Real Time American Option Pricing

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March 2015

Partly based on joint work with D. Offengenden and M. Lake

First:

- Congratulations to Steve!

Agenda

- Motivation/Problem
- American options with Smooth Dividends
- Fixed-Point formulations for Exercise Boundary
- Test results
- American put with Dividends: Basic Properties
- Integral Equations for Boundary
- Boundary Results
- American Call Options with Dividends
- Extensions

Motivation/Problem (1)

- For this occasion, an old-school problem seemed in order. Sometimes, a problem that appears to be a little old in the tooth still has some surprises left.
- We will show how applying modern computational finance methods (MSCF-style) can improve the efficiency of American option pricing algorithms by at least 4 orders of magnitude.
- The method we will discuss, can produce better precision than a 1M x 1M (!!) modern finite difference grid (12+ hours of work), in about 1/10 seconds.
- And it can calculate in the order of 100,000 prices at the same precision as a 10,000-step binomial tree.
- Without cheating: no parallel processing, no caching, ..

Motivation/Problem (2)

- The application we have in mind is real-time risk management exchange-traded options in the US and Asia, where pricing/quotation standards all revolve around Black-Scholes modeling.
- Our approach is to apply careful optimization on integral equation(s) for the American exercise boundary, a method that has often been neglected in favor of the more popular tree, lattice, and Monte Carlo methods (not to mention method of lines, convolution methods, and more).
- Our primary application is options on futures, but we also discuss options on underlyings with discrete dividends.
- Reference paper (ALO): Andersen, L., M. Lake, and D. Offengenden, "High Performance American Option Pricing," ssrn.com

Setup (1)

- First, for reference consider an underlying security with value $S(t)$ with "classical" dynamics:

$$dS(t)/S(t) = (r - q)\ dt + \sigma\, dW(t),$$

where r, q, σ are constants. (Can handle time-dependence, but makes notation annoying)

- Introduce a K-strike, T-maturity American put option, paying $(K - S(\nu))^+$ if exercised time $\nu \in [0, T]$. American call can be found by put-call symmetry.
- Well-known that the optimal strategy is to exercise when $S(t) \le S_T^*(t)$ for some deterministic, T indexed exercise boundary S_T^*, satisfying

$$S_T^*(t) = \begin{cases} K, & t = T, \\ K \min(1, r/q), & t = T- . \end{cases}$$

Setup (2)

- For time-homogenous arguments (as here) common to write $S_T^*(t) = B(T-t) = B(\tau)$
- If also $V(T-t, S)$ is the time t price of the American put for $S(t) = S$, then for $S > B$,

$$V_\tau - (r-q)V_S - \frac{1}{2}S^2\sigma^2 V_{SS} + rV = 0, \quad V(0,S) = (K-S)^+\,, \tag{1}$$

subject to the *value match condition*

$$V(\tau, B(\tau)) = K - B(\tau) \tag{2}$$

and the *smooth pasting condition*

$$V_S(\tau, B(\tau)) = -1. \tag{3}$$

Setup (3)

- Differentiating with respect to τ and using the smooth pasting condition:

$$V_\tau(\tau, B(\tau)) = 0. \tag{4}$$

- And using the PDE shows that

$$V_{SS}(\tau, B(\tau)) = \frac{2\left(rK - qB(\tau)\right)}{B(\tau)^2\sigma^2}. \tag{5}$$

Pricing given Boundary

- From the basic PDE and (2)-(3), it has classically been shown that the American put price must satisfy ($S \leq B$)

$$
\begin{aligned}
V(\tau, S) = v(\tau, S) &+ \int_0^\tau rKe^{-r(\tau-u)}\Phi\left(-d_-\left(\tau-u, S/B(u)\right)\right) du \\
&- \int_0^\tau qSe^{-q(\tau-u)}\Phi\left(-d_+\left(\tau-u, S/B(u)\right)\right) du \qquad (6)
\end{aligned}
$$

- Where Φ is the Gaussian CDF, and

$$
d_\pm(s, x) = \frac{\ln x + s\left(r - q \pm \frac{1}{2}\sigma^2\right)}{\sigma\sqrt{s}}
$$

and $v(\tau, S)$ is the European put price (Black-Scholes formula).

Location of Boundary (1)

- To use the integral pricing expression (6), we need to locate the optimal exercise boundary B.
- We have several possibilities here. The most obvious equation (most common in Finance) arises when one sets $S = B(\tau)$ in (6):

$$K - B(\tau) = v\left(\tau, B(\tau)\right) + \int_0^\tau rKe^{-r(\tau-u)}\Phi\left(-d_-\left(\tau-u, B(\tau)/B(u)\right)\right)\,du$$
$$-\int_0^\tau qB(\tau)e^{-q(\tau-u)}\Phi\left(-d_+\left(\tau-u, B(\tau)/B(u)\right)\right)\,du. \qquad (7)$$

- We may, however, also use the smooth pasting equation (3) or the equations for V_{SS} or for V_τ to derive alternative equations.
- The numerical solution of all these boundary equations is traditionally done using a *direct quadrature method*, on an equidistant grid. Numerous reference works in this area.

Location of Boundary (2)

- A few recent papers suggest that to use a fixed point iteration, rather than direct quadrature. Here we write

$$B(\tau) = Ke^{-(r-q)\tau} \frac{N(\tau, B)}{D(\tau, B)}$$

where N and D are functionals.

- This suggests an algorithm where we iterate, starting from a guess,

$$B^{(j)}(\tau) = Ke^{-(r-q)\tau} \frac{N(\tau, B^{(j-1)})}{D(\tau, B^{(j-1)})}, \quad j = 1, 2, \ldots, m.$$

- Rewriting basic equations (such as 7) for the exercise boundary into fixed-point format can be done numerous ways, but only a few ones define efficient contraction mappings.

Location of Boundary (3)

- N and D depend on which boundary formulation we use. For the smooth pasting boundary equation, we get *fixed point system A:*

$$N(\tau, B) = \frac{\phi\left(d_-(\tau, B(\tau)/K)\right)}{\sigma\sqrt{\tau}} + r\int_0^\tau \frac{e^{ru}}{\sigma\sqrt{\tau-u}}\phi\left(d_-\left(\tau-u, B(\tau)/B(u)\right)\right)\,du,$$

$$D(\tau, B) = \frac{\phi\left(d_+(\tau, B(\tau)/K)\right)}{\sigma\sqrt{\tau}} + \Phi\left(d_+(\tau, B(\tau)/K)\right)$$
$$+ q\left(\int_0^\tau e^{qu}\Phi\left(d_+\left(\tau-u, B(\tau)/B(u)\right)\right)\,du + \int_0^\tau \frac{e^{qu}}{\sigma\sqrt{\tau-u}}\phi\left(d_+\left(\tau-u, B(\tau)/B(u)\right)\right)\right).$$

- The value match integral equation leads to *fixed point system B:*

$$N(\tau, B) = \Phi\left(d_-(\tau, B(\tau)/K)\right) + r\int_0^\tau e^{ru}\left(\Phi\left(d_-\left(\tau-u, B(\tau)/B(u)\right)\right)\right)\,du,$$
$$D(\tau, B) = \Phi\left(d_+(\tau, B(\tau)/K)\right) + q\int_0^\tau e^{qu}\left(\Phi\left(d_+\left(\tau-u, B(\tau)/B(u)\right)\right)\right)\,du.$$

Collocation & Interpolation (1)

- ALO shows how to run the fixed point iteration in a modern manner, using a relaxed Jacobi-Newton iteration.
- But the fixed point systems cannot practically be solved for all τ simultaneously, so we need a way to discretize the system.
- A common approach involves discretizing τ to a grid, $\{\tau_i\}_{i=1}^n$ and enforcing the fixed point condition at these points only. Other points on the $B(\tau)$ curve are found by polynomial interpolation; integrals can be resolved by (say) Gauss-Legendre integration.
- This is known as the *collocation method*.
- ALO shows how this method is very effective, *if done right*.

Collocation & Interpolation (2)

- **WRONG**: a) interpolate on B directly; b) use an equidistant grid; .
- **RIGHT**: a) interpolate on a transformed function $H(\sqrt{\tau}) = \ln B(\tau)/X$, $X = K \min(1, r/q)$; and b) Use Chebyshev spacing in $\sqrt{\tau}$ domain.
- Justification and full boundary algorithm is given in (excruciating) detail in ALO.
- Let m: number of iterations; n: number of collocation points; l: number of Gauss-Legendre points in the numerical integration. Then computational cost is of order

$$c_1 \cdot lmn^2 + c_2 \cdot lmn$$

where first term is from interpolation and second from integration. $c_2 \gg c_1$.

Sample Tests + Speed (1)

- **Precision test.** Use high number of collocation and integration nodes to a high-precision estimate (for benchmark purposes).

Method	Dimensions	American Premium	Error	Timing (sec)
FP-A	$(l = 1024,\ m = 16,\ n = 32)$	0.10695270275	-	1.40E-01
PDE	100 x 100	0.10279251763	4.16E-03	3.10E-03
PDE	500 x 500	0.10672868802	3.94E-03	9.50E-03
PDE	1,000 x 1,000	0.10689130239	1.63E-04	3.07E-02
PDE	5,000 x 5,000	0.10694949491	5.82E-05	7.83E-01
PDE	10,000 x 10,000	0.10695176844	2.27E-06	3.40E+00
PDE	50,000 x 50,000	0.10695264506	8.77E-07	9.27E+01
PDE	100,000 x 100,000	0.10695268484	3.98E-08	4.17E+02
PDE	250,000 x 250,000	0.10695271369	2.88E-08	2.87E+03
PDE	500,000 x 500,000	0.10695270841	5.28E-09	1.14E+04

Table 1: $S = K = 100$, $r = 1 = 5\%$, $T = 1$, $\sigma = 0.25$. 3.33GHz PC.

- Additional tests show the result for FP-A in table is accurate to about 12 digits. Would theoretically need a 10Mx10M PDE solver for this.

Sample Tests + Speed (2)

- **Speed test.** 1,675 different options, $T \in [0, 3]$.

	Bin 100	Bin 1,000	Bin 10,000
RMSE	2.1E-02	2.0E-03	2.1E-04
RRMSE	3.2E-03	2.7E-04	3.0E-05
Options/sec	12,900	800	?

- Algo FP-A, various combinations of l, m, n. No caching, single CPU.

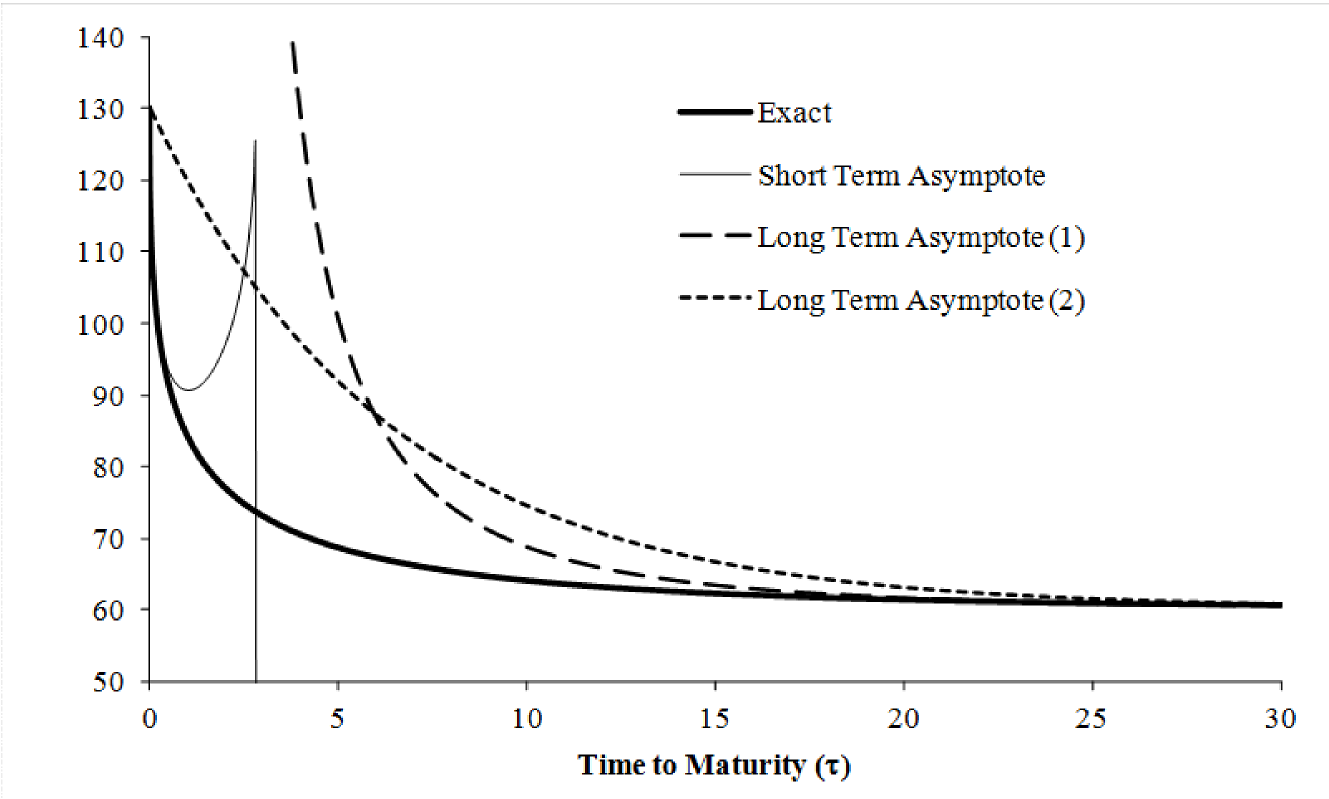
	(m,n):	(1,4)	(2,4)	(1,6)	(2,6)	(3,6)	(4,6)	(2,10)	(3,10)	(4,10)
l=5	RMSE	3.1E-04	3.1E-04	8.3E-05	4.4E-05	5.6E-05	5.7E-05	5.2E-05	6.4E-05	6.6E-05
	RRMSE	2.0E-05	1.8E-05	3.0E-06	1.5E-06	1.7E-06	1.7E-06	1.6E-06	1.8E-06	1.9E-06
	Options/sec	79,700	61,200	61,200	45,500	36,000	29,900	29,300	22,600	18,500
l=7	RMSE	3.4E-04	3.3E-04	8.4E-05	1.5E-05	1.3E-05	1.4E-05	7.3E-06	1.6E-05	1.8E-05
	RRMSE	2.0E-05	1.9E-05	2.6E-06	6.6E-07	6.8E-07	6.4E-07	3.6E-07	5.3E-07	5.6E-07
	Options/sec	74,500	55,300	57,200	39,500	30,900	25,500	25,500	19,200	15,400
l=15	RMSE	3.5E-04	3.4E-04	8.9E-05	2.3E-05	1.3E-05	1.2E-05	1.6E-05	4.2E-06	3.0E-06
	RRMSE	2.0E-05	1.9E-05	2.6E-06	7.6E-07	6.5E-07	6.5E-07	4.8E-07	3.7E-07	3.5E-07
	Options/sec	56,300	37,500	41,250	26,300	19,300	15,200	15,500	11,100	8,700

More on tests

- We also ran tests against other methods, including the related fixed-point methods in Kim *et al* (2013) and Cortazar *et al* (2013).
- We are always *far* better than any convergent method
- Robustness tested on 10,000's of options (see ALO)
- For the case $r = q$, fixed point system A is about 5-10 times more efficient that fixed point system B.
- For otther configurations, FP-A and FP-B are about equal – but FP-A is more robust, especially for convection-dominated dynamics.

Boundary Asymptotes

- Boundary Asymptotes $r = 5\%$, $\mu = 0$, $\sigma = 0.25\%$, $K = 130$.



- Neither short nor long-dated asymptotes have wide range. Short asymptote ceases to exist after 2.8 years.

Setup with Dividends (1)

- Now extend the process for $S(t)$ to the RCLL process

$$dS(t) = \mu S(t)\,dt + \sigma S(t)\,dW(t) - \sum_{i=1}^{d} D_i \cdot 1_{t_i=t}$$

- ..where $\{t_i\}_{i=1}^{d}$ is a set of discrete dividend dates.
- Note that we, unlike existing literature, do NOT force $\mu = r$. This ensures that we can model the fact that repo rates for stocks (as observed in forward prices, say) often differ from (OIS) discount rates.
- Also, it allows us to use a mixed discrete-continuous dividends model. But it is a fair bit of a complication..

Setup with Dividends (2)

- Here we focus on the *proportional* specification

$$D_i = c_i S(t_i-).$$

- This is convenient for several reasons, including the fact that there are no problem with crossing of zero + forward stock prices are easy to compute:

$$F(t,u) = \mathbb{E}\left(S(u) | S(t)\right) = S(t)e^{\mu(u-t)}G(t,u)$$

- ..where

$$G(t,u) \triangleq \prod_{t_i \in (t,u]} (1 - c_i).$$

American Put

- Even with discrete dividends, there is again an optimal boundary $S_T^*(t)$, below which the American put option should be exercised. **Note**: we don't shift to $B(T-t)$ and $\tau = T - t$ notation here, since the problem is no longer time-stationary.

- Above the exercise boundary and away from the dividend dates, the put option price $P(t, S)$ satisfies the usual Black-Scholes PDE:

$$\frac{\partial P}{\partial t} + \mu S \frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = rP, \quad t \notin \{t_i\}_{i=1}^d, \quad S > S_T^*(t),$$

- Across each dividend date, the put option does not lose exercise value (see below), wherefore we may impose the simple jump-type continuity condition $P\left(t_i-, S(t_i-)\right) = P\left(t_i+, S(t_i+)\right)$, i.e.

$$P\left(t_i-, S\right) = P\left(t_i+, S(1-c_i)\right), \quad i = 1, 2, \ldots, d.$$

Boundary (1)

- To characterize the boundary, we have as before *value match* and *smooth pasting*

$$P(t, S_T^*(t)) = K - S_T^*(t),$$
$$\frac{\partial P(t,S)}{\partial S}|_{S=S_T^*(t)} = -1, \quad t \notin \{t_i\}_{i=1}^d.$$

- To further characterize the boundary, consider that it can never be optimal to exercise the option just prior to a dividend date, wherefore

Lemma 1. *The American put exercise boundary $S_T^*(t)$ satisfies*

$$S_T^*(t_i-) = 0, \quad i = 1, \dots, d, \tag{8}$$

and

$$S_T^*(T-) = \begin{cases} K \min\left(1, \frac{r}{r-\mu}\right), & r > \mu, \\ K & r \leq \mu \end{cases} \tag{9}$$

Boundary (2)

- We can also use carry arguments to prove:

Lemma 2. *For* $i = 1, \ldots, d$ *define*

$$t_i^* = \begin{cases} \max\left(t_i + \frac{\ln(1-c_i)}{\mu}, t_{i-1}\right), & \mu > 0, \\ t_{i-1}, & \mu \leq 0, \end{cases}$$

where necessarily $t_i^* \in [t_{i-1}, t_i)$. *For* $t \in [t_{i-1}, t_i)$ *we then have*

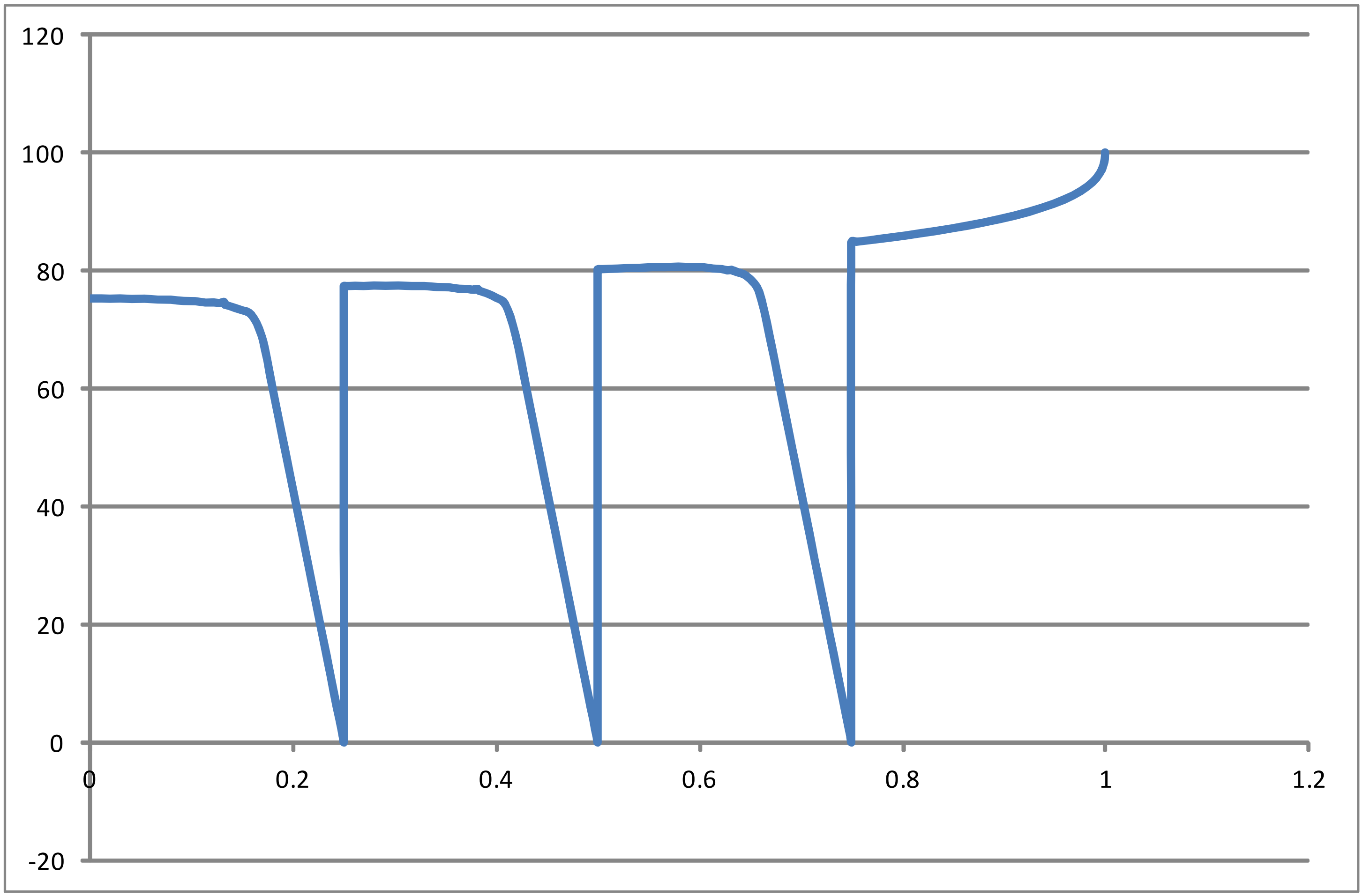
$$S_T^*(t) \leq \begin{cases} K \frac{1-e^{-r(t_i-t)}}{1-e^{(\mu-r)(t_i-t)}(1-c_i)}, & t \in (t_i^*, t_i), \\ K, & t \in [t_{i-1}, t_i^*]. \end{cases} \quad (10)$$

In particular, (8) holds for $t \uparrow t_i$.

- The upper bound is a very good proxy for the boundary close to dividend dates

Boundary Shape

- Boundary shape for American Put w. 3 Proportional Dividends



American Put Option Price (1)

- We need an equation for the American put option price, given the boundary. Here it is:

  Proposition 1. *Let $p\,(t, S)$ be the time t price of a European put option with maturity T and strike K. For the dividend-paying stock $S(t)$, the American put option price P is given by*

$$P(t, S) = p(t, S) + rK \int_t^T e^{-r(u-t)} \mathbb{E}\left(1_{\{S(u)<S_T^*(u)\}} | S(t) = S\right) du$$
$$- (r - \mu) \int_t^T e^{-r(u-t)} \mathbb{E}_t \left(1_{\{S(u)<S_T^*(u)\}} S(u) | S(t) = S\right) du, \quad (11)$$

  for all $S \geq S_T^*(t)$.

- The result is a generalization of Goetsche and Vellekoep (Math. Finance, 2011) to cover the (practically important) case $\mu \neq r$.

American Put Option Price (2)

- Proposition holds for a large class of dividends. For the proportional dividend type, we have *explicitly* (with $q \triangleq r - \mu$):

$$P(t,S) = p(t,S) + rK \int_t^T e^{-r(u-t)} \Phi\left(-d_-(S/S_T^*(u);t,u)\right)\, du$$
$$- qS \int_t^T e^{-q(u-t)} G(t,u) \Phi\left(-d_+(S/S_T^*(u);t,u)\right)\, du, \quad (12)$$

for $S \geq S_T^*(t)$.

- Here we have redefined

$$d_\pm(z;t,T) = \frac{\ln z + \mu(T-t) + \ln G(t,T) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}.$$

American Put Option Price (3)

- Proof of Proposition is instructional, so we can give a sketch:
- Define $H(t) = e^{-rt}P(t, S(t))$. When S is below the boundary, $H(t) = e^{-rt}(K - S(t))$ and

$$dH(t) = -e^{-rt}dS(t) - re^{-rt}\left(K - S(t)\right)\, dt, \quad S(t) < S_T^*(t).$$

- When S is above boundary, H is a martingale and (by BS equation)

$$dH(t) = \sigma e^{-rt}S(t)\frac{\partial P}{\partial S}dW(t)$$

- Right AT the boundary, there would normally (by Tanaka's rule) be a local time contribution to dH, but due to smooth pasting, it vanishes.
- And crossing dividend dates add no terms to dH.
- Collecting, integrating to find $H(T) - H(t)$ + forming time t expectations pops out the result.

Boundary Fixed Point Formulation (1)

- By setting $S = S_T^*(t)$ in (12), we get an integral equation for $S_T^*(t)$:

$$K - S_T^*(t) = p\left(t, S_T^*(t)\right) + rK \int_t^T e^{-r(u-t)} \Phi\left(-d_-(S_T^*(t)/S_T^*(u); t, u)\right) du$$
$$- qS_T^*(t) \int_t^T e^{-q(u-t)} G(t,u) \Phi\left(-d_+(S_T^*(t)/S_T^*(u); t, u)\right) du.$$

- We can, after some work, arrange this for a fixed-point iteration

$$S_T^*(t) = K \frac{N_T(t, S_T^*)}{D_T(t, S_T^*)} \tag{13}$$

where

$$N_T(t, S_T^*) = e^{-r(T-t)} \Phi\left(d_-(S_T^*(t)/K; t, T)\right)$$
$$+ r \int_t^{T-} e^{-r(u-t)} \Phi\left(d_-(S_T^*(t)/S_T^*(u); t, u)\right) du$$

Boundary Fixed Point Formulation (2)

- ..and

$$
\begin{aligned}
D_T(t, S_T^*) = e^{-q(T-t)} G(t,T) \Phi(d_+(S_T^*(t)/K; t, T)) \\
+ q \int_t^{T-} e^{-q(u-t)} G(t,u) \Phi\left(d_+(S_T^*(t)/S_T^*(u); t, u)\right) du \\
- \sum_{t_i \in [t,T)} e^{-q(t_i - t)} \left(G(t,t_i) - G(t,t_{i-1})\right).
\end{aligned}
$$

Numerical Algorithm

- The fixed point system can be executed using a variation of the algorithm in ALO.
- In particular, given Lemma 2's "periodic" constraint $S_T^*(t_i-) = 0$, $i = 1, \ldots, d$, it makes sense to break the problem into d sub-problems, one per dividend period, and use ALO algorithm backwards to time 0 from time t_d.
- While Chebyshev spacing is still needed for the collocation scheme, we generally do not need to be as careful with boundary transformations for $t < t_d$.
- With d dividends, the effort of the scheme is (better than) $d + 1$ times that of the regular scheme. For single stocks, d is normally 4 times/year.

American Call Options (1)

- Unlike the case for smooth dividends, there are no obvious parity results to extract American call prices from puts.
- American Calls, in fact, are very different from puts, and there are situations where the exercise boundary is completely degenerate, except for a few points.
- In this case, the American option price integral changes from being an integral in time along the boundary, to being (a convolution) of integrals in asset space.
- For instance, in the case where $\mu \geq r$, it is easy to see that the only possible exercise dates are at t_i-, $i = 1, \ldots, d$, just before each dividend.

American Call Options (2)

- In this case, we can introduce an exercise boundary (**above** which to exercise) as

$$S_T^*(t) = \begin{cases} \infty, & t \notin \{t_i\} \cup T \\ B_i, & t \in \{t_i\} \\ K, & t = T \end{cases}$$

- That is, the American option effectively becomes a Bermudan one.
- For the American call option, continuity is *not* necessarily preserved as time passes through an exercise date – so no continuity condition similar to that of a put holds.
- Indeed, if $S(t_i-) \geq B_i$, there will be a *loss of exercise value* as time moves from t_i- to t_i+.

American Call – Jump Condition

- We can capture this as

$$C\left(t_i-, S\right) = \begin{cases} C\left(t_i+, S(1-c_i)\right), & S < B_i, \\ S - K, & S \geq B_i. \end{cases}$$

- Or equivalently

$$C\left(t_i-, S\right) = \max\left(C\left(t_i+, S(1-c_i)\right), S - K\right).$$

- When we attempt to repeat the proof of the American put valuation formula, these jump conditions add a new type of term to the formulas.

American Call Valuation Formulas (1)

- The same basic method now leads to

  Proposition 2. *Let $c\,(t,S)$ be the time t price of a European call option with maturity T and strike K. Assume that $\mu \geq r$. For the dividend-paying stock $S(t)$, the American call option price is given by*

$$
\begin{aligned}
C(t,S) = c\,(t,S) + &\sum_{t_i>t} e^{-r(t_i-t)}\mathbb{E}\left(1_{S(t_i-)\geq B_i} C(t_i+, S(t_i+)) | S(t)=S\right) \\
&- \sum_{t_i>t} e^{-r(t_i-t)}\mathbb{E}\left(1_{S(t_i-)\geq B_i}\left(S(t_i-)-K\right)|S(t)=S\right). \qquad (14)
\end{aligned}
$$

- Here, an irritating fact is the dependence on $C\,(t_i+, S(1-c_i))$, which is not known explicitly.

American Call Valuation Formulas (2)

- For the location of B_i, we may write

$$C\left(t_i+, B_i(1-c_i)\right) = B_i - K$$

which *also* depends on $C\left(t_i+, S(1-c_i)\right)$.

- In practice, we need to rely on a lattice/integration method on the $\{t_i\}$ grid, such as Fast Gauss Transform, to uncover $C\left(t_i+, S(1-c_i)\right)$. We are forced to move closer to traditional methods for American options.
- We note, however, that the representation in Proposition gives a static hedge for the American Call, but that is another story...

American Call Valuation Formulas (3)

- The case where $\mu < r$ becomes a **hybrid**: the exercise strategy will come into existence between the exercise dates, and the valuation expression will contain elements of "vertical" (asset) integration around discrete dividend dates; and "horizontal" (time) integration around discrete dividend dates.
- We omit the equations; they are easy (but lengthy) extensions of the case $\mu \geq r$.
- The topology of the resulting exercise boundary can be complicated, depending on the size of $r - \mu$.
- Still outstanding question: why are calls so difficult?