Continuous Time Finance and Mean-Variance in Post Retirement Planning

Presented at Conference in Honor of Steven’s 65th Birthday

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“Society needs scientists willing to explain the content and consequence of their work to the public. These are modern Renaissance men and women...who understand science beyond a superficial level.”

~ Steven
About GuidedChoice

- GuidedChoice mission: Financial freedom for all
- Provide confidence and peace of mind from real financial facts and reliable projections

GuidedSavings℠  GuidedChoiceIRA℠  GuidedSpending℠
Current 401(k) Market

- 5 trillion dollars
- 70 million participants

A. Half of participants outsource their asset allocation to target date funds
B. Women tend to take less risk
C. Those with supplemental pension plans invest more conservatively than those without
D. Young participants are not taking enough risk
E. Naïve diversification
Total Risk Distribution

![Graph showing total risk distribution over participant age categories.](GUIDEDCHOICE.COM)
Mean-Variance Distribution: 1989-2013 Returns
Performance

10-year Risk/Return Analysis ending 2013
GuidedChoice Asset Class Portfolios (net) vs. Target Date Fund Averages

GuidedChoice Allocation Frontier
TDF Average Frontier

GC performance reflects GuidedChoice asset allocations to representative index returns, rebalanced monthly, and are net of an additional 60bp, accrued quarterly.

Sources: GuidedChoice, Morningstar
Spending Problem: Practitioner’s Solution

- 4% Rule: Withdraw constant dollars equal to 4% initial wealth
- \( dX_t = \mu \pi_t X_t dt - C \times dt + \sigma \pi_t X_t dw_t \)
- Probability \((X_{30} \leq 0) \leq 0.06\)
- Historically with \(\pi = 0.5\)
- An acceptable and good solution
Other Market Condition

**Probability of Failure vs. Equity Allocation**

4% Rule, $T=30$, $\mu=6\%$

- **Y-axis**: Probability of Failure
- **X-axis**: Equity Allocation

The graph illustrates the relationship between the probability of failure and equity allocation under the specified conditions.
Spending Problem: Mathematician’s Solution

- \( dp_t = p_t (\mu dt + \sigma d\omega_t) \)
- \( E \int_0^\tau e^{-\beta t} U(c_t) dt \)
- \( U(c) = \frac{c^{1-\gamma}}{1-\gamma} \)
- \( c^*_s = \frac{1}{A(s)} X_s, A(s) \overset{\text{def}}{=} \int_s^T e^{-k(\gamma)(T-t)} dt \)
- \( k(\gamma) = \frac{\beta}{\gamma} + r \left(1 - \frac{1}{\gamma}\right) + 0.5 \times \theta^2 \times \frac{1}{\gamma} \times \left(1 - \frac{1}{\gamma}\right) \)
- \( \pi^*_s = \frac{1}{\gamma} \frac{\mu - r}{\sigma^2} = \frac{1}{\gamma} \frac{\bar{\mu}}{\sigma^2} = \frac{\theta}{\gamma \sigma} \) (Merton Line)
Practice Questions

What is my risk aversion parameter?
What is my utility discount rate?

Typical question: If offered the choices listed below, which option would you select?

- Accept $1M now
- Play a game where you have a 90% chance of winning $2M and a 10% chance of winning nothing
- Play a game where you have a 50% chance of winning $5M and a 50% chance of winning nothing
Experiment’s Solution

- For any annuitization rate/portfolio policy \((g, T, \pi)\),

\[
AF(t) \overset{\text{def}}{=} \frac{1}{\int_t^T e^{-g(T-s)} ds} = \frac{g}{1-e^{-g(T-t)}}
\]

- \(c_t = AF(t) \times X_t\)

- \(c_t = x \times AF(0) \times e^{(\pi \mu - g)t - 0.5(\pi \sigma)^2 t + \pi \sigma W_t}\)
Mean-Variance Analysis

Average Spending-Variability Tradeoff Curve
Life Expectancy 30 Yrs, μ=8%,6%

- Standard Deviation vs. Average Spending per Dollar
- 1% g, 8% μ and 1% g, 6% μ curves
- Points for different allocation: 25/75, 50/50, 75/25, 100% equity, 100% bond
Probability of Spending Less

\[
\begin{align*}
\bullet & \quad P(c_t < \kappa \times x \times AF(0)) \\
\bullet & \quad = P\left( e^{(\pi \mu - g) t - 0.5(\pi \sigma)^2 t + \pi \sigma W_t} < \kappa \right) \\
\bullet & \quad = 1 - \Phi \left( \frac{-\ln(\kappa) + (\pi \mu - g) t - 0.5(\pi \sigma)^2 t}{\pi \sigma \sqrt{t}} \right) \\
\bullet & \quad = \max_{t \geq 0} P(c_t < \kappa \times x \times AF(0))
\end{align*}
\]
Probability of Spending Less

Risk of Ever Having to Spend Less than 75% of Initial Spending

Equity Weight

Probability

25% 50% 75% 100%

μ-6%, g-1% μ-6%, g-2% μ-6%, g-1% μ-6%, g-2%
More **Interesting Problem**

- Budget constraints: \( \underline{c} \leq c_s \leq \overline{c}, 0 \leq s \leq T \)
- No bankruptcy implies:
  
  Initial Wealth \( x \geq \text{Present Value}(\underline{c}) \)

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- No analytical solution for \( U(c) = \frac{c^{1-\gamma}}{1-\gamma} \)
Portfolio Insurance and Profit-Taking

- \( U(c) = -c \ast (\bar{c} - c)^{-\gamma} + \frac{(c - \bar{c})^{1-\gamma}}{1-\gamma} \)
- \( I(y) = \bar{c} + (y + (\Delta c)^{-\gamma}) \frac{1}{\gamma}, \forall y \geq 0 \)
- \( c_s = \bar{c} + (\Delta c)(\lambda(t, x) \ast \Delta c)\gamma \ast \zeta(t, s) + 1) \frac{1}{\gamma} \)
- \( E \int_t^T \zeta(t, \tau) (\lambda(t, x)\zeta(t, \tau) + (\Delta c)^{-\gamma})^{-\frac{1}{\gamma}} d\tau = x - \bar{c} \ast (T - t) \)
Selection of **Risk Aversion Parameter**

![Diagram showing Average Spending and Equity Weight](image-url)
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THANK YOU & Happy Birthday Steven!