptions above hold true. Then \( u'' \) exists and

\[
c_1 < -\frac{xu''(x)}{u'(x)} < c_2, \quad x > 0.
\]

where \( c_1 \) and \( c_2 \) are the same constants as in the assumption on \( U \).

Remark Both assumptions are essential for the assertion of the theorem to hold true.

Other results:

1. The existence of the derivative processes \( X'(x) \) and \( Y'(y) \) of the solutions to the primal and dual problems.

2. The computation of \( p'(x) \) and \( D(x) \) in general case (when \( R(x) \) does not exist).
Summary

- For non replicable (in practice, all) contingent claims the fair prices depend on the trading volume.

- The following conditions are equivalent:
  - Approximate utility based prices have nice qualitative properties
  - Risk-tolerance wealth processes exist.
  - Minimal martingale measures do not depend on initial capital.

- We need to solve the mean-variance hedging problem, where the risk-tolerance wealth process plays the role of the numeraire.