Real Time American Option Pricing

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Partly based on joint work with D. Offengenden and M. Lake
First:

- Congratulations to Steve!
Agenda

- Motivation/Problem
- American options with Smooth Dividends
- Fixed-Point formulations for Exercise Boundary
- Test results
- American put with Dividends: Basic Properties
- Integral Equations for Boundary
- Boundary Results
- American Call Options with Dividends
- Extensions
For this occasion, an old-school problem seemed in order. Sometimes, a problem that appears to be a little old in the tooth still has some surprises left.

We will show how applying modern computational finance methods (MSCF-style) can improve the efficiency of American option pricing algorithms by at least 4 orders of magnitude.

The method we will discuss, can produce better precision than a 1M x 1M (!!!) modern finite difference grid (12+ hours of work), in about 1/10 seconds.

And it can calculate in the order of 100,000 prices at the same precision as a 10,000-step binomial tree.

Without cheating: no parallel processing, no caching, ..
The application we have in mind is real-time risk management exchange-traded options in the US and Asia, where pricing/quotations standards all revolve around Black-Scholes modeling.

Our approach is to apply careful optimization on integral equation(s) for the American exercise boundary, a method that has often been neglected in favor of the more popular tree, lattice, and Monte Carlo methods (not to mention method of lines, convolution methods, and more).

Our primary application is options on futures, but we also discuss options on underlyings with discrete dividends.

First, for reference consider an underlying security with value \( S(t) \) with “classical” dynamics:

\[
dS(t)/S(t) = (r - q) \, dt + \sigma \, dW(t),
\]

where \( r, q, \sigma \) are constants. (Can handle time-dependence, but makes notation annoying)

Introduce a \( K \)-strike, \( T \)-maturity American put option, paying \((K - S(\nu))^+\) if exercised time \( \nu \in [0, T] \). American call can be found by put-call symmetry.

Well-known that the optimal strategy is to exercise when \( S(t) \leq S_T^*(t) \) for some deterministic, \( T \) indexed exercise boundary \( S_T^* \), satisfying

\[
S_T^*(t) = \begin{cases} 
K, & t = T, \\
K \min(1, r/q), & t = T - .
\end{cases}
\]
Setup (2)

- For time-homogenous arguments (as here) common to write
  \[ S_T^*(t) = B(T - t) = B(\tau) \]

- If also \( V(T - t, S) \) is the time \( t \) price of the American put for \( S(t) = S \), then for \( S > B \),
  \[
  V_\tau - (r - q)V_S - \frac{1}{2}S^2\sigma^2V_{SS} + rV = 0, \quad V(0, S) = (K - S)^+, \quad (1)
  \]
  subject to the value match condition
  \[
  V(\tau, B(\tau)) = K - B(\tau) \quad (2)
  \]
  and the smooth pasting condition
  \[
  V_S(\tau, B(\tau)) = -1. \quad (3)
  \]
Differentiating with respect to $\tau$ and using the smooth pasting condition:

$$V_\tau(\tau, B(\tau)) = 0. \quad (4)$$

And using the PDE shows that

$$V_{SS}(\tau, B(\tau)) = \frac{2(rK - qB(\tau))}{B(\tau)^2\sigma^2}. \quad (5)$$
Pricing given Boundary

From the basic PDE and (2)-(3), it has classically been shown that the American put price must satisfy \((S \leq B)\)

\[
V(\tau, S) = v(\tau, S) + \int_0^\tau rKe^{-r(\tau-u)}\Phi(-d_- (\tau - u, S/B(u))) \, du \\
- \int_0^\tau qSe^{-q(\tau-u)}\Phi(-d_+ (\tau - u, S/B(u))) \, du \tag{6}
\]

Where \(\Phi\) is the Gaussian CDF, and

\[
d_{\pm} (s, x) = \frac{\ln x + s (r - q \pm \frac{1}{2} \sigma^2)}{\sigma \sqrt{s}}
\]

and \(v(\tau, S)\) is the European put price (Black-Scholes formula).
Location of Boundary (1)

To use the integral pricing expression (6), we need to locate the optimal exercise boundary $B$.

We have several possibilities here. The most obvious equation (most common in Finance) arises when one sets $S = B(\tau)$ in (6):

$$K - B(\tau) = v(\tau, B(\tau)) + \int_0^\tau rKe^{-r(\tau-u)}\Phi(-d_- (\tau - u, B(\tau)/B(u))) \, du$$

$$- \int_0^\tau qB(\tau)e^{-q(\tau-u)}\Phi(-d_+ (\tau - u, B(\tau)/B(u))) \, du. \tag{7}$$

We may, however, also use the smooth pasting equation (3) or the equations for $V_{SS}$ or for $V_\tau$ to derive alternative equations.

The numerical solution of all these boundary equations is traditionally done using a direct quadrature method, on an equidistant grid. Numerous reference works in this area.
A few recent papers suggest that to use a fixed point iteration, rather than direct quadrature. Here we write

\[ B(\tau) = Ke^{-(r-q)\tau} \frac{N(\tau, B)}{D(\tau, B)} \]

where \( N \) and \( D \) are functionals.

This suggests an algorithm where we iterate, starting from a guess,

\[ B^{(j)}(\tau) = Ke^{-(r-q)\tau} \frac{N(\tau, B^{(j-1)})}{D(\tau, B^{(j-1)})}, \quad j = 1, 2, \ldots, m. \]

Rewriting basic equations (such as 7) for the exercise boundary into fixed-point format can be done numerous ways, but only a few ones define efficient contraction mappings.
Location of Boundary (3)

\(N\) and \(D\) depend on which boundary formulation we use. For the smooth pasting boundary equation, we get **fixed point system A**:

\[
N(\tau, B) = \frac{\phi(d_-(\tau, B(\tau)/K))}{\sigma \sqrt{\tau}} + r \int_0^\tau \frac{e^{ru}}{\sigma \sqrt{\tau - u}} \phi\left(d_-(\tau - u, B(\tau)/B(u))\right) \, du,
\]

\[
D(\tau, B) = \frac{\phi(d_+(\tau, B(\tau)/K))}{\sigma \sqrt{\tau}} + \Phi\left(d_+(\tau, B(\tau)/K)\right)
+ q \left(\int_0^\tau e^{qu} \Phi\left(d_+(\tau - u, B(\tau)/B(u))\right) \, du + \int_0^\tau \frac{e^{qu}}{\sigma \sqrt{\tau - u}} \phi\left(d_+(\tau - u, B(\tau)/B(u))\right)\right).
\]

The value match integral equation leads to **fixed point system B**:

\[
N(\tau, B) = \Phi\left(d_-(\tau, B(\tau)/K)\right) + r \int_0^\tau e^{ru} \left(\Phi\left(d_-(\tau - u, B(\tau)/B(u))\right)\right) \, du,
\]

\[
D(\tau, B) = \Phi\left(d_+(\tau, B(\tau)/K)\right) + q \int_0^\tau e^{qu} \left(\Phi\left(d_+(\tau - u, B(\tau)/B(u))\right)\right) \, du.
\]
Collocation & Interpolation (1)

- ALO shows how to run the fixed point iteration in a modern manner, using a relaxed Jacobi-Newton iteration.

- But the fixed point systems cannot practically be solved for all $\tau$ simultaneously, so we need a way to discretize the system.

- A common approach involves discretizing $\tau$ to a grid, $\{\tau_i\}_{i=1}^n$ and enforcing the fixed point condition at these points only. Other points on the $B(\tau)$ curve are found by polynomial interpolation; integrals can be resolved by (say) Gauss-Legendre integration.

- This is known as the *collocation method*.

- ALO shows how this method is very effective, *if done right*. 
Collocation & Interpolation (2)

- **WRONG**: a) interpolate on $B$ directly; b) use an equidistant grid.
- **RIGHT**: a) interpolate on a transformed function $H(\sqrt{\tau}) = \ln B(\tau)/X$, $X = K \min(1, r/q)$; and b) Use Chebyshev spacing in $\sqrt{\tau}$ domain.

Justification and full boundary algorithm is given in (excruciating) detail in ALO.

Let $m$: number of iterations; $n$: number of collocation points; $l$: number of Gauss-Legendre points in the numerical integration. Then computational cost is of order

$$c_1 \cdot lmn^2 + c_2 \cdot lmn$$

where first term is from interpolation and second from integration. $c_2 \gg c_1$. 
**Sample Tests + Speed (1)**

- **Precision test.** Use high number of collocation and integration nodes to a high-precision estimate (for benchmark purposes).

<table>
<thead>
<tr>
<th>Method</th>
<th>Dimensions</th>
<th>American Premium</th>
<th>Error</th>
<th>Timing (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>FP-A</td>
<td>( l = 1024, m = 16, n = 32 )</td>
<td>0.10695270275</td>
<td>-</td>
<td>1.40E-01</td>
</tr>
<tr>
<td>PDE</td>
<td>100 x 100</td>
<td>0.10279251763</td>
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<td>3.10E-03</td>
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<td>500 x 500</td>
<td>0.10672868802</td>
<td>3.94E-03</td>
<td>9.50E-03</td>
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<tr>
<td>PDE</td>
<td>1,000 x 1,000</td>
<td>0.10689130239</td>
<td>1.63E-04</td>
<td>3.07E-02</td>
</tr>
<tr>
<td>PDE</td>
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<td>5.82E-05</td>
<td>7.83E-01</td>
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<tr>
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<td>3.40E+00</td>
</tr>
<tr>
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<td>9.27E+01</td>
</tr>
<tr>
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<td>3.98E-08</td>
<td>4.17E+02</td>
</tr>
<tr>
<td>PDE</td>
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<tr>
<td>PDE</td>
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<td>0.10695270841</td>
<td>5.28E-09</td>
<td>1.14E+04</td>
</tr>
</tbody>
</table>

**Table 1:** \( S = K = 100, \ r = 1 = 5\%, \ T = 1, \sigma = 0.25. \) **3.33GHz** PC.

- **Additional tests** show the result for FP-A in table is accurate to about 12 digits. Would theoretically need a 10Mx10M PDE solver for this.
**Speed test.** 1,675 different options, \( T \in [0, 3] \).

<table>
<thead>
<tr>
<th></th>
<th>Bin 100</th>
<th>Bin 1,000</th>
<th>Bin 10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>RMSE</td>
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<td>2.0E-03</td>
<td>2.1E-04</td>
</tr>
<tr>
<td>RRMSE</td>
<td>3.2E-03</td>
<td>2.7E-04</td>
<td>3.0E-05</td>
</tr>
<tr>
<td>Options/sec</td>
<td>12,900</td>
<td>800</td>
<td>?</td>
</tr>
</tbody>
</table>

**Algo FP-A**, various combinations of \( l, m, n \). No caching, single CPU.

<table>
<thead>
<tr>
<th>(m,n):</th>
<th>(1,4)</th>
<th>(2,4)</th>
<th>(1,6)</th>
<th>(2,6)</th>
<th>(3,6)</th>
<th>(4,6)</th>
<th>(2,10)</th>
<th>(3,10)</th>
<th>(4,10)</th>
</tr>
</thead>
<tbody>
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<td>l=5</td>
<td>RMSE</td>
<td>3.1E-04</td>
<td>3.1E-04</td>
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<td>4.4E-05</td>
<td>5.6E-05</td>
<td>5.7E-05</td>
<td>5.2E-05</td>
<td>6.4E-05</td>
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<tr>
<td></td>
<td>RRMSE</td>
<td>2.0E-05</td>
<td>1.8E-05</td>
<td>3.0E-06</td>
<td>1.5E-06</td>
<td>1.7E-06</td>
<td>1.7E-06</td>
<td>1.6E-06</td>
<td>1.8E-06</td>
</tr>
<tr>
<td></td>
<td>Options/sec</td>
<td>79,700</td>
<td>61,200</td>
<td>61,200</td>
<td>45,500</td>
<td>36,000</td>
<td>29,900</td>
<td>29,300</td>
<td>22,600</td>
</tr>
<tr>
<td>l=7</td>
<td>RMSE</td>
<td>3.4E-04</td>
<td>3.3E-04</td>
<td>8.4E-05</td>
<td>1.5E-05</td>
<td>1.3E-05</td>
<td>1.4E-05</td>
<td>7.3E-06</td>
<td>1.6E-05</td>
</tr>
<tr>
<td></td>
<td>RRMSE</td>
<td>2.0E-05</td>
<td>1.9E-05</td>
<td>2.6E-06</td>
<td>6.6E-07</td>
<td>6.8E-07</td>
<td>6.4E-07</td>
<td>3.6E-07</td>
<td>5.3E-07</td>
</tr>
<tr>
<td></td>
<td>Options/sec</td>
<td>74,500</td>
<td>55,300</td>
<td>57,200</td>
<td>39,500</td>
<td>30,900</td>
<td>25,500</td>
<td>25,500</td>
<td>19,200</td>
</tr>
<tr>
<td>l=15</td>
<td>RMSE</td>
<td>3.5E-04</td>
<td>3.4E-04</td>
<td>8.9E-05</td>
<td>2.3E-05</td>
<td>1.3E-05</td>
<td>1.2E-05</td>
<td>1.6E-05</td>
<td>4.2E-06</td>
</tr>
<tr>
<td></td>
<td>RRMSE</td>
<td>2.0E-05</td>
<td>1.9E-05</td>
<td>2.6E-06</td>
<td>7.6E-07</td>
<td>6.5E-07</td>
<td>6.5E-07</td>
<td>4.8E-07</td>
<td>3.7E-07</td>
</tr>
<tr>
<td></td>
<td>Options/sec</td>
<td>56,300</td>
<td>37,500</td>
<td>41,250</td>
<td>26,300</td>
<td>19,300</td>
<td>15,200</td>
<td>15,500</td>
<td>11,100</td>
</tr>
</tbody>
</table>
More on tests

- We also ran tests against other methods, including the related fixed-point methods in Kim et al (2013) and Cortazar et al (2013).
- We are always far better than any convergent method.
- Robustness tested on 10,000's of options (see ALO).
- For the case $r = q$, fixed point system A is about 5-10 times more efficient than fixed point system B.
- For other configurations, FP-A and FP-B are about equal – but FP-A is more robust, especially for convection-dominated dynamics.
Boundary Asymptotes

Boundary Asymptotes $r = 5\%, \mu = 0, \sigma = 0.25\%, K = 130$.

Neither short nor long-dated asymptotes have wide range. Short asymptote ceases to exist after 2.8 years.
Setup with Dividends (1)

Now extend the process for $S(t)$ to the RCLL process

$$dS(t) = \mu S(t) \, dt + \sigma S(t) \, dW(t) - \sum_{i=1}^{d} D_i \cdot 1_{t_i = t}$$

..where $\{t_i\}_{i=1}^{d}$ is a set of discrete dividend dates.

Note that we, unlike existing literature, do NOT force $\mu = r$. This ensures that we can model the fact that repo rates for stocks (as observed in forward prices, say) often differ from (OIS) discount rates.

Also, it allows us to use a mixed discrete-continuous dividends model. But it is a fair bit of a complication..
Here we focus on the proportional specification

\[ D_i = c_i S(t_i-) \].

This is convenient for several reasons, including the fact that there are no problem with crossing of zero + forward stock prices are easy to compute:

\[ F(t, u) = \mathbb{E}(S(u)|S(t)) = S(t)e^{\mu(u-t)}G(t, u) \]

..where

\[ G(t, u) \triangleq \prod_{t_i \in (t, u]} (1 - c_i). \]
American Put

Even with discrete dividends, there is again an optimal boundary $S_{T}^{*}(t)$, below which the American put option should be exercised.

**Note:** we don’t shift to $B(T - t)$ and $\tau = T - t$ notation here, since the problem is no longer time-stationary.

Above the exercise boundary and away from the dividend dates, the put option price $P(t, S)$ satisfies the usual Black-Scholes PDE:

$$\frac{\partial P}{\partial t} + \mu S \frac{\partial P}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} = rP, \quad t \notin \{t_i\}_{i=1}^d, \quad S > S_{T}^{*}(t),$$

Across each dividend date, the put option does not lose exercise value (see below), wherefore we may impose the simple jump-type continuity condition $P(t_i-, S(t_i-)) = P(t_i+, S(t_i+))$, i.e.

$$P(t_i-, S) = P(t_i+, S(1 - c_i)), \quad i = 1, 2, \ldots, d.$$
Boundary (1)

To characterize the boundary, we have as before value match and smooth pasting

\[ P(t, S^*_T(t)) = K - S^*_T(t), \]
\[ \frac{\partial P(t, S)}{\partial S} \bigg|_{S=S^*_T(t)} = -1, \quad t \notin \{t_i\}_{i=1}^d. \]

To further characterize the boundary, consider that it can never be optimal to exercise the option just prior to a dividend date, wherefore

**Lemma 1.** The American put exercise boundary \( S^*_T(t) \) satisfies

\[ S^*_T(t_i-) = 0, \quad i = 1, \ldots, d, \tag{8} \]

and

\[ S^*_T(T-) = \begin{cases} 
K \min \left(1, \frac{r}{r-\mu}\right), & r > \mu, \\
K & r \leq \mu 
\end{cases} \tag{9} \]
We can also use carry arguments to prove:

**Lemma 2.** For $i = 1, \ldots, d$ define

$$t_i^* = \begin{cases} 
\max \left( t_i + \frac{\ln(1-c_i)}{\mu}, t_{i-1} \right), & \mu > 0, \\
\quad t_{i-1}, & \mu \leq 0,
\end{cases}$$

where necessarily $t_i^* \in [t_{i-1}, t_i)$. For $t \in [t_{i-1}, t_i)$ we then have

$$S^*_T(t) \leq \begin{cases} 
K \frac{1-e^{-r(t_i-t)}}{1-e^{(\mu-r)(t_i-t)}(1-c_i)}, & t \in (t_i^*, t_i), \\
K, & t \in [t_{i-1}, t_i^*],
\end{cases}$$

In particular, (8) holds for $t \uparrow t_i$.

The upper bound is a very good proxy for the boundary close to dividend dates.
Boundary Shape

Boundary shape for American Put w. 3 Proportional Dividends
We need an equation for the American put option price, given the boundary. Here it is:

**Proposition 1.** Let $p(t, S)$ be the time $t$ price of a European put option with maturity $T$ and strike $K$. For the dividend-paying stock $S(t)$, the American put option price $P$ is given by

\[
P(t, S) = p(t, S) + rK \int_t^T e^{-r(u-t)} \mathbb{E} \left( 1 \{ S(u) < S_T^*(u) \} | S(t) = S \right) \, du
\]

\[- (r - \mu) \int_t^T e^{-r(u-t)} \mathbb{E}_t \left( 1 \{ S(u) < S_T^*(u) \} S(u) | S(t) = S \right) \, du, \quad (11)
\]

for all $S \geq S_T^*(t)$.

The result is a generalization of Goetsche and Vellekoep (Math. Finance, 2011) to cover the (practically important) case $\mu \neq r$. 

---

American Put Option Price (1)


**American Put Option Price (2)**

- Proposition holds for a large class of dividends. For the proportional dividend type, we have *explicitly* (with \( q \triangleq r - \mu \)):

\[
P(t, S) = p(t, S) + rK \int_t^T e^{-r(u-t)} \Phi (-d_-(S/S^*_T(u); t, u)) \, du \\
- qS \int_t^T e^{-q(u-t)} G(t, u) \Phi (-d_+(S/S^*_T(u); t, u)) \, du,
\]

for \( S \geq S^*_T(t) \).

- Here we have redefined

\[
d_{\pm}(z; t, T) = \frac{\ln z + \mu(T - t) + \ln G(t, T) \pm \frac{1}{2} \sigma^2(T - t)}{\sigma \sqrt{T - t}}.
\]
Proof of Proposition is instructional, so we can give a sketch:

Define $H(t) = e^{-rt} P(t, S(t))$. When $S$ is below the boundary, $H(t) = e^{-rt}(K - S(t))$ and

$$dH(t) = -e^{-rt} dS(t) - re^{-rt} (K - S(t)) \, dt, \quad S(t) < S_T^*(t).$$

When $S$ is above boundary, $H$ is a martingale and (by BS equation)

$$dH(t) = \sigma e^{-rt} S(t) \frac{\partial P}{\partial S} dW(t)$$

Right AT the boundary, there would normally (by Tanaka’s rule) be a local time contribution to $dH$, but due to smooth pasting, it vanishes.

And crossing dividend dates add no terms to $dH$.

Collecting, integrating to find $H(T) - H(t) + \text{forming time } t$ expectations pops out the result.
Boundary Fixed Point Formulation (1)

By setting $S = S^*_T(t)$ in (12), we get an integral equation for $S^*_T(t)$:

$$K - S^*_T(t) = p(t, S^*_T(t)) + rK \int_t^T e^{-r(u-t)} \Phi \left(-d_-(S^*_T(t)/S^*_T(u); t, u)\right) du$$

$$- qS^*_T(t) \int_t^T e^{-q(u-t)} G(t, u) \Phi \left(-d_+(S^*_T(t)/S^*_T(u); t, u)\right) du.$$

We can, after some work, arrange this for a fixed-point iteration

$$S^*_T(t) = K \frac{N_T(t, S^*_T)}{D_T(t, S^*_T)}$$

(13)

where

$$N_T(t, S^*_T) = e^{-r(T-t)} \Phi \left(d_-(S^*_T(t)/K; t, T)\right)$$

$$+ r \int_t^T e^{-r(u-t)} \Phi \left(d_-(S^*_T(t)/S^*_T(u); t, u)\right) du.$$
Boundary Fixed Point Formulation (2)

..and

\[
D_T(t, S_T^*) = e^{-q(T-t)} G(t, T) \Phi \left( d_+ \left( \frac{S_T^*(t)}{K}; t, T \right) \right) \\
+ q \int_t^{T-} e^{-q(u-t)} G(t, u) \Phi \left( d_+ \left( \frac{S_T^*(t)}{S_T^*(u)}; t, u \right) \right) du \\
- \sum_{t_i \in [t, T)} e^{-q(t_i-t)} (G(t, t_i) - G(t, t_{i-1})) .
\]
Numerical Algorithm

- The fixed point system can be executed using a variation of the algorithm in ALO.

- In particular, given Lemma 2’s “periodic” constraint $S_{T_i}^*(t_i-) = 0, \quad i = 1, \ldots, d$, it makes sense to break the problem into $d$ sub-problems, one per dividend period, and use ALO algorithm backwards to time 0 from time $t_d$.

- While Chebyshev spacing is still needed for the collocation scheme, we generally do not need to be as careful with boundary transformations for $t < t_d$.

- With $d$ dividends, the effort of the scheme is (better than) $d + 1$ times that of the regular scheme. For single stocks, $d$ is normally 4 times/year.
Unlike the case for smooth dividends, there are no obvious parity results to extract American call prices from puts.

American Calls, in fact, are very different from puts, and there are situations where the exercise boundary is completely degenerate, except for a few points.

In this case, the American option price integral changes from being an integral in time along the boundary, to being (a convolution) of integrals in asset space.

For instance, in the case where \( \mu \geq r \), it is easy to see that the only possible exercise dates are at \( t_i^- \), \( i = 1, \ldots, d \), just before each dividend.
American Call Options (2)

In this case, we can introduce an exercise boundary (above which to exercise) as

\[ S^*_T(t) = \begin{cases} 
\infty, & t \notin \{t_i\} \cup T \\
B_i, & t \in \{t_i\} \\
K, & t = T 
\end{cases} \]

That is, the American option effectively becomes a Bermudan one.

For the American call option, continuity is *not* necessarily preserved as time passes through an exercise date – so no continuity condition similar to that of a put holds.

Indeed, if \( S(t_i-) \geq B_i \), there will be a *loss of exercise value* as time moves from \( t_i- \) to \( t_i+ \).
American Call – Jump Condition

- We can capture this as

\[
C(t_i-, S) = \begin{cases} 
C(t_i+, S(1 - c_i)), & S < B_i, \\
S - K, & S \geq B_i.
\end{cases}
\]

- Or equivalently

\[
C(t_i-, S) = \max (C(t_i+, S(1 - c_i)), S - K).
\]

- When we attempt to repeat the proof of the American put valuation formula, these jump conditions add a new type of term to the formulas.
The same basic method now leads to

**Proposition 2.** Let $c(t, S)$ be the time $t$ price of a European call option with maturity $T$ and strike $K$. Assume that $\mu \geq r$. For the dividend-paying stock $S(t)$, the American call option price is given by

$$C(t, S) = c(t, S) + \sum_{t_i > t} e^{-r(t_i - t)} \mathbb{E} \left( 1_{S(t_i^-) \geq B_i} C(t_i^+, S(t_i^+)) \middle| S(t) = S \right)$$

$$- \sum_{t_i > t} e^{-r(t_i - t)} \mathbb{E} \left( 1_{S(t_i^-) \geq B_i} (S(t_i^-) - K) \middle| S(t) = S \right). \tag{14}$$

Here, an irritating fact is the dependence on $C(t_i^+, S(1 - c_i))$, which is not known explicitly.
For the location of $B_i$, we may write

$$C(t_i+, B_i(1 - c_i)) = B_i - K$$

which also depends on $C(t_i+, S(1 - c_i))$.

In practice, we need to rely on a lattice/integration method on the $\{t_i\}$ grid, such as Fast Gauss Transform, to uncover $C(t_i+, S(1 - c_i))$. We are forced to move closer to traditional methods for American options.

We note, however, that the representation in Proposition gives a static hedge for the American Call, but that is another story...
American Call Valuation Formulas (3)

- The case where $\mu < r$ becomes a **hybrid**: the exercise strategy will come into existence between the exercise dates, and the valuation expression will contain elements of “vertical” (asset) integration around discrete dividend dates; and “horizontal” (time) integration around discrete dividend dates.

- We omit the equations; they are easy (but lengthy) extensions of the case $\mu \geq r$.

- The topology of the resulting exercise boundary can be complicated, depending on the size of $r - \mu$.

- Still outstanding question: why are calls so difficult?