Optimal Control of McKean-Vlasov Dynamics

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CREDITS

Joint Work with

François Delarue (Nice)

(series of papers and forthcoming book)

Classical Stochastic Differential Control

$$\inf_{\alpha \in \mathbb{A}} \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)dt + g(X_T, \mu_T)\right]$$

subject to $\quad dX_t = b(t, X_t, \alpha_t)dt + \sigma(t, X_t, \alpha_t)dW_t; \quad X_0 = x_0.$

- **Analytic Approach** (by PDEs)
  - HJB equation
- **Probabilistic Approaches** (by FBSDEs)
  1. Represent value function as solution of a BSDE
  2. Represent the gradient of the value function as solution of a FBSDE (Stochastic Maximum Principle)

I. First Probabilistic Approach

Assumptions

- σ is uncontrolled
- σ is invertible

Reduced **Hamitonian**

$$H(t,x,y,\alpha) = b(t,x,\alpha)\cdot y + f(t,x,\alpha)$$

For each control $\boldsymbol{\alpha}$ solve **BSDE**

$$dY_t = -H(t,X_t,Z_t\sigma(t,X_t)^{-1},\alpha_t)dt + Z_t\cdot dW_t, \qquad Y_T = g(X_T)$$

Then

$$Y_0^{\boldsymbol{\alpha}} = J(\boldsymbol{\alpha}) = \mathbb{E}\left[\int_0^T f(t,X_t,\alpha_t)dt + g(X_T,\mu_T)\right]$$

So by **comparison theorems** for BSDEs, optimal control $\hat{\boldsymbol{\alpha}}$ given by:

$$\hat{\alpha}_t = \hat{\alpha}(t,X_t,Z_t\sigma(t,X_t)^{-1}), \quad \text{with} \quad \hat{\alpha}(t,x,y)\in \text{argmin}_{\alpha\in A}H(t,x,y,\alpha)$$

and $Y_0^{\boldsymbol{\alpha}} = J(\hat{\boldsymbol{\alpha}})$

II. Pontryagin Stochastic Maximum Approach

Assumptions

- Coefficients b, σ and f differentiable
- f convex in (x, α) and g convex

Hamitonian

$$H(t, x, y, z, \alpha) = b(t, x, \alpha) \cdot y + \sigma(t, x, \alpha) \cdot z + f(t, x, \alpha)$$

For each control $\boldsymbol{\alpha}$ solve **BSDE** for the adjoint processes $\mathbf{Y} = (Y_t)_t$ and $\mathbf{Z} = (Z_t)_t$

$$dY_t = -\partial_x H(t, X_t, Y_t, Z_t, \alpha_t)dt + Z_t \cdot dW_t, \qquad Y_T = \partial_x g(X_T)$$

Then, optimal control $\hat{\boldsymbol{\alpha}}$ given by:

$$\hat{\alpha}_t = \hat{\alpha}(t, X_t, Y_t, Z_t), \quad \text{with} \quad \hat{\alpha}(t, x, y, z) \in \text{argmin}_{\alpha \in A} H(t, x, y, z, \alpha)$$

and $Y_0^{\hat{\boldsymbol{\alpha}}} = J(\hat{\boldsymbol{\alpha}})$

SUMMARY

In both cases (σ uncontrolled), need to **solve a FBSDE**

$$\begin{cases} dX_t = B(t, X_t, Y_t, Z_t)dt + \Sigma(t, X_t)dW_t, \\ dY_t = -F(t, X_t, Y_t, Z_t)dt + Z_t dW_t \end{cases}$$

First Approach

$$\begin{aligned} B(t,x,y,z) &= b\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1})\big), \\ F(t,x,y,z) &= -f\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1}\big) \\ &\qquad - \big(z\sigma(t,x,)^{-1}\big) \cdot b\big(t,x,\hat{\alpha}(t,x,z\sigma(t,x)^{-1})\big). \end{aligned}$$

Second Approach

$$\begin{aligned} B(t,x,y,z) &= b(t,x,\hat{\alpha}(t,x,y)), \\ F(t,x,y,z) &= -\partial_x f(t,x,\hat{\alpha}(t,x,y)) - y \cdot \partial_x b(t,x,\hat{\alpha}(t,x,y)). \end{aligned}$$

Propagation of Chaos & McKean-Vlasov SDEs

System of N particles $X_t^{N,i}$ at time t with **symmetric (Mean Field)** interactions

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N})dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N})dW_t^i, \quad i = 1, \cdots, N$$

where $\overline{\mu}^N_{X_t^N}$ is the empirical measure $\overline{\mu}^N_{\mathbf{x}} = \frac{1}{N}\sum_{i=1}^N \delta_{x^i}$

Large population asymptotics ($N \to \infty$)

1. The N processes $(X_t^{N,i})_{0 \le t \le T}$ for $i = 1, \cdots, N$ become asymptotically **i.i.d.**
2. Each of them is (asymptotically) distributed as the solution of the McKean-Vlasov SDE

$$dX_t = b(t, X_t, \mathcal{L}(X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t))dW_t$$

Forward SDEs of McKean-Vlasov Type

$$dX_t = B(t, X_t, \mathcal{L}(X_t))dt + \Sigma(t, X_t, \mathcal{L}(X_t))dW_t, \qquad T \in [0, T].$$

Assumption. There exists a constant $c \geq 0$ such that

(A1) For each $(x, \mu) \in \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$, the processes $B(\cdot, \cdot, x, \mu) : \Omega \times [0, T] \ni (\omega, t) \mapsto B(\omega, t, x, \mu)$ and $\Sigma(\cdot, \cdot, x, \mu) : \Omega \times [0, T] \ni (\omega, t) \mapsto \Sigma(\omega, t, x, \mu)$ are $\mathbb{F}$-progressively measurable and belong to $\mathbb{H}^{2,d}$ and $\mathbb{H}^{2,d\times d}$ respectively.

(A2) $\forall t \in [0, T]$, $\forall x, x' \in \mathbb{R}^d$, $\forall \mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$, with probability 1 under $\mathbb{P}$,

$$|B(t, x, \mu) - B(t, x', \mu')| + |\Sigma(t, x, \mu) - \Sigma(t, x', \mu')| \leq c\big[|x - x'| + W_2(\mu, \mu')\big],$$

where W_2 denotes the 2-Wasserstein distance on the space $\mathcal{P}_2(\mathbb{R}^d)$.

Result. if $X_0 \in L^2(\Omega, \mathcal{F}_0, \mathbb{P}; \mathbb{R}^d)$, then there exists a unique solution $\mathbf{X} = (X_t)_{0\leq t\leq T}$ in $\mathbb{S}^{2,d}$ s.t. for every $p \in [1, 2]$

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} |X_t|^p\Big] < +\infty.$$

Sznitmann

Controlling Large Symmetric Populations

Assume Mean Field Interactions

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dW_t^i \quad i = 1, \cdots, N$$

Assume distributed strategies

$$\alpha_t^i = \phi(t, X_t^{N,i})$$

Assume population is large (i.e. $N = \infty$)

1. The N state processes evolve independently of each other
2. Controlling each of them reduces to the optimal control problem

$$\inf_{\phi \in \Phi} \mathbb{E}\Big[\int_0^T f(t, X_t, \mathcal{L}(X_t), \phi(t, X_t))dt + g(X_T, \mathcal{L}(X_T))\Big]$$

s.t. $dX_t = b(t, X_t, \mathcal{L}(X_t), \phi(t, X_t))dt + \sigma(t, X_t, \mathcal{L}(X_t), \phi(t, X_t))dW_t \quad X_0 = x_0.$

Control of a McKean-Vlasov SDE (Markovian - closed loop)

Control of McKean-Vlasov Dynamics

Mathematical Formulation

1. **State dynamics** given by an SDE of McKean - Vlasov type

$$dX_t = b(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + \sigma(t, X_t, \mathcal{L}(X_t), \alpha_t)dW_t$$

2. **Objective function** to minimize of the McKean-Vlasov type

$$J(\alpha) = \mathbb{E}\Big[\int_0^T f(t, X_t, \mathcal{L}(X_t), \alpha_t)dt + g(X_T, \mathcal{L}(X_T))\Big]$$

Could use **open loop** formulation.

Control of McKean - Vlasov SDEs

State at time t, say $(X_t, \mathcal{L}(X_t))$ is **infinite dimensional**

Analytic Approach

- **Infinite dimensional HJB** equations (**Crandall, Lions, Ishii**?)

Probabilistic Approaches

1. **McKean - Vlasov FBSDEs !**
2. Pontryagin maximum principle approach
    - How should we **differentiate the Hamiltonian w.r.t. the measure?**

More to come

N-PLAYER STOCHASTIC DIFFERENTIAL GAMES

Assume Mean Field Interactions (symmetric game)

$$dX_t^{N,i} = b(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + \sigma(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dW_t^i \quad i = 1, \cdots, N$$

Assume player i tries to minimize

$$J^i(\boldsymbol{\alpha}^1, \cdots, \boldsymbol{\alpha}^N) = \mathbb{E}\left[\int_0^T f(t, X_t^{N,i}, \overline{\mu}^N_{X_t^N}, \alpha_t^i)dt + g(X_T, \overline{\mu}^N_{X_T^N})\right]$$

Search for Nash equilibria

- Very difficult in general, even if N is small
- ϵ-Nash equilibria? Still hard.
- How about in the limit $N \to \infty$?

Mean Field Games Lasry - Lions, Caines-Huang-Malhamé

MFG Paradigm

A **typical** agent plays against a **continuum** of players whose states he/she feels through their **distribution** μ_t at time t

1. For each **Fixed** measure flow (μ_t) in $\mathcal{P}(\mathbb{R})$, solve the **standard stochastic control problem**

$$\hat{\alpha} = \arg\inf_{\alpha} \mathbb{E}\left\{ \int_0^T f(t, X_t, \mu_t, \alpha_t)dt + g(X_T, \mu_T) \right\}$$

   subject to

$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t$$

2. **Fixed Point Problem**: determine (μ_t) so that

$$\forall t \in [0, T], \quad \mathcal{L}(X_t) = \mu_t \quad a.s.$$

Once this is done one expects that, if $\hat{\alpha}_t = \phi(t, X_t)$,

$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$

form an **approximate Nash equilibrium** for the game with N players.

I. Value Function Representation: Prep.

Recall

$$\sigma(t,x,\mu,\alpha)=\sigma(t,x) \text{ **uniformly Lip-1 and uniformly elliptic**}$$

$$H(t,x,\mu,y,\alpha)=y\cdot b(t,x,\mu,\alpha)+f(t,x,\mu,\alpha)$$

and

$$\hat{\alpha}(t,x,\mu,y)\in \arg \in_{\alpha\in A} H(t,x,\mu,y,\alpha).$$

(A.1) b is affine in α: $b(t,x,\mu,\alpha)=b_1(t,x,\mu)+b_2(t)\alpha$ with b_1 and b_2 bounded.

(A.2) Running cost f strongly convex

$$f(t,x',\mu,\alpha')-f(t,x,\mu,\alpha)-\langle(x'-x,\alpha'-\alpha),\partial_{(x,\alpha)}f(t,x,\mu,\alpha)\rangle\geq\lambda|\alpha'-\alpha|^2.$$

Then

$\hat{\alpha}(t,x,\mu,y)$ ***is unique and***

$$[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\ni(t,x,\mu,y)\to\hat{\alpha}(t,x,\mu,y)$$

is measurable, locally bounded and Lipschitz-continuous with respect to (x,y)***, uniformly in*** $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$

I. Value Function Representation: Cont.

If $A \subset \mathbb{R}^k$ is **bounded** (not really needed), if $\mathbf{X}^{t,x} = (X_s^{t,x})_{t \le s \le T}$ is the unique strong solution of $dX_t = \sigma(t, X_t)dW_t$ over $[t, T]$ s.t. $X_t^{t,x} = x$, and if $(\hat{\boldsymbol{Y}}^{t,x}, \hat{\boldsymbol{Z}}^{t,x})$ is a solution of the BSDE

$$
\begin{aligned}
d\hat{Y}_s^{t,x} = &-H(t, X_s^{t,x}, \mu_s, \hat{Z}_s^{t,x}\sigma(s, X_s^{t,x})^{-1}, \hat{\alpha}(s, X_s^{t,x}, \mu_s, \hat{Z}_s^{t,x}\sigma(s, X_s^{t,x})^{-1}))ds \\
&- \hat{Z}_s^{t,x} dW_s, \qquad t \le s \le T,
\end{aligned}
$$

with $\hat{Y}_T = g(X_T^{t,x}, \mu_T)$, then

$$
\hat{\alpha}_t = \hat{\alpha}(s, X_s^{t,x}, \mu_s, \hat{Z}_s^{t,x}\sigma(s, X_s^{t,x})^{-1})
$$

is an optimal control over the interval $[t, T]$ and the value of the problem is given by:

$$
V(t, x) = \hat{Y}_t^{t,x}.
$$

The value function appears as the decoupling field of an FBSDE.

Fixed Point Step $\Longrightarrow$ McKean-Vlasov FBSDE

Starting from $t = 0$ and dropping the superscript ^{t,x}

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}))dt + \sigma(t, X_t)dW_t \\ dY_t = -H(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}, \hat{\alpha}(t, X_t, \mu_t, Z_t\sigma(t, X_t)^{-1}))dt - Z_t dW_t, \end{cases}$$

for $0 \leq t \leq T$, with $\hat{Y}_T = g(X_T, \mu_T)$.

Implementing the **fixed point step**

$$\mu_t \quad \hookrightarrow \quad \mathcal{L}(X_t)$$

gives an **FBSDE of McKean-Vlasov** type.

II. Pontryagin Stochastic Maximum Principle

Freeze $\mu = (\mu_t)_{0 \le t \le T}$,

Recall (reduced) Hamiltonian

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)$$

Adjoint processes

Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ and the corresponding controlled state process $X^{\alpha} = (X_t^{\alpha})_{0 \le t \le T}$, any couple $(Y_t, Z_t)_{0 \le t \le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H(t, X_t^{\alpha}, \mu_t, Y_t, \alpha_t)dt + Z_t dW_t \\ Y_T = \partial_x g(X_T^{\alpha}, \mu_T) \end{cases}$$

Stochastic Control Step

Determine

$$\hat{\alpha}(t, x, \mu, y) = \arg\inf_{\alpha} H(t, x, \mu, y, \alpha)$$

Inject in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t, X_t, \mu_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + \sigma(t, X_t)dW_t \\ dY_t = -\partial_x H(t, X, \mu_t, Y_t, \hat{\alpha}(t, X_t, \mu_t, Y_t))dt + Z_t dW_t \end{cases}$$

with $X_0 = x_0$ and $Y_T = \partial_x g(X_T, \mu_T)$

Standard **FBSDE** (for each **fixed** $t \hookrightarrow \mu_t$)

Fixed Point Step

Solve the **fixed point problem**

$$(\mu_t)_{0\le t\le T} \quad \longrightarrow \quad (X_t)_{0\le t\le T} \quad \longrightarrow \quad (\mathcal{L}(X_t))_{0\le t\le T}$$

Note: if we enforce $\mu_t = \mathcal{L}(X_t)$ for all $0 \le t \le T$ in FBSDE we have

$$\begin{cases} dX_t = b(t, X_t, \mathcal{L}(X_t), \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t))dt + \sigma(t, X_t)dW_t, \\ dY_t = -\partial_x H(t, X_t^\alpha, \mathcal{L}(X_t), Y_t, \hat{\alpha}(t, X_t, \mathcal{L}(X_t), Y_t))dt + Z_t dW_t \end{cases}$$

with

$$X_0 = x_0 \qquad \text{and} \qquad Y_T = \partial_x g(X_T, \mathcal{L}(X_T))$$

FBSDE of McKean-Vlasov type !!!

Very difficult

FBSDEs of McKean - Vlasov Type

In both probabilistic approaches to the MFG problem the problem reduces to the solution of an FBSDE

$$\begin{cases} dX_t = B(t, X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + \Sigma(t, X_t, \mathcal{L}(X_t))dW_t, \\ dY_t = -F(t, X_t, \mathcal{L}(X_t), Y_t, Z_t)dt + Z_t dW_t \end{cases}$$

with in the first approach

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, z\sigma(t, x)^{-1})), \\ F(t, x, \mu, y, z) = -f(t, x, \mu, \hat{\alpha}(t, x, \mu, z\sigma(t, x)^{-1}) - z\sigma(t, x)^{-1} b(t, x, \mu, \hat{\alpha}(t, x, \mu, z \end{cases}$$

and in the second:

$$\begin{cases} B(t, x, \mu, y, z) = b(t, x, \mu, \hat{\alpha}(t, x, \mu, y)), \\ F(t, x, \mu, y, z) = -\partial_x f(t, x, \mu, \hat{\alpha}(t, x, \mu, y)) - y\partial_x b(t, x, \mu, \hat{\alpha}(t, x, \mu, y)). \end{cases}$$

A Typical Existence Result

$$\begin{cases} dX_t = B\big(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t)}\big)dt + \Sigma\big(t, X_t, Y_t, \mathbb{P}_{(X_t,Y_t)}\big)dW_t \\ dY_t = -F\big(t, X_t, Y_t, Z_t, \mathbb{P}_{(X_t,Y_t)}\big)dt + Z_t dW_t, \quad 0 \le t \le T, \end{cases}$$

with $X_0 = x_0$ and $Y_T = G(X_T, \mathcal{L}(X_T))$.

Assumptions

(A1). B, F, Σ and G are continuous in μ and uniformly (in μ) Lipschitz in (x, y, z)

(A2). Σ and G are bounded and

$$\begin{cases} |B(t,x,y,z,\mu)| \le L\left[1 + |x| + |y| + |z| + \left(\int_{\mathbb{R}^d \times \mathbb{R}^p} |(x',y')|^2 d\mu(x',y')\right)^{1/2}\right], \\ |F(t,x,y,z,\mu)| \le L\left[1 + |y| + \left(\int_{\mathbb{R}^d \times \mathbb{R}^p} |y'|^2 d\mu(x',y')\right)^{1/2}\right]. \end{cases}$$

(A3). Σ is uniformly elliptic

$$\Sigma(t,x,y,\mu)\Sigma(t,x,y,\mu)^\dagger \ge L^{-1} I_d$$

and $[0,T] \ni t \hookrightarrow \Sigma(t,0,0,\delta_{(0,0)})$ is also assumed to be continuous.

Under (A1–3), there exists a solution $(\mathbf{X}, \mathbf{Y}, \mathbf{Z}) \in \mathbb{S}^{2,d} \times \mathbb{S}^{2,p} \times \mathbb{H}^{2,p\times m}$

MEAN FIELD GAMES WITH A COMMON NOISE

Starting with a finite player game, i.e.

Simultaneous Minimization of

$$J^i(\boldsymbol{\alpha}) = \mathbb{E}\left\{\int_0^T f(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + g(X_T, \overline{\mu}_T^N)\right\}, \quad i = 1, \cdots, N$$

under **constraints** (dynamics of players private states)

$$dX_t^i = b(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dt + \sigma(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dW_t^i + \sigma^0(t, X_t^i, \overline{\mu}_t^N, \alpha_t^i)dW_t^0$$

for i.i.d. Wiener processes W_t^k for $k = 0, 1, \cdots, N$.

Large Game Asymptotics (cont.)

Conditional Law of Large Numbers

- If we consider **exchangeable equilibriums**, $(\alpha_t^1, \cdots, \alpha_t^N)$, then
  - By **de Finetti LLN**

$$\lim_{N\to\infty} \overline{\mu}_t^N = \mathbb{P}_{X_t^1 | \mathcal{F}_t^0}$$

  - **Dynamics** of player 1 (or any other player) becomes

$$dX_t^1 = b(t, X_t^1, \mu_t, \alpha_t^1)dt + \sigma(t, X_t^1, \mu_t, \alpha_t^1)dW_t + \sigma^0(t, X_t^1, \mu_t, \alpha_t^1)dW_t^0;$$

with $\mu_t = \mathbb{P}_{X_t^1 | \mathcal{F}_t^0}$.

  - **Cost** to player 1 (or any other player) becomes

$$\mathbb{E}\left\{\int_0^T f(t, X_t, \mu_t, \alpha_t^1)dt + g(X_T, \mu_T)\right\}$$

MFG with Common Noise Paradigm

1. For each **Fixed** measure valued $(\mathcal{F}_t^0)$-adapted process (μ_t) in $\mathcal{P}(\mathbb{R})$, solve the standard **stochastic control problem**

$$\hat{\alpha} = \arg\inf_{\alpha} \mathbb{E}\left\{\int_0^T f(t, X_t, \mu_t, \alpha_t)dt + g(X_T, \mu_T)\right\}$$

subject to

$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t, \mu_t, \alpha_t)dW_t + \sigma^0(t, X_t, \mu_t, \alpha_t)dW_t^0;$$

2. **Fixed Point Problem**: determine (μ_t) so that

$$\forall t \in [0, T], \quad \mathbb{P}_{X_t | \mathcal{F}_t^0} = \mu_t \quad a.s.$$

Once this is done one expects that, if $\hat{\alpha}_t = \phi(t, X_t)$, for N player game,

$$\alpha_t^{j*} = \phi^*(t, X_t^j), \qquad j = 1, \cdots, N$$

form an **approximate Nash equilibrium** for the game with N players.

Ex: Pontryagin Stochastic Maximum Principle

Freeze $\mu = (\mu_t)_{0 \le t \le T}$, write (reduced) Hamiltonian

$$H(t, x, \mu, y, \alpha) = b(t, x, \mu, \alpha) \cdot y + f(t, x, \mu, \alpha)$$

Standard definition

Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ and the corresponding controlled state process $X^\alpha = (X_t^\alpha)_{0 \le t \le T}$, any couple $(Y_t, Z_t)_{0 \le t \le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H(t, X_t^\alpha, \mu_t, Y_t, \alpha_t)dt + Z_t dW_t + Z_t^0 dW_t^0 \\ Y_T = \partial_x g(X_T^\alpha, \mu_T) \end{cases}$$

is called a set of **adjoint processes**

Stochastic Control Step Solution

Determine

$$\hat{\alpha}(t,x,\mu,y) = \arg\inf_{\alpha} H(t,x,\mu,y,\alpha)$$

Inject in **FORWARD** and **BACKWARD** dynamics and **SOLVE**

$$\begin{cases} dX_t = b(t,X_t,\mu_t,\hat{\alpha}(t,X_t,\mu_t,Y_t))dt + \sigma(t,X_t)dW_t + \sigma^0(t,X_t)dW_t^0, \\ dY_t = -\partial_x H^{\mu_t}(t,X,Y_t,\hat{\alpha}(t,X_t,\mu_t,Y_t))dt + Z_t dW_t + Z_t^0 dW_t^0 \end{cases}$$

with $X_0 = x_0$ and $Y_T = \partial_x g(X_T,\mu_T)$

Standard **FBSDE** (for each **fixed** $t \hookrightarrow \mu_t$)

Fixed Point Step

Solve the **fixed point problem**

$$(\mu_t)_{0\le t\le T} \quad \longrightarrow \quad (X_t)_{0\le t\le T} \quad \longrightarrow \quad (\mathbb{P}_{X_t|\mathcal{F}_t^0})_{0\le t\le T}$$

Note: if we enforce $\mu_t = \mathbb{P}_{X_t|\mathcal{F}_t^0}$ for all $0 \le t \le T$ in FBSDE we have

$$\begin{cases} dX_t = b(t, X_t, \mathbb{P}_{X_t|\mathcal{F}_t^0}, \hat{\alpha}^{\mathbb{P}_{X_t|\mathcal{F}_t^0}}(t, X_t, Y_t))dt + \sigma(t, X_t)dW_t + \sigma(t, X_t) \circ dW_t^0, \\ dY_t = -\partial_x H^{\mathbb{P}_{X_t|\mathcal{F}_t^0}}(t, X_t^\alpha, Y_t, \hat{\alpha}^{\mathbb{P}_{X_t|\mathcal{F}_t^0}}(t, X_t, Y_t))dt + Z_t dW_t + Z_t^0 dW_t^0, \end{cases}$$

with

$$X_0 = x_0 \qquad \text{and} \qquad Y_T = \partial_x g(X_T, \mathbb{P}_{X_T|\mathcal{F}_T^0})$$

FBSDE of Conditional McKean-Vlasov type !!!

Very difficult

Several Approches

- **Relaxed Controls** (**R.C. - Delarue - Lacker**)
- **FBSDEs of Conditional McKean-Vlasov Type** (**RC - Delarue**)
  - SDEs of Conditional McKean-Vlasov Type (**RC - Zhu**)
  - Conditional Propagation of Chaos (**RC - Zhu**)
  - Existence for a finite common noise (Schauder Theorem)
  - Weak Solution by Limiting arguments
  - Uniqueness via Monotonicity or Strong Convexity
  - Strong Solution via extension of Yamada-Watanabe

Back to Control of McKean - Vlasov Dynamics

Say using Pontryaging Maximum Principle

Differentiability and Convexity of $\mu \hookrightarrow h(\mu)$

- Notions of differentiability for functions defined on spaces of measures from theory of optimal transportation, gradient flows, etc) studied by **Ambrosio, De Giorgi, Otto, Villani**, etc
- Tailored made notion (**Lions**' Collège de France Lectures, **Cardaliaguet**)

Lift a function $\mu \hookrightarrow h(\mu)$ to $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ into

$$X \hookrightarrow \tilde{h}(X) = h(\tilde{\mathbb{P}}_X)$$

and say

h is differentiable at μ if $\tilde{h}$ is Fréchet differentiable at X whenever $\tilde{\mathbb{P}}_X = \mu$.

A function g on $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ is said to be **convex** if for every (x, μ) and (x', μ') in $\mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d)$ we have

$$g(x', \mu') - g(x, \mu) - \partial_x g(x, \mu) \cdot (x' - x) - \tilde{\mathbb{E}}[\partial_\mu g(x, \tilde{X}) \cdot (\tilde{X}' - \tilde{X})] \geq 0$$

whenever $\tilde{\mathbb{P}}_{\tilde{X}} = \mu$ and $\tilde{\mathbb{P}}_{\tilde{X}'} = \mu'$

POTENTIAL GAMES

Start with **Mean Field Game** à la **Lasry-Lions**

$$\inf_{\boldsymbol{\alpha}=(\alpha_t)_{0\le t\le T},\ dX_t=\alpha_t dt+\sigma dW_t} \mathbb{E}\Big[\int_0^T \big[\frac{1}{2}|\alpha_t|^2 + f(t,X_t,\mu_t)\big]\,dt + g(X_T,\mu_T)\Big]$$

such that f and g are differentiable w.r.t. x s.t. there exist differentiable functions F and G

$$\partial_x f(t,x,\mu) = \partial_\mu F(t,\mu)(x) \quad \text{and} \quad \partial_x g(x,\mu) = \partial_\mu G(\mu)(x) \tag{1}$$

Solving this **MFG** is equivalent to solving the **central planner** optimization problem

$$\inf_{\boldsymbol{\alpha}=(\alpha_t)_{0\le t\le T},\ dX_t=\alpha_t dt+\sigma dW_t} \mathbb{E}\Big[\int_0^T \big[\frac{1}{2}|\alpha_t|^2 + F(t,\mathcal{L}(X_t))\big]\,dt + G(\mathcal{L}(X_T))\Big]$$

Special case of **McKean-Vlasov optimal control**

THE ADJOINT EQUATIONS

Lifted Hamiltonian

$$\tilde{H}(t, x, \tilde{X}, y, \alpha) = H(t, x, \mu, y, \alpha)$$

for any random variable $\tilde{X}$ with distribution μ.

Given an admissible control $\boldsymbol{\alpha} = (\alpha_t)_{0 \le t \le T}$ and the corresponding controlled state process $\mathbf{X}^\alpha = (X_t^\alpha)_{0 \le t \le T}$, any couple $(Y_t, Z_t)_{0 \le t \le T}$ satisfying:

$$\begin{cases} dY_t = -\partial_x H(t, X_t^\alpha, \mathbb{P}_{X_t^\alpha}, Y_t, \alpha_t)dt + Z_t dW_t \\ \qquad\qquad -\tilde{\mathbb{E}}[\partial_\mu H(t, \tilde{X}_t, X, \tilde{Y}_t, \tilde{\alpha}_t)]|_{X = X_t^\alpha} dt \\ Y_T = \partial_x g(X_T^\alpha, \mathbb{P}_{X_T^\alpha}) + \tilde{\mathbb{E}}[\partial_\mu g(x, \tilde{X}_t)]|_{x = X_T^\alpha} \end{cases}$$

where $(\tilde{\alpha}, \tilde{X}, \tilde{Y}, \tilde{Z})$ is an independent copy of (α, X^α, Y, Z), is called a set of **adjoint processes**

BSDE of Mean Field type according to Buckhdan-Li-Peng !!!

Extra terms in red are the ONLY difference between MFG and Control of McKean-Vlasov dynamics !!!

Pontryagin Maximum Principle (Sufficiency)

Assume

1. Coefficients continuously differentiable with bounded derivatives;
2. Terminal cost function g is convex;
3. α admissible control, X corresponding dynamics, (Y, Z) adjoint processes and

$$(x, \mu, \alpha) \hookrightarrow H(t, x, \mu, Y_t, Z_t, \alpha)$$

   is $dt \otimes d\mathbb{P}$ a.e. **convex**,

then, if moreover

$$H(t, X_t, \mathbb{P}_{X_t}, Y_t, Z_t, \alpha_t) = \inf_{\alpha \in A} H(t, X_t, \mathbb{P}_{X_t}, Y_t, \alpha), \quad \text{a.s.}$$

Then α is an optimal control, i.e.

$$J(\alpha) = \inf_{\overline{\alpha} \in \mathcal{A}} J(\overline{\alpha})$$

Scalar Interactions

$$b(t,x,\mu,\alpha) = \tilde{b}(t,x,\langle\psi,\mu\rangle,\alpha) \quad \sigma(t,x,\mu,\alpha) = \tilde{\sigma}(t,x,\langle\phi,\mu\rangle,\alpha)$$
$$f(t,x,\mu,\alpha) = \tilde{f}(t,x,\langle\gamma,\mu\rangle,\alpha) \quad g(x,\mu) = \tilde{g}(x,\langle\zeta,\mu\rangle)$$

- ψ, ϕ, γ and ζ **differentiable** with at most quadratic growth at ∞,
- $\tilde{b}$, $\tilde{\sigma}$ and $\tilde{f}$ **differentiable** in $(x,r) \in \mathbb{R}^d \times \mathbb{R}$ for $t,\alpha)$ fixed
- $\tilde{g}$ **differentiable** in $(x,r) \in \mathbb{R}^d \times \mathbb{R}$.

Recall that the adjoint process satisfies

$$Y_T = \partial_x g(X_T, \mathbb{P}_{X_T}) + \tilde{\mathbb{E}}[\partial_\mu g(\tilde{X}_T, \mathbb{P}_{\tilde{X}_T})(X_T)].$$

but since

$$\partial_\mu g(x,\mu)(x') = \partial_r \tilde{g}\big(x, \langle\zeta,\mu\rangle\big)\partial\zeta(x'),$$

the terminal condition reads

$$Y_T = \partial_x \tilde{g}\big(X_T, \mathbb{E}[\zeta(X_T)]\big) + \tilde{\mathbb{E}}\big[\partial_r \tilde{g}\big(\tilde{X}_T, \mathbb{E}[\zeta(X_T)]\big)\big]\partial\zeta(X_T)$$

Convexity in μ follows convexity of $\tilde{g}$

Scalar Interactions (cont.)

$$H(t,x,\mu,y,z,\alpha)=\tilde{b}(t,x,\langle\psi,\mu\rangle,\alpha)\cdot y+\tilde{\sigma}(t,x,\langle\phi,\mu\rangle,\alpha)\cdot z+\tilde{f}(t,x,\langle\gamma,\mu\rangle,\alpha).$$

$\partial_\mu H(t,x,\mu,y,z,\alpha)$ can be identified wih

$$\begin{aligned}\partial_\mu H(t,x,\mu,y,z,\alpha)(x') = &\left[\partial_r\tilde{b}(t,x,\langle\psi,\mu\rangle,\alpha)\cdot y\right]\partial\psi(x')\\ &+\left[\partial_r\tilde{\sigma}(t,x,\langle\phi,\mu\rangle,\alpha)\cdot z\right]\partial\phi(x')\\ &+\partial_r\tilde{f}(t,x,\langle\gamma,\mu\rangle,\alpha)\,\partial\gamma(x')\end{aligned}$$

and the adjoint equation rewrites:

$$\begin{aligned}dY_t = &-\Big\{\partial_x\tilde{b}(t,X_t,\mathbb{E}[\psi(X_t)],\alpha_t)\cdot Y_t+\partial_x\tilde{\sigma}(t,X_t,\mathbb{E}[\phi(X_t)],\alpha_t)\cdot Z_t\\ &\qquad+\partial_x\tilde{f}(t,X_t,\mathbb{E}[\gamma(X_t)],\alpha_t)\Big\}dt+Z_t dW_t\\ &-\Big\{\tilde{\mathbb{E}}\left[\partial_r\tilde{b}(t,\tilde{X}_t,\mathbb{E}[\psi(\tilde{X}_t)],\tilde{\alpha}_t)\cdot\tilde{Y}_t\right]\partial\psi(X_t)+\tilde{\mathbb{E}}\left[\partial_r\tilde{\sigma}(t,\tilde{X}_t,\mathbb{E}[\phi(\tilde{X}_t)],\tilde{\alpha}_t)\cdot\tilde{Z}_t\right]\partial\phi(X_t)\\ &\qquad+\tilde{\mathbb{E}}\left[\partial_r\tilde{f}((t,\tilde{X}_t,\mathbb{E}[\gamma(\tilde{X}_t)],\tilde{\alpha}_t)\right]\partial\gamma(X_t)\Big\}dt\end{aligned}$$

Anderson - Djehiche

Solution of the McKV Control Problem

Assume

- $b(t, x, \mu, \alpha) = b_0(t) \int_{\mathbb{R}^d} x d\mu(x) + b_1(t)x + b_2(t)\alpha$
  with b_0, b_1 and b_2 is $\mathbb{R}^{d\times d}$-valued and are bounded.
- f and g as in MFG problem.

There exists a solution $(X_t, Y_t, Z_t)_0$ of the McKean-Vlasov FBSDE

$$\begin{cases} dX_t = b_0(t)\mathbb{E}(X_t)dt + b_1(t)X_t dt + b_2(t)\hat{\alpha}(t, X_t, \mathbb{P}_{X_t}, Y_t)dt + \sigma dW_t, \\ dY_t = -\partial_x H\big(t, X_t, \mathbb{P}_{X_t}, Y_t, \hat{\alpha}_t\big)dt \\ \qquad\qquad - \mathbb{E}\big[\partial_\mu \underline{H}(t, X_t', X_t, Y_t', \hat{\alpha}_t')\big]dt + Z_t dW_t. \end{cases}$$

with $Y_t = u(t, X_t, \mathbb{P}_{X_t})$ for a function

$$u : [0, T] \times \mathbb{R}^d \times \mathcal{P}_1(\mathbb{R}^d) \ni (t, x, \mu) \mapsto u(t, x, \mu)$$

uniformly of Lip-1 and with linear growth in x.

A Finite Player Approximate Equilibrium

For N independent Brownian motions $(W^1, \dots, W^N)$ and for a square integrable exchangeable process $\beta = (\beta^1, \dots, \beta^N)$, consider the system

$$dX_t^i = \frac{1}{N} b_0(t) \sum_{j=1}^N X_t^j + b_1(t) X_t^i + b_2(t)\beta_t^i + \sigma dW_t^i, \quad X_0^i = \xi_0^i,$$

and define the common cost

$$J^N(\beta) = \mathbb{E}\left[\int_0^T f(s, X_s^i, \bar{\mu}_s^N, \beta_s^i) ds + g(X_T^1, \bar{\mu}_T^N)\right], \quad \text{with } \bar{\mu}_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{X_t^i}.$$

Then, there exists a sequence $(\epsilon_N)_{N\geq 1}$, $\epsilon_N \searrow 0$, s.t. **for all** $\beta = (\beta^1, \dots, \beta^N)$,

$$J^N(\beta) \geq J^N(\alpha) - \epsilon_N,$$

where, $\alpha = (\alpha^1, \cdots, \alpha^N)$ with

$$\alpha_t^i = \hat{\alpha}(s, \tilde{X}_t^i, u(t, \tilde{X}_t^i), \mathbb{P}_{X_t})$$

where X and u are from the solution to the **controlled McKean Vlasov problem**, and $(\tilde{X}^1, \dots, \tilde{X}^N)$ is the state of the system controlled by α, i.e.

$$d\tilde{X}_t^i = \frac{1}{N}\sum_{j=1}^N b_0(t)\tilde{X}_t^j + b_1(t)\tilde{X}_t^i + b_2(t)\hat{\alpha}(s, \tilde{X}_s^i, u(s, \tilde{X}_s^i), \mathbb{P}_{X_s}) + \sigma dW_t^i, \quad \tilde{X}_0^i = \xi_0^i.$$