EXPLOSIONS AND ARBITRAGE

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PART ONE: A CLASSICAL SETTING

DISTRIBUTION OF THE TIME-TO-EXPLOSION FOR LINEAR DIFFUSIONS
I.1: STOCHASTIC DIFFERENTIAL EQUATION

\[ dX(t) = s(X(t)) \left[ dW(t) + b(X(t))dt \right], \quad X(0) = \xi \in \mathcal{I} \]

The state-space is an open subinterval \( \mathcal{I} = (\ell, r) \subseteq \mathbb{R} \) of the real line. Here \( W(\cdot) \) is standard Brownian motion, and \( b : \mathcal{I} \to \mathbb{R}, \ s : \mathcal{I} \to \mathbb{R} \setminus \{0\} \) are measurable functions.

**Standing Assumption:** The function \( 1/s^2(\cdot) \) and the local mean/over/variance (or “signal-to-noise ratio”) function

\[ f(\cdot) := \frac{b(\cdot)}{s(\cdot)} = \frac{b(\cdot)s(\cdot)}{s^2(\cdot)} \]

are locally integrable over \( \mathcal{I} \).
. Under these conditions, there exists a weak solution of the above SDE, defined up until the so-called “explosion time”

$$S := \lim_{n \to \infty} \uparrow S_n, \quad S_n = \inf \{ t \geq 0 : X(t) \notin (\ell_n, r_n) \}$$

for $\ell_n \downarrow \ell$, $r_n \uparrow r$. This solution is unique in distribution.

(ENGELBERT & SCHMIDT 1984, 1991.)

We know that $\mathbb{P}(S = \infty) = 1$ holds under the familiar linear growth conditions of the ITÔ theory, when $\mathcal{I} = \mathbb{R}$. 

More generally, fixing a reference point \( c \in \mathcal{I} \) and introducing the “FELLER function”

\[
v(x) := \int_c^x \int_c^y \exp \left( -2 \int_z^y f(u) \, du \right) \frac{dz}{s^2(z)} \, dy, \quad x \in \mathcal{I},
\]

we have: \( \mathbb{P}(S = \infty) = 1 \) if and only if

\[
v(\ell+) = v(r-) = \infty.
\]

This is the classical FELLER test for explosions.

**QUESTION** (posed to us by Marc YOR):

*If this condition fails and \( \mathbb{P}(S < \infty) > 0 \), what can we say about the distribution function \( \mathbb{P}(S \leq T), \ 0 < T < \infty \) of the explosion time?*
I.2: A GENERALIZED GIRSANOV / McKEAN IDENTITY

Let us consider the diffusion in natural scale

\[ dX^0(t) = \mathfrak{s}(X^0(t)) \, dW^\circ(t), \quad X(0) = \xi \in \mathcal{I} \]

with explosion time \( S^\circ \); clearly, \( Q(S^\circ = \infty) = 1 \) if \( \mathcal{I} = \mathbb{R} \). Here \( W^\circ(\cdot) \) is Brownian motion under another probability measure \( Q \) (possibly on a different probability space).

Suppose that the mean/variance function \( f(\cdot) \) is locally \textit{square-integrable} on \( \mathcal{I} \), and define the exponential \( Q \)–local martingale

\[
L(\cdot; X^0) := \exp \left\{ \int_0^\cdot b(X^0(t)) \, dW^\circ(t) - \frac{1}{2} \int_0^\cdot b^2(X^0(t)) \, dt \right\}
\]

\[
= \exp \left\{ \int_0^\cdot f(X^0(t)) \, dX^0(t) - \frac{1}{2} \int_0^\cdot b^2(X^0(t)) \, dt \right\} \quad \text{on} \ [0, S^\circ).
\]
Then for $T \in (0, \infty)$ and bounded, $\mathcal{B}_T$–measurable $h_T : \Omega \to \mathbb{R}$,

$$
\mathbb{E}^P[h_T(X) \cdot 1_{\{S>T\}}] = \mathbb{E}^Q[L(T; X^0) h_T(X^0) \cdot 1_{\{S^0>T\}}].
$$

**A couple of early lessons from this identity.** Suppose $X(\cdot)$ is non-explosive: $\mathbb{P}(S = \infty) = 1$.

Then

$$
\mathbb{E}^P[h_T(X)] = \mathbb{E}^Q[L(T; X^0) h_T(X^0) \cdot 1_{\{S^0>T\}}].
$$

In particular, the exponential process $L(\cdot; X^0) 1_{\{S^0>\cdot\}}$ is then a true $\mathbb{Q}$–martingale; and for every $T \in (0, \infty)$ we have

$$
\mathbb{E}^P\left( \frac{1}{L(T; X)} \right) = \mathbb{Q}(S^0 > T).
$$
\[
\frac{d\mathbb{P}}{d\mathbb{Q}} \bigg|_{\mathcal{F}(T)} = L(T; X^o) \cdot 1_{\{S^o > T\}}.
\]

Please also note that, always under \(\mathbb{P}(S = \infty) = 1\), the exponential process
\[
\frac{1}{L(\cdot; X)} = \exp \left\{ - \int_0^\cdot f(X(t)) \, dX(t) + \frac{1}{2} \int_0^\cdot b^2(X(t)) \, dt \right\}
\]
\[
= \exp \left\{ - \int_0^\cdot b(X(t)) \, dW(t) - \frac{1}{2} \int_0^\cdot b^2(X(t)) \, dt \right\}
\]
is a strictly positive \(\mathbb{P}\)-local martingale (and supermartingale).

It is a true \(\mathbb{P}\)-martingale, if and only if we have, in addition, \(\mathbb{Q}(S^o = \infty) = 1\).
. When $f(\cdot)$ is actually continuous and continuously differentiable on $\mathcal{I}$, the above expression gives

$$P_\xi(S > T) = E^Q \left[ \exp \left( \int_\xi^{X^o(T)} f(z) \, dz - \int_0^T V(X^o(t)) \, dt \right) \cdot 1\{S^o > T\} \right]$$

where

$$V(x) := \frac{1}{2} s^2(x) \left( f^2(x) + f'(x) \right).$$

. And in a totally "symmetrical" fashion:

$$Q_\xi(S^o > T) = E^P \left[ \exp \left( - \int_\xi^{X(T)} f(z) \, dz + \int_0^T V(X(t)) \, dt \right) \cdot 1\{S > T\} \right].$$
I.3: RESULTS: We have the following, general results.

**PROPOSITION 1: Positivity, Full Support.** The function

\[ [0, \infty) \times I \ni (T, \xi) \mapsto U(T, \xi) := \mathbb{P}_\xi(S > T) \in (0, 1] \]

is (strictly positive and) continuous;
as well as strictly decreasing in \( T \) (***) , when \( \mathbb{P}_\xi(S < \infty) > 0 \).

(***) Last result – strict decrease – needs the
**local square-integrability of** \( 1/\xi^2(\cdot) \) on \( I \)
(with the possible exception of finitely many points).
This assumption guarantees that “the diffusion can
reach far away points fast, with positive probability”.

. It has been removed very recently, in work of
Cameron BRUGGEMAN and Johannes RUF.
PROPOSITION 2: The continuous function $U(\cdot, \cdot)$ of

$$[0, \infty) \times \mathcal{I} \ni (T, \xi) \mapsto U(T, \xi) := \mathbb{P}_\xi(S > T) \in (0, 1]$$

is dominated by every nonnegative, classical (super)solution of the Cauchy problem

$$\begin{align*}
\frac{\partial U}{\partial \tau}(\tau, x) &= \frac{s^2(x)}{2} \frac{\partial^2 U}{\partial x^2}(\tau, x) + b(x)s(x) \frac{\partial U}{\partial x}(\tau, x), \quad \tau > 0, \ x \in \mathcal{I} \\
U(0+, x) &= 1, \quad x \in \mathcal{I}.
\end{align*}$$

. Please note that this characterization is impervious to the boundary behavior of the diffusion $X(\cdot)$ at the endpoints of its state-space $\mathcal{I} = (\ell, r)$. 
PROPOSITION 3: Minimality. Suppose that both functions $s(\cdot), b(\cdot)$ are locally Hölder-continuous on $I$.

Then $U(\cdot, \cdot)$ solves this Cauchy problem, and is its smallest nonnegative classical (super)solution.

And if $U(\cdot, \cdot) \equiv 1$ (i.e., if our SDE is non-explosive), then the above Cauchy problem has a unique bounded classical solution, namely, $U(\cdot, \cdot) \equiv 1$.

RECENT WORK: Important generalizations of these results in the viscosity and generalized solution framework, when the functions $s(\cdot), b(\cdot)$ are simply continuous, have been carried out – and in several dimensions – by Ms. Yinghui WANG (2014).
PROPOSITION 4: A Generalized FELLER Test.
The following conditions are equivalent:

(i) The diffusion $X(\cdot)$ has no explosions, i.e., $\mathbb{P}(S = \infty) = 1$;
(ii) $v(\ell+) = v(r-) = \infty$ hold for the “Feller test” function;
(iii) The truncated exponential $\mathbb{Q}$-supermartingale

$$L^b(\cdot; X^o) = \exp \left( \int_0^\cdot b(X^o(t))dW^o(t) - \frac{1}{2} \int_0^\cdot b^2(X^o(t))dt \right) 1_{\{S^o > \cdot\}}$$

is a true $\mathbb{Q}$-martingale.

. If the functions $s(\cdot)$ and $b(\cdot)$ are locally Hölder-continuous on $I$, then the conditions (i)–(iii) are equivalent to:

(iv) The smallest nonnegative classical solution of the above Cauchy problem is $U(\cdot, \cdot) \equiv 1$;
(iv)' The unique bounded classical solution of the Cauchy problem is $U(\cdot, \cdot) \equiv 1$. 
I.4: AN EXAMPLE: Bessel Process in dimension $\delta \in (1, 2)$.

$$dX(t) = \frac{\delta - 1}{2X(t)} \, dt + dW(t), \quad X(0) = \xi \in \mathcal{I} = (0, \infty).$$

The solution of this equation does not explode to infinity, but reaches the origin in finite time: $\mathbb{P}(S < \infty) = 1$. We have

$$f(x) = \frac{1/2 - \nu}{x}, \quad V(x) = \frac{\nu^2 - 1/4}{2 x^2}$$

for $\nu = 1 - (\delta/2)$. With

$$X^o(t) = \xi + W(t), \quad S^o = \inf\{t \geq 0 : X^o(t) = 0\},$$

the representation

$$\mathbb{P}_\xi(S > T) = \mathbb{E}_Q^\xi\left[\exp\left(\int_{\xi}^{X^o(T)} f(z) \, dz - \int_0^T V(X^o(t)) \, dt\right) \cdot 1_{\{S^o > T\}}\right]$$


\[ P(S > T) = \mathbb{E}^Q \left[ \left( \frac{X^0(T)}{\xi} \right)^{-2\nu} \cdot \left( \frac{X^0(T)}{\xi} \right)^{\nu + 1/2} \exp \left( \frac{1/4 - \nu^2}{2} \int_0^T \frac{dt}{(X^0(t))^2} \right) \cdot 1_{\{S^0 > T\}} \right] \]

\[ = \mathbb{E}^{Q^\nu} \left[ \left( \frac{X^0(T)}{\xi} \right)^{-2\nu} \right]. \]

Here \( Q^\nu \) is the probability measure under which the auxiliary diffusion \( X^0(\cdot) = \xi + W(\cdot) \) is Bessel process in dimension \( 2\nu + 2 = 4 - \delta > 2 \).
With the modified Bessel function of the second type

\[ I_\nu(u) := \sum_{n \in \mathbb{N}_0} \frac{(u/2)^{\nu+2n}}{n! \Gamma(n + \nu + 1)} \]

this gives

\[
\mathbb{P}(S > T) = \frac{1}{T} \xi^\nu \exp\left(\frac{-\xi^2}{2T}\right) \int_0^\infty x^{1-\nu} \exp\left(\frac{-x^2}{2T}\right) I_\nu \left(\frac{\xi x}{T}\right) \, dx.
\]

Algebraic manipulation leads now to a simple proof of

\[
U(T, \xi) = \mathbb{P}_\xi(S > T) = \mathbb{P}\left(\xi < \frac{\xi^2}{2T}\right) = H\left(\frac{\xi^2}{2T}\right),
\]

a result of Ronald GETOOR (1979), where

\[
H(u) := \frac{1}{\Gamma(\nu)} \int_0^u t^{\nu-1} \exp(-t) \, dt.
\]
The resulting function

\[ U(T, \xi) = P_\xi(S > T) = \frac{1}{\Gamma(\nu)} \int_0^{\xi^2/(2T)} t^{\nu-1} \exp(-t) \, dt \]

is the smallest nonnegative classical solution of the Cauchy problem

\[
\frac{\partial U}{\partial T}(T, \xi) = \frac{1}{2} \frac{\partial^2 U}{\partial \xi^2}(T, \xi) + \frac{\delta - 1}{2 \xi} \frac{\partial U}{\partial \xi}(T, \xi), \quad (T, \xi) \in (0, \infty) \times \mathcal{I},
\]

\[ U(0^+, \xi) = 1, \quad \xi \in \mathcal{I}. \]

Many more such (one-dimensional) examples are possible; a small parlor game.
PART TWO: A MORE ELABORATE SETTING

OPTIMAL ARBITRAGE RELATIVE TO THE MARKET PORTFOLIO
II.1: PRELIMINARIES

Filtered probability space $\left(\Omega, \mathcal{F}, P\right)$, $\mathcal{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}$. Vector $\mathbf{x}(\cdot) = (X_1(\cdot), \cdots, X_n(\cdot))'$ of strictly positive and continuous semimartingales; these represent the capitalizations of assets in a large equity market, say $n = 8,000$.

Then

$$X(\cdot) := X_1(\cdot) + \cdots + X_n(\cdot)$$

is the total capitalization, and

$$Z_1(\cdot) := \frac{X_1(\cdot)}{X(\cdot)}, \quad \cdots, \quad Z_n(\cdot) := \frac{X_n(\cdot)}{X(\cdot)},$$

the corresponding relative market weights.
The vector $\mathcal{Z}(\cdot) = (Z_1(\cdot), \cdots, Z_n(\cdot))^\prime$ of these weights is a semimartingale with values in the interior $\Delta^0$ of the simplex

$$\Delta := \left\{ (z_1, \cdots, z_n)^\prime \in [0,1]^n : \sum_{i=1}^{n} z_i = 1 \right\};$$

$\Gamma := \Delta \setminus \Delta^0$ will be the boundary of $\Delta$. We shall denote $(z_1, \cdots, z_n)^\prime =: \mathbf{z}$.

**II.2: PORTFOLIO** \(\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))^\prime\) is an $\mathcal{F}$−progr. measurable process, such that \((\pi_i/X_i)(\cdot) \in \mathcal{L}(X_i), i = 1, \cdots, n\).

We call this portfolio **strict**, if \(\sum_{i=1}^{n} \pi_i(\cdot) \equiv 1\).

We denote the resulting collections by $\Pi$ (resp., $\Pi_{\text{str}}$).

Here $\pi_i(t)$ stands for the proportion of wealth \(V^{\pi}(t)\) that gets invested at time $t > 0$ in the $i^{\text{th}}$ asset, for each $i = 1, \cdots, n$. 
Dynamics of wealth corresponding to portfolio $\pi(\cdot)$ is multiplicative in the initial wealth, and is given by

$$\frac{\mathrm{d} V^\pi(t)}{V^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{\mathrm{d} X_i(t)}{X_i(t)}, \quad V^\pi(0) = 1.$$

Scaling: If we start instead with initial capital $\nu > 0$, then the corresponding wealth is $\nu V^\pi(\cdot)$.

A strict portfolio will be called “long-only”, if $\pi_1(\cdot) \geq 0, \ldots, \pi_n(\cdot) \geq 0$.

The most conspicuous strict long-only portfolio is the **Market Portfolio** $Z(\cdot) = (Z_1(\cdot), \ldots, Z_n(\cdot))'$ itself. This takes values in $\Delta^0$, and generates wealth proportional to the total market capitalization at all times:

$$V^Z(\cdot) = X(\cdot)/X(0).$$
II.3: ARBITRAGE

Given a horizon $T \in (0, \infty)$ and two portfolios $\pi(\cdot)$ and $\rho(\cdot)$, we say that $\pi(\cdot)$ is arbitrage relative to $\rho(\cdot)$ over $[0, T]$, if

$$\mathbb{P}(V^\pi(T) \geq V^\rho(T)) = 1 \quad \text{and} \quad \mathbb{P}(V^\pi(T) > V^\rho(T)) > 0.$$  

• When in fact $\mathbb{P}(V^\pi(T) > V^\rho(T)) = 1$, we call such relative arbitrage **strong**.

• We recover the “classical” notion of arbitrage (relative to cash) by taking $\rho(\cdot) \equiv 0$, thus $V^\rho(\cdot) \equiv 1$. 

¶ We shall be interested in **performance with respect to the market**, so we consider for any given portfolio \( \pi(\cdot) \in \Pi \)

\[
Y^\pi(\cdot) := \frac{V^\pi(\cdot)}{V^Z(\cdot)}, \quad \text{with} \quad \frac{dY^\pi(t)}{Y^\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{dZ_i(t)}{Z_i(t)},
\]

its relative performance. Equivalently, write

\[
\frac{dY^\pi(t)}{Y^\pi(t)} = \sum_{i=1}^n \pi_i(t) \frac{dZ_i(t)}{Z_i(t)} = \sum_{i=1}^n \psi_i(t) dZ_i(t),
\]

with the portfolio proportions expressed as

\[
\pi_i(t) = Z_i(t) \psi_i(t), \quad i = 1, \ldots, n.
\]

The process \( \psi(\cdot) = (\psi_1(\cdot), \ldots, \psi_n(\cdot))' \) in this scheme of things “generates” the portfolio process \( \pi(\cdot) = (\pi_1(\cdot), \ldots, \pi_n(\cdot))' \).
II.4: RELATIVE ARBITRAGE FUNCTION

The smallest amount of relative initial wealth required at \( t = 0 \), in order to attain at time \( t = T \) relative wealth of (at least) 1 with respect to the market, \( \mathbb{P} \)-a.s.:

\[
U(T, z) := \inf \left\{ q \in (0, 1] : \exists \pi(\cdot) \in \Pi \text{ s.t. } \mathbb{P}\left( q \frac{V^\pi(T)}{V^Z(T)} \geq 1 \right) = 1 \right\}.
\]

Equivalently, \( 1/U(T, z) \) gives the maximal relative amount by which the market portfolio can be outperformed over \([0, T]\).

We have: \( 0 < U(T, z) \leq 1 \).

We shall try to characterize this function.
The strict inequality $U(T, z) > 0$ is a consequence of conditions to be imposed below. These amount to NUIP (No Unbounded Increasing Profits): “Absence of Egregious Arbitrages”.

- When $U(T, z) = 1$, it is not possible strongly to outperform (“beat”) the market strongly over $[0, T]$.

- When $U(T, z) < 1$, there exists for every $q \in [U(T, z), 1)$ a portfolio $\pi^q(\cdot) \in \Pi$ such that $q Y \pi^q(T) \geq 1$, i.e.,

$$\frac{V \pi^q(T)}{V Z(T)} \geq \frac{1}{q} > 1,$$

holds $\mathbb{P} – \text{a.s.}$

*Strong arbitrage relative to the market portfolio $Z(\cdot)$ exists then over the time-horizon $[0, T]$.\)

¶ In order to be able to say something about this function $U(\cdot, \cdot)$, we need a “Model”: I.e., some specification of dynamics.
II.5: MARKET WEIGHT “MODEL”

Hybrid MARKOV/ITO-process dynamics for the $\Delta^0$–valued relative market weights $Z(\cdot) = \left(Z_1(\cdot), \cdots, Z_n(\cdot)\right)$, of the form

$$dZ(t) = s(Z(t)) \left(dW(t) + \vartheta(t) \, dt\right), \quad Z(0) = z \in \Delta^0.$$

Here $W(\cdot)$ is an $n$–dimensional $\mathbb{P}$–Brownian motion; the relative drift process $\vartheta(\cdot)$ is $\mathbb{F}$–progressively measurable and satisfies

$$\int_0^T \left\| \vartheta(t) \right\|^2 \, dt < \infty, \quad \mathbb{P} – \text{a.s.}$$

for every $T \in (0, \infty)$.
Whereas \( s(\cdot) = (s_{i\nu}(\cdot))_{1 \leq i, \nu \leq n} \) is a matrix-valued function with \( s_{i\nu} : \Delta \rightarrow \mathbb{R} \) continuous,

\[
\sum_{i=1}^{n} s_{i\nu}(\cdot) \equiv 0, \quad \nu = 1, \cdots, n.
\]

We shall assume that the corresponding covariance matrix

\[
a(z) := s(z)s'(z), \quad z \in \Delta
\]

has rank \( n - 1 \), \( \forall \ z \in \Delta^o \);

as well as rank \( k - 1 \) in the interior \( \Delta^o \) of every

sub-simplex \( \mathcal{D} \subset \Gamma \) in \( k \) dimensions, \( k = 1, \cdots, n - 1 \).

- The quantity \( U(T, z) \) is a number in the interval \( (0,1] \).

So it is the probability of some event. 

\textit{Which event? Under what probability measure?}

We shall try to answer these questions.
II.6: NUMÉRAIRE PORTFOLIO, LOG-OPTIMALITY

Recall the relative portfolio dynamics in the form
\[
\frac{d Y^\pi(t)}{Y^\pi(t)} = \sum_{i=1}^{n} \pi_i(t) \frac{dZ_i(t)}{Z_i(t)} = \sum_{i=1}^{n} \psi_i^{(\pi)}(t) dZ_i(t)
\]
where we are expressing the portfolio proportions as
\[
\pi_i(t) = Z_i(t) \psi_i^{(\pi)}(t), \quad i = 1, \ldots, n.
\]

The market portfolio \( \pi(\cdot) \equiv Z(\cdot) \) is generated by \( \psi^{(\pi)}(\cdot) \equiv 1 \).
Recall

\[ d\mathbb{Z}(t) = s(\mathbb{Z}(t)) \left( dW(t) + \varphi(t) \, dt \right), \quad Z(0) = z \in \Delta^o. \]

- Now, for any two portfolios \( \pi(\cdot), \nu(\cdot) \) with corresponding scaled relative weights \( \psi_{i}^{(\pi)}(\cdot) \) and \( \psi_{i}^{(\nu)}(\cdot) \) as above, simple calculus gives

\[
d \left( \frac{Y^{\pi}(t)}{Y^{\nu}(t)} \right) = \left( \frac{Y^{\pi}(t)}{Y^{\nu}(t)} \right) \left( \psi^{(\pi)}(t) - \psi^{(\nu)}(t) \right)' \left[ d\mathbb{Z}(t) - a(\mathbb{Z}(t)) \psi^{(\nu)}(t) \, dt \right].
\]

Thus, the finite-variation part of this expression vanishes, \textbf{IFF} the portfolio \( \nu(\cdot) \) has scaled relative weights that satisfy the “\textit{perfect balance}” condition

\[
(s(\mathbb{Z}(\cdot)))' \psi^{(\nu)}(\cdot) = \varphi(\cdot).
\]
With $\nu(\cdot) \equiv \nu^P(\cdot)$ selected this way, namely

$$(s(Z(\cdot)))' \psi(\nu)(\cdot) = \vartheta(\cdot) :$$

. For any given portfolio $\pi(\cdot) \in \Pi$, the ratio

$$Y^\pi(\cdot)/Y^\nu^P(\cdot) = V^\pi(\cdot)/V^\nu^P(\cdot)$$

is a positive local martingale – thus also a supermartingale.

- We say that this portfolio $\nu^P(\cdot)$ has the “numéraire property”, and that the ratio $1/Y^\nu^P(\cdot) \equiv VZ(\cdot)/V^\nu^P(\cdot)$ is a “deflator” in this market.

No arbitrage relative to a portfolio with the numéraire property is possible, over ANY finite time-horizon.
. And if \( \psi(\cdot) \equiv 0 \), i.e.,

\[
d\mathcal{Z}(t) = \mathcal{S}(\mathcal{Z}(t)) \, dW(t),
\]

then the **market portfolio** \( \mathcal{Z}(\cdot) \) **ITSELF** has the numéraire property.

Because then we can take \( \psi^{(\nu)}(\cdot) \equiv 1 \), thus \( \nu(\cdot) \equiv \mathcal{Z}(\cdot) \).

\[ \blacksquare \] **Indeed:** “You cannot beat the market” portfolio, when it has the numéraire property.

**But this property is (very) special.**
**Relative Log-Optimality** of the numéraire portfolio $\nu^\mathbb{P} (\cdot)$:

For every portfolio $\pi (\cdot) \in \Pi$ and time-horizon $T \in (0, \infty )$, we have

$$\mathbb{E}^\mathbb{P} \left[ \log Y^\pi (T) \right] \leq \mathbb{E}^\mathbb{P} \left[ \log Y^{\nu^\mathbb{P}} (T) \right] = \frac{1}{2} \mathbb{E}^\mathbb{P} \int_0^T \| \vartheta (t) \|^2 dt .$$

Recall:

$$Y^\pi (\cdot) := \frac{V^\pi (\cdot)}{V Z (\cdot)} , \quad Y^{\nu^\mathbb{P}} (\cdot) := \frac{V^{\nu^\mathbb{P}} (\cdot)}{V Z (\cdot)}$$

keep track of the relative performance of $\pi (\cdot)$ (resp., $\nu^\mathbb{P} (\cdot)$) with respect to the market.
The “deflator” process

\[
\frac{1}{Y^{\nu^P}(\cdot)} \equiv \frac{1}{L(\cdot)} := \exp \left\{ - \int_0^\cdot \vartheta'(t) \, dW(t) - \frac{1}{2} \int_0^\cdot \|\vartheta(t)\|^2 \, dt \right\},
\]

i.e., the performance \( V^{\mathcal{Z}}(\cdot) / V^{\nu^P}(\cdot) \) of the market relative to the numéraire portfolio \( \nu^P(\cdot) \), is a strictly positive \( \mathbb{P} \)-local martingale and a supermartingale.

We need not assume – and are not assuming – \( a \ priori \), that this local martingale is a true martingale.

But we \( ARE \) assuming that it is strictly positive. This is guaranteed by the assumption that, for every \( T \in (0, \infty) \),

\[
\int_0^T \|\vartheta(t)\|^2 \, dt < \infty \quad \text{holds} \quad \mathbb{P} - \text{a.s.}
\]
Thanks to this assumption there is in this model, as we shall see, *No Unbounded Increasing Profit*.

“*No Arbitrage of the First Kind*”,
“*No Egregious Arbitrage*”,
“*No Scalable Arbitrage*”. 
II.7: \( U(\cdot, \cdot) \) AND THE FÖLLMER “EXIT MEASURE”

Under “canonical” conditions on the filtered space \((\Omega, \mathcal{F}), \mathcal{F} = \{\mathcal{F}(t)\}_{0 \leq t < \infty}\), there exists a probability measure \( Q \), under which

\[
W^O(\cdot) := W(\cdot) + \int_0^\cdot \vartheta(t) \, dt
\]

is Brownian motion (the so-called FÖLLMER exit measure; I learned all I know about this from some beautiful notes of my student Gordan ZITKOVIC dated Thu. September 27, 2001.)

And the performance of the numéraire portfolio \( \nu^\mathbb{P}(\cdot) \) relative to the market, i.e., the reciprocal

\[
\frac{V^{\nu^\mathbb{P}}(\cdot)}{V^\mathbb{Z}(\cdot)} = Y^{\nu^\mathbb{P}}(\cdot) \equiv L(\cdot) = \exp \left\{ \int_0^\cdot \vartheta'(t) \, dW^O(t) - \frac{1}{2} \int_0^\cdot \|\vartheta(t)\|^2 \, dt \right\}
\]
of our deflator process, is a $\mathbb{Q}$–martingale; indeed,

$$\mathbb{P}(A) = \int_A L(T) \, d\mathbb{Q}, \quad A \in \mathcal{F}(T); \quad \forall \ T \in (0, \infty).$$

• Whereas the market-weight process $\mathcal{Z}(\cdot)$ is a $\mathbb{Q}$–martingale and Markov process, with values in $\Delta$ and “purely diffusive” $\mathbb{Q}$–dynamics

$$d\mathcal{Z}(t) = s(\mathcal{Z}(t)) \, dW^o(t), \quad \mathcal{Z}(0) = z \in \Delta^o.$$

Thus, the market portfolio $\mathcal{Z}(\cdot)$ has the numéraire property under the exit measure $\mathbb{Q}$:

$$\mathcal{Z}(\cdot) \equiv \nu_{\mathbb{Q}}(\cdot).$$
• If we consider the first time ("explosion", or rather implosion) 

\[ S := \inf \{ t \geq 0 : \mathcal{Z}(t) \in \Gamma \} \]

\( \mathcal{Z}(\cdot) \) reaches the boundary \( \Gamma \) of the unit simplex \( \Delta \), the arbitrage function is represented in the already familiar form 

\[
U(T, z) = \mathbb{E}_{P^z} \left[ \frac{1}{L(T)} \right] = Q_z(S > T), \quad (T, z) \in (0, \infty) \times \Delta^o.
\]

The relative arbitrage function \( U(T, z) \) emerges as the probability under the FÖLLMER measure, that \( \mathcal{Z}(\cdot) \) has not reached the boundary \( \Gamma \) of the simplex by time \( t = T \), when started at initial configuration \( z \). Tail-distribution of the "explosion" time.
Please think of the passage from the original measure $\mathbb{P}$ to the FÖLLMER measure $\mathbb{Q}$, as a Girsanov-like change of probability that “removes the drift” in the dynamics

$$d\mathcal{Z}(t) = s(\mathcal{Z}(t)) \left( dW(t) + \vartheta(t)\, dt \right),$$

when all we can say about the exponential (“deflator”) process

$$\frac{1}{L(\cdot)} = \exp \left\{ - \int_0^\cdot \vartheta'(t)\, dW(t) - \frac{1}{2} \int_0^\cdot \|\vartheta(t)\|^2 \, dt \right\} \equiv \frac{1}{Y^{\nu_P}(\cdot)}$$

is that it is a local martingale under $\mathbb{P}$ (\textbf{strict}, when $U(T, z) < 1$).
The process \( L(\cdot) \) *can in principle* reach the origin with positive \( Q \)–probability, so this is in general *not* an equivalent change of measure:

**We have** \( P \ll Q \), **but not necessarily** \( Q \ll P \).

Nonetheless, the process \( Z(\cdot) \) of market weights is a *\( Q \)–martingale* with values in the unit simplex – and now with the possibility of reaching its faces.

(Thus, we can think of the FÖLLMER measure \( Q \) as an Ersatz “martingale measure” for the model under consideration.)
II.8: \( U(\cdot, \cdot) \) AS SMALLEST SUPERSOLUTION

Under regularity conditions on the covariance structure \( a(\cdot) \) and on the relative drift \( \vartheta(\cdot) \), the arbitrage function \( U(\cdot, \cdot) \) is of class \( C^{1,2} \) on \((0, \infty) \times \Delta^o\), and satisfies there the equation

\[
 D_\tau U(\tau, z) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(z) D_{ij}^2 U(\tau, z),
\]

or

\[
 D_\tau U = \frac{1}{2} \text{Tr}(a D^2 U).
\]

Further, \( U(\cdot, \cdot) \) is also the smallest nonnegative supersolution of this equation, subject to

\[
 U(0+, \cdot) \equiv 1.
\]
• Please note that this equation

\[ D_\tau U = \frac{1}{2} \text{Tr}(a D^2 U) \]

involves only the covariance structure of the assets.

• The only rôle the relative drift \( \vartheta(\cdot) \) plays in this context, is to keep the market weight process \( \mathcal{Z}(\cdot) \) in the interior of the unit simplex, \( \mathbb{P}\)–a.e. (Once again, this characterization is completely impervious to boundary conditions on the faces of the simplex.)

• With Knightian uncertainty about the covariance \( a(\cdot) \) and the relative drift \( \vartheta(\cdot) \), this equation becomes fully nonlinear (of HJB-Pucci type), as in the work of Terry Lyons (1995).

• Great generalizations of these results, in the context of viscosity solutions of the fully nonlinear PDE’s, appear in very recent work by Ms. Yinghui Wang (2015).
II.9: CONDITIONING, CLASS \( \mathcal{P} \)

Let us consider the collection \( \mathcal{P} \) of probability measures \( P \ll Q \) with \( P(\mathcal{Z}(t) \in \Delta^o, \forall 0 \leq t \leq T) = 1 \). (Our original measure \( P \) belongs to this collection.) We single out an element of \( \mathcal{P} \) via

\[
P_*(A) := Q( A \mid S > T ), \quad A \in \mathcal{F}(T).
\]

(1)

This is the conditioning of the FÖLLMERER measure \( Q \) on the set \( \{ \mathcal{Z}(\cdot) \text{ has not reached the boundary of the simplex by time } T \} \).

Elementary computations give, \( Q \)-a.s.:

\[
\frac{d P_*}{d Q} \bigg|_{\mathcal{F}(t)} = \frac{U(T - t, \mathcal{Z}(t))}{U(T, z)} 1_{\{S > t\}} =: \frac{\hat{Y}(t)}{\hat{Y}(0)}, \quad 0 \leq t \leq T
\]
\[ \frac{dP_*}{d\mathcal{Q}} \bigg|_{\mathcal{F}(t)} = \frac{U(T-t, Z(t))}{U(T, z)} 1_{\{S > t\}} =: \frac{\hat{Y}(t)}{\hat{Y}(0)}, \quad 0 \leq t \leq T \]

with the \( \mathcal{Q} \)-martingale

\[ \hat{Y}(t) := U(T-t, Z(t)) 1_{\{S > t\}} \equiv q Y^{\hat{\pi}}(t) \quad \text{for} \quad q = U(T, z), \]

and with the \textit{functionally-generated} portfolio in \( \Pi_{str} \):

\[ \hat{\pi}_i(t) = Z_i(t) \cdot D_i \log U(T-t, Z(t)). \quad (2) \]

- This portfolio has the numéraire property under the conditioning \( P_* \) of the FÖLLMER measure:

\[ \hat{\pi}(\cdot) \equiv \nu^{P_*}(\cdot). \]
Whenever \( U(T, z) < 1 \), this portfolio implements the best achievable arbitrage under the original probability measure \( \mathbb{P} \); that is,

\[
\frac{V^{\hat{\pi}}(T)}{V^Z(T)} = \frac{1}{U(T, z)} > 1 \quad \text{holds } \mathbb{P} \text{ – a.s.}
\]

**II.10: A RECIPE**

We can characterize the portfolio \( \hat{\pi}(\cdot) \) of (2) that implements the optimal arbitrage over a given time-horizon \([0, T]\) as follows, given the market weight covariance structure under the original probability measure \( \mathbb{P} \) (and nothing else...):
• **FIRST**, find a probability measure $\mathbb{Q}$ under which the market weights are martingales, as in

$$dZ(t) = s(Z(t)) \, dW^0(t), \quad Z(0) = z \in \Delta^o,$$

and compute the function $U(T, z) = \mathbb{Q}_z(S > T)$.

• **SECONDLY**, construct the measure $\mathbb{P}_\star$ by conditioning $\mathbb{Q}$ on the event $\{S > T\}$ as in $\mathbb{P}_\star(A) := \mathbb{Q}(A \mid S > T)$, $A \in \mathcal{F}(T)$.

• **FINALLY**, construct the portfolio $\hat{\pi}(\cdot)$ that maximizes expected log-return (equiv., has the numéraire property) under $\mathbb{P}_\star$.

This portfolio is generated by the vector process of log-derivatives, i.e., is given by the recipe

$$\hat{\pi}_i(t) = Z_i(t) \cdot D_i \log U(T - t, Z(t)), \quad i = 1, \ldots, n.$$
II.12: MINIMAL ENERGY AND ENTROPY

With

\[ H_T(P | Q) := \mathbb{E}^P \left[ \log \left( \frac{dP}{dQ} \right|_{\mathcal{F}(T)} \right) \right] = \frac{1}{2} \mathbb{E}^P \int_0^T \| \vartheta^P(t) \|^2 \, dt \]

we have the “minimum entropy and energy” properties

\[
\log \left( \frac{1}{U(T, z)} \right) = H_T(P^* | Q) = \min_{P \in \mathcal{P}} H_T(P | Q)
\]

\[ = \frac{1}{2} \mathbb{E}^{P^*} \int_0^T \| \vartheta^{P^*}(t) \|^2 \, dt = \min_{P \in \mathcal{P}} \frac{1}{2} \mathbb{E}^P \int_0^T \| \vartheta^P(t) \|^2 \, dt. \]

We call \( P^* \) “minimal energy” measure in \( \mathcal{P} \).

Has relative risk process \( \vartheta^{P^*}(\cdot) \) that keeps the market weights strictly positive throughout \([0, T]\) by expending minimal energy.
This minimal entropy function

\[ \mathcal{H}(\tau, z) := \log \left( \frac{1}{U(T, z)} \right) = H_T(\mathbb{P}_\tau | \mathbb{Q}) \]

solves the HJB equation for this problem

\[ D_\tau \mathcal{H}(\tau, z) = \frac{1}{2} \text{Tr}(a(z) D^2 \mathcal{H}(\tau, z)) \]

\[ + \min_{\theta \in \mathbb{R}^n} \left[ (D\mathcal{H}(\tau, z))' s(z) \theta + \frac{1}{2} \|\theta\|^2 \right], \]

which is of course a semilinear equation

\[ D_\tau \mathcal{H}(\tau, z) = \frac{1}{2} \text{Tr}(a(z) D^2 \mathcal{H}(\tau, z)) - \frac{1}{2} (D\mathcal{H}(\tau, z))' s(z) (D\mathcal{H}(\tau, z)). \]
II.13: A STOCHASTIC GAME

The pair \((\mathcal{P}_*, \hat{\pi}(\cdot))\) of (1), (2) is a saddle point in \(\mathcal{P} \times \Pi\) for the zero-sum stochastic game with value

\[
\log \left( \frac{1}{U(T, z)} \right) = \mathbb{E}^{\mathbb{P}^*} \left[ \log Y^{\hat{\pi}}(T) \right] =
\]

\[
= \min_{\mathbb{P} \in \mathcal{P}} \max_{\pi(\cdot) \in \Pi} \mathbb{E}^{\mathbb{P}} \left[ \log Y^{\pi}(T) \right] = \max_{\pi(\cdot) \in \Pi} \min_{\mathbb{P} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}} \left[ \log Y^{\pi}(T) \right];
\]

and for every \((\mathbb{P}, \pi(\cdot)) \in \mathcal{P} \times \Pi\) we have the saddle

\[
\mathbb{E}^{\mathbb{P}} \left[ \log Y^{\pi}(T) \right] \geq \mathbb{E}^{\mathbb{P}^*} \left[ \log Y^{\hat{\pi}}(T) \right] =
\]

\[
= \log \left( \frac{1}{U(T, z)} \right) \geq \mathbb{E}^{\mathbb{P}^*} \left[ \log Y^{\pi}(T) \right].
\]
II.14: A SUFFICIENT CONDITION AND A TOY MODEL

It can be shown that a sufficient condition for $U(T, z) < 1$ is that there exist a real constant $h > 0$ for which

$$
\sum_{i=1}^{n} z_i \left( \frac{a_{ii}(z)}{z_i^2} \right) \geq h, \quad \forall \ z \in \Delta^o.
$$

(3)

The weighted relative variance of log-returns in (3) is a measure of the market’s “intrinsic” (or “average relative”) variance; condition (3) posits a positive lower bound on this quantity as sufficient for $U(T, z) < 1$.

Under the condition (3), very simple long-only portfolios can be designed, that lead to arbitrage over sufficiently long horizons.
For instance, given any real number $T > (2 \log n)/h$, there is $c > 0$ sufficiently large, so that the portfolio

$$
\pi_i(t) = \frac{Z_i(t)(c - \log Z_i(t))}{\sum_{j=1}^{n} Z_j(t)(c - \log Z_j(t))}, \quad i = 1, \ldots, n
$$

is strong arbitrage relative to the market portfolio $\mathcal{Z}(\cdot)$ over the time-horizon $[0, T]$.

. OPEN QUESTION: Is arbitrage relative to the market possible under condition (3) over arbitrary time-horizons?

(A few additional examples exist, under different structural conditions, and with the equally-weighted portfolio playing a very important rôle. Would be nice to have more of them ... .)

. Very recent development: Counterexample by Johannes RUF.
II.15: A CONCRETE TOY-EXAMPLE

A concrete example where the condition

$$\sum_{i=1}^{n} \frac{a_{ii}(z)}{z_i} \geq h, \quad \forall \ z \in \Delta^o$$

of (3) is satisfied concerns the “Volatility-Stabilized” Model

$$d \log X_i(t) = \left( \kappa/Z_i(t) \right) dt + \left( 1/\sqrt{Z_i(t)} \right) dW_i(t), \quad i = 1, \ldots, n$$

with constant $\kappa \geq 1/2$, or equivalently for the market weights

$$dZ_i(t) = \kappa \left( 1 - n Z_i(t) \right) dt + \sqrt{Z_i(t)} dW_i(t) - Z_i(t) \sum_{k=1}^{n} \sqrt{Z_k(t)} dW_k(t)$$

$$= \kappa \left( 1 - n Z_i(t) \right) dt + \sqrt{Z_i(t)} \sqrt{1 - Z_i(t)} dW_i^\#(t).$$
The variances in this last diffusion equation

\[ dZ_i(t) = \kappa \left(1 - n \, Z_i(t)\right) dt + \sqrt{Z_i(t)} \sqrt{1 - Z_i(t)} \, dW_i^\#(t) \]

(in which the \( W_i^\#(\cdot), \ i = 1, \cdots, n \) are correlated BM’s) are of \textit{WRIGHT-FISHER} type

\[ a_{ii}(z) = z_i(1 - z_i); \]

so the condition

\[ \sum_{i=1}^{n} \frac{a_{ii}(z)}{z_i} \geq h, \quad \forall \ z \in \Delta^o \]

of (3) holds as equality, in fact with \( h = n - 1 \geq 1. \)
Here, and indeed in any setting of the form
\[
d \log X_i(t) = \beta_i(t) \, dt + \left(1/\sqrt{Z_i(t)}\right) \, dW_i(t), \quad i = 1, \ldots, n,
\]
the market CAN be outperformed over arbitrary time horizons (A. BANNER & D. FERNHOLZ (2008), R. PICKOVÁ (2014)).

In this case, one can “compute” the relative arbitrage function
\[
U(T, z) = \mathbb{E}^p \left[ \frac{z_1 \cdots z_n}{Z_1(T) \cdots Z_n(T)} \right] \cdot \mathbb{E}^p \left[ e^{-(n-1)(\gamma T+W(T))} \right],
\]
because S. PAL (2011) has computed the joint distribution of the weights \( Z_1(T), \ldots, Z_n(T) \) fairly explicitly (Dirichlet). Here
\[
\gamma = \kappa n - \frac{1}{2}.
\]
• Under the FÖLLMER measure $\mathbb{Q}$, each weight $Z_i(\cdot)$ is a WRIGHT-FISHER diffusion in natural scale, and reaches an endpoint of $(0,1)$ in finite expected time $S_i = \inf\{t \geq 0 : Z_i(t) = 0\}$:

$$dZ_i(t) = \kappa (1 - n Z_i(t)) dt + \sqrt{Z_i(t)} \sqrt{1 - Z_i(t)} dW^\#_i(t)$$

$$= \sqrt{Z_i(t)} \sqrt{1 - Z_i(t)} dW^o_i(t).$$

For us, of course, the time of interest is

$$S = \min_{1 \leq i \leq n} S_i.$$

Eventually all but one of the $Z_i(\cdot)$’s “perish”, and one of them emerges as the survivor.

Think of a catalytic reaction involving $n$ compounds with nucleation/condensation (very recent work of C.LANDIM et al., May 2015); or of a gladiatorial fight in the Colosseum.


FURTHER BIBLIOGRAPHY


THANK YOU FOR YOUR ATTENTION

HAPPY BIRTHDAY, STEVE !!!!

ΠΟΛΥΧΡΟΝΙΟΣ !!!!