Over-the-Counter market models

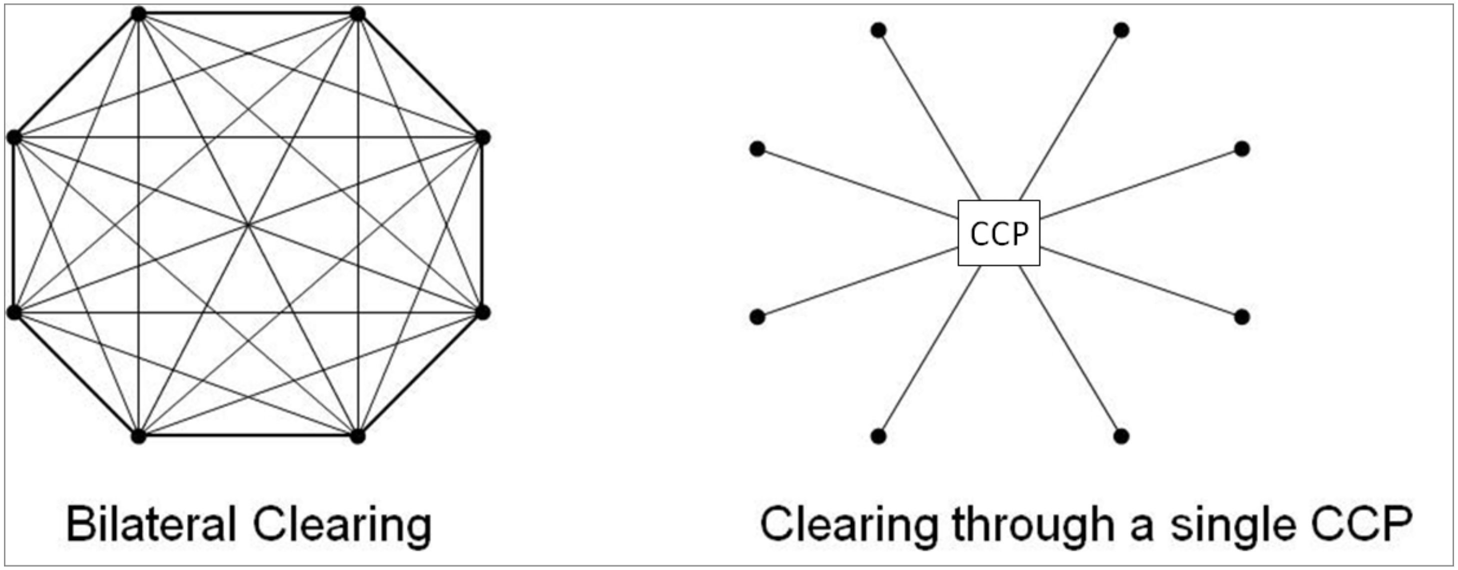
Alain Bélanger

Joint work with Gaston Giroux & Ndouné Ndouné

Université de Sherbrooke

Conference in honor of Steve Shreve, June 1-5, 2015

·What is an Over-the-Counter ('OTC') market?



Assets traded: Corporate bonds, derivatives, Asset-Backed Securities, etc.

Size: At the end of June 2014, BIS estimate for OTC derivatives was USD 691 trillion notional outstanding for a market value of USD 17 trillion. To this day, 27% of Credit-Default Swaps are traded via a Central-Clearing-Party.

Modelling efforts:

Pioneering work: D. Duffie, N. Gârleanu and L.H. Pedersen, (2005) *Over-the-Counter Markets*, Econometrica 73, 1815-1847.

Recent monograph: D. Duffie, *Dark Markets*: *Asset Pricing and Information Transmission*, Princeton Lecture Series, 2012.

·What are we trying to capture in the modelling effort of these "search and bargaining" markets?

Equilibrium; Price formation mechanisms; Liquidity, efficiency; Resilience to shocks/stability.

The focus of this talk is more foundational.

We will present our approach, inspired by the work of Mark Kac (in statistical physics), to derive the evolution equation of a model using asymptotic independence in a large interacting set.

The articles of D. Duffie and his collaborators use the notion of independent random matching of D. Duffie and Y. Sun (2007) Annals of Applied Probability,17, 386-389 and (2012) Journal of economic Theory, 147 no.3, 1105-1139.

Roadmap of the presentation

1. The market as a large interacting set of agents

2. The solution of the evolution equation

3. Extension to the case where there are two types of interactions

4. Further work (inasmuch as time permits).

1 The market as a large interacting set of agents

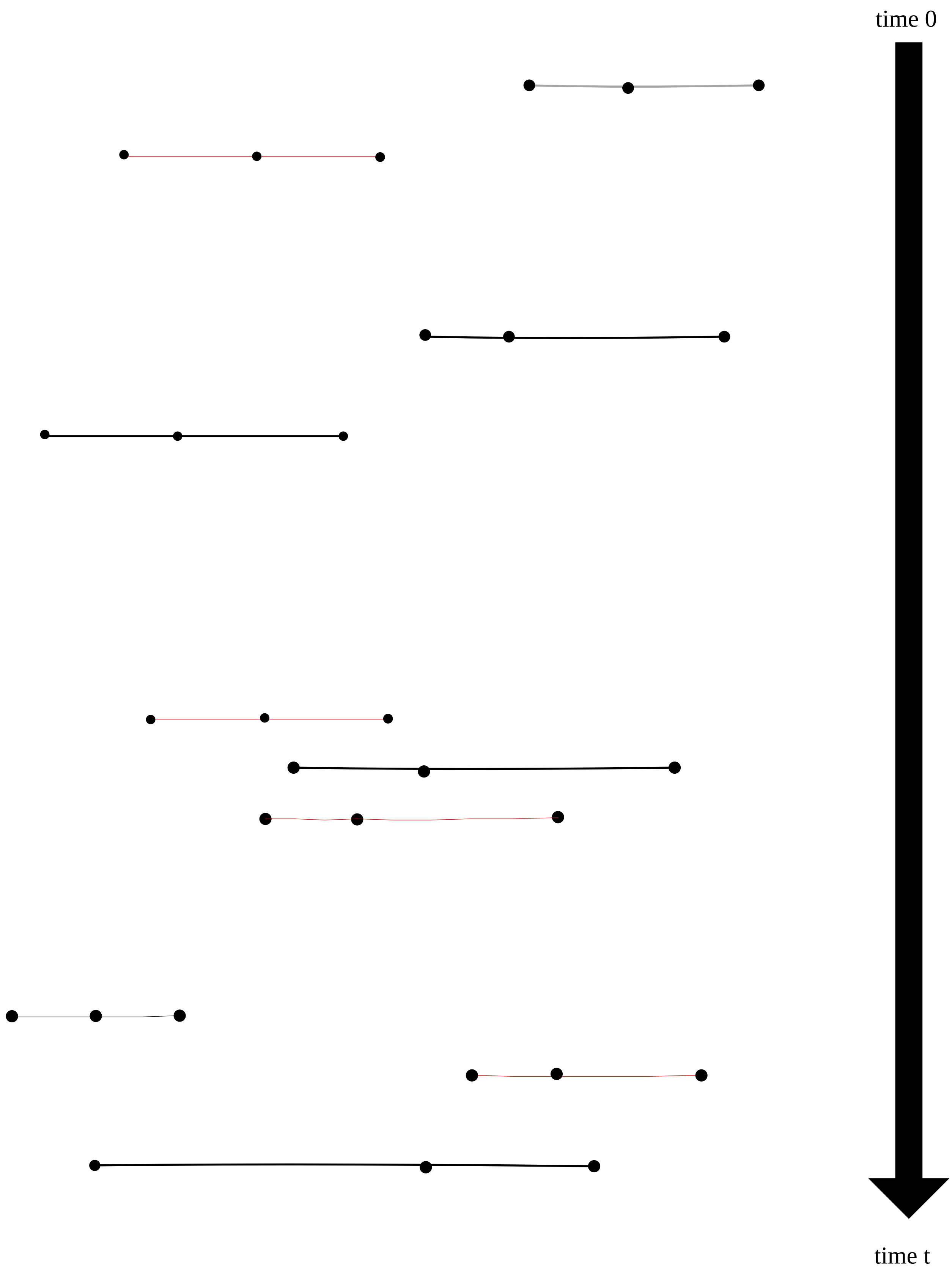
We have N investors in our market (with N fixed but large).

(We will eventually think about $N \to \infty$.)

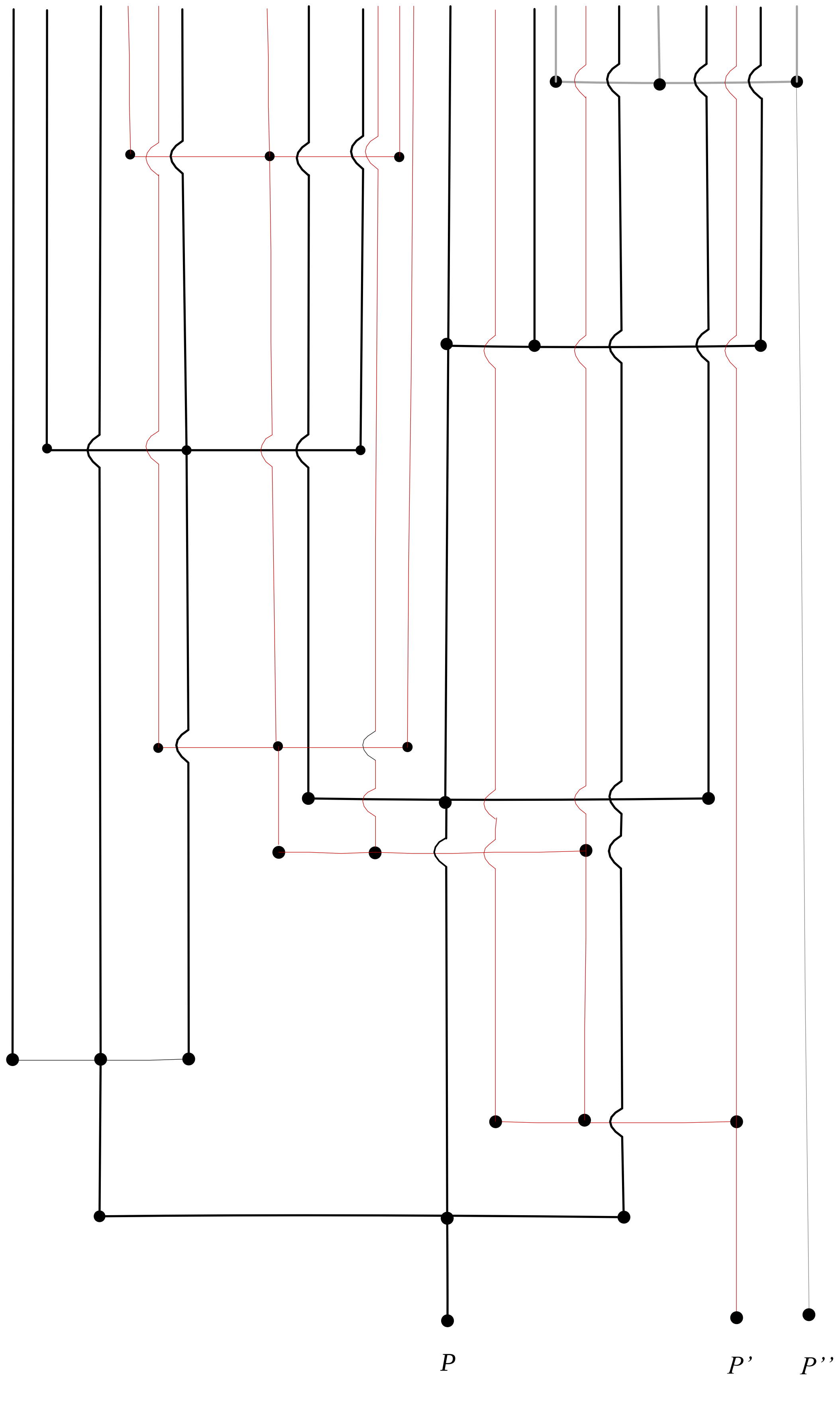
These investors/agents meet/interact at unexpected times in groups of m, fixed, $m \geq 2$.

We suppose the intensity of the meetings to be $\lambda \frac{N}{m}$ then each agent has a meeting rate equal to $\lambda \left(= \lambda \frac{N}{m} \left(1 - \binom{N-1}{m} \binom{N}{m}^{-1} \right) \right)$.

A word about the parameter $\lambda > 0$, and the case $\lambda = 1$.



Meetings in an OTC market (with three investors per meeting)



P
P’
P’’

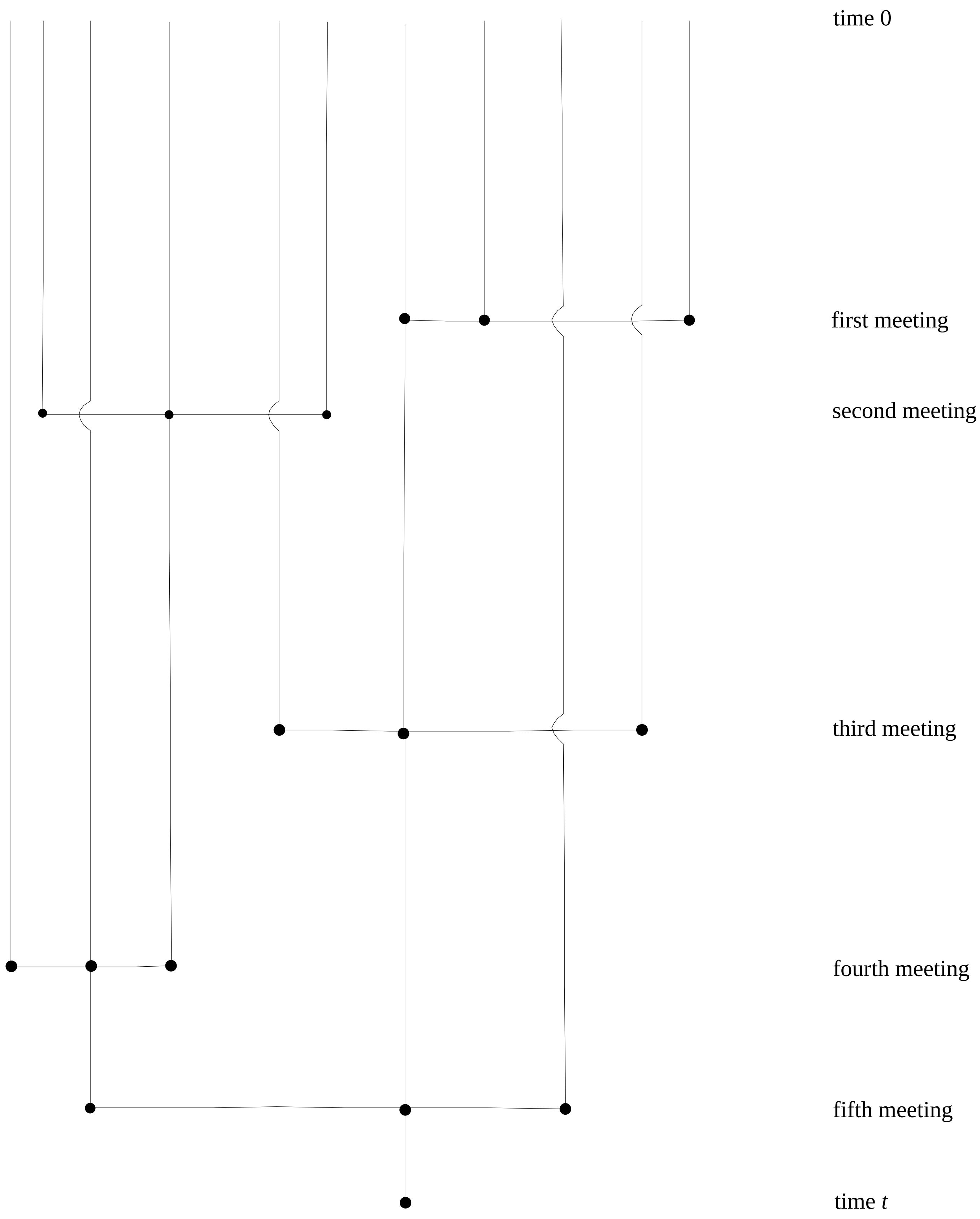


Figure 1: Sample random tree representing the history of P when there are five meetings up to time t (with 3 investors per meeting)

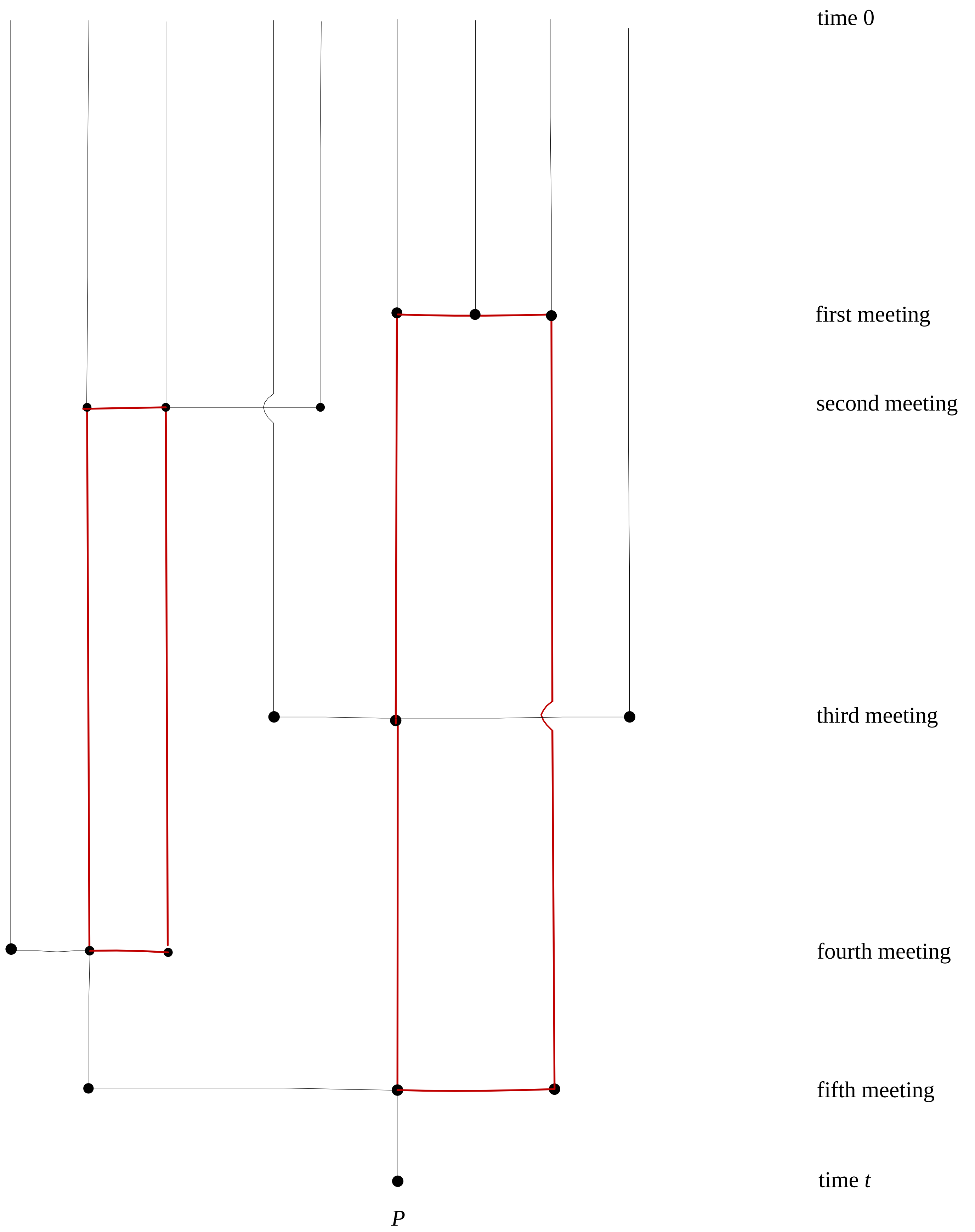


Figure 1: Sample random graph with cycles.

Removing cycles up to time 0 we get a tree with n branchings, say, which has the same law as the law of a tree obtained by a pure-birth process. Namely, the tree starting at P's vertical line at time t with intensity 1 and which at time 0 has intensity $(m-1)n+1$ and that same number of leaves.

Between two branchings of this process, a graph representing P's meeting history can have a random number of additional horizontal lines following a Poisson law of parameter at most $\frac{N}{m}\left(\binom{(m-1)n+1}{2}\binom{N}{m}^{-1}\right)$.

Thus we have that the mean number of redundant horizontal lines is bounded above by $\sum_{n\geq 0}\frac{N}{m}\left(\binom{(m-1)n+1}{2}\binom{N}{m}^{-1}\right)p_{N,n}(t)$, where $p_{N,n}(t)$ is the probability of having n branchings up to time t of the pure birth process with successive branching waiting times following exponential laws of parameter

$$
\begin{aligned}
\lambda_{N,n} &= \frac{N}{m}((m-1)n+1)\begin{pmatrix} N-((m-1)n+1) \\ m-1 \end{pmatrix}\begin{pmatrix} N \\ m \end{pmatrix}^{-1} \\
&= ((m-1)n+1)\frac{\binom{N-((m-1)n+1)}{m-1}}{\binom{N-1}{m-1}} \qquad (*) \\
&\leq (m-1)n+1 \triangleq \lambda_n \text{ (call } p_n, \text{ the associated probabilities)}
\end{aligned}
$$

Then $p_{N,n}(t)$ is stochastically smaller than the law obtained with the intensities λ_n, which in turn are less than the intensities $\overline{\lambda}_n = m(n+1)$. Its transition kernel is then obtained by solving Kolmogorov's affine system of equations:

$$\frac{d\overline{p}_t(0)}{dt} = -m\overline{p}_t(0)$$
$$\frac{d\overline{p}_t(n)}{dt} = mn\overline{p}_t(n-1) - m(n+1)\overline{p}_t(n) \quad ; n \geq 1.$$

Thus the latter intensities give us a geometric law $\overline{p}_t(n) = e^{-mt}(1-e^{-mt})^n$. Since geometric laws have finite moments of all orders, the mean number of redundant horizontal lines is bounded above by a quantity converging to 0.

Thus, after having specified the initial agents' states and their interaction kernels, we can approximate $P's$ law, denoted by $\mu_t^{*,N}$, using the tree obtained from removing all redundant horizontal lines from its graph.

The interactions

We suppose that all components take their values in a measurable space, $(E, \mathcal{E})$, (one can think of $(\mathbb{R}, B(\mathbb{R}))$ or of a **discrete set of states**) and their interactions are given by a symmetric probability kernel Q on the product space $(E^m, \mathcal{E}^{\otimes m})$ for $m \geq 2$.

That is: the function $Q(x_1, x_2, ..., x_m; C_1 \times \cdots \times C_m)$

1) is measurable in $(x_1, x_2, ..., x_m)$;

2) is a probability measure in $(C_1 \times \cdots \times C_m)$; and

3) satisfies $Q(x_1, x_2, ..., x_m; C_1 \times \cdots \times C_m) = Q(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(m)}; C_{\sigma(1)} \times C_{\sigma(2)} \times \cdots \times C_{\sigma(m)})$ for any permutation σ of $\{1, 2, ..., m\}$.

Example 1: Duffie-Gârleanu-Pedersen (partial model)

$E = \{(l,n),(l,o),(h,n),(h,o)\}$ describes the state space.

The kernel implementing the trading of the asset is defined by

$$Q_2((l,o),(h,n);C_1 \times C_2) = \delta_{(l,n)}(C_1)\delta_{(h,o)}(C_2) = Q_2((h,n),(l,o);C_2 \times C_1)$$

and $Q_2(\cdot,\cdot;C_1 \times C_2\}) = 0$ otherwise.

Example 2: Duffie-Malamud-Manso (2009, Econometrica, 77, 1513-1574)

Information percolation in meetings involving m agents (partial model).

The state space is $E = \mathbb{N}$ and the kernel implementing perfect information transmission is

$$\begin{aligned}&Q_p(n_1, n_2, ..., n_m; l_1, l_2, ..., l_m)\\ = \ &\delta_{n_1+n_2+...+n_m}(l_1)\delta_{n_1+n_2+...+n_m}(l_2)\cdots\delta_{n_1+n_2+...+n_m}(l_m)\end{aligned}$$

For each integer N, we consider an interacting set of N components which interact by groups of m according to the kernel Q.

$$\mu^{\circ m}(C) \triangleq \int_{E^m} \mu(dx_1)\mu(dx_2)...\mu(dx_m)Q(x_1, x_2, ..., x_m; C \times E^{m-1}) \text{ for } C \in \mathcal{E}.$$

The probability law $\mu^{\circ m}$ is the law of a component after the interaction of m i.i.d. components with law μ. We can think of it as the law at the root of the m-ary tree with only one interaction. We will look at all the trees representing the interaction history of a component up to time t.

So for a tree, A, with more than one interaction, we divide the tree in m subtrees at that last interaction and continue recursively up to time 0 to define $\mu^{\circ_m A}$. Let $\mathbb{A}_n$ be the set of all trees with n interactions (a.k.a. nodes), each node producing m branches.

If $A_n \in \mathbb{A}_n$, then $\mu^{\circ_m A_n}$ denotes the law obtained by iteration of μ°_m} through the successive nodes of the tree when we place random variables of law μ on each leaf of A_n.

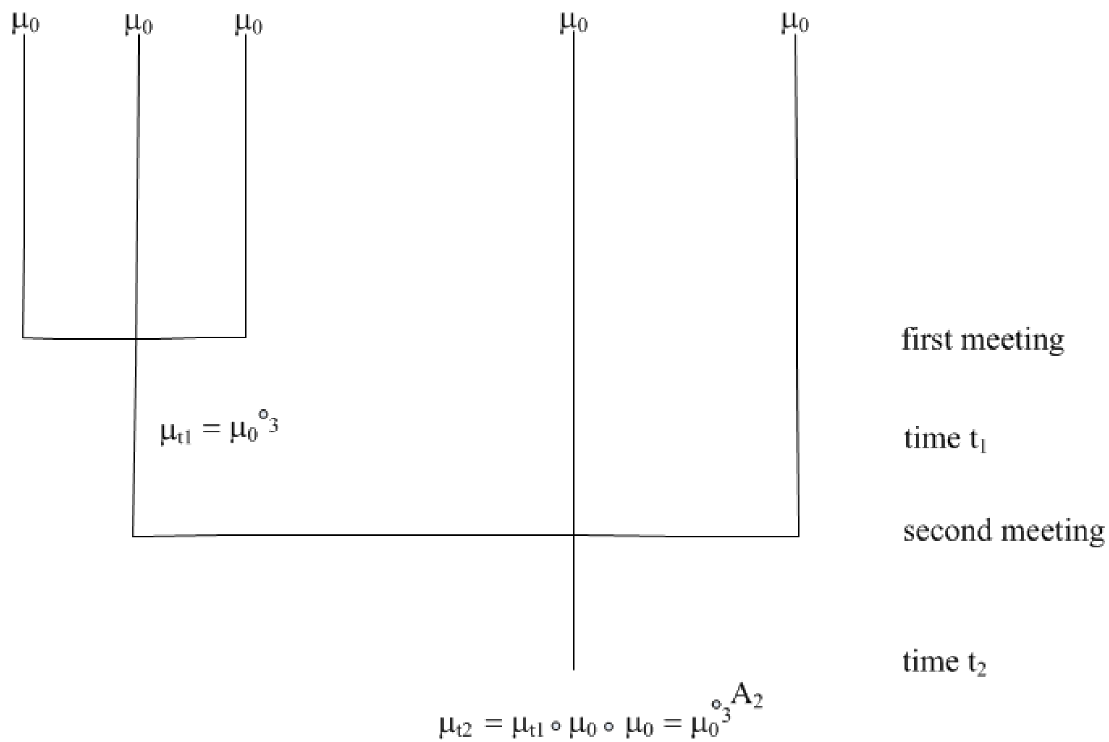


Figure 1: Simple interaction tree with only two meetings and $m = 3$.

Conditioning on the number of interactions up to time t, and then by the component's history we have

$$\mu_t^{*,N} = \sum_{n \geq 0} p_{n,N}(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

And we let

$$\mu_t \triangleq \sum_{n \geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n} \qquad (1)$$

where $\#_m(n) = \prod_{k=1}^{n-1} ((m-1)k + 1)$ denotes the number of trees with n nodes, taking into account their branching orders.

Proposition 1 *The sequence of laws $\mu_t^{*,N}$ converges to μ_t as N increases.*

Proof. By Kurtz (1969) we have that $p_{N,n}(t) \to p_n(t)$ as N increases. But $(p_n(t))_{n\geq 0}$ is a probability law, so for $\epsilon > 0$, there exists $n(\epsilon)$ such that

$\sum_{n \geq n(\epsilon)} p_n(t) < \epsilon$. Now let $N(\epsilon)$ be such that $N > N(\epsilon)$ implies that $|p_{N,n}(t) - p_n(t)| < \frac{\epsilon}{n(\epsilon)}$ for $0 \leq n \leq n(\epsilon)$.We then have for $C \in \mathcal{E}$ and $N > N(\epsilon)$

$$|\mu_t^{*,N}(C) - \mu_t(C)| \leq \sum_{n=0}^{n(\epsilon)} |p_{N,n}(t) - p_n(t)| + 2\epsilon \leq 3\epsilon$$

since $\frac{1}{\#m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ m A_n}(C) \leq 1$ and $(p_{N,n}(t))_{n\geq 0}$ are probability laws. Our claim is proved. ■

Lemma 2 $p_n(t) = \frac{\#_m(n)}{(m-1)^n n!} e^{-t}(1 - e^{-(m-1)t})^n$.

We need to solve the affine Kolmogorov system of equations:

$$\begin{aligned} \frac{dp_t(0)}{dt} &= -p_t(0) \\ \frac{dp_t(n)}{dt} &= ((m-1)(n-1)+1)p_t(n-1) - ((m-1)n+1)p_t(n) \quad ; n \geq 1. \end{aligned}$$

Proof. Proceeding by induction we have:

$$\begin{aligned} p_t(0) &= -e^{-t} \\ p_t(n) &= ((m-1)(n-1)+1)e^{-(n(m-1)+1)t} \int_0^t e^{(n(m-1)+1)s} p_s(n-1) ds \end{aligned}$$

To prove the lemma it suffices to note that

$$\#_m(n) = \#_m(n-1)((n-1)(m-1)+1)$$

and that

$$e^{(n(m-1)+1)s}e^{-s}(1-e^{-(m-1)s})^{n-1} = e^{(m-1)s}(e^{(m-1)s}-1)^{n-1}$$

is the derivative of $\frac{1}{(m-1)n}(e^{(m-1)s}-1)^n$. ■

$$\mu_t = \sum_{n\geq 0} p_n(t) \frac{1}{\#_m(n)} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n} \quad (1)$$

$$= \sum_{n\geq 0} \frac{e^{-t}(1-e^{-(m-1)t})^n}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

Remark 3 *We call this law an extended Wild sum and note that the convex combination we obtain for the case* $m = 2$ *is indeed the Wild sum,* $\mu_t = \sum_{n\geq 0} e^{-t}(1-e^{-t})^n \frac{1}{n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$*, which has been well-known in the statistical physics of gases since the work of Kac (1956), "Foundations of kinetic theory". Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability.*

2. The solution of the evolution equation.

The kernel Q allows us to describe the macroscopic evolution of the system with an associated system of non-linear differential equations via the evolution of the law of a component. This probability law, denoted μ_t, evolves with time and is in fact the solution of the ODE system with initial condition:

$$\frac{d\mu_t}{dt} = \lambda \left(\mu_t^{\circ m} - \mu_t \right) \; ; \mu_0 = \mu$$

Theorem 4 *The convex combination,*

$$\mu_t = \sum_{n\geq 0} \frac{e^{-t}(1-e^{-(m-1)t})^n}{(m-1)^n n!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

is the solution of the ODE system with initial condition

$$\frac{d\mu_t}{dt} = \mu_t^{\circ m} - \mu_t \,; \mu_0 = \mu.$$

Proof. We can differentiate μ_t term by term to obtain:

$$-\mu_t + e^{-mt} \sum_{n\geq 1} (1-e^{-(m-1)t})^{n-1} \frac{1}{(m-1)^{n-1}(n-1)!} \sum_{A_n \in \mathbb{A}_n} \mu^{\circ_m A_n}$$

Thus we need to show that: (2)

$$\mu_t^{\circ m}(C) = e^{-mt} \sum_{n\geq 0} (1-e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} \sum_{A_{n+1} \in \mathbb{A}_{n+1}} \mu^{\circ_m A_{n+1}}(C)$$

Starting with the definition, we have that the LHS of (2) is equal to

$$\int_{\mathbb{R}^m} \left(\sum_{i_1 \geq 0} e^{-t} (1 - e^{-(m-1)t})^{i_1} \frac{1}{(m-1)^{i_1} i_1!} \sum_{A_{i_1} \in \mathbb{A}_{i_1}} \mu^{\circ m A_{i_1}}(dx_1) \right) \ldots$$

$$\ldots \left(\sum_{i_m \geq 0} e^{-t} (1 - e^{-(m-1)t})^{i_m} \frac{1}{(m-1)^{i_m} i_m!} \sum_{A_{i_m} \in \mathbb{A}_{i_m}} \mu^{\circ m A_{i_m}}(dx_m) \right) \ldots$$

$$\ldots Q(x_1, \ldots, x_m; C \times E^{m-1})$$

which is equal to

$$\int_{\mathbb{R}^m} e^{-mt} \left\{ \sum_{n\geq 0} (1-e^{-(m-1)t})^n \sum_{i_1+...+i_m=n} \frac{1}{(m-1)^n i_1!...i_m!} \cdots \right.$$

$$\left. \left(\sum_{A_{i_1}\in \mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) .. \left(\sum_{A_{i_m}\in \mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) \right\} Q(x_1,..,x_m; C\times E^{m-1}$$

which in turn is equal to

$$\int_{\mathbb{R}^m} e^{-mt} \left(\sum_{n\geq 0} (1-e^{-(m-1)t})^n \frac{1}{(m-1)^n n!} F(i_1,...,i_m,n,\mu,A_{i_1},....,A_{i_m,}Q,C) \right)$$

where

$$F(i_1,...,i_m,n,\mu,A_{i_1},....,A_{i_m,}Q,C) = ...$$

$$\sum_{i_1+...+i_m=n} \binom{n}{i_1}\binom{n-i_1}{i_2}....\binom{i_{m-1}+i_m}{i_{m-1}} \left(\sum_{A_{i_1}\in\mathbb{A}_{i_1}} \mu^{\circ_m A_{i_1}}(dx_1) \right) ...$$

$$... \left(\sum_{A_{i_m}\in\mathbb{A}_{i_m}} \mu^{\circ_m A_{i_m}}(dx_m) \right) Q(x_1,...,x_m; C\times E^{m-1})$$

And this last expression is a decomposition of the trees $A_{n+1} \in \mathbb{A}_{n+1}$ appearing in the RHS of (2) in m subtrees after the first node (taking the branching order into account). The two expressions are therefore equal and this proves the theorem. ■

Remark 5 *H.P. McKean Jr. (1967), An exponential formula for solving Boltzmann's equation for a Maxwellian gas, J. of Comb. Theory, 2, no. 3, 358-382.*

Remark 6 *In Duffie-Giroux-Manso (2010), Information Percolation, AEJ: Microeconomic Theory, 2, 10-11, the term* $a_{(m-1)n+1}$ *can now be given explicitly as* $\frac{1}{(m-1)^n n!}$.

Remark 7 *In Bélanger-Giroux, (2013), Some new results on information percolation, Stoch. Systems, vol. 3, 1-10, we used the explicit form of the solution for a collision model to show its stability.*

3. Extension to the case where there are two types of interactions.

Example 1: Duffie-Gârleanu-Pedersen (full-scale*)

$$
\begin{aligned}
\frac{d\mu_t(l,n)}{dt} &= -\gamma_u\mu_t(l,n) + \gamma_d\mu_t(h,n) + \lambda\mu_t(h,n)\mu_t(l,o) \\
\frac{d\mu_t(h,n)}{dt} &= \gamma_u\mu_t(l,n) - \gamma_d\mu_t(h,n) - \lambda\mu_t(h,n)\mu_t(l,o) \\
\frac{d\mu_t(h,o)}{dt} &= \gamma_u\mu_t(l,o) - \gamma_d\mu_t(h,o) + \lambda\mu_t(h,n)\mu_t(l,o) \\
\frac{d\mu_t(l,o)}{dt} &= -\gamma_u\mu_t(l,o) + \gamma_d\mu_t(h,o) - \lambda\mu_t(h,n)\mu_t(l,o)
\end{aligned}
$$

Example 2: Duffie-Malamud-Manso (full-scale):

$$\frac{d\mu_t(n)}{dt} = \eta(\pi(n) - \mu_t(n)) + \sum_{n_1+\ldots+n_m=n} \mu_t(n_1)\cdots\mu_t(n_m)h(n_1, \ldots, n_m)$$
$$- \sum_{n_2,\ldots,n_m} \mu_t(n)\cdots\mu_t(n_m)h(n, n_2, \ldots, n_m)$$

Remark: Our next result allows us, in particular, to state the existence of a global solution of a large number of non-linear differential systems of a finite number of equations.

If we come back to the DGP model, we have $E = \{(l,n),(l,o),(h,n),(h,o)\}$ we have defined the binary kernel Q_2 and we have a unary kernel defined by

$$
\begin{aligned}
Q_1((l,n);C) &= \delta_{(h,n)}(C); Q_1((l,o);C) = \delta_{(h,0)}(C); \\
Q_1((h,n);C) &= \delta_{(l,n)}(C); Q_1((h,o);C) = \delta_{(l,0)}(C)
\end{aligned}
$$

And we define

$$\nu^{\circ_1}(C) = \nu(l,n)\delta_{(h,n)}(C) + \nu(h,n)\delta_{(l,n)}(C) + \nu(l,o)\delta_{(h,o)}(C) + \nu(h,o)\delta_{(l,o)}(C)$$

In order to define the kernel and the operation in this way, we have made the (temporary) assumption that $\gamma_u = \gamma_d \triangleq \gamma$.

Then $q_p(t) = \exp(-\gamma t)\frac{(\gamma t)^p}{p!}$ is the probability of having p autonomous movements up to time t.

Let $\mathbb{K}_n^p$ denote the set of all arrangements of p undistinguishable objects in n boxes. Then $|\mathbb{K}_n^p| = \binom{n+p-1}{n-1}$.

If $A_n \in \mathbb{A}_n$ then A_n has $mn+1$ branches.

If $\sigma \in \mathbb{K}_{mn+1}^p$ let A_n^σ denote the tree obtained by placing the p autonomous movements according to the arrangement σ.

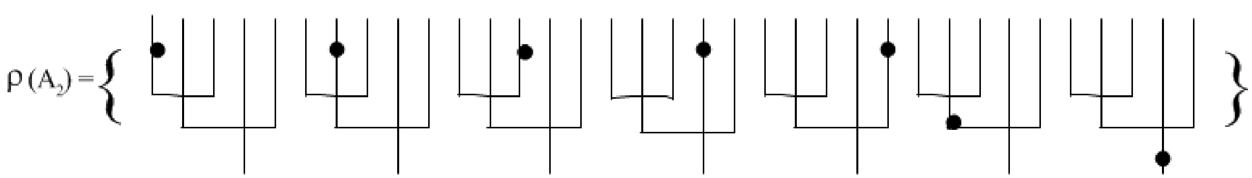


Figure 3: Interaction trees for m=3, A_2 and p=1.

If we call these 7 arrangements resp. $\{\sigma_i\}_{i=1}^{7}$ then $\left(\nu^{\circ_3 A_2}\right)^{\circ 1} = \sum_{i=1}^{7} \nu^{\circ_3 A_2^{\sigma_i}}$.

Proposition 8 *The convex combination,*

$$\nu_t = \sum_{n \geq 0} p_n(t) \frac{1}{\#m(n)} \sum_{A_n \in \mathbb{A}_n} \left[\sum_{p=0}^{\infty} \frac{q_p(t)}{\binom{mn+p}{mn}} \sum_{\sigma \in \mathbb{K}_{mn+1}^{p}} \nu^{\circ_m A_n^{\sigma}} \right]$$

is the solution of the ODE system with initial condition

$$\frac{d\nu_t}{dt} = \lambda \left(\nu_t^{\circ m} - \nu_t\right) + \gamma \left(\nu_t^{\circ 1} - \nu_t\right); \nu_0 = \nu.$$

Remark 9 *In the DGP model, the binary interaction (i.e. $m = 2$) gives a simplification of the formula. But removing the equality of intensities for up and down movements (i.e. allowing $\gamma_u \neq \gamma_d$) complicates a bit its last term. Let $\gamma \triangleq \gamma_u + \gamma_d$.*

$$
\begin{aligned}
\nu_t \;=\; & \sum_{n\geq 0} \frac{e^{-\lambda t}(1-e^{-\lambda t})}{n!} \ldots \\
& \sum_{A_n \in \mathbb{A}_n} \left[\sum_{p=0}^{\infty} \frac{e^{-\gamma t}}{p!\binom{2n+p}{2n}} \left[\sum_{k=0}^{p} \frac{(\gamma_u t)^k \, (\gamma_d t)^{n-k}}{\binom{p}{k}} \sum_{\substack{\sigma_u \in K_{2n+1}^{k} \\ \sigma_d \in K_{2n+1}^{n-k}}} \nu^{\circ_2 A_n^{\sigma_u \cup \sigma_d}} \right] \right]
\end{aligned}
$$

4. Further work:

The study of Duffie-Garleanu-Pedersen OTC market models with several assets.

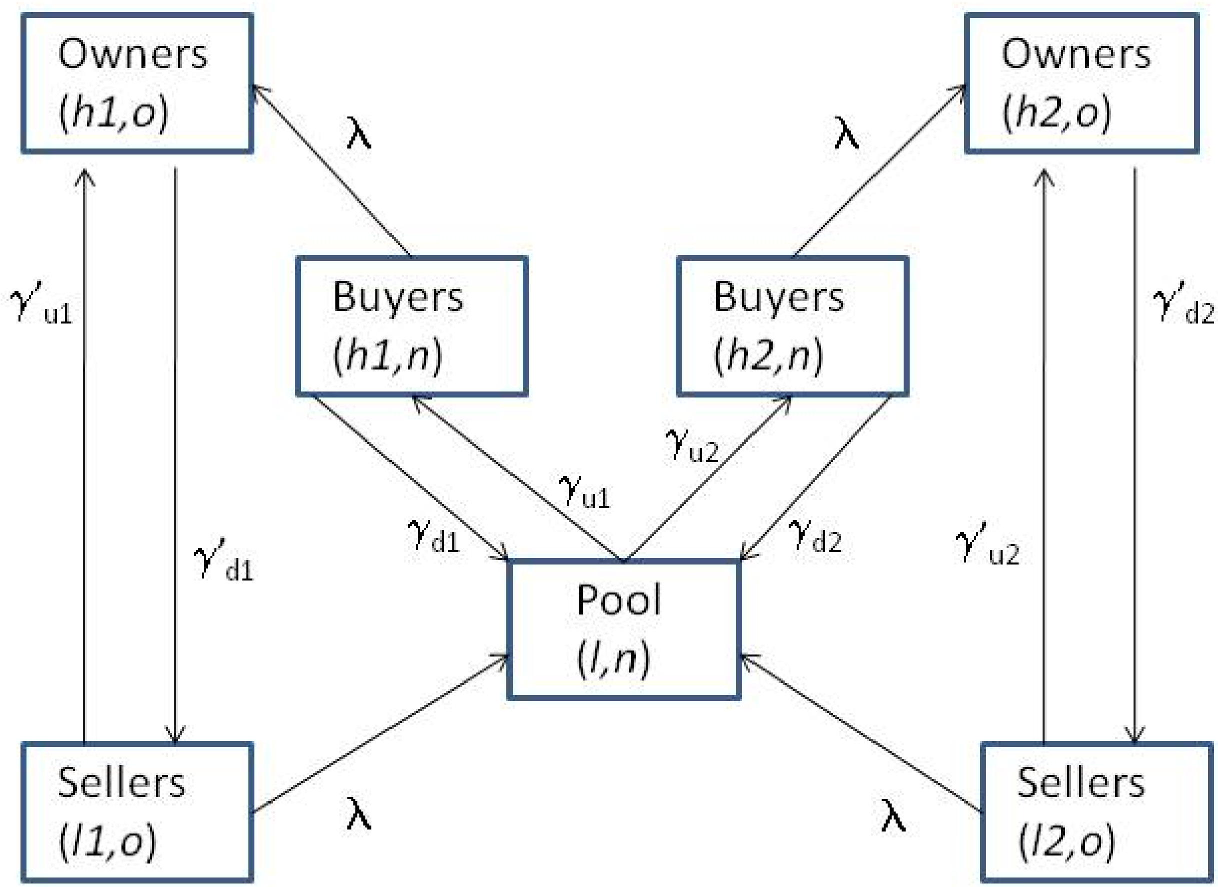


Figure 1: DGP model with two sub-markets

$$\begin{aligned}
\frac{d\mu_t(l,n)}{dt} &= -\gamma_{u1}\mu_t(l,n) + \gamma_{d1}\mu_t(h1,n) + \lambda\mu_t(h1,n)\mu_t(l1,o) \\
&\quad -\gamma_{u2}\mu_t(l,n) + \gamma_{d2}\mu_t(h2,n) + \lambda\mu_t(h2,n)\mu_t(l2,o) \\
\frac{d\mu_t(hi,n)}{dt} &= \gamma_{ui}\mu_t(l,n) - \gamma_{di}\mu_t(hi,n) - \lambda\mu_t(hi,n)\mu_t(li,o) \\
\frac{d\mu_t(hi,o)}{dt} &= \gamma'_{ui}\mu_t(li,o) - \gamma'_{di}\mu_t(hi,o) + \lambda\mu_t(hi,n)\mu_t(li,o) \\
\frac{d\mu_t(li,o)}{dt} &= -\gamma'_{ui}\mu_t(li,o) + \gamma'_{di}\mu_t(hi,o) - \lambda\mu_t(hi,n)\mu_t(li,o)
\end{aligned}$$

where $i = 1, 2$.With the following three market constraints:

$$\begin{aligned}\mu_t(h1, o) + \mu_t(l1, o) &= m_1 \\ \mu_t(h2, o) + \mu_t(l2, o) &= m_2 \\ \mu_t(l, n) + \sum_{i=1}^{2} [\mu_t(li, o) + \mu_t(hi, n) + \mu_t(hi, o)] &= 1\end{aligned}$$

where the parameters m_1 and m_2 are positive and such that $m_1 + m_2 < 1$.

Thank you!