EXPLOSIONS AND ARBITRAGE

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PART ONE: A CLASSICAL SETTING

DISTRIBUTION OF THE TIME-TO-EXPLOSION FOR LINEAR DIFFUSIONS

I.1: STOCHASTIC DIFFERENTIAL EQUATION

$$\mathrm{d}X(t) = \mathfrak{s}(X(t))\left[\mathrm{d}W(t) + \mathfrak{b}(X(t))\mathrm{d}t\right], \qquad X(0) = \xi \in \mathcal{I}$$

The state-space is an open subinterval $\mathcal{I} = (\ell, r) \subseteq \mathbb{R}$ of the real line. Here $W(\cdot)$ is standard Brownian motion, and $\mathfrak{b} : \mathcal{I} \to \mathbb{R}$, $\mathfrak{s} : \mathcal{I} \to \mathbb{R} \setminus \{0\}$ are measurable functions.

Standing Assumption: The function $1/\mathfrak{s}^2(\cdot)$ and the local mean/over/variance (or "signal-to-noise ratio") function

$$\mathfrak{f}(\cdot) := \frac{\mathfrak{b}(\cdot)}{\mathfrak{s}(\cdot)} = \frac{\mathfrak{b}(\cdot)\,\mathfrak{s}(\cdot)}{\mathfrak{s}^2(\cdot)}$$

are locally integrable over $\mathcal{I}$.

. Under these conditions, there exists a weak solution of the above SDE, defined up until the so-called "explosion time"

$$\mathcal{S} := \lim_{n\to\infty} \uparrow \mathcal{S}_n\,, \quad \mathcal{S}_n = \inf\{t \ge 0 : X(t) \notin (\ell_n, r_n)\}$$

for $\ell_n \downarrow\downarrow \ell$, $r_n \uparrow\uparrow r$. This solution is unique in distribution.

(ENGELBERT & SCHMIDT 1984, 1991.)

We know that $\mathbb{P}(\mathcal{S} = \infty) = 1$ holds under the familiar linear growth conditions of the ITÔ theory, when $\mathcal{I} = \mathbb{R}$.

More generally, fixing a reference point $c \in \mathcal{I}$ and introducing the "FELLER function"

$$v(x) := \int_c^x \int_c^y \exp\left(-2\int_z^y \mathfrak{f}(u)\mathrm{d}u\right)\frac{\mathrm{d}z}{\mathfrak{s}^2(z)}\,\mathrm{d}y\,, \quad x \in \mathcal{I}\,,$$

we have: $\mathbb{P}(\mathcal{S} = \infty) = 1$ if and only if

$$v(\ell+) = v(r-) = \infty\,.$$

This is the classical FELLER test for explosions.

QUESTION (posed to us by Marc YOR):

. *If this condition fails and* $\mathbb{P}(\mathcal{S} < \infty) > 0$*,*
what can we say about the distribution function
$\mathbb{P}(\mathcal{S} \le T)$, $0 < T < \infty$ *of the explosion time?*

I.2: A GENERALIZED GIRSANOV / McKEAN IDENTITY

Let us consider the diffusion in natural scale

$$\mathrm{d}X^o(t) = \mathfrak{s}(X^o(t))\,\mathrm{d}W^o(t)\,, \quad X(0) = \xi \in \mathcal{I}$$

with explosion time $\mathcal{S}^o$; clearly, $\mathbb{Q}(\mathcal{S}^o = \infty) = 1$ if $\mathcal{I} = \mathbb{R}$. Here $W^o(\cdot)$ is Brownian motion under another probability measure $\mathbb{Q}$ (possibly on a different probability space).

Suppose that the mean/variance function $\mathfrak{f}(\cdot)$ is locally *square-integrable* on $\mathcal{I}$, and define the exponential $\mathbb{Q}$–local martingale

$$L(\cdot\,; X^o) := \exp\left\{\int_0^\cdot \mathfrak{b}(X^o(t))\,\mathrm{d}W^o(t) - \frac{1}{2}\int_0^\cdot \mathfrak{b}^2(X^o(t))\,\mathrm{d}t\right\}$$

$$= \exp\left\{\int_0^\cdot \mathfrak{f}(X^o(t))\,\mathrm{d}X^o(t) - \frac{1}{2}\int_0^\cdot \mathfrak{b}^2(X^o(t))\,\mathrm{d}t\right\} \quad \text{on } [0, \mathcal{S}^o).$$

Then for $T \in (0, \infty)$ and bounded, $\mathcal{B}_T-$measurable $h_T : \Omega \to \mathbb{R}$,

$$\boxed{\mathbb{E}^{\mathbb{P}}\Big[\, h_T(X) \cdot \mathbf{1}_{\{\mathcal{S} > T\}} \Big] = \mathbb{E}^{\mathbb{Q}}\Big[\, L(T; X^o)\, h_T(X^o) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}} \,\Big] .}$$

A couple of early lessons from this identity. Suppose $X(\cdot)$ is non-explosive: $\mathbb{P}(\mathcal{S} = \infty) = 1$.
Then

$$\mathbb{E}^{\mathbb{P}}\Big[\, h_T(X) \Big] = \mathbb{E}^{\mathbb{Q}}\Big[\, L(T; X^o)\, h_T(X^o) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}} \,\Big] .$$

In particular, the exponential process $L(\cdot\,; X^o)\, \mathbf{1}_{\{\mathcal{S}^o > \,\cdot\}}$ is then a true $\mathbb{Q}-$martingale; and for every $T \in (0, \infty)$ we have

$$\boxed{\mathbb{E}^{\mathbb{P}}\left(\frac{1}{L(T; X)} \right) = \mathbb{Q}\big(\mathcal{S}^o > T \big)}$$

$$“ \quad \left.\frac{\mathrm{d}\,\mathbb{P}}{\mathrm{d}\,\mathbb{Q}}\right|_{\mathcal{F}(T)} = L(T; X^o) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}} \, . \text{ ”}$$

. Please also note that, always under $\mathbb{P}(\mathcal{S} = \infty) = 1$, the exponential process

$$\frac{1}{L(\cdot\,; X)} = \exp\left\{ - \int_0^{\cdot} \mathfrak{f}(X(t))\, \mathrm{d}X(t) + \frac{1}{2} \int_0^{\cdot} \mathfrak{b}^2(X(t))\, \mathrm{d}t \right\}$$

$$= \exp\left\{ - \int_0^{\cdot} \mathfrak{b}(X(t))\, \mathrm{d}W(t) - \frac{1}{2} \int_0^{\cdot} \mathfrak{b}^2(X(t))\, \mathrm{d}t \right\}$$

is a strictly positive $\mathbb{P}-$local martingale (and supermartingale).

. It is a true $\mathbb{P}-$martingale, if and only if we have, in addition, $\mathbb{Q}(\mathcal{S}^o = \infty) = 1$.

. When $\mathfrak{f}(\cdot)$ is actually continuous and continuously differentiable on $\mathcal{I}$, the above expression gives

$$\mathbb{P}_\xi(\mathcal{S} > T) = \mathbb{E}^{\mathbb{Q}}\left[\exp\left(\int_\xi^{X^o(T)} \mathfrak{f}(z)\,\mathrm{d}z - \int_0^T V(X^o(t))\mathrm{d}t\right) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}}\right]$$

where

$$V(x) := \frac{1}{2}\,\mathfrak{s}^2(x)\left(\mathfrak{f}^{\,2}(x) + \mathfrak{f}'(x)\right).$$

. And in a totally "symmetrical" fashion:

$$\mathbb{Q}_\xi(\mathcal{S}^o > T) = \mathbb{E}^{\mathbb{P}}\left[\exp\left(-\int_\xi^{X(T)} \mathfrak{f}(z)\,\mathrm{d}z + \int_0^T V(X(t))\mathrm{d}t\right) \cdot \mathbf{1}_{\{\mathcal{S} > T\}}\right].$$

I.3: RESULTS: We have the following, general results.

PROPOSITION 1: Positivity, Full Support. *The function*

$$[0,\infty) \times \mathcal{I} \ \ni \ (T,\xi) \longmapsto U(T,\xi) \ := \ \mathbb{P}_\xi(\mathcal{S} > T) \ \in \ (0,1]$$

is (strictly positive and) continuous;
as well as strictly decreasing in T $(***)$, when $\mathbb{P}_\xi(\mathcal{S} < \infty) > 0$.

(***) Last result – strict decrease – needs the
local square-integrability of $1/\mathfrak{s}^2(\cdot)$ on $\mathcal{I}$
(with the possible exception of finitely many points).
This assumption guarantees that *"the diffusion can reach far away points fast, with positive probability"*.

. It has been removed very recently, in work of Cameron BRUGGEMAN and Johannes RUF.

PROPOSITION 2: *The continuous function* $U(\cdot\,,\cdot)$ *of*

$$[0,\infty) \times \mathcal{I} \ni (T,\xi) \longmapsto U(T,\xi) := \mathbb{P}_\xi(\mathcal{S} > T) \in (0,1]$$

is dominated by every nonnegative, classical (super)solution of the Cauchy problem

$$\frac{\partial \mathcal{U}}{\partial \tau}(\tau,x) = \frac{\mathfrak{s}^2(x)}{2}\frac{\partial^2 \mathcal{U}}{\partial x^2}(\tau,x) + \mathfrak{b}(x)\mathfrak{s}(x)\frac{\partial \mathcal{U}}{\partial x}(\tau,x)\,, \quad \tau > 0\,,\ x \in \mathcal{I}$$

$$\mathcal{U}(0+,x) = 1\,, \qquad x \in \mathcal{I}.$$

. Please note that this characterization is impervious to the boundary behavior of the diffusion $X(\cdot)$ at the endpoints of its state-space $\mathcal{I} = (\ell, r)$.

PROPOSITION 3: Minimality. *Suppose that both functions* $\mathfrak{s}(\cdot)$, $\mathfrak{b}(\cdot)$ *are locally Hölder-continuous on* $\mathcal{I}$.

. *Then* $U(\cdot\,,\cdot)$ *solves this Cauchy problem, and is its smallest nonnegative classical (super)solution.*

. *And if* $U(\cdot\,,\cdot) \equiv 1$ *(i.e., if our SDE is non-explosive), then the above Cauchy problem has a unique bounded classical solution, namely,* $\mathcal{U}(\cdot\,,\cdot) \equiv 1$.

RECENT WORK: Important generalizations of these results in the viscosity and generalized solution framework, when the functions $\mathfrak{s}(\cdot)$, $\mathfrak{b}(\cdot)$ are simply continuous, have been carried out – and in several dimensions – by Ms. Yinghui WANG (2014).

PROPOSITION 4: A Generalized FELLER Test.
The following conditions are equivalent:

(i) *The diffusion* $X(\cdot)$ *has no explosions, i.e.,* $\mathbb{P}(\mathcal{S} = \infty) = 1$;
(ii) $v(\ell+) = v(r-) = \infty$ *hold for the "Feller test" function;*
(iii) *The truncated exponential* $\mathbb{Q}$*-supermartingale*

$$L^\flat(\cdot\,; X^o) = \exp\left(\int_0^{\cdot} \mathfrak{b}(X^o(t))\mathrm{d}W^o(t) - \frac{1}{2}\int_0^{\cdot} \mathfrak{b}^2(X^o(t))\mathrm{d}t\right) \mathbf{1}_{\{\mathcal{S}^o > \cdot\}}$$

is a true $\mathbb{Q}$*-martingale.*

. If the functions $\mathfrak{s}(\cdot)$ and $\mathfrak{b}(\cdot)$ are locally Hölder-continuous on $\mathcal{I}$, then the conditions (i)–(iii) are equivalent to:

(iv) The *smallest nonnegative* classical solution of the above Cauchy problem is $\mathcal{U}(\cdot\,,\cdot) \equiv 1$;
(iv)$'$ The *unique bounded* classical solution of the Cauchy problem is $\mathcal{U}(\cdot\,,\cdot) \equiv 1$.

I.4: AN EXAMPLE: Bessel Process in dimension $\delta \in (1,2)$.

$$\mathrm{d}X(t) = \frac{\delta - 1}{2X(t)}\,\mathrm{d}t + \mathrm{d}W(t), \qquad X(0) = \xi \in \mathcal{I} = (0,\infty).$$

The solution of this equation does not explode to infinity, but reaches the origin in finite time: $\mathbb{P}(\mathcal{S} < \infty) = 1$. We have

$$\mathfrak{f}(x) = \frac{1/2 - \nu}{x}, \qquad V(x) = \frac{\nu^2 - 1/4}{2\,x^2}$$

for $\nu = 1 - (\delta/2)$. With

$$X^o(t) = \xi + W(t)\,, \qquad \mathcal{S}^o = \inf\{t \geq 0 : X^o(t) = 0\}\,,$$

the representation

$$\mathbb{P}_\xi(\mathcal{S} > T) = \mathbb{E}^{\mathbb{Q}}\Big[\exp\Big(\int_\xi^{X^o(T)} \mathfrak{f}(z)\,\mathrm{d}z - \int_0^T V(X^o(t))\mathrm{d}t\Big) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}}\Big]$$

gives

$$\mathbb{P}(\mathcal{S} > T) = \mathbb{E}^{\mathbb{Q}}\Bigg[\left(\frac{X^o(T)}{\xi} \right)^{-2\nu} \cdot$$

$$\cdot \left(\frac{X^o(T)}{\xi} \right)^{\nu+1/2} \exp\left(\frac{1/4 - \nu^2}{2} \int_0^T \frac{\mathrm{d}t}{(X^o(t))^2} \right) \cdot \mathbf{1}_{\{\mathcal{S}^o > T\}} \Bigg]$$

$$= \mathbb{E}^{\mathbb{Q}^\nu} \left[\left(\frac{X^o(T)}{\xi} \right)^{-2\nu} \right].$$

Here $\mathbb{Q}^\nu$ is the probability measure under which the auxiliary diffusion $X^o(\cdot) = \xi + W(\cdot)$ is Bessel process in dimension

$$2\nu + 2 = 4 - \delta > 2\,.$$

With the modified Bessel function of the second type

$$I_\nu(u) := \sum_{n\in\mathbb{N}_0} \frac{(u/2)^{\nu+2n}}{n!\,\Gamma(n+\nu+1)}$$

this gives

$$\mathbb{P}(\mathcal{S} > T) \;=\; \frac{1}{T}\,\xi^\nu \exp\left(\frac{-\xi^2}{2T}\right) \int_0^\infty x^{1-\nu} \exp\left(\frac{-x^2}{2T}\right) I_\nu\left(\frac{\xi x}{T}\right) \mathrm{d}x\,.$$

. Algebraic manipulation leads now to a simple proof of

$$U(T,\xi) \;=\; \mathbb{P}_\xi\left(\mathcal{S} > T\right) \;=\; \mathbb{P}\left(\mathfrak{G} < \frac{\xi^2}{2T}\right) \;=\; H\left(\frac{\xi^2}{2T}\right),$$

a result of Ronald GETOOR (1979), where

$$H(u) \;:=\; \frac{1}{\Gamma(\nu)} \int_0^u \, t^{\,\nu-1} \exp(-t)\,\mathrm{d}t\,.$$

- The resulting function

$$U(T,\xi) = \mathbb{P}_{\xi}\left(\mathcal{S} > T\right) = \frac{1}{\Gamma(\nu)} \int_0^{\xi^2/(2T)} t^{\nu-1} \exp(-t)\, \mathrm{d}t$$

is the smallest nonnegative classical solution of the Cauchy problem

$$\frac{\partial \mathcal{U}}{\partial T}(T,\xi) = \frac{1}{2}\frac{\partial^2 \mathcal{U}}{\partial \xi^2}(T,\xi) + \frac{\delta - 1}{2\,\xi}\,\frac{\partial \mathcal{U}}{\partial \xi}(T,\xi)\,, \quad (T,\xi) \in (0,\infty) \times \mathcal{I},$$

$$\mathcal{U}(0+,\xi) = 1\,, \quad \xi \in \mathcal{I}\,.$$

. Many more such (one-dimensional) examples are possible; a small parlor game.

PART TWO: A MORE ELABORATE SETTING

OPTIMAL ARBITRAGE RELATIVE TO THE MARKET PORTFOLIO

II.1: PRELIMINARIES

Filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$. Vector $\mathfrak{X}(\cdot) = (X_1(\cdot), \cdots, X_n(\cdot))'$ of *strictly positive and continuous semimartingales;* these represent the capitalizations of assets in a large equity market, say $n = 8,000$.

. Then

$$X(\cdot) := X_1(\cdot) + \cdots + X_n(\cdot)$$

is the total capitalization, and

$$Z_1(\cdot) := \frac{X_1(\cdot)}{X(\cdot)}, \quad \cdots \quad , Z_n(\cdot) := \frac{X_n(\cdot)}{X(\cdot)},$$

the corresponding relative *market weights.*

The vector $\mathcal{Z}(\cdot) = (Z_1(\cdot), \cdots, Z_n(\cdot))'$ of these weights is a semimartingale with values in the interior Δ^o of the simplex

$$\Delta := \left\{ (z_1, \cdots, z_n)' \in \left[0, 1\right]^n : \sum_{i=1}^{n} z_i = 1 \right\};$$

$\Gamma := \Delta \setminus \Delta^o$ will be the boundary of Δ.
We shall denote $(z_1, \cdots, z_n)' =: \mathbf{z}$.

II.2: PORTFOLIO $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$ is an $\mathbb{F}-$progr. measurable process, such that $(\pi_i/X_i)(\cdot) \in \mathcal{L}(X_i)$, $i = 1, \cdots, n$.

We call this portfolio **strict,** if $\sum_{i=1}^{n} \pi_i(\cdot) \equiv 1$.

We denote the resulting collections by $\mathbf{\Pi}$ (resp., $\mathbf{\Pi_{str}}$).

Here $\pi_i(t)$ stands for the **proportion** of *wealth* $V^\pi(t)$ that gets invested at time $t > 0$ in the i^{th} asset, for each $i = 1, \cdots, n$.

Dynamics of wealth corresponding to portfolio $\pi(\cdot)$ is multiplicative in the initial wealth, and is given by

$$\frac{\mathsf{d}\, V^{\pi}(t)}{V^{\pi}(t)} = \sum_{i=1}^{n} \pi_i(t)\, \frac{\mathsf{d} X_i(t)}{X_i(t)}\,, \qquad V^{\pi}(0) = 1\$\,.$$

Scaling: If we start instead with initial capital $v > 0$, then the corresponding wealth is $v\, V^{\pi}(\cdot)$.

. A strict portfolio will be called "long-only", if

$$\pi_1(\cdot) \geq 0\,, \cdots\,, \pi_n(\cdot) \geq 0\,.$$

The most conspicuous strict long-only portfolio is the **Market Portfolio** $\mathcal{Z}(\cdot) = (Z_1(\cdot), \cdots, Z_n(\cdot))'$ itself. This takes values in Δ^o, and generates wealth proportional to the total market capitalization at all times:

$$V^{\mathcal{Z}}(\cdot) = X(\cdot)/X(0)\,.$$

II.3: ARBITRAGE

Given a horizon $T \in (0, \infty)$ and two portfolios $\pi(\cdot)$ and $\rho(\cdot)$, we say that $\pi(\cdot)$ is *arbitrage relative to* $\rho(\cdot)$ over $[0, T]$, if

$$\mathbb{P}\Big(V^{\pi}(T) \geq V^{\rho}(T)\Big) = 1 \quad \text{and} \quad \mathbb{P}\Big(V^{\pi}(T) > V^{\rho}(T)\Big) > 0\,.$$

• When in fact $\mathbb{P}\Big(V^{\pi}(T) > V^{\rho}(T)\Big) = 1$, we call such relative arbitrage *strong*.

• We recover the "classical" notion of arbitrage (relative to cash) by taking $\rho(\cdot) \equiv 0$, thus $V^{\rho}(\cdot) \equiv 1$.

¶ We shall be interested in **performance with respect to the market**, so we consider for any given portfolio $\pi(\cdot) \in \mathbf{\Pi}$

$$Y^\pi(\cdot) := \frac{V^\pi(\cdot)}{V^{\mathcal{Z}}(\cdot)}, \qquad \text{with} \qquad \frac{\mathrm{d}\, Y^\pi(t)}{Y^\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{\mathrm{d} Z_i(t)}{Z_i(t)},$$

its relative performance. Equivalently, write

$$\frac{\mathrm{d}\, Y^\pi(t)}{Y^\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{\mathrm{d} Z_i(t)}{Z_i(t)} = \sum_{i=1}^n \Psi_i(t)\, \mathrm{d} Z_i(t),$$

with the portfolio proportions expressed as

$$\pi_i(t) = Z_i(t)\, \Psi_i(t), \qquad i = 1, \cdots, n.$$

The process $\Psi(\cdot) = (\Psi_1(\cdot), \cdots, \Psi_n(\cdot))'$ in this scheme of things "generates" the portfolio process $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$.

II.4: RELATIVE ARBITRAGE FUNCTION

The smallest amount of relative initial wealth required at $t = 0$, in order to attain at time $t = T$ relative wealth of (at least) 1 with respect to the market, $\mathbb{P}-$a.s.:

$$U(T, \mathbf{z}) := \inf \left\{ q \in (0,1] \; : \; \exists \; \pi(\cdot) \in \mathbf{\Pi} \;\; \text{s.t.} \; \mathbb{P}\Big(q \, \frac{V^{\pi}(T)}{V^{\mathcal{Z}}(T)} \geq 1 \Big) = 1 \right\}.$$

. Equivalently, $1/U(T, \mathbf{z})$ gives the maximal relative amount by which the market portfolio can be outperformed over $[0, T]$.

We have: $0 < U(T, \mathbf{z}) \leq 1$.

We shall try to characterize this function.

The strict inequality $U(T, \mathbf{z}) > 0$ is a consequence of conditions *to be imposed below*. These amount to *NUIP* (No Unbounded Increasing Profits): *"Absence of Egregious Arbitrages"*.

• When $U(T, \mathbf{z}) = 1$, it is not possible strongly to outperform ("beat") the market strongly over $[0, T]$.

• When $U(T, \mathbf{z}) < 1$, there exists for every $q \in [U(T, \mathbf{z}), 1)$ a portfolio $\pi^q(\cdot) \in \mathbf{\Pi}$ such that $q\, Y^{\pi^q}(T) \geq 1$, i.e.,

$$\frac{V^{\pi^q}(T)}{V^{\mathcal{Z}}(T)} \geq \frac{1}{q} > 1\,, \quad \text{holds } \mathbb{P}-\text{a.s.}$$

Strong arbitrage relative to the market portfolio $\mathcal{Z}(\cdot)$ exists then over the time-horizon $[0, T]$.

¶ In order to be able to say something about this function $U(\cdot\,,\cdot)$, we need a "Model": I.e., some specification of dynamics.

II.5: MARKET WEIGHT "MODEL"

Hybrid *MARKOV/ITÔ-process* dynamics for the Δ^o−valued relative market weights $\mathcal{Z}(\cdot) = \big(Z_1(\cdot), \cdots, Z_n(\cdot)\big)$, of the form

$$\mathrm{d}\mathcal{Z}(t) = \mathfrak{s}\big(\mathcal{Z}(t)\big)\big(\mathrm{d}W(t) + \vartheta(t)\,\mathrm{d}t\big)\,, \qquad Z(0) = \mathbf{z} \in \Delta^o.$$

Here $W(\cdot)$ is an n−dimensional $\mathbb{P}$−Brownian motion; the relative drift process $\vartheta(\cdot)$ is $\mathbb{F}$−progressively measurable and satisfies

$$\boxed{\int_0^T \big\|\vartheta(t)\big\|^2 \mathrm{d}t < \infty\,, \quad \mathbb{P}-\text{a.s.}}$$

for every $T \in (0, \infty)$.

Whereas $\mathfrak{s}(\cdot) = (\mathfrak{s}_{i\nu}(\cdot))_{1 \le i,\nu \le n}$ is a matrix-valued function with $\mathfrak{s}_{i\nu} : \Delta \to \mathbb{R}$ continuous,

$$\sum_{i=1}^{n} \mathfrak{s}_{i\nu}(\cdot) \equiv 0\,, \qquad \nu = 1, \cdots, n\,.$$

. We shall assume that the corresponding *covariance matrix*

$$\mathrm{a}(\mathbf{z}) := \mathfrak{s}(\mathbf{z})\,\mathfrak{s}'(\mathbf{z})\,, \qquad \mathbf{z} \in \Delta$$

has rank $n-1$, $\forall\ \mathbf{z} \in \Delta^o$;
as well as rank $k-1$ in the interior $\mathfrak{D}^o$ of every sub-simplex $\mathfrak{D} \subset \Gamma$ in k dimensions, $k = 1, \cdots, n-1$.

• The quantity $U(T, \mathbf{z})$ is a number in the interval (0,1]. So it is the probability of some event.
Which event? Under what probability measure?
We shall try to answer these questions.

II.6: NUMÉRAIRE PORTFOLIO, LOG-OPTIMALITY

Recall the relative portfolio dynamics in the form

$$\frac{\mathrm{d}\, Y^{\pi}(t)}{Y^{\pi}(t)} = \sum_{i=1}^{n} \pi_i(t) \, \frac{\mathrm{d} Z_i(t)}{Z_i(t)} = \sum_{i=1}^{n} \Psi_i^{(\pi)}(t) \, \mathrm{d} Z_i(t)$$

where we are expressing the portfolio proportions as

$$\pi_i(t) = Z_i(t) \, \Psi_i^{(\pi)}(t) \, , \qquad i = 1, \cdots, n \, .$$

The market portfolio $\pi(\cdot) \equiv Z(\cdot)$ is generated by $\Psi^{(\pi)}(\cdot) \equiv 1$.

Recall

$$\mathrm{d}\mathcal{Z}(t) = \mathfrak{s}\Big(\mathcal{Z}(t)\Big)\Big(\mathrm{d}W(t) + \vartheta(t)\,\mathrm{d}t\Big)\,, \qquad Z(0) = \mathbf{z} \in \Delta^o.$$

• Now, for any two portfolios $\pi(\cdot)$, $\nu(\cdot)$ with corresponding scaled relative weights $\Psi_i^{(\pi)}(\cdot)$ and $\Psi_i^{(\nu)}(\cdot)$ as above, simple calculus gives

$$\mathrm{d}\left(\frac{Y^\pi(t)}{Y^\nu(t)}\right) = \left(\frac{Y^\pi(t)}{Y^\nu(t)}\right)\Big(\Psi^{(\pi)}(t) - \Psi^{(\nu)}(t)\Big)'\Big[\mathrm{d}\mathcal{Z}(t) - \mathrm{a}\Big(\mathcal{Z}(t)\Big)\Psi^{(\nu)}(t)\mathrm{d}t\Big].$$

Thus, the finite-variation part of this expression vanishes, **IFF** the portfolio $\nu(\cdot)$ has scaled relative weights that satisfy the *"perfect balance" condition*

$$\Big(\mathfrak{s}(\mathcal{Z}(\cdot))\Big)'\,\Psi^{(\nu)}(\cdot) = \vartheta(\cdot)\,.$$

With $\nu(\cdot) \equiv \nu^{\mathbb{P}}(\cdot)$ selected this way, namely

$$\Big(\mathfrak{s}(\mathcal{Z}(\cdot))\Big)' \, \Psi^{(\nu)}(\cdot) \; = \; \vartheta(\cdot) \; :$$

. *For any given portfolio* $\pi(\cdot) \in \mathbf{\Pi}$, the ratio

$$Y^{\pi}(\cdot) \Big/ Y^{\nu^{\mathbb{P}}}(\cdot) \; = \; V^{\pi}(\cdot) \Big/ V^{\nu^{\mathbb{P}}}(\cdot)$$

is a positive local martingale – thus also *a supermartingale.*

- We say that this portfolio $\nu^{\mathbb{P}}(\cdot)$ has the "numéraire property", and that the ratio $1 \Big/ Y^{\nu^{\mathbb{P}}}(\cdot) \equiv V^{\mathcal{Z}}(\cdot) \Big/ V^{\nu^{\mathbb{P}}}(\cdot)$ is a "deflator" in this market.

No arbitrage relative to a portfolio with the numéraire property is possible, over ANY finite time-horizon.

. And if $\vartheta(\cdot) \equiv 0$, i.e.,

$$\mathrm{d}\mathcal{Z}(t) = \mathfrak{s}\Big(\mathcal{Z}(t)\Big)\,\mathrm{d}W(t)\,,$$

then the **market portfolio** $\mathcal{Z}(\cdot)$ **ITSELF has the numéraire property.**

Because then we can take $\Psi^{(\nu)}(\cdot) \equiv \mathbf{1}$, thus $\nu(\cdot) \equiv Z(\cdot)$.

¶ **Indeed:** *"You cannot beat the market"* portfolio, when it has the numéraire property.

But this property is (very) special.

Relative Log-Optimality of the numéraire portfolio $\nu^{\mathbb{P}}(\cdot)$:

For every portfolio $\pi(\cdot) \in \mathbf{\Pi}$ and time-horizon $T \in (0, \infty)$, we have

$$\mathbb{E}^{\mathbb{P}}\Big[\log Y^{\pi}(T)\Big] \le \mathbb{E}^{\mathbb{P}}\Big[\log Y^{\nu^{\mathbb{P}}}(T)\Big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \Big\|\vartheta(t)\Big\|^2 \mathrm{d}t\,.$$

Recall:

$$Y^{\pi}(\cdot) := \frac{V^{\pi}(\cdot)}{V^{\mathcal{Z}}(\cdot)}\,, \qquad Y^{\nu^{\mathbb{P}}}(\cdot) := \frac{V^{\nu^{\mathbb{P}}}(\cdot)}{V^{\mathcal{Z}}(\cdot)}$$

keep track of the relative performance of $\pi(\cdot)$ (resp., $\nu^{\mathbb{P}}(\cdot)$) with respect to the market.

. The "deflator" process

$$\frac{1}{Y^{\nu^{\mathbb{P}}}(\cdot)} \equiv \frac{1}{L(\cdot)} := \exp\left\{- \int_0^{\cdot} \vartheta'(t)\,\mathrm{d}W(t) - \frac{1}{2}\int_0^{\cdot} \left\|\vartheta(t)\right\|^2 \mathrm{d}t \right\},$$

i.e., the performance $V^{\mathcal{Z}}(\cdot) \big/ V^{\nu^{\mathbb{P}}}(\cdot)$ of the market relative to the numéraire portfolio $\nu^{\mathbb{P}}(\cdot)$, is a strictly positive $\mathbb{P}-$local martingale and a supermartingale.

. We need not assume – and are not assuming – *a priori,* that this local martingale is a true martingale.

But we *ARE* assuming that it is strictly positive. This is guaranteed by the assumption that, for every $T \in (0, \infty)$,

$$\boxed{\int_0^T \left\|\vartheta(t)\right\|^2 \mathrm{d}t < \infty \quad \text{holds} \quad \mathbb{P}-\text{a.s.}}$$

Thanks to this assumption there is in this model, as we shall see, *No Unbounded Increasing Profit.*

"No Arbitrage of the First Kind",
"No Egregious Arbitrage",
"No Scalable Arbitrage".

II.7: $\mathrm{U}(\cdot\,,\cdot)$ AND THE FÖLLMER "EXIT MEASURE"

Under "canonical" conditions on the filtered space $(\Omega, \mathcal{F})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$, there exists a probability measure $\mathbb{Q}$, under which

$$W^o(\cdot) := W(\cdot) + \int_0^\cdot \vartheta(t)\,\mathrm{d}t$$

is Brownian motion (the so-called *FÖLLMER exit measure;*
I learned all I know about this from some beautiful notes of my student Gordan ZITKOVIĆ dated Thu. September 27, 2001.)

. And the performance of the numéraire portfolio $\nu^{\mathbb{P}}(\cdot)$ relative to the market, i.e., the reciprocal

$$\frac{V^{\nu^{\mathbb{P}}}(\cdot)}{V^{\mathcal{Z}}(\cdot)} = Y^{\nu^{\mathbb{P}}}(\cdot) \equiv L(\cdot) = \exp\left\{ \int_0^\cdot \vartheta'(t)\,\mathrm{d}W^o(t) - \frac{1}{2}\int_0^\cdot \left\|\vartheta(t)\right\|^2 \mathrm{d}t \right\}$$

of our deflator process, is a $\mathbb{Q}-$martingale; indeed,

$$\mathbb{P}(A) = \int_A L(T)\, \mathrm{d}\mathbb{Q}\,, \quad A \in \mathcal{F}(T)\,; \qquad \forall \;\; T \in (0, \infty).$$

- Whereas the market-weight process $\mathcal{Z}(\cdot)$ is a $\mathbb{Q}-$martingale and Markov process, with values in $\boldsymbol{\Delta}$ and "purely diffusive" $\mathbb{Q}-$dynamics

$$\mathrm{d}\mathcal{Z}(t) \,=\, \mathsf{s}\Big(\mathcal{Z}(t)\Big)\, \mathrm{d}W^o(t)\,, \qquad \mathcal{Z}(0) = \mathbf{z} \in \boldsymbol{\Delta}^o.$$

Thus, *the market portfolio* $\mathcal{Z}(\cdot)$ *has the numéraire property under the exit measure* $\mathbb{Q}$:

$$\mathcal{Z}(\cdot) \equiv \nu^{\mathbb{Q}}(\cdot)\,.$$

- If we consider the first time ("explosion", or rather **implosion**)

$$\mathcal{S} := \inf \left\{ t \geq 0 \,:\, \mathcal{Z}(t) \in \Gamma \right\}$$

$\mathcal{Z}(\cdot)$ reaches the boundary Γ of the unit simplex Δ, the arbitrage function is represented in the already familiar form

$$\boxed{U(T, \mathbf{z}) = \mathbb{E}^{\mathbb{P}_{\mathbf{z}}}\left[\frac{1}{L(T)} \right] = \mathbb{Q}_{\mathbf{z}}\big(\mathcal{S} > T \big)\,,} \qquad (T, \mathbf{z}) \in (0, \infty) \times \Delta^{o}\,.$$

The relative arbitrage function $U(T, \mathbf{z})$ emerges as *the probability under the FÖLLMER measure, that $\mathcal{Z}(\cdot)$ has not reached the boundary Γ of the simplex by time $t = T$*, when started at initial configuration $\mathbf{z}$. Tail-distribution of the "explosion" time.

• Please think of the passage from the original measure $\mathbb{P}$ to the FÖLLMER measure $\mathbb{Q}$, as a Girsanov-like change of probability that "removes the drift" in the dynamics

$$\mathrm{d}\mathcal{Z}(t) = \mathfrak{s}\Big(\mathcal{Z}(t)\Big)\Big(\mathrm{d}W(t) + \vartheta(t)\,\mathrm{d}t\Big),$$

when all we can say about the exponential ("deflator") process

$$\frac{1}{L(\cdot)} = \exp\left\{-\int_0^{\cdot} \vartheta'(t)\,\mathrm{d}W(t) - \frac{1}{2}\int_0^{\cdot} \Big\|\vartheta(t)\Big\|^2 \mathrm{d}t\right\} \equiv \frac{1}{Y^{\nu^{\mathbb{P}}}(\cdot)}$$

is that it is a local martingale under $\mathbb{P}$ (**strict**, when $U(T,\mathbf{z}) < 1$).

The process $L(\cdot)$ *can in principle* reach the origin with positive $\mathbb{Q}-$probability, so this is in general *not* an equivalent change of measure:

We have $\mathbb{P} \ll \mathbb{Q}$, but not necessarily $\mathbb{Q} \ll \mathbb{P}$.

. Nonetheless, the process $\mathcal{Z}(\cdot)$ of market weights is a $\mathbb{Q}-$martingale with values in the unit simplex – and now with the possibility of reaching its faces.

(Thus, we can think of the FÖLLMER measure $\mathbb{Q}$ as an Ersatz "martingale measure" for the model under consideration.)

II.8: $\mathrm{U}(\cdot,\cdot)$ AS SMALLEST SUPERSOLUTION

Under regularity conditions on the covariance structure $\mathsf{a}(\cdot)$ and on the relative drift $\vartheta(\cdot)$, the arbitrage function $U(\cdot\,,\cdot)$ is of class $\mathcal{C}^{1,2}$ on $(0,\infty)\times\Delta^o$, and satisfies there the equation

$$D_\tau U(\tau,\mathbf{z}) \;=\; \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathsf{a}_{ij}(\mathbf{z})\,D_{ij}^2 U(\tau,\mathbf{z})\,,$$

or

$$D_\tau U = \frac{1}{2}\mathrm{Tr}\Big(\mathsf{a}\,D^2 U\Big)\,.$$

Further, $U(\cdot\,,\cdot)$ is also the **smallest nonnegative supersolution** of this equation, subject to

$$U(0+\,,\cdot)\equiv 1\,.$$

• Please note that this equation

$$D_\tau U = \frac{1}{2}\mathrm{Tr}\Big(\mathrm{a}\, D^2 U\Big)$$

involves only the covariance structure of the assets.

. The only rôle the relative drift $\vartheta(\cdot)$ plays in this context, is to keep the market weight process $\mathcal{Z}(\cdot)$ in the interior of the unit simplex, $\mathbb{P}-$a.e. (Once again, this characterization is completely impervious to boundary conditions on the faces of the simplex.)

. With Knightian uncertainty about the covariance a$(\cdot)$ and the relative drift $\vartheta(\cdot)$, this equation becomes fully nonlinear (of HJB-PUCCI type), as in the work of Terry LYONS (1995).

. Great generalizations of these results, in the context of viscosity solutions of the fully nonlinear PDE's, appear in very recent work by Ms. Yinghui WANG (2015).

II.9: CONDITIONING, CLASS $\mathfrak{P}$

Let us consider the collection $\mathfrak{P}$ of probability measures $\mathbb{P} \ll \mathbb{Q}$ with $\mathbb{P}(\mathcal{Z}(t) \in \Delta^o, \ \forall\ 0 \le t \le T) = 1$. (Our original measure $\mathbb{P}$ belongs to this collection.) We single out an element of $\mathfrak{P}$ via

$$\mathbb{P}_\star(A) := \mathbb{Q}\,(\,A \mid \mathcal{S} > T\,)\,, \qquad A \in \mathcal{F}(T)\,. \tag{1}$$

This is the conditioning of the FÖLLMER measure $\mathbb{Q}$ on the set $\{\mathcal{Z}(\cdot)$ has not reached the boundary of the simplex by time $T\}$.

Elementary computations give, $\mathbb{Q}-$a.s.:

$$\frac{\mathrm{d}\,\mathbb{P}_\star}{\mathrm{d}\,\mathbb{Q}}\bigg|_{\mathcal{F}(t)} = \frac{U(T-t,\mathcal{Z}(t))}{U(T,\mathbf{z})}\,\mathbf{1}_{\{\mathcal{S}>t\}} =: \frac{\widehat{Y}(t)}{\widehat{Y}(0)}, \qquad 0 \le t \le T$$

$$\frac{\mathsf{d}\,\mathbb{P}_\star}{\mathsf{d}\,\mathbb{Q}}\bigg|_{\mathcal{F}(t)} = \frac{U(T-t,\mathcal{Z}(t))}{U(T,\mathbf{z})}\,\mathbf{1}_{\{\mathcal{S}>t\}} =: \frac{\widehat{Y}(t)}{\widehat{Y}(0)}, \qquad 0 \le t \le T$$

with the $\mathbb{Q}$−martingale

$$\widehat{Y}(t) := U\Big(T-t,\mathcal{Z}(t)\Big)\,\mathbf{1}_{\{\mathcal{S}>t\}} \equiv q\,Y^{\widehat{\pi}}(t) \qquad \text{for} \qquad q = U(T,\mathbf{z})\,,$$

and with the **functionally-generated** portfolio in $\mathbf{\Pi_{str}}$:

$$\widehat{\pi}_i(t) \;=\; Z_i(t)\cdot D_i \log U\Big(T-t,\mathcal{Z}(t)\Big)\,. \tag{2}$$

• This portfolio has the numéraire property under the conditioning $\mathbb{P}_\star$ of the FÖLLMER measure:

$$\widehat{\pi}(\cdot) \;\equiv\; \nu^{\mathbb{P}_\star}(\cdot)\,.$$

. Whenever $U(T, \mathbf{z}) < 1$, this portfolio implements the best achievable arbitrage under the **original** probability measure $\mathbb{P}$; that is,

$$\frac{V^{\widehat{\pi}}(T)}{V^{\mathcal{Z}}(T)} = \frac{1}{U(T, \mathbf{z})} > 1 \quad \text{holds} \quad \mathbb{P} - \text{a.s.}$$

II.10: A RECIPE

We can characterize the portfolio $\widehat{\pi}(\cdot)$ of (2) that implements the optimal arbitrage over a given time-horizon $[0, T]$ as follows, *given the market weight covariance structure under the original probability measure* $\mathbb{P}$ (and nothing else...):

• *FIRST, find a probability measure* $\mathbb{Q}$ *under which the market weights are martingales, as in*

$$\mathrm{d}\mathcal{Z}(t) = \mathfrak{s}\Big(\mathcal{Z}(t)\Big)\,\mathrm{d}\,W^{o}(t)\,, \qquad \mathcal{Z}(0) = \mathbf{z} \in \Delta^{o},$$

and compute the function $U(T, \mathbf{z}) = \mathbb{Q}_{\mathbf{z}}(\mathcal{S} > T)\,.$

• *SECONDLY, construct the measure* $\mathbb{P}_{\star}$ *by conditioning* $\mathbb{Q}$ *on the event* $\{\mathcal{S} > T\}$ *as in* $\mathbb{P}_{\star}(A) := \mathbb{Q}\,(\,A \mid \mathcal{S} > T\,)\,, \quad A \in \mathcal{F}(T)\,.$

• *FiINALLY, construct the portfolio* $\widehat{\pi}(\cdot)$ *that maximizes expected log-return (equiv., has the numéraire property) under* $\mathbb{P}_{\star}$ *.*

This portfolio is generated by the vector process of log-derivatives, i.e., is given by the recipe

$$\widehat{\pi}_i(t) = Z_i(t) \cdot D_i \log U\Big(T - t, \mathcal{Z}(t)\Big), \qquad i = 1, \cdots, n\,.$$

II.12: MINIMAL ENERGY AND ENTROPY

With

$$H_T(\mathbb{P}\,|\,\mathbb{Q}) := \mathbb{E}^{\mathbb{P}}\Big[\log\Big(\big(\mathrm{d}\,\mathbb{P}/\mathrm{d}\,\mathbb{Q}\big)\Big|_{\mathcal{F}(T)}\Big)\Big] = \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \big\|\vartheta^{\mathbb{P}}(t)\big\|^2\,\mathrm{d}t$$

we have the "minimum entropy and energy" properties

$$\log\big(1/U(T,\mathbf{z})\big) = H_T(\mathbb{P}_\star\,|\,\mathbb{Q}) = \min_{\mathbb{P}\in\mathfrak{P}} H_T(\mathbb{P}\,|\,\mathbb{Q})$$

$$= \frac{1}{2}\,\mathbb{E}^{\mathbb{P}_\star}\int_0^T \big\|\vartheta^{\mathbb{P}_\star}(t)\big\|^2\,\mathrm{d}t = \min_{\mathbb{P}\in\mathfrak{P}} \frac{1}{2}\,\mathbb{E}^{\mathbb{P}}\int_0^T \big\|\vartheta^{\mathbb{P}}(t)\big\|^2\,\mathrm{d}t\,.$$

We call $\mathbb{P}_\star$ **"minimal energy"** measure in $\mathfrak{P}$.

Has relative risk process $\vartheta^{\mathbb{P}_\star}(\cdot)$ that keeps the market weights strictly positive throughout $[0,T]$ by expending minimal energy.

This minimal entropy function

$$\mathcal{H}(\tau, \mathbf{z}) \;:=\; \log\left(1 \big/ U(T, \mathbf{z})\right) \;=\; H_T(\, \mathbb{P}_\star \,|\, \mathbb{Q}\,)$$

solves the HJB equation for this problem

$$D_\tau \mathcal{H}(\tau, \mathbf{z}) = \frac{1}{2}\,\mathsf{Tr}\Big(\mathsf{a}(\mathbf{z})\, D^2\mathcal{H}(\tau, \mathbf{z})\Big) + \min_{\theta \in \mathbb{R}^n}\left[\,\Big(D\mathcal{H}(\tau, \mathbf{z})\Big)'\mathsf{s}(\mathbf{z})\,\theta + \frac{1}{2}\,\big\|\theta\big\|^2\,\right],$$

which is of course a semilinear equation

$$D_\tau \mathcal{H}(\tau, \mathbf{z}) = \frac{1}{2}\,\mathsf{Tr}\Big(\mathsf{a}(\mathbf{z})\, D^2\mathcal{H}(\tau, \mathbf{z})\Big) - \frac{1}{2}\Big(D\mathcal{H}(\tau, \mathbf{z})\Big)'\mathsf{s}(\mathbf{z})\Big(D\mathcal{H}(\tau, \mathbf{z})\Big).$$

II.13: A STOCHASTIC GAME

The pair $(\mathbb{P}_\star, \widehat{\pi}(\cdot))$ of (1), (2) is a saddle point in $\mathfrak{P} \times \mathbf{\Pi}$ for the zero-sum stochastic game with value

$$\log\Big(1/U(T,\mathbf{z})\Big) = \mathbb{E}^{\mathbb{P}_\star}\Big[\,\log Y^{\widehat{\pi}}(T)\,\Big] =$$

$$= \min_{\mathbb{P}\in\mathfrak{P}}\ \max_{\pi(\cdot)\in\mathbf{\Pi}}\ \mathbb{E}^{\mathbb{P}}\Big[\,\log Y^{\pi}(T)\,\Big] = \max_{\pi(\cdot)\in\mathbf{\Pi}}\ \min_{\mathbb{P}\in\mathfrak{P}}\ \mathbb{E}^{\mathbb{P}}\Big[\,\log Y^{\pi}(T)\,\Big];$$

and for every $(\mathbb{P}, \pi(\cdot)) \in \mathfrak{P} \times \mathbf{\Pi}$ we have the saddle

$$\mathbb{E}^{\mathbb{P}}\Big[\,\log Y^{\widehat{\pi}}(T)\,\Big] \ge \mathbb{E}^{\mathbb{P}_\star}\Big[\,\log Y^{\widehat{\pi}}(T)\,\Big] =$$

$$= \log\Big(1/U(T,\mathbf{z})\Big) \ge \mathbb{E}^{\mathbb{P}_\star}\Big[\log Y^{\pi}(T)\Big].$$

II.14: A SUFFICIENT CONDITION AND A TOY MODEL

It can be shown that a *sufficient condition* for $U(T, \mathbf{z}) < 1$ is that there exist a real constant $h > 0$ for which

$$\sum_{i=1}^{n} z_i \left(\frac{\mathsf{a}_{ii}(\mathbf{z})}{z_i^2} \right) \geq h\,, \qquad \forall \;\; \mathbf{z} \in \Delta^o\,. \tag{3}$$

The weighted relative variance of log-returns in (3) is a measure of the market's "intrinsic" (or "average relative") variance; condition (3) posits a positive lower bound on this quantity as sufficient for $U(T, \mathbf{z}) < 1$.

. *Under the condition (3), very simple long-only portfolios can be designed, that lead to arbitrage over sufficiently long horizons.*

For instance, given any real number $T > (2 \log n)/h$, there is $c > 0$ sufficiently large, so that the portfolio

$$\pi_i(t) = \frac{Z_i(t)(c - \log Z_i(t))}{\sum_{j=1}^{n} Z_j(t)(c - \log Z_j(t))}, \qquad i = 1, \cdots, n$$

is strong arbitrage relative to the market portfolio $\mathcal{Z}(\cdot)$ over the time-horizon $[0, T]$.

. **OPEN QUESTION: Is arbitrage relative to the market possible under condition (3) over arbitrary time-horizons ?**

(A few additional examples exist, under different structural conditions, and with the *equally-weighted portfolio* playing a very important rôle. Would be nice to have more of them ... .)

. Very recent development: Counterexample by Johannes RUF.

II.15: A CONCRETE TOY-EXAMPLE

A concrete example where the condition

$$\sum_{i=1}^{n} \frac{\mathsf{a}_{ii}(\mathbf{z})}{z_i} \geq h \,, \qquad \forall \;\; \mathbf{z} \in \Delta^o$$

of (3) is satisfied concerns the "Volatility-Stabilized" Model

$$\boxed{\mathrm{d} \log X_i(t) = \Big(\kappa / Z_i(t) \Big) \, \mathrm{d}t + \Big(1 \Big/ \sqrt{Z_i(t)} \; \Big) \, \mathrm{d}W_i(t) \,, \quad i = 1, \cdots, n}$$

with constant $\kappa \geq 1/2$, or equivalently for the market weights

$$\mathrm{d}Z_i(t) = \kappa \Big(1 - n \, Z_i(t) \Big) \mathrm{d}t + \sqrt{Z_i(t)} \, \mathrm{d}W_i(t) - Z_i(t) \sum_{k=1}^{n} \sqrt{Z_k(t)} \, \mathrm{d}W_k(t)$$

$$= \kappa \Big(1 - n \, Z_i(t) \Big) \mathrm{d}t + \sqrt{Z_i(t)} \, \sqrt{1 - Z_i(t)} \, \mathrm{d}W_i^{\#}(t) \,.$$

The variances in this last diffusion equation

$$\mathrm{d}Z_i(t) = \kappa\Big(1 - n\,Z_i(t)\Big)\mathrm{d}t + \sqrt{Z_i(t)}\,\sqrt{1 - Z_i(t)}\,\mathrm{d}W_i^{\#}(t)$$

(in which the $W_i^{\#}(\cdot)\,,\ i = 1, \cdots, n$ are correlated BM's) are of *WRIGHT-FISHER type*

$$\mathsf{a}_{ii}(\mathbf{z}) \;=\; z_i(1 - z_i)\,;$$

so the condition

$$\sum_{i=1}^{n} \frac{\mathsf{a}_{ii}(\mathbf{z})}{z_i} \geq h\,, \qquad \forall \;\; \mathbf{z} \in \mathbf{\Delta}^o$$

of (3) holds as equality, in fact with $h = n - 1 \geq 1$.

. Here, and indeed in any setting of the form

$$\mathrm{d}\log X_i(t) = \beta_i(t)\,\mathrm{d}t + \left(1\Big/\sqrt{Z_i(t)}\,\right)\mathrm{d}W_i(t)\,, \qquad i = 1, \cdots, n\,,$$

the market CAN be outperformed over arbitrary time horizons (A. BANNER & D. FERNHOLZ (2008), R. PICKOVÁ (2014)).

- In this case, one can "compute" the relative arbitrage function

$$U(T, \mathbf{z}) \;=\; \mathbb{E}^{\mathbb{P}}\Big[\,\frac{z_1 \cdots z_n}{Z_1(T) \cdots Z_n(T)}\,\Big] \cdot \mathbb{E}^{\mathbb{P}}\Big[\,e^{-(n-1)(\gamma\, T + W(T))}\,\Big]\,,$$

because S. PAL (2011) has computed the joint distribution of the weights $Z_1(T), \cdots, Z_n(T)$ fairly explicitly (Dirichlet). Here

$$\gamma \;=\; \kappa\, n - \frac{1}{2}\,.$$

• Under the FÖLLMER measure $\mathbb{Q}$, each weight $Z_i(\cdot)$ is a WRIGHT-FISHER diffusion in natural scale, and reaches an end-point of (0,1) in finite expected time $\mathcal{S}_i = \inf\{t \geq 0 : Z_i(t) = 0\}$:

$$\mathrm{d}Z_i(t) = \kappa\Big(1 - n\,Z_i(t)\Big)\mathrm{d}t + \sqrt{Z_i(t)}\,\sqrt{1 - Z_i(t)}\,\mathrm{d}W_i^{\#}(t)$$

$$= \sqrt{Z_i(t)}\,\sqrt{1 - Z_i(t)}\,\mathrm{d}W_i^{o}(t)\,.$$

For us, of course, the time of interest is

$$\mathcal{S} = \min_{1 \leq i \leq n} \mathcal{S}_i\,.$$

Eventually all but one of the $Z_i(\cdot)$'s "perish", and one of them emerges as **the** survivor.

. Think of a catalytic reaction involving n compounds with nucleation/condensation (very recent work of C.LANDIM et al., May 2015); or of a gladiatorial fight in the Colosseum.

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THANK YOU FOR YOUR ATTENTION

HAPPY BIRTHDAY, STEVE !!!!

ΠΟΛΥΧΡΟΝΙΟΣ !!!!