

EXPLOSIONS AND ARBITRAGE

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ABSTRACT

The FELLER and KHAS'MINSKII tests provide conditions, under which a given diffusion process $X(\cdot)$ in a domain D can have explosions. If such explosions happen with positive probability, what is the distribution $F(t, x) = \mathbb{P}_x(S \leq t)$, $0 \leq t < \infty$ of the explosion time $S = \inf\{t \geq 0 : X(t) \notin D\}$, as a function of the starting position $X(0) = x \in D$? We answer this question in some generality for one-dimensional diffusions in the spirit of MCKEAN (1969), provide specific examples for which explicit computations are possible, and characterize the “survival” function $U(t, x) = 1 - F(t, x)$ as the smallest nonnegative solution of a linear parabolic PDE. (This part of the talk is ongoing joint work with JOHANNES RUF.)

A very similar situation develops when one studies the possibility of outperforming the market portfolio in the context of a MARKOVIAN model with n assets. There again, the highest return $\mathcal{U}(t, x)$ that can be achieved relative to the market over a given time horizon $[0, t]$, using nonanticipative investment rules and starting with an initial configuration of asset prices $X(0) = x \in (0, \infty)^n$, is characterized as the smallest nonnegative solution of an appropriate linear parabolic PDE.

The connection with explosions also exists, but is now somewhat subtler: it involves an auxiliary probability measure \mathbb{Q} , the so-called FÖLLMER measure, with respect to which the “real” probability measure \mathbb{P} is absolutely continuous and under which the relative market weights become martingales. When outperformance of the market portfolio is possible, these two measures are not equivalent, and strict local martingales enter the picture. In the presence of uncertainty about the local characteristics of this MARKOVIAN model, $\mathcal{U}(t, x)$ becomes the smallest nonnegative supersolution of a fully nonlinear partial differential equation of parabolic, in particular HJB-PUCCI, type. (This part of the talk reports on joint work with DANIEL FERNHOLZ.)

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