A new perspective on the fundamental theorem of asset pricing for large financial markets

Josef Teichmann
(based on joint work with Christa Cuchiero and Irene Klein)

ETH Zürich

June 3, 2015
Problem formulation and motivation

- The question we deal with stems from financial markets with a potentially uncountably infinite number of tradeable assets.
Problem formulation and motivation

- The question we deal with stems from financial markets with a potentially uncountably infinite number of tradeable assets.

- Examples: Term structure models for
  - bond markets,
  - forward prices in commodity markets,
  - call options (with even a continuum of strikes), etc.
Problem formulation and motivation

- The question we deal with stems from financial markets with a potentially uncountably infinite number of tradeable assets.

- Examples: Term structure models for
  - bond markets,
  - forward prices in commodity markets,
  - call options (with even a continuum of strikes), etc.

- Usual assumption to preclude arbitrage is to suppose the existence of an equivalent (local/\(\sigma\)-) martingale measure for the (uncountably many) discounted assets.
Problem formulation and motivation

- The question we deal with stems from financial markets with a potentially uncountably infinite number of tradeable assets.

- **Examples:** Term structure models for
  - bond markets,
  - forward prices in commodity markets,
  - call options (with even a continuum of strikes), etc.

- Usual assumption to preclude arbitrage is to suppose the existence of an equivalent (local/ $\sigma$-) martingale measure for the (uncountably many) discounted assets.

- In contrast to classical small financial markets, this property has not been characterized in an economically satisfying way, (only the Kreps-Yan theorem involving weak-$*$-closures is available in this setting).
Goal and outline of today’s talk

Version of the fundamental theorem of asset pricing (FTAP) for large financial markets:

Certain economically meaningful “No asymptotic arbitrage” condition $\iff \exists$ an equivalent separating measure for the large financial market.
Goal and outline of today’s talk

Version of the fundamental theorem of asset pricing (FTAP) for large financial markets:

Certain economically meaningful “No asymptotic arbitrage” condition \[\iff\exists\text{ an equivalent separating measure for the large financial market}\]

Exemplary large financial market (LFM) model with countably many assets on one fixed probability space as in the work of M. De Donno, P. Guasoni and M. Pratelli (2005)

- Formulation of the setting and the main result
- Relation to the literature
Goal and outline of today’s talk

Version of the fundamental theorem of asset pricing (FTAP) for large financial markets:

Certain economically meaningful “No asymptotic arbitrage” condition

\[ \Leftrightarrow \]

\[ \exists \text{ an equivalent separating measure for the large financial market} \]

1. Exemplary large financial market (LFM) model with countably many assets on one fixed probability space as in the work of M. De Donno, P. Guasoni and M. Pratelli (2005)

   - Formulation of the setting and the main result
   - Relation to the literature

2. Extension of the abstract portfolio wealth process setting introduced by Y. Kabanov (1997) to LFM to embed uncountably many assets (bond markets)
Goal and outline of today’s talk

Version of the fundamental theorem of asset pricing (FTAP) for large financial markets:

Certain economically meaningful “No asymptotic arbitrage” condition \[ \iff \]
\[ \exists \text{ an equivalent separating measure for the large financial market} \]

1. Exemplary large financial market (LFM) model with countably many assets on one fixed probability space as in the work of M. De Donno, P. Guasoni and M. Pratelli (2005)
   - Formulation of the setting and the main result
   - Relation to the literature

2. Extension of the abstract portfolio wealth process setting introduced by Y. Kabanov (1997) to LFMs to embed uncountably many assets (bond markets)

3. On \((\sigma)\)-martingale measures in large financial markets
Setting and notation

- Finite time horizon $[0, 1]$
- One fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$
- Financial market with countably many (discounted) assets modeled by a sequence of $\mathbb{R}$-valued semimartingales $(S^n_t)_{t\in[0,1], n\in\mathbb{N}}$
- Vector of first $n$ assets: $S^n = (S^1, \ldots, S^n)\top$
- 1-admissible portfolio wealth processes in the small financial market $n$:

$$\mathcal{X}_1^n = \{(H \cdot S^n) | H \in \mathcal{H}_1^n\},$$

with

$$\mathcal{H}_\lambda^n = \{H | \mathbb{R}^n\text{-valued, predictable, } S^n\text{-integrable, } \lambda\text{-admissible process}\}.$$

As usual $\lambda$-admissibility means $(H \cdot S^n)_t \geq -\lambda$ for all $t \in [0, 1]$. 
Generalized strategies in the large financial market

- From a sequence of small financial markets to a large financial market:
  - generalized stochastic integration with respect to a sequence of semimartingales using so-called generalized strategies as integrands (introduced by M. De Donno et M. Pratelli (2006))
Generalized strategies in the large financial market

- From a sequence of small financial markets to a large financial market:
  
  - generalized stochastic integration with respect to a sequence of semimartingales using so-called generalized strategies as integrands (introduced by M.De Donno et M.Pratelli (2006))

- The precise definitions are introduced via the Emery topology on the space of $\mathbb{R}$-valued semimartingales $S$:

\[
d_S(X_1, X_2) := \sup_{K \in \mathcal{bE}, \|K\|_\infty \leq 1} E\left[\left(\|K \cdot (X_1 - X_2)\|_1^* \wedge 1\right)^*\right],
\]

where $|X|_1^* = \sup_{t \leq 1} |X_t|$ and $\mathcal{bE}$ denotes the set of simple predictable strategies. The Emery topology makes the space of semimartingales a complete metric space.
Generalized strategies in the large financial market

- From a sequence of small financial markets to a large financial market:
  - generalized stochastic integration with respect to a sequence of semimartingales using so-called generalized strategies as integrands (introduced by M.De Donno et M.Pratelli (2006))

- The precise definitions are introduced via the Emery topology on the space of $\mathbb{R}$-valued semimartingales $S$:
  \[
d_S(X_1, X_2) := \sup_{K \in b\mathcal{E}, \|K\|_\infty \leq 1} E \left[ |(K \bullet (X_1 - X_2))|_1^* \wedge 1 \right],
\]
  where $|X|_1^* = \sup_{t \leq 1} |X_t|$ and $b\mathcal{E}$ denotes the set of simple predictable strategies. The Emery topology makes the space of semimartingales a complete metric space.

- Two semimartingales (portfolios) are close in the Emery topology if all their increments are close or in more financial terms if all investments in the difference of two portfolios are small.
Definition of (admissible) generalized strategies

**Definition**

For each $n \in \mathbb{N}$, let $H^n$ be an $\mathbb{R}^n$-valued predictable $S^n$-integrable process. A sequence $(H^n)_{n \in \mathbb{N}}$ is called generalized strategy if $(H^n \cdot S^n)$ converges in the Emery topology to a semimartingale

$$Z := \mathbb{S}-\lim (H^n \cdot S^n),$$

which is called generalized stochastic integral and where $\mathbb{S}$-lim denotes the limit in the Emery topology.
Exemplary LFM with countably many assets

Generalized strategies

Definition of (admissible) generalized strategies

Definition

1. For each $n \in \mathbb{N}$, let $H^n$ be an $\mathbb{R}^n$-valued predictable $S^n$-integrable process. A sequence $(H^n)_{n \in \mathbb{N}}$ is called generalized strategy if $(H^n \cdot S^n)$ converges in the Emery topology to a semimartingale

$$Z := \mathcal{S}\text{-lim}(H^n \cdot S^n),$$

which is called generalized stochastic integral and where $\mathcal{S}\text{-lim}$ denotes the limit in the Emery topology.

2. Let $\lambda > 0$. A generalized strategy $(H^n)$ is called $\lambda$-admissible if each element of the sequence $H^n$ is $\lambda$-admissible.
Definition of (admissible) generalized strategies

Definition

1. For each $n \in \mathbb{N}$, let $H^n$ be an $\mathbb{R}^n$-valued predictable $S^n$-integrable process. A sequence $(H^n)_{n \in \mathbb{N}}$ is called generalized strategy if $(H^n \cdot S^n)$ converges in the Emery topology to a semimartingale

$$Z := S\text{-lim}(H^n \cdot S^n),$$

which is called generalized stochastic integral and where $S\text{-lim}$ denotes the limit in the Emery topology.

2. Let $\lambda > 0$. A generalized strategy $(H^n)$ is called $\lambda$-admissible if each element of the sequence $H^n$ is $\lambda$-admissible.

- Considering generalized strategies, means including portfolios $Z$ of which the difference between the increments of $Z$ and those of a small market portfolio can be made arbitrarily small in the Emery metric.
- $\Rightarrow$ Economically meaningful to include these limits in no arbitrage requirements
Admissible generalized portfolio processes in the LFM

**Definition**

1. Consider the set

\[ \mathcal{X}_1 = \bigcup_{n \geq 1} \mathcal{X}_1^n \]

\[ = \left\{ S\text{-lim}(H^n \circ S^n) \mid (H^n) \text{ 1-admissible generalized strategy} \right\}, \]

where \((\cdot)^S\) denotes the closure in the Emery-topology. The elements of \(\mathcal{X}_1\) are called **1-admissible generalized portfolio wealth processes in the large financial market**.
Admissible generalized portfolio processes in the LFM

**Definition**

1. Consider the set

\[ X_1 = \bigcup_{n \geq 1} X_1^n = \{ \mathbf{S} \text{-} \lim (H^n \bullet S^n) \mid (H^n) \text{ 1-admissible generalized strategy} \}, \]

where \( (\cdot)^S \) denotes the closure in the Emery-topology. The elements of \( X_1 \) are called **1-admissible generalized portfolio wealth processes in the large financial market**.

2. We denote by \( X' \) the set \( X' := \bigcup_{\lambda > 0} \lambda X_1 \) and call its elements **admissible generalized portfolio wealth processes in the large financial market**.
Admissible generalized portfolio processes in the LFM

**Definition**

1. Consider the set

   \[ X_1 = \bigcup_{n \geq 1} X^n_1 \]

   \[ = \{ S\text{-lim}(H^n \bullet S^n) \mid (H^n) \text{ 1-admissible generalized strategy} \}, \]

   where \((\cdot)^S\) denotes the closure in the Emery-topology. The elements of \(X_1\) are called **1-admissible generalized portfolio wealth processes** in the large financial market.

2. We denote by \(X\) the set \(X := \bigcup_{\lambda > 0} \lambda X_1\) and call its elements **admissible generalized portfolio wealth processes** in the large financial market.

3. We denote by \(K_0\), respectively \(K_1^0\) the evaluations of elements of \(X\), respectively \(X_1\), at terminal time \(T = 1\).
No asymptotic free lunch with vanishing risk (NAFLVR)

Goal: Economically meaningful no asymptotic arbitrage condition for the large financial market, which allows to conclude the existence of a separating measure for the whole market.
No asymptotic free lunch with vanishing risk (NAFLVR)

- Goal: *Economically meaningful no asymptotic arbitrage condition* for the large financial market, which allows *to conclude the existence of a separating measure* for the whole market.

- In perfect *analogy to the notion of (NFLVR)* in the classical setting of small financial markets, we define (NAFLVR) as follows:
No asymptotic free lunch with vanishing risk (NAFLVR)

- Goal: Economically meaningful no asymptotic arbitrage condition for the large financial market, which allows to conclude the existence of a separating measure for the whole market.

- In perfect analogy to the notion of (NFLVR) in the classical setting of small financial markets, we define (NAFLVR) as follows:

**Definition (NAFLVR)**

The set $\mathcal{X}$ is said to satisfy no asymptotic free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$ and $\overline{C}$ denotes the norm closure in $L^\infty$. 
No free lunch and equivalent separating measures

**Definition**

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$. 
No free lunch and equivalent separating measures

Definition

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

- Under the condition
  
  $$\overline{C}^* \cap L_\geq^\infty = \{0\},$$

  where $\overline{C}^*$ denotes the weak-\*\*-closure in $L^\infty$, the (ESM) property is a consequence of the Kreps-Yan Theorem (80, 81), which in turn follows from Hahn-Banach’s Theorem. Condition (NFL) is the classical no free lunch condition for the abstract set $C$. 
No free lunch and equivalent separating measures

**Definition**

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

- Under the condition

$$\overline{C}^* \cap L^\infty_{\geq 0} = \{0\},$$

where $\overline{C}^*$ denotes the weak-*$*$-closure in $L^\infty$, the (ESM) property is a consequence of the Kreps-Yan Theorem (80, 81), which in turn follows from Hahn-Banach’s Theorem. Condition (NFL) is the classical no free lunch condition for the abstract set $C$.

- It is clear that (NFL) $\Rightarrow$ (NAFLVR). The goal is to show the reverse implication, that is...
A fundamental theorem of asset pricing for large financial markets

Theorem (C. Cuchiero, I. Klein, JT (2014))

Under (NAFLVR), \( C = \overline{C}^* \), i.e., the cone \( C \) is already weak-*-closed and (NFL) holds.

From this, we obtain immediately...
A fundamental theorem of asset pricing for large financial markets

Theorem (C. Cuchiero, I. Klein, JT (2014))

Under (NAFLVR), \( C = \overline{C}^* \), i.e., the cone \( C \) is already weak-\(*\)-closed and (NFL) holds.

From this, we obtain immediately...

Theorem (Fundamental Theorem of Asset Pricing (C. Cuchiero, I. Klein, JT (2014)))

(NAFLVR) \( \iff \) (ESM), i.e., \( \exists Q \sim P \) such that \( E_Q[X_1] \leq 0 \) for all \( X \in \mathcal{X} \).
Remarks on the proof

- The methods used in C. Cuchiero and JT (2014) also work in this large financial market setting.
Remarks on the proof

- The methods used in C. Cuchiero and JT (2014) also work in this large financial market setting.

- The so-called concatenation property of $\mathcal{X}_1$ does not hold, namely on the dense set (w.r.t. the Emery topology) $\bigcup_{n \geq 1} \mathcal{X}_1^n$:
  
  - For all $X, Y \in \bigcup_{n \geq 1} \mathcal{X}_1^n$ and all bounded predictable strategies $H, G \geq 0$, with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \bigcup_{n \geq 1} \mathcal{X}_1^n$, but not necessarily on the closure.
Remarks on the proof

- The methods used in C. Cuchiero and JT (2014) also work in this large financial market setting.

- The so-called concatenation property of $\mathcal{X}_1$ does not hold, namely on the dense set (w.r.t. the Emery topology) $\bigcup_{n \geq 1} \mathcal{X}_1^n$:
  - For all $X, Y \in \bigcup_{n \geq 1} \mathcal{X}_1^n$ and all bounded predictable strategies $H, G \geq 0$, with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \bigcup_{n \geq 1} \mathcal{X}_1^n$, but not necessarily on the closure.

- The other crucial properties of $\mathcal{X}_1$ (axiomatized by Y. Kabanov (1997)), namely convexity, boundedness from below by $-1$ and closedness in the Emery topology, hold true.
Remarks on the proof

- The methods used in C. Cuchiero and JT (2014) also work in this large financial market setting.

- The so-called concatenation property of $\mathcal{X}_1$ does \textit{not} hold, namely on the dense set (w.r.t. the Emery topology) $\bigcup_{n \geq 1} \mathcal{X}_1^n$:
  - For all $X, Y \in \bigcup_{n \geq 1} \mathcal{X}_1^n$ and all bounded predictable strategies $H, G \geq 0$, with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \bigcup_{n \geq 1} \mathcal{X}_1^n$, but not necessarily on the closure.

- The other crucial properties of $\mathcal{X}_1$ (axiomatized by Y. Kabanov (1997)), namely convexity, boundedness from below by $-1$ and closedness in the Emery topology, hold true.

- Adding admissible generalized portfolios to $\bigcup_{n \geq 1} \mathcal{X}_1^n$ such that $\mathcal{X}_1$ is closed in the Emery topology is the crucial insight.
Connection to no asymptotic arbitrage of the first kind

- The notion of arbitrage of the first kind (Ingersoll (1987)) was introduced in the context of LFM by Y. Kabanov and D. Kramkov (1994).
- It describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small risk.
Exemplary LFM with countably many assets

Further “no arbitrage” notions and relations to the literature

Connection to no asymptotic arbitrage of the first kind

- The notion of arbitrage of the first kind (Ingersoll (1987)) was introduced in the context of LFMIs by Y. Kabanov and D. Kramkov (1994).
- It describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small risk.

Definition (NAA1)

There exists an asymptotic arbitrage of the first kind (AA1) if there exist some $\alpha > 0$ and sequences $\varepsilon_n \to 0$, $c_n \to \infty$ and a sequence of strategies $(H^n)$ such that for each $n \in \mathbb{N}$

1. $(H^n \cdot S^n)_t \geq -\varepsilon_n$ for all $t \in [0, 1]$
2. $P[(H^n \cdot S^n)_1 \geq c_n] \geq \alpha$,

No asymptotic arbitrage of the first kind (NAA1) holds if there exists no (AA1).
Connection to no asymptotic arbitrage of the first kind

- The notion of arbitrage of the first kind (Ingersoll (1987)) was introduced in the context of LFMs by Y. Kabanov and D. Kramkov (1994).
- It describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small risk.

Definition (NAA1)

There exists an asymptotic arbitrage of the first kind (AA1) if there exist some \( \alpha > 0 \) and sequences \( \epsilon_n \to 0, \ c_n \to \infty \) and a sequence of strategies \((H^n)\) such that for each \( n \in \mathbb{N} \)

1. \( (H^n \bullet S^n)_t \geq -\epsilon_n \) for all \( t \in [0, 1] \)
2. \( P[(H^n \bullet S^n)_1 \geq c_n] \geq \alpha \),

No asymptotic arbitrage of the first kind (NAA1) holds if there exists no (AA1).

Remark

\( (NAA1) \Leftrightarrow K_0^1 \) is a bounded subset of \( L^0 \) (NUPBR)
(NA) in large financial markets

No arbitrage (NA) in large financial markets is defined completely analogously to small markets:

Definition (NA)

The set $X$ is said to satisfy no arbitrage if $K_0 \cap L_0 \geq 0 = \{0\}$.

This means that almost surely nonnegative terminal values of admissible generalized portfolios (and not only of portfolios in small markets) have to be almost surely equal to zero.

Similarly, to small markets we have...
(NA) in large financial markets

No arbitrage (NA) in large financial markets is defined completely analogously to small markets:

**Definition (NA)**

The set $\mathcal{X}$ is said to satisfy no arbitrage if $K_0 \cap L_{\geq 0}^0 = \{0\}$. This means that almost surely nonnegative terminal values of admissible generalized portfolios (and not only of portfolios in small markets) have to be almost surely equal to zero. Similarly, to small markets we have...
(NA) in large financial markets

No arbitrage (NA) in large financial markets is defined completely analogously to small markets:

**Definition (NA)**

The set $\mathcal{X}$ is said to satisfy no arbitrage if $K_0 \cap L_{\geq 0}^0 = \{0\}$.

- This means that almost surely nonnegative terminal values of admissible generalized portfolios (and not only of portfolios in small markets) have to be almost surely equal to zero.
Exemplary LFM with countably many assets

Further “no arbitrage” notions and relations to the literature

(NA) in large financial markets

No arbitrage (NA) in large financial markets is defined completely analogously to small markets:

**Definition (NA)**
The set $\mathcal{X}$ is said to satisfy no arbitrage if $K_0 \cap L_{\geq 0}^0 = \{0\}$.

- This means that almost surely nonnegative terminal values of admissible generalized portfolios (and not only of portfolios in small markets) have to be almost surely equal to zero.

- Similarly, to small markets we have...

**Proposition**
$(NA) + (NAA1) \iff (NAFLVR)$
“(NAFLVR)” without Emery closure

Consider $\bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} \mathcal{X}_1^n$ the set of admissible portfolios in all small markets $n$, but without the closure in the Emery-topology. We use calligraphic red letters for quantities referring to these portfolios, e.g., $\mathcal{K}_0 = \{X_1 \mid X \in \bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} \mathcal{X}_1^n\}$. 
“(NAFLVR)” without Emery closure

Consider $\bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} X^n_1$ the set of admissible portfolios in all small markets $n$, but without the closure in the Emery-topology. We use calligraphic red letters for quantities referring to these portfolios, e.g.,

$$\mathcal{K}_0 = \{X_1 \mid X \in \bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} X^n_1\}.$$

Consider the analogous notion of (NAFLVR) for this set i.e.,

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $C = (\mathcal{K}_0 - L^0_{\geq 0}) \cap L^\infty$. 
“(NAFLVR)” without Emery closure

Consider $\bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} X^n_1$ the set of admissible portfolios in all small markets $n$, but without the closure in the Emery-topology. We use calligraphic red letters for quantities referring to these portfolios, e.g., $\mathcal{K}_0 = \{X_1 \mid X \in \bigcup_{\lambda > 0} \lambda \bigcup_{n \geq 1} X^n_1\}$.

Consider the analogous notion of (NAFLVR) for this set i.e., $\overline{C} \cap L^\infty_{\geq 0} = \{0\}$, where $C = (\mathcal{K}_0 - L^0_{\geq 0}) \cap L^\infty$.

Proposition

Suppose every small market satisfies (NFLVR). Then

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\} \iff (NAA1)$$
Remarks

- *(NAA1) + (NFLVR) in every small market* is one important no arbitrage concept in the LFM literature (e.g., Y. Kabanov and D. Kramkov (1998)).
Remarks

- (NAA1) + (NFLVR) in every small market is one important no arbitrage concept in the LFM literature (e.g., Y. Kabanov and D. Kramkov (1998)).

- There are examples for which (NAA1) + (NFLVR) holds for every small market, but we do not get the existence of an equivalent separating measure.
Remarks

- (NAA1) + (NFLVR) in every small market is one important no arbitrage concept in the LFM literature (e.g., Y. Kabanov and D. Kramkov (1998)).

- There are examples for which (NAA1) + (NFLVR) holds for every small market, but we do not get the existence of an equivalent separating measure.

- The problem is that the norm closure of $C$ is too small, taking the weak*-closure of $C$, which corresponds to the (NAFL) condition, however yields the desired result.
Remarks

- **(NAA1) + (NFLVR) in every small market** is one important no arbitrage concept in the LFM literature (e.g., Y. Kabanov and D. Kramkov (1998)).

- There are examples for which (NAA1) + (NFLVR) holds for every small market, but we do not get the existence of an equivalent separating measure.

- The problem is that the norm closure of $C$ is too small, taking the weak*-closure of $C$, which corresponds to the (NAFL) condition, however yields the desired result.

- The crucial issue to obtain an FTAP without using weak-∗-closures is to strengthen the no arbitrage condition, i.e., not only requiring "No arbitrage" for each small market, but also for the portfolios obtained via generalized strategies in the large market.
Remarks

- (NAA1) + (NFLVR) in every small market is one important no arbitrage concept in the LFM literature (e.g., Y. Kabarov and D. Kramkov (1998)).
- There are examples for which (NAA1) + (NFLVR) holds for every small market, but we do not get the existence of an equivalent separating measure.
- The problem is that the norm closure of $C$ is too small, taking the weak*-closure of $C$, which corresponds to the (NAFL) condition, however yields the desired result.
- The crucial issue to obtain an FTAP without using weak-*-closures is to strengthen the no arbitrage condition, i.e., not only requiring "No arbitrage" for each small market, but also for the portfolios obtained via generalized strategies in the large market.
- This is precisely achieved by taking the Emery closure $\mathcal{X}_1$ of $\bigcup_{n \geq 1} \mathcal{X}_1^n$ and considering $\mathcal{X} = \bigcup_{\lambda > 0} \lambda \mathcal{X}_1$, which is equivalent to weak-*-closing the set $C$.

Josef Teichmann (ETH Zürich)
Abstract portfolio wealth process setting including uncountably many assets

- In order to allow for a unified treatment of different financial markets, involving e.g. a continuum of assets such as in the case of bond markets, we adapt the abstract portfolio wealth process setting introduced Y. Kabanov (1997) to large financial markets.

- **Notation:**
  - $I \subseteq [0, \infty)$: parameter space which can be any subset, countable or uncountable of $[0, \infty)$.
  - For each $n \in \mathbb{N}$, define
    $$A^n = \{\text{some/all subsets } A \subseteq I, \text{ such that } |A| = n \},$$
    such that if $A^1, A^2 \in \bigcup_{n \geq 1} A^n$, then $A^1 \cup A^2 \in \bigcup_{n \geq 1} A^n$. 

Example

- Markets consisting of on a continuum of tradeable assets, such as bonds, modeled by families of semimartingales \((S^\alpha_t)_{0 \leq t \leq 1, \alpha \in [0, T^*]}\) correspond to
  
  - \(I = [0, T^*],\)
  
  - \(\mathcal{A}^n = \{\text{all subsets } A \subseteq [0, T^*] | |A| = n\},\) where \(\alpha \in [0, T^*]\) can e.g. be thought of as the maturity of a bond.

- For \(A := \{\alpha_1, \ldots, \alpha_n\} \in \mathcal{A}^n\) and \(\alpha_1, \ldots, \alpha_n \in [0, T^*]\), define

\[
\mathcal{X}_1^A = \{(H^A \bullet S^A) | H^A \text{ is } \mathbb{R}^n\text{-valued, predictable, } S^A\text{-integrable, 1-adm.}\},
\]

where \(S^A = (S^{\alpha_1}, \ldots, S^{\alpha_n}).\)

- \(\mathcal{X}_1^n = \bigcup_{A \in \mathcal{A}^n} \mathcal{X}_1^A\) are then the 1-admissible portfolio processes built by strategies that include at most \(n\) assets.
Definition of 1-admissible portfolio wealth processes

Definition (1-admissible portfolio processes in small markets)

1. For each $A \in \bigcup_{n \geq 1} A^n$ we consider a convex set $\mathcal{X}_1^A \subset S$ of semimartingales
   - starting at 0,
   - bounded from below by $-1$,
   - satisfying the following concatenation property: for all bounded, predictable strategies $H, G \geq 0, X, Y \in \mathcal{X}_1^A$ with $HG = 0$ and $Z = (H \cdot X) + (G \cdot Y) \geq -1$, it holds that $Z \in \mathcal{X}_1^A$.
   - For $A^1, A^2 \in \bigcup_{n \geq 1} A^n$ with $A^1 \subseteq A^2$ we have that $\mathcal{X}_1^{A^1} \subseteq \mathcal{X}_1^{A^2}$.

and call its elements 1-admissible portfolio wealth processes in the small financial market $A$. 
Definition of 1-admissible portfolio wealth processes

Definition (1-admissible portfolio processes in small markets)

1. For each $A \in \bigcup_{n \geq 1} A^n$ we consider a convex set $\mathcal{X}_1^A \subset S$ of semimartingales
   - starting at 0,
   - bounded from below by $-1$,
   - satisfying the following concatenation property: for all bounded, predictable strategies $H, G \geq 0, X, Y \in \mathcal{X}_1^A$ with $HG = 0$ and $Z = (H \cdot X) + (G \cdot Y) \geq -1$, it holds that $Z \in \mathcal{X}_1^A$.
   - For $A^1, A^2 \in \bigcup_{n \geq 1} A^n$ with $A^1 \subseteq A^2$ we have that $\mathcal{X}_1^{A^1} \subseteq \mathcal{X}_1^{A^2}$, and call its elements 1-admissible portfolio wealth processes in the small financial market $A$.

2. For each $n \in \mathbb{N}$, we denote by $\mathcal{X}_1^n$ the set $\mathcal{X}_1^n := \bigcup_{A \in A^n} \mathcal{X}_1^A$ corresponding to 1-admissible portfolio wealth processes with respect to strategies that include at most $n$ assets (but all possible different choices of $n$ assets).
FTAP in the abstract portfolio wealth process setting

As in the exemplary large financial market with countably many assets, we define completely analogously

- $\mathcal{X}_1 = \bigcup_{n \geq 1} \mathcal{X}_1^n$, $\mathcal{X} := \bigcup_{\lambda > 0} \lambda \mathcal{X}_1$
- $K_0 = \{X_1 | X \in \mathcal{X}\}$, $C = K_0 - L_{\geq 0} \cap L^\infty$
- (NAFLVR): $\overline{C} \cap L_{\geq 0}^\infty = \{0\}$

and get in this abstract setting
FTAP in the abstract portfolio wealth process setting

As in the exemplary large financial market with countably many assets, we define completely analogously

\[ X_1 = \bigcup_{n \geq 1} X_1^n, \quad \mathcal{X} := \bigcup_{\lambda > 0} \lambda X_1 \]

\[ K_0 = \{ X_1 | X \in \mathcal{X} \}, \quad C = K_0 - L_{\geq 0} \cap L^\infty \]

(NAFLVR): \( \overline{C} \cap L_{\geq 0}^\infty = \{0\} \)

and get in this abstract setting

Theorem (FTAP (C. Cuchiero, I. Klein, JT))

(NAFLVR) \( \iff \) (ESM), i.e., \( \exists Q \sim P \) such that \( E[X_1] \leq 0 \) for all \( X \in \mathcal{X} \).
On \((\sigma-)\)martingale measures in large financial markets

- In the case of (possibly uncountably many) locally bounded assets, equivalent separating measures correspond to equivalent local martingale measures. Hence in this case (NAFLVR) is equivalent to the existence of a local martingale measures.

- In contrast to classical small financial markets (NAFLVR) however does not necessarily imply the existence of a \(\sigma\)-martingale measure in the non-locally bounded case (counterexample in a one period market).
Conclusion

- Formulation of the notion of no asymptotic free lunch with vanishing risk (NAFLVR) by considering the Emery-closure of 1-admissible portfolio wealth processes in small markets (corresponding to generalized integrals in the setting of M. De Donno et al. with 1-admissible generalized strategies.)

- Proof of a version of the fundamental theorem of asset pricing (FTAP) in markets with an (even uncountably) infinite number of assets, i.e., (NAFLVR) ⇔ (ESM), in particular in the case of locally bounded assets (NAFLVR) ⇔ (ELMM).

- In the non locally bounded case, (NAFLVR) does not yield the existence of an equivalent (σ)-martingale measure in general.
Thanks for your inspiring works as researcher and teacher, Steve!

Ad multos annos!