

# On the Utility Based Pricing of Contingent Claims in Incomplete Markets

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### **Abstract**

We study the uniqueness of the utility based price of contingent claims in a semimartingale model of incomplete financial market. In particular, we obtain that a necessary and sufficient condition for all bounded contingent claims to admit a unique utility based price is that the solution to the dual problem defines an equivalent local martingale measure.

# Introduction

As is well known, in a complete financial market every contingent claim can be perfectly replicated by a controlled portfolio of the traded securities and therefore admits a well-defined arbitrage free price. In an incomplete market, to every contingent claim is associated an interval of arbitrage free prices and, unless the contingent claim is replicable in which case this set consists of a single point, arbitrage arguments alone are not sufficient to determine a unique price. Since the endpoints of the arbitrage free interval coincide with the sub and super replication costs of the contingent claim, the buying or selling of the claim at any price in the interior of this set leads to a possible loss at the maturity. Hence, in this case the choice of a price can only be made with respect to some risk functional representing the preferences and endowments of the agent under consideration.

Relying on this observation different utility based valuation approaches have been developed and studied by Hodges and Neuberger [10], Davis [2], Karatzas and Kou [14], Frittelli [8], Foldes [7] and Kallsen [13] among others. We study in this paper the concept of utility based price which was introduced in finance by Davis [2]. The basic idea underlying this valuation principle goes back to the classic works of Sir Hicks [9] and consists in choosing the price in such a way that, given his utility function and current endowment, the agent is locally indifferent to a small position in the claim. In other words, the amount  $p$  is said to be a *utility based price* if, given the possibility of *buy & hold* trading at that price, the agent's optimal demand for the contingent claim is equal to zero.

While utility based prices exist under very minimal assumptions on the financial market model, the agent's preferences and the contingent claim, their uniqueness remains a more delicate question. In our main theorem we show that the uniqueness of the utility based price is closely related to the property that contingent claims be dominated by the terminal wealth of a portfolio such that the product of its capital process and the minimal solution of a dual problem is a uniformly integrable martingale. As a corollary to this result, we obtain that a necessary and sufficient condition for *all* bounded contingent claims to admit a unique utility based price is that the minimal solution to the dual problem defines an equivalent local martingale measure.

The rest of the paper is organized as follows. In Section 1 we present the model of financial market and recall the definition of various classes of trading strategies. In Section 2 we state and discuss our main results. Section

3 contains all the proofs.

## 1 The Model

We consider a finite horizon model of a financial market which consists in  $d + 1$  securities: one savings account and  $d$  stocks. As usual in mathematical finance, we shall assume that the price process of the savings account is normalized to one. The price process  $S := (S^i)_{i=1}^n$  of the stocks is assumed to be a locally bounded semimartingale on a given filtered probability space  $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$  where the filtration  $\mathbf{F} := (\mathcal{F}_t)_{t \in [0, T]}$  satisfies the usual conditions and we let  $\mathcal{F}_T = \mathcal{F}$ . Hereafter we denote by  $\mathbf{L}^0$  the set of  $\mathcal{F}$ -measurable random variables.

A probability measure  $\mathbb{Q}$  is called an equivalent local martingale measure if it is equivalent to  $\mathbb{P}$  and if  $S$  is a local martingale under  $\mathbb{Q}$ . We denote by  $\mathcal{M}$  the family of every such measures and assume that

$$\mathcal{M} \neq \emptyset. \quad (1)$$

This rather mild condition is essentially equivalent to the absence of arbitrage opportunities in the financial market, see [3, 5] for precise statements and further references.

A self financing portfolio is defined by a pair  $(x, H)$  where  $x \in \mathbf{R}$  represents the initial capital and  $H = (H^i)_{i=1}^d$  is a predictable and  $S$ -integrable process specifying the number of shares of each of the stocks held in the portfolio. The value process of a self financing portfolio evolves in time as the stochastic integral of the process  $H$  with respect to the stock price:

$$X_t := x + (H \cdot S)_t = x + \int_0^t H_\tau dS_\tau, \quad t \in [0, T]. \quad (2)$$

A process  $X$  is called *admissible* if it is the value of a self financing portfolio and is almost surely non negative. For every  $x > 0$  we denote by  $\mathcal{X}(x)$  the class of admissible processes whose initial value is equal to  $x$ , that is

$$\mathcal{X}(x) := \left\{ X \in \mathbf{R}_+ : X \text{ satisfies (2) for some } H \text{ and } X_0 = x \right\}. \quad (3)$$

We shall use a shorter notation  $\mathcal{X}$  for the set  $\mathcal{X}(1)$ . A process  $X \in \mathcal{X}(x)$  is said to be *maximal* if its terminal value cannot be dominated by that of any

other process in  $\mathcal{X}(x)$ , that is if  $X' \in \mathcal{X}(x)$  and  $X_T \leq X'_T$  imply  $X' = X$ . Finally, a process  $X$  is said to be *acceptable* if it admits a decomposition of the form  $X = X' - X''$  where  $X'$  is admissible and  $X''$  is maximal. For details on maximal and acceptable processes we refer to [4].

We consider a price taking agent whose preferences over terminal wealth are represented by a utility function  $U : (0, \infty) \rightarrow \mathbf{R}$  which is assumed to be strictly increasing, strictly concave, continuously differentiable and to satisfy the Inada conditions:

$$U'(0) := \lim_{x \rightarrow 0} U'(x) = \infty, \quad U'(\infty) := \lim_{x \rightarrow \infty} U'(x) = 0. \quad (4)$$

In what follows we set  $U(0) := \lim_{x \rightarrow 0} U(x)$  and  $U(x) = -\infty$  for all  $x < 0$ .

The *convex conjugate* function of the agent's utility function is defined to be the Legendre transform of the convex function  $-U(-\cdot)$ , that is

$$V(y) := \sup_{x > 0} \left\{ -xy + U(x) \right\}, \quad y > 0. \quad (5)$$

It is well known that under the Inada conditions (4), the conjugate function is a continuously differentiable, strictly decreasing and strictly convex function satisfying  $-V'(0) = \infty$ ,  $V'(\infty) = 0$  and  $V(0) = U(\infty)$ ,  $V(\infty) = U(0)$  as well as the following bidual relation

$$U(x) := \inf_{y > 0} \left\{ xy + V(y) \right\}, \quad x > 0. \quad (6)$$

We also note that under the Inada conditions (4), the agent's marginal utility is the inverse of minus the derivative of the convex conjugate:  $(U')^{-1} = -V'$ .

## 2 Main Results

Assume that the agent has some initial capital  $x > 0$ . In accordance with the model set forth in the previous section, the maximal expected utility that this agent can achieve by trading in the financial market is given by

$$u(x) := \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)], \quad x > 0. \quad (7)$$

Let  $\mathcal{Y}$  denote the set of non negative semimartingales  $Y$  with initial value one and such that for any  $X \in \mathcal{X}$  the product  $XY$  is a supermartingale:

$$\mathcal{Y} := \left\{ Y \in \mathbf{R}_+ : Y_0 = 1 \text{ and } XY \text{ is a supermartingale } \forall X \in \mathcal{X} \right\}.$$

Note that since  $1 \in \mathcal{X}$  the semimartingales in  $\mathcal{Y}$  are supermartingales. Define a non increasing convex function by setting

$$v(y) = \inf_{Y \in \mathcal{Y}} \mathbb{E}[V(yY_T)], \quad y > 0. \quad (8)$$

Hereafter we shall assume that  $v$  is finitely valued on  $(0, \infty)$ , that is

$$v(y) < \infty, \quad y > 0. \quad (9)$$

Under this assumption the value function  $u$  is strictly increasing, strictly concave and continuously differentiable and  $v$  is its convex conjugate, that is

$$v(y) = \sup_{x > 0} \{-xy + u(x)\}. \quad (10)$$

The solutions  $\widehat{X}(x)$  and  $\widehat{Y}(y)$  to, respectively, (7) and (8) exist and are unique for any  $x > 0$  and  $y > 0$ . Furthermore, if  $y = u'(x)$  then

$$U'(\widehat{X}_T(x)) = y\widehat{Y}_T(y), \quad \mathbb{E}[\widehat{Y}_T(y)\widehat{X}_T(x)] = x. \quad (11)$$

For details on these results as well as for sufficient condition guaranteeing the validity of (9), we refer to the papers [15, 16].

Let now  $B \in \mathbf{L}^0$  denote a European contingent claim which matures at the terminal time  $T$  of the model. For a pair  $(x, q) \in \mathbf{R}^2$  we denote by  $\mathcal{X}(x, q|B)$  the set of acceptable processes with initial value  $x$  and whose terminal value dominates the random variable  $-qB$ , that is

$$\mathcal{X}(x, q|B) := \left\{ X \text{ is acceptable with } X_0 = x \text{ and } X_T + qB \geq 0 \right\}. \quad (12)$$

Note that with  $x > 0$  and  $q = 0$  this set coincides with the set of admissible processes with initial capital equal to  $x$ .

**Definition 1** Let  $B \in \mathbf{L}^0$  and  $x > 0$ . A real number  $p$  is called a *utility based price for  $B$  given the initial capital  $x$*  if

$$\mathbb{E}\left[U\left(X_T + qB\right)\right] \leq u(x), \quad q \in \mathbf{R}, X \in \mathcal{X}(x - qp, q|B). \quad (13)$$

The interpretation of this definition is that  $p$  is a utility based price for a contingent claim if, given the possibility of buy & hold trading in the claim at that price, the agent's optimal demand is equal to zero.

The following theorem provides sufficient conditions for the uniqueness of the utility based price and constitutes our main result.

**Theorem 1** *Assume that the conditions (1), (4), (9) hold true. Fix  $x > 0$ , define  $y = u'(x)$  and let  $\widehat{Y}(y)$  denote the corresponding solution to (8). Then for any maximal admissible process  $\widetilde{X} \in \mathcal{X}$  we have:*

- (i) *If the product process  $\widehat{Y}(y)\widetilde{X}$  is a uniformly integrable martingale then every contingent claim with the property that  $|B| \leq c\widetilde{X}_T$  for some  $c > 0$  admits a unique utility based price with respect to the initial capital  $x$  which is given by*

$$p(B|x) := \mathbb{E} \left[ \widehat{Y}_T(y)B \right] = \frac{1}{u'(x)} \mathbb{E} \left[ U' \left( \widehat{X}_T(x) \right) B \right]. \quad (14)$$

- (ii) *If the product process  $\widehat{Y}(y)\widetilde{X}$  fails to be a, that is if there exists a constant  $\delta > 0$  such that*

$$\mathbb{E} \left[ \widehat{Y}_T(y)\widetilde{X}_T \right] = 1 - \delta,$$

*then there exists a contingent claim  $0 \leq B \leq \widetilde{X}_T$  and a constant  $\alpha \geq 0$  such that every  $\alpha \leq \pi \leq \alpha + \delta$  is a utility based price for  $B$ .*

The proof of the theorem will be given in the next section. We conclude this section with an important corollary and some remarks.

**Corollary 1** *Assume that the conditions of Theorem 1 hold true. Fix an arbitrary  $x > 0$ , define  $y = u'(x)$  and let  $\widehat{Y}(y)$  denote the corresponding solution to (8). Then the following assertions are equivalent:*

- (i) *The dual optimizer  $\widehat{Y}(y)$  is the density process of an equivalent local martingale measure  $\widehat{\mathbb{Q}}(y)$  with respect to  $\mathbb{P}$ .*
- (ii) *Every bounded contingent claim admits a unique utility based price given the initial capital  $x$ .*

*Moreover, if any of the above assertions hold true, then the utility based price of any bounded contingent claim  $B$  has the representation:*

$$p(B|x) = \mathbb{E}_{\widehat{\mathbb{Q}}(y)}[B]. \quad (15)$$

*Proof.* The result follows from Theorem 1 by taking  $\widetilde{X} \equiv 1$ . ■

**Remark 1** Let  $\widehat{X} \in \mathcal{X}$  be a maximal admissible process, fix a contingent claim  $|B| \leq \widehat{X}_T$  and denote by  $\mathcal{B}$  the interior of the set of points  $(x, q)$  such that the family  $\mathcal{X}(x, q|B)$  is non empty. With this notation, the function defined by

$$u(x, q) := \sup_{X \in \mathcal{X}(x, q|B)} \mathbb{E} \left[ U \left( X_T + qB \right) \right], \quad (x, q) \in \mathcal{B} \quad (16)$$

represents the maximal expected utility that an agent whose endowment is given by  $x$  units of cash and  $q$  units of the contingent claim can achieve by trading in the financial market, see [11].

Using standard arguments from the theory of convex functions we deduce that the set of utility based prices for the contingent claim is given by

$$\mathcal{P}(B|x) := \left\{ p = \frac{r}{y} : (y, r) \in \partial u(x, 0) \right\} \quad (17)$$

where  $\partial u$  denotes the subdifferential of the function  $u$ . Uniqueness of the utility based price for the contingent claim  $B$  given an initial capital  $x > 0$  is thus equivalent to the differentiability of  $u$  at the point  $(x, 0)$  and our main result can be seen as providing sufficient conditions for this property to hold. If the function  $u$  is indeed differentiable at the point  $(x, 0)$ , then it follows from (17) that the contingent claim's utility based price is uniquely given by

$$p(B|x) = \frac{u_q(x, 0)}{u_x(x, 0)}. \quad (18)$$

**Remark 2** If  $B$  is replicable in the sense that there exists a maximal acceptable process  $X$  whose terminal value is equal to  $B$ , then  $B$  admits a unique utility based price which is independent of both the initial capital and the utility function and is given by  $b := X_0$ . Indeed, using the fact that the set of acceptable processes is a cone, we deduce that  $u(z, q) = u(z + qb)$  holds for all  $z + qb > 0$  and the desired result now follows from Definition 1 and the increase of the agent's utility function.

Relying on the above observation, one may be tempted to think that for a replicable claim  $B \geq 0$  the utility based price is always given by equation (14). Unfortunately, this guess is wrong as can be seen by taking  $B = X_T$  for some maximal admissible process such that  $\mathbb{E}[\widehat{Y}_T(y)X_T] < X_0$ . For example, in the setting of [15, Example 5.1] one may take  $B = 1$ . Note that in this case the representation (18) for the utility based price still holds true.

**Remark 3** In Theorem 1 above we considered the case of a utility function  $U : (0, \infty) \rightarrow \mathbf{R}$ , satisfying  $U'(0) = \infty$  so that we extended  $U$  to the whole real line by letting  $U(x) = -\infty$ , for  $x < 0$  (see (4)).

Another important class of utility functions are those  $U : \mathbf{R} \rightarrow \mathbf{R}$  which are finitely valued on  $\mathbf{R}$ , and satisfy — apart from increasingness, strict concavity and differentiability — the Inada conditions  $U'(-\infty) = \infty$  and  $U'(\infty) = 0$ . The prime example of such functions is the exponential utility function  $U(x) = -\exp(-x)$ .

For this latter class of utility functions it was shown in [1] (see also [18]) that — under mild regularity conditions — the dual optimizer  $\widehat{Y}_T(y)$  to (8) equals the density of a probability measure  $\widehat{Q}(y) \in \mathcal{M}$ . In other words, condition (i) of Corollary 1 is automatically satisfied in this context. Using similar arguments as in the proof of Theorem 1 below one may therefore show that, in the setting of [1] or [18], every bounded contingent claim  $B$  admits a unique utility based price (w.r. to a fixed given initial capital  $x$ ). In other words the “kink phenomenon” described in Remark 1 only appears in the present case of utility functions satisfying (4) and not for utility functions which are finitely valued on all of  $\mathbf{R}$ .

### 3 Proof of the main theorem

For the convenience of the reader we start by recalling some results on maximal processes.

**Lemma 1** ([4]) *Assume (1) and let  $(X^n)_{n \geq 1}$  be a sequence of maximal admissible processes. Then the set of probability measures  $\mathbb{Q} \in \mathcal{M}$  under which all these processes are uniformly integrable martingales is non empty and dense in  $\mathcal{M}$  with respect to the variation norm.*

**Lemma 2** *Assume (1). Let  $\widetilde{X} \in \mathcal{X}$ ,  $Y \in \mathcal{Y}$ ,  $B \in \mathbf{L}^0$  be such that  $|B| \leq \widetilde{X}_T$  and  $x > 0$ ,  $q \in \mathbf{R}$  be such that the set  $\mathcal{X}(x, q|B)$  is not empty. If the product  $Y\widetilde{X}$  is a uniformly integrable martingale, then*

$$\mathbb{E}[Y_T X_T] \leq x, \quad X \in \mathcal{X}(x, q|B).$$

*Proof.* Fix  $X \in \mathcal{X}(x, q|B)$  and consider the process  $X + |q|\widetilde{X}$ . As is easily seen from the assumption made on the contingent claim, this process has a positive terminal value. On the other hand, the process  $X$  being acceptable

it can be written as  $X' - X''$  where  $X'$  is admissible and  $X''$  is maximal. By Lemma 1 we have that there exists at least one probability measure  $\mathbb{Q} \in \mathcal{M}$  under which  $X''$  is a uniformly integrable martingale. Using this in conjunction with the fact that admissible processes are supermartingales under  $\mathbb{Q}$  we obtain

$$0 \leq \mathbb{E}_{\mathbb{Q}} \left[ \left( X_T + |q| \tilde{X}_T \right) \middle| \mathcal{F}_t \right] \leq X_t + |q| \tilde{X}_t$$

and conclude that  $X + |q| \tilde{X}$  is an admissible process. The definition of  $\mathcal{Y}$  and the assumption of the lemma then imply that

$$\begin{aligned} \mathbb{E} \left[ Y_T X_T \right] &= \mathbb{E} \left[ Y_T \left( X_T + |q| \tilde{X}_T \right) \right] - |q| \cdot \mathbb{E} \left[ Y_T \tilde{X}_T \right] \\ &= \mathbb{E} \left[ Y_T \left( X_T + |q| \tilde{X}_T \right) \right] - |q| \leq x. \end{aligned}$$

■

*Proof of Theorem 1.(i).* Let  $B \in \mathbf{L}^0$  be a European contingent claim satisfying the assumption of the statement and assume without loss of generality that  $c = 1$ . Fix also  $x > 0$  and set  $y = u'(x)$ . We start by showing that the quantity  $p = p(B|x)$  defined by (14) is indeed a utility based price for the contingent claim. Using (6) in conjunction with (14) and (10) we obtain that

$$\begin{aligned} \mathbb{E} \left[ U \left( X_T + qB \right) \right] &\leq \mathbb{E} \left[ V \left( y \hat{Y}_T(y) \right) + y \hat{Y}_T(y) \left( X_T + qB \right) \right] \\ &= v(y) + y \left( \mathbb{E} \left[ \hat{Y}_T(y) X_T \right] + qp \right) \\ &= u(x) + y \left( \mathbb{E} \left[ \hat{Y}_T(y) X_T \right] - (x - qp) \right) \end{aligned}$$

holds for all  $q \in \mathbf{R}$ ,  $X \in \mathcal{X}(x - qp, q|B)$  and the desired inequality (13) now follows from Lemma 2.

We start the proof of uniqueness by showing that  $p$  defines the minimal utility based price in the sense that is for any  $\pi < p$  one can find a positive number  $q$  such that inequality

$$u(x - q\pi, q) = \sup_{X \in \mathcal{X}(x - q\pi, q|B)} \mathbb{E} \left[ U \left( X_T + qB \right) \right] > u(x) \quad (19)$$

holds true. As is easily seen, the existence of such a positive number will follow once we have shown that

$$D(x, \pi) := \liminf_{q \searrow 0} \left\{ \frac{u(x - q\pi, q) - u(x)}{q} \right\} > 0. \quad (20)$$

Let  $0 \leq (q_n)_{n=1}^\infty \leq \frac{x}{1+\pi}$  be an arbitrary sequence of positive numbers decreasing to zero and define a sequence of acceptable processes by setting

$$X^n := \widehat{X}(x - q_n(1 + \pi)) + q_n \widetilde{X}$$

where  $\widehat{X}(x - q_n(1 + \pi))$  is the optimal wealth process for the no-contingent claim problem (7) with initial capital  $x - q_n(1 + \pi)$ . As is easily seen, this process belongs to the set  $\mathcal{X}(x - q_n\pi, q_n|B)$  for each  $n \geq 1$  and it thus follows from the concavity of the agent's utility function that we have

$$\begin{aligned} u(x - q_n\pi, q_n) &\geq \mathbb{E} \left[ U \left( X_T^n + q_n B \right) \right] \\ &\geq \mathbb{E} \left[ U \left( \widehat{X}_T(x - q_n(1 + \pi)) \right) + q_n \left( \widetilde{X}_T + B \right) U' \left( X_T^n + q_n B \right) \right] \\ &= u(x - q_n(1 + \pi)) + \mathbb{E} \left[ q_n \left( \widetilde{X}_T + B \right) U' \left( X_T^n + q_n B \right) \right]. \end{aligned}$$

Using the above inequality in conjunction with (14), the definition of  $\partial_q^+(x, \pi)$  and the differentiability of the value function  $u = u(x)$  we obtain

$$\begin{aligned} D(x, \pi) &\geq -u'(x)(1 + \pi) + \liminf_{n \rightarrow \infty} \mathbb{E} \left[ \left( \widetilde{X}_T + B \right) U' \left( X_T^n + q_n B \right) \right] \\ &\geq -u'(x)(1 + \pi) + \mathbb{E} \left[ \left( \widetilde{X}_T + B \right) U' \left( \widehat{X}_T(x) \right) \right] \\ &= u'(x)(p - \pi) > 0. \end{aligned}$$

where the second inequality follows from the assumption  $|B| \leq \widetilde{X}_T$ , the non negativity of  $U'$ , the fact that  $X_T^n$  converges to  $\widehat{X}_T(x)$  in probability and Fatou's Lemma.

We thus have shown that for any contingent claim  $B'$  dominated by  $\widetilde{X}_T$  the quantity  $p(B'|x)$  defined by (14) determines the lower bound of utility based prices. In particular, we have that  $-p = p(-B|x)$  is the minimal utility based price for  $-B$ . However, as one can easily see from Definition 1, this implies that  $p$  is the maximal utility based price for  $B$ . This finishes the proof of uniqueness. ■

*Proof of Theorem 1.(ii).* Assume that the product  $\widehat{Y}(y)\widetilde{X}$  is not a uniformly integrable martingale. Since this process is a supermartingale by definition of the set  $\mathcal{Y}$  it follows that there exists  $0 < \delta < 1$  such that

$$\mathbb{E} \left[ \widehat{Y}_T(y) \widetilde{X}_T \right] = 1 - \delta < 1. \quad (21)$$

Let  $\mathcal{M}'$  be the set of equivalent probability measures  $\mathbb{Q} \in \mathcal{M}$  under which the maximal admissible process  $\tilde{X}$  is a uniformly integrable martingale. As is easily seen, this set is closed under countable convex combinations and by Lemma 1 it is dense in  $\mathcal{M}$  with respect to the variation norm. It thus follows from [15, Propositions 3.1&3.2] that

$$\sup_{\mathbb{Q} \in \mathcal{M}'} \mathbb{E}_{\mathbb{Q}}[X_T] = \sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}_{\mathbb{Q}}[X_T] = \sup_{Y \in \mathcal{Y}} \mathbb{E}[Y_T X_T]$$

holds for all admissible processes  $X \in \mathcal{X}$  and

$$\begin{aligned} v(y) &= \min_{Y \in \mathcal{Y}} \mathbb{E}[V(yY_T)] \\ &= \mathbb{E}\left[V\left(y\hat{Y}_T(y)\right)\right] = \inf_{\mathbb{Q} \in \mathcal{M}'} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]. \end{aligned}$$

Let now  $(\mathbb{Q}^n)_{n=1}^{\infty}$  be an arbitrary minimizing sequence for the above problem, that is a sequence in  $\mathcal{M}'$  with the property that

$$\lim_{n \rightarrow \infty} \mathbb{E}\left[V\left(y\frac{d\mathbb{Q}^n}{d\mathbb{P}}\right)\right] = v(y) = \mathbb{E}\left[V\left(y\hat{Y}_T(y)\right)\right]. \quad (22)$$

Passing if necessary to a subsequence and relying on Komlòs lemma, we may assume without loss of generality that the sequence  $(\frac{d\mathbb{Q}^n}{d\mathbb{P}})_{n=1}^{\infty}$  converges in probability to a non negative function  $Y_*$ . Applying [15, Lemma 3.4] and taking into account the uniqueness of the solution to (8) we deduce that  $Y_* = \hat{Y}_T(y)$ . Using this fact in conjunction with (21) and the definition of the set  $\mathcal{M}'$ , we conclude that even though it is convergent in probability and bounded in expectations, the sequence

$$(Z^n)_{n=1}^{\infty} := \left(\tilde{X}_T \frac{d\mathbb{Q}^n}{d\mathbb{P}}\right)_{n=1}^{\infty}$$

is not uniformly integrable. Therefore, by Rosenthal's subsequence splitting lemma (see for example [6, 12, 17]) we can find a subsequence  $(Z^k)_{k=1}^{\infty}$  and a sequence of pairwise disjoint measurable sets  $(A_k)_{k=1}^{\infty}$  with the properties that

$$\lim_{k \rightarrow \infty} \mathbb{E}\left[1_{A_k} Z^k\right] = \delta, \quad (23)$$

and for any bounded random variable  $R$

$$\lim_{k \rightarrow \infty} \mathbb{E}\left[\left(1_{A_k^c} Z^k - \hat{Y}_T(y)\tilde{X}_T\right) R\right] = 0. \quad (24)$$

After these lengthy preparations we now turn to the construction of a contingent claim satisfying the assertions of the statement. Let  $\mathcal{E}$  denote the set of even natural numbers and define

$$A := \bigcup_{k \in \mathcal{E}} A_k \subset \Omega.$$

For our contingent claim we take the positive random variable  $B := 1_A \tilde{X}_T$  and finally, for the non negative constant  $\alpha$  we take

$$\alpha := \mathbb{E}[\widehat{Y}_T(y)B] = \mathbb{E}[1_A \widehat{Y}_T(y) \tilde{X}_T].$$

In order to complete our proof, it is now sufficient to show that the inequality  $u(x - qp, q) \leq u(x)$  holds true for all strictly positive  $q$  when  $p = \alpha$  and for all strictly negative  $q$  when  $p = \alpha + \delta$ . To this end, we start by observing that it follows from (5), the definition of  $\mathcal{M}'$  and Lemma 2 that

$$\begin{aligned} \mathbb{E}\left[U\left(X_T + qB\right)\right] &\leq \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] + y \mathbb{E}\left[\frac{d\mathbb{Q}^k}{d\mathbb{P}}\left(X_T + qB\right)\right] \\ &\leq \mathbb{E}\left[V\left(y \frac{d\mathbb{Q}^k}{d\mathbb{P}}\right)\right] + y(x - qp) + q \mathbb{E}\left[\frac{d\mathbb{Q}^k}{d\mathbb{P}}B\right] \end{aligned} \quad (25)$$

holds for all  $(q, p) \in \mathbf{R}^2$ ,  $X \in \mathcal{X}(x - qp, q|B)$  and  $k \geq 1$ .

Now let  $p = \alpha$  and fix an arbitrary  $q > 0$ . Taking limits on both sides of the above expression as  $k \notin \mathcal{E} \rightarrow \infty$  and using (22) in conjunction with the definition of the set  $A$  and (24) with  $R = 1_A$  we obtain that

$$\begin{aligned} \mathbb{E}\left[U\left(X_T + qB\right)\right] &\leq v(y) + y(x - q\alpha) + \lim_{k \notin \mathcal{E} \rightarrow \infty} \mathbb{E}\left[yq\left(1_A Z^k\right)\right] \\ &= u(x) - yq\alpha + \lim_{k \notin \mathcal{E} \rightarrow \infty} \mathbb{E}\left[yq\left(1_{A \cap A_k} Z^k + 1_{A \cap A_k^c} Z^k\right)\right] \\ &= u(x) - yq\alpha + \lim_{k \notin \mathcal{E} \rightarrow \infty} \mathbb{E}\left[yq\left(1_{A \cap A_k^c} Z^k\right)\right] = u(x) \end{aligned}$$

holds for all  $X \in \mathcal{X}(x - q\alpha, q|B)$  and the desired inequality follows. Similarly, fix an arbitrary  $q < 0$  and let  $p = \alpha + \delta =: \beta$ . Taking limits on both sides of (25) as  $k \in \mathcal{E}$  increases to infinity and using (22) in conjunction with the

definition of  $A$ , (23) and (24) with  $R = 1_A$  we obtain that

$$\begin{aligned} \mathbb{E}\left[U\left(X_T + qB\right)\right] - u(x) &\leq yq \left(-\beta + \lim_{k \in \mathcal{E} \rightarrow \infty} \mathbb{E}\left[1_A Z^k\right]\right) \\ &= yq \left(-\beta + \lim_{k \in \mathcal{E} \rightarrow \infty} \mathbb{E}\left[1_{A \cap A_k} Z^k + 1_{A \cap A_k^c} Z^k\right]\right) \\ &= yq \left(-\beta + \lim_{k \in \mathcal{E} \rightarrow \infty} \mathbb{E}\left[1_{A_k} Z^k + 1_{A \cap A_k^c} Z^k\right]\right) = 0 \end{aligned}$$

holds for all  $X \in \mathcal{X}(x - q\beta, q|B)$  and our proof is complete. ■

## References

- [1] F. Bellini, M. Frittelli. On the existence of minimax martingale measures. *Mathematical Finance*, 12(1):1–21, 2002.
- [2] M.H.A. Davis. Option pricing in incomplete markets. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, pages 216–226. Cambridge University Press, 1997.
- [3] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300:463–520, 1994.
- [4] F. Delbaen and W. Schachermayer. The Banach space of workable contingent claims in arbitrage theory. *Annales de l'Institut Henri Poincaré: Probabilités et Statistiques*, 33:113–44, 1997.
- [5] F. Delbaen and W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. *Mathematische Annalen*, 312:215–250, 1998.
- [6] J. Diestel and J.J. Uhl, *Vector Measures*, Mathematical Surveys, American Mathematical Society, 1977.
- [7] L. Foldes. Valuation and martingale properties of shadow prices: An exposition. *Journal of Economic Dynamics and Control*, 24:1641–1701, 2000.
- [8] M. Frittelli. Introduction to a theory of value coherent with the no-arbitrage principle. *Finance and Stochastics*, 4(3):275–297, 2000.

- [9] Sir J. R. Hicks. *A Revision of Demand Theory*. Oxford University Press, Oxford, 1986 (First Edition 1956).
- [10] S.D. Hodges and A. Neuberger. Optimal replication of contingent claims under transaction costs. *The Review of Futures Markets*, 8:222–239, 1989.
- [11] J. Hugonnier and D. Kramkov. Optimal investment with random endowments in incomplete markets. To Appear in *The Annals of Applied Probability*, 2002.
- [12] M. Kadeč and A. Pelczynski. Basic sequences, biorthogonal systems and norming sets in Banach and Fréchet spaces. *Studia Mathematica*, 25:297–323, 1965.
- [13] J. Kallsen. Derivative pricing based on local utility maximization. *Finance and Stochastics*, 6:115–140, 2002.
- [14] I. Karatzas and S. G. Kou. On the pricing of contingent claims under constraints. *The Annals of Applied Probability*, 6(2):321–69, 1996.
- [15] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9(3):904–950, 1999.
- [16] D.O. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. The Appear in *The Annals of Applied Probability*, 2001.
- [17] H. P. Rosenthal. Sous espaces de  $L^1$ . Lectures Notes, Department of Mathematics, Université Paris 6–Pierre & Marie Curie, 1979.
- [18] W. Schachermayer. Optimal Investment in Incomplete Markets when Wealth may Become Negative. *The Annals of Applied Probability*, 11(3):694–734, 2001.