Chapter 7

Electromagnetism

In this chapter, we discuss the worldpaths of charged particles in constant electromagnetic fields. We will see that skew lineon fields provide a suitable context in which to discuss electromagnetism. We begin with some basic definitions and algebraic preliminaries in §**7.1**, and proceed in §**7.2** to analyze in some detail skew lineons in an inner-product space of signature (3,1). Finally, we apply these ideas in §**7.3** in describing worldpaths of charged particles.

7.1 Skew Lineons

We will see later that skew lineon fields are useful for describing electromagnetic fields in a relativistic world. Although we will not yet offer a formal justification for this observation, it will be beneficial to address at this point some basic properties of skew lineons.

Let a finite-dimensional inner-product space $\mathcal{V}$ be given.

7100 Definition: *We say that* $\mathbf{W} \in \mathsf{Lin}\,\mathcal{V}$ *is* **skew** *if*

$$\mathbf{b} \cdot \mathbf{W}\mathbf{a} = -\mathbf{a} \cdot \mathbf{W}\mathbf{b} \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{V}.$$

We denote by $\mathsf{Skew}\,\mathcal{V}$ *the set of all skew lineons; i.e., all members of* $\mathsf{Lin}\,\mathcal{V}$ *that are skew.*

7101 Theorem: *Let* $\mathbf{W} \in \operatorname{Lin} \mathcal{V}$ *be given. Then* $\mathbf{W}$ *is skew if and only if*

$$\mathbf{a} \cdot \mathbf{W}\mathbf{a} = 0 \quad \text{for all } \mathbf{a} \in \mathcal{V}.$$

Proof: Suppose that $\mathbf{W}$ is skew. Then for all $\mathbf{a} \in \mathcal{V}$, we have

$$\mathbf{W}\mathbf{a} \cdot \mathbf{a} = -\mathbf{a} \cdot \mathbf{W}\mathbf{a},$$

and hence $\mathbf{a} \cdot \mathbf{W}\mathbf{a} = 0$.

Now suppose that $\mathbf{c} \cdot \mathbf{W}\mathbf{c} = 0$ for all $\mathbf{c} \in \mathcal{V}$, and let $\mathbf{a}, \mathbf{b} \in \mathcal{V}$ be given. Then

$$\begin{aligned} 0 &= (\mathbf{a} + \mathbf{b}) \cdot \mathbf{W}(\mathbf{a} + \mathbf{b}) \\ &= \mathbf{a} \cdot \mathbf{W}\mathbf{a} + \mathbf{a} \cdot \mathbf{W}\mathbf{b} + \mathbf{b} \cdot \mathbf{W}\mathbf{a} + \mathbf{b} \cdot \mathbf{W}\mathbf{b} \\ &= \mathbf{a} \cdot \mathbf{W}\mathbf{b} + \mathbf{b} \cdot \mathbf{W}\mathbf{a}. \end{aligned}$$

Hence,

$$\mathbf{b} \cdot \mathbf{W}\mathbf{a} = -\mathbf{a} \cdot \mathbf{W}\mathbf{b}.$$

As $\mathbf{a}, \mathbf{b} \in \mathcal{V}$ were arbitrary, we see that $\mathbf{W}$ is skew. ◊

7102 Proposition: *If* $\mathbf{W}$ *is skew, and if* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are given subspaces of* $\mathcal{V}$ *that satisfy*

$$\mathbf{W}_{>}(\mathcal{U}_1) \subset \mathcal{U}_2,$$

then

$$\mathbf{W}_{>}(\mathcal{U}_2^{\perp}) \subset \mathcal{U}_1^{\perp}.$$

Proof: Suppose that $\mathcal{U}_1$ and $\mathcal{U}_2$ satisfy $\mathbf{W}_{>}(\mathcal{U}_1) \subset \mathcal{U}_2$, and let $\mathbf{u} \in \mathbf{W}_{>}(\mathcal{U}_2^{\perp})$ be given. The we may choose $\mathbf{b} \in \mathcal{U}_2^{\perp}$ such that $\mathbf{W}\mathbf{b} = \mathbf{u}$. Then for all $\mathbf{v} \in \mathcal{U}_1$, we have

$$\mathbf{u} \cdot \mathbf{v} = \mathbf{W}\mathbf{b} \cdot \mathbf{v} = -\mathbf{b} \cdot \mathbf{W}\mathbf{v} = 0$$

since $\mathbf{b} \in \mathcal{U}_2^{\perp}$ and $\mathbf{W}\mathbf{v} \in \mathbf{W}_{>}(\mathcal{U}_1) \subset \mathcal{U}_2$. Since $\mathbf{v} \in \mathcal{U}_1$ was arbitrary, we have $\mathbf{u} \in \mathcal{U}_1^{\perp}$; as $\mathbf{u}$ was arbitrary, we have $\mathbf{W}_{>}(\mathcal{U}_2^{\perp}) \subset \mathcal{U}_1^{\perp}$. ◊

7103 Corollary: *Let* $\mathbf{W} \in \operatorname{Skew} \mathcal{V}$ *be given. If a given subspace* $\mathcal{U}$ *of* $\mathcal{V}$ *is a* $\mathbf{W}$*-space (see* **Def. D18** *of Appendix D), then so is* $\mathcal{U}^{\perp}$*.*

Proof: If a subspace $\mathcal{U}$ of $\mathcal{V}$ is a $\mathbf{W}$-space, we may put $\mathcal{U}_1 := \mathcal{U}_2 := \mathcal{U}$ in the previous Proposition. The result is immediate. $\Diamond$

7104 Proposition: *Let* $\mathbf{W} \in \mathsf{Skew}\,\mathcal{V}$ *be given. Then* $\mathsf{Null}\,\mathbf{W} = (\mathsf{Rng}\,\mathbf{W})^{\perp}$, *and hence* $\mathsf{Rng}\,\mathbf{W} = (\mathsf{Null}\,\mathbf{W})^{\perp}$.

Proof: We apply **Prop. 7102** with $\mathcal{U}_1 := \mathcal{V}$ and $\mathcal{U}_2 := \mathsf{Rng}\,\mathbf{W}$. Clearly, $\mathbf{W}_{>}(\mathcal{V}) \subset \mathsf{Rng}\,\mathbf{W}$, and hence we conclude that

$$\mathbf{W}_{>}((\mathsf{Rng}\,\mathbf{W})^{\perp}) \subset \mathcal{V}^{\perp} = \{\mathbf{0}\},$$

and therefore $(\mathsf{Rng}\,\mathbf{W})^{\perp} \subset \mathsf{Null}\,\mathbf{W}$.

Applying **Prop. 7102** again with $\mathcal{U}_1 := \mathsf{Null}\,\mathbf{W}$ and $\mathcal{U}_2 := \{\mathbf{0}\}$, we see that $\mathbf{W}_{>}(\mathsf{Null}\,\mathbf{W}) \subset \{\mathbf{0}\}$ implies

$$\mathsf{Rng}\,\mathbf{W} = \mathcal{W}_{>}(\mathcal{V}) = \mathcal{W}_{>}(\{\mathbf{0}\}^{\perp}) \subset (\mathsf{Null}\,\mathbf{W})^{\perp},$$

and hence, by **Prop. 5102**, $\mathsf{Null}\,\mathbf{W} = (\mathsf{Null}\,\mathbf{W})^{\perp\perp}$. As a result, we have $\mathsf{Null}\,\mathbf{W} \subset (\mathsf{Rng}\,\mathbf{W})^{\perp}$. Thus, we have $\mathsf{Null}\,\mathbf{W} = (\mathsf{Rng}\,\mathbf{W})^{\perp}$, from which it immediately follows, again by **Prop. 5102**, that $\mathsf{Rng}\,\mathbf{W} = (\mathsf{Null}\,\mathbf{W})^{\perp}$. $\Diamond$

7105 Definition: *For all* $\mathbf{a}, \mathbf{b} \in \mathcal{V}$, *we define the* **exterior product of a and b**, *denoted by* $\mathbf{a} \wedge \mathbf{b}$, *by*

$$\mathbf{a} \wedge \mathbf{b} := \mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}.$$

(See **Def. D22** *of Appendix D for a definition of "*$\mathbf{a} \otimes \mathbf{b}$*").*

7106 Proposition: *Let* $\mathbf{a}, \mathbf{b} \in \mathcal{V}$ *be given. Then* $\mathbf{a} \wedge \mathbf{b}$ *is a skew lineon.*

Proof: The proof of the linearity of $\mathbf{a} \wedge \mathbf{b}$ follows immediately from **Prop. D23** of Appendix D.

Now let $\mathbf{v} \in \mathcal{V}$ be given. Then

$$\begin{aligned} \mathbf{v} \cdot (\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a})(\mathbf{v}) &= \mathbf{v} \cdot ((\mathbf{b} \cdot \mathbf{v})\mathbf{a} - (\mathbf{a} \cdot \mathbf{v})\mathbf{b}) \\ &= (\mathbf{b} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{a}) - (\mathbf{a} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{b}) \\ &= 0. \end{aligned}$$

Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, we conclude from **Prop. 7101** that $\mathbf{a} \otimes \mathbf{b} - \mathbf{b} \otimes \mathbf{a}$ is skew. $\Diamond$

7107 Theorem: *Assume that* $\dim \mathcal{V} = 2$, *and let* $\mathbf{W} \in \mathsf{Skew}\, \mathcal{V}$ *be given. Also, let an orthonormal list-basis* $(\mathbf{b}_1, \mathbf{b}_2)$ *in the sense of* **Prop. 5110** *be given. Then there is* $\omega \in \mathbb{R}$ *such that* $\mathbf{W}\mathbf{b}_1 = \omega \mathbf{b}_2$.

If $\mathsf{sig}\mathcal{V} = (1,1)$, *the matrix of* $\mathbf{W}$ *relative to* $(\mathbf{b}_1, \mathbf{b}_2)$ *is*

$$[\mathbf{W}] = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix}, \tag{71.1}$$

and we have $\mathbf{W}^2 = \omega^2 \mathbf{1}_{\mathcal{V}}$ *and* $\mathbf{W} = \omega(\mathbf{b}_1 \wedge \mathbf{b}_2)$.

If $\mathsf{sig}\mathcal{V} = (2,0)$, *the matrix of* $\mathbf{W}$ *relative to* $(\mathbf{b}_1, \mathbf{b}_2)$ *is*

$$[\mathbf{W}] = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}, \tag{71.2}$$

and we have $\mathbf{W}^2 = -\omega^2 \mathbf{1}_{\mathcal{V}}$ *and* $\mathbf{W} = \omega(\mathbf{b}_1 \wedge \mathbf{b}_2)$.

Proof: Since $\mathbf{W}$ is skew, we have $\mathbf{b}_1 \cdot \mathbf{W}\mathbf{b}_1 = 0$ and hence $\mathbf{W}\mathbf{b}_1 \in \{\mathbf{b}_1\}^\perp = \mathbb{R}\mathbf{b}_2$. Hence we can determine $\omega \in \mathbb{R}$ such that $\mathbf{W}\mathbf{b}_1 = \omega\mathbf{b}_2$. Similarly, we can determine $\omega' \in \mathbb{R}$ such that $\mathbf{W}\mathbf{b}_2 = \omega'\mathbf{b}_1$. We have

$$\omega\mathbf{b}_2 \cdot \mathbf{b}_2 = \mathbf{b}_2 \cdot \mathbf{W}\mathbf{b}_1 = -\mathbf{b}_1 \cdot \mathbf{W}\mathbf{b}_2 = -\omega'\mathbf{b}_1 \cdot \mathbf{b}_1. \tag{71.3}$$

If $\mathsf{sig}\mathcal{V} = (1,1)$, we have $\mathbf{b}_1 \cdot \mathbf{b}_1 = -1$ and $\mathbf{b}_2 \cdot \mathbf{b}_2 = 1$, and then (71.3) yields $\omega' = \omega$, so that the the matrix of $\mathbf{W}$ is (71.1). If $\mathsf{sig}\mathcal{V} = (2,0)$ we have $\mathbf{b}_1 \cdot \mathbf{b}_1 = \mathbf{b}_2 \cdot \mathbf{b}_2 = 1$ and (71.3) yields $\omega' = -\omega$, so that the matrix of $\mathbf{W}$ is given by (71.2).

The remainder of the proof is left as an Exercise. ◊

Remark: If $\mathsf{sig}\mathcal{V} = (1,1)$ and $(\mathbf{d}, \mathbf{e})$ is an orthonormal list-basis of $\mathcal{V}$ such that the matrix of $\mathbf{W}$ relative to this list-basis is given by

$$[\mathbf{W}] = \begin{bmatrix} 0 & \omega \\ \omega & 0 \end{bmatrix},$$

then the matrix of $\mathbf{W}$ relative to the orthonormal list-basis $(\mathbf{d}, -\mathbf{e})$ or $(-\mathbf{d}, \mathbf{e})$ is given by

$$[\mathbf{W}] = \begin{bmatrix} 0 & -\omega \\ -\omega & 0 \end{bmatrix}.$$

7108 Proposition: *Suppose that $\mathcal{V}$ is a three-dimensional genuine inner-product space. Let a non-zero $\mathbf{H} \in \mathsf{Skew}\,\mathcal{V}$ be given. Then $\dim \mathsf{Null}\,\mathbf{H} = 1$; we call $\mathsf{Null}\,\mathbf{H}$ the* **axis of H**. *Also, $\mathsf{Rng}\,\mathbf{H} = (\mathsf{Null}\,\mathbf{H})^{\perp}$ and $\dim \mathsf{Rng}\,\mathbf{H} = 2$. Moreover, for every $\mathbf{u} \in \mathsf{Rng}\,\mathbf{H}$ with $|\mathbf{u}| = 1$, there is exactly one $\mathbf{v} \in \mathsf{Rng}\,\mathbf{H}$ with $|\mathbf{v}| = 1$ and exactly one $\alpha \in \mathbb{P}^{\times}$ such that $\mathbf{u} \cdot \mathbf{v} = 0$ and*

$$\mathbf{H} = \alpha(\mathbf{v} \wedge \mathbf{u}).$$

Further, given $\mathbf{b} \in \{\mathbf{u}, \mathbf{v}\}^{\perp}$ such that $|\mathbf{b}| = 1$, so that $(\mathbf{b}, \mathbf{u}, \mathbf{v})$ is an orthonormal basis, the matrix of $\mathbf{H}$ relative to this basis is given by

$$[\mathbf{H}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{bmatrix}.$$

Finally, α is independent of $\mathbf{u} \in \mathsf{Rng}\,\mathbf{H}$, and is given by $\alpha = \sqrt{-\frac{1}{2}\mathsf{tr}\,(\mathbf{H}^2)}$ (see **Prop. D21** *of Appendix D).*

Proof: Let $\mathbf{u} \in \mathsf{Rng}\,\mathbf{H}$ with $|\mathbf{u}| = 1$ be given (since $\mathsf{Rng}\,\mathbf{H} \neq \emptyset$, such $\mathbf{u}$ do exist). Choose $\mathbf{w} \in \mathcal{V}$ such that $\mathbf{u} = \mathbf{H}\mathbf{w}$. Since $\mathbf{H}$ is skew, we have

$$\mathbf{w} \cdot \mathbf{H}\mathbf{u} = -\mathbf{u} \cdot \mathbf{H}\mathbf{w} = -\mathbf{u} \cdot \mathbf{u} = -1$$

and hence $\mathbf{H}\mathbf{u} \neq \mathbf{0}$. We put

$$\alpha := \frac{1}{|\mathbf{H}\mathbf{u}|}, \quad \mathbf{v} := \frac{1}{\alpha}\mathbf{H}\mathbf{u}.$$

Then $|\mathbf{v}| = 1$, and $\mathbf{v} \cdot \mathbf{u} = \frac{1}{\alpha}(\mathbf{H}\mathbf{u}) \cdot \mathbf{u} = 0$ as $\mathbf{H}$ is skew. Since $\dim \mathcal{V} = 3$, we may determine $\mathbf{b} \in \mathcal{V}$ such that $(\mathbf{b}, \mathbf{u}, \mathbf{v})$ is an orthonormal list-basis of $\mathcal{V}$.

We have $(\mathbf{H}\mathbf{b}) \cdot \mathbf{b} = 0$ and $(\mathbf{H}\mathbf{b}) \cdot \mathbf{u} = -\mathbf{b} \cdot \mathbf{H}\mathbf{u} = -\alpha\mathbf{b} \cdot \mathbf{v} = 0$. Hence we have $\mathbf{H}\mathbf{b} \in \{\mathbf{b}, \mathbf{u}\}^{\perp} = \mathbb{R}\mathbf{v}$, *i.e.,* we may choose $\gamma \in \mathbb{R}$ such that $\mathbf{H}\mathbf{b} = \gamma\mathbf{v}$. It follows that

$$\gamma\mathbf{w} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{H}\mathbf{b} = -\mathbf{b} \cdot \mathbf{H}\mathbf{w} = -\mathbf{b} \cdot \mathbf{u} = 0.$$

On the other hand, we have

$$1 = \mathbf{u} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{H}\mathbf{w} = -\mathbf{w} \cdot \mathbf{H}\mathbf{u} = -\alpha\mathbf{w} \cdot \mathbf{v}$$

and henve $\mathbf{w} \cdot \mathbf{v} \neq 0$. We conclude that $\gamma = 0$ and hence $\mathbf{Hb} = \mathbf{0}$.

Since $\mathbf{b} \cdot \mathbf{Hv} = 0$, $\mathbf{u} \cdot \mathbf{Hv} = -\mathbf{v} \cdot \mathbf{Hu} = -\alpha$, and $\mathbf{v} \cdot \mathbf{Hv} = 0$, and since $(\mathbf{b}, \mathbf{u}, \mathbf{v})$ is an orthonormal list-basis of $\mathcal{V}$, we have $\mathbf{Hv} = -\alpha\mathbf{u}$. Recalling that $\mathbf{Hu} = \alpha\mathbf{v}$ and $\mathbf{Hb} = \mathbf{0}$, it follows that the matrix of $\mathbf{H}$ has the form given in the Proposition. Moreover, it is clear that $\mathsf{Null}\ \mathbf{H} = \mathbb{R}\mathbf{b}$ and $\mathsf{Rng}\ \mathbf{H} = \mathsf{Lsp}\ \{\mathbf{u}, \mathbf{v}\}$. The remaining assertions of the Proposition easily follow. ◊

7.2 The Structure of Skew Lineons

In this section, we assume that $\mathcal{V}$ is a given inner-product space of signature $(3, 1)$. We also assume that $\mathbf{F} \in \mathsf{Skew}\ \mathcal{V}$ is given. Below, we analyze the structure of $\mathbf{F}$ in some detail. We use a terminology that suggests the interpretation of $\mathbf{F}$ as the value of an electromagnetic field, as will be explained in the next section.

7200 Definition: *Let* $\mathbf{d} \in \mathcal{F}_1$ *be given, and put* $\mathcal{W} := \{\mathbf{d}\}^{\perp}$. *We define the* **electric part of F relative to d**, *denoted by* $\mathsf{E}_{\mathbf{d}}$, *by*

$$\mathsf{E}_{\mathbf{d}} := \mathbf{Fd}.$$

We define the **magnetic part of F relative to d**, *denoted by* $\mathbf{H}_{\mathbf{d}}$, *by*

$$\mathbf{H}_{\mathbf{d}} := \mathbf{PF}|_{\mathcal{W}},$$

where $\mathbf{P} : \mathcal{V} \to \mathcal{W}$ *is the orthogonal projection of* $\mathcal{V}$ *onto* $\mathcal{W}$ *(i.e., the projection of* $\mathcal{V}$ *onto* $\mathcal{W}$ *along* $\mathcal{W}^{\perp}$*; see* **Def. D16** *of Appendix D) so that* $\mathbf{P}(\xi\mathbf{d} + \mathbf{w}) = \mathbf{w}$ *for all* $\xi \in \mathbb{R}$ *and* $\mathbf{w} \in \mathcal{W}$.

Now assume that $\mathbf{d} \in \mathcal{F}_1$ is given and put $\mathcal{W} := \{\mathbf{d}\}^{\perp}$. When confusion is not likely, we write "E" for "$\mathsf{E}_{\mathbf{d}}$" and "$\mathbf{H}$" for "$\mathbf{H}_{\mathbf{d}}$". We also use "electric part" for "electric part of $\mathbf{F}$ relative to $\mathbf{d}$" and "magnetic part" for "magnetic part of $\mathbf{F}$ relative to $\mathbf{d}$" when unambiguous. We adopt these conventions at this point.

Since $\mathbf{F}$ is skew, we must have $\mathbf{d} \cdot \mathbf{Fd} = 0$ by **Thm. 7101**, and hence $\mathbf{d} \cdot \mathsf{E} = 0$. Thus, we see that $\mathsf{E} \in \mathcal{W}$, and consequently the electric part is spacelike or zero. $|\mathsf{E}|$ is called the **intensity** of the electric part, and when $|\mathsf{E}| \neq 0$, $\mathsf{E}/|\mathsf{E}|$ is called the **direction** of the electric part.

Although it is simple to see what the direction of the electric part of $\mathbf{F}$ is relative to $\mathbf{d}$, some additional work is needed to formulate the definition of an axis of the magnetic part of $\mathbf{F}$ relative to $\mathbf{d}$. Our first task will be to show that $\mathbf{H}$ is skew. To this end, let $\mathbf{a} \in \mathcal{W}$ be given. Since $\mathbf{a} \in \mathcal{W}$, we have $\mathbf{Pa} = \mathbf{a}$ and $\mathbf{Ha} = \mathbf{PFa}$. Now $\mathbf{F}$ is skew and $\mathbf{P}$ is a projection, so that

$$0 = \mathbf{a} \cdot \mathbf{Fa} = \mathbf{Pa} \cdot \mathbf{Fa} = \mathbf{a} \cdot \mathbf{PFa} = \mathbf{a} \cdot \mathbf{Ha}.$$

Since $\mathbf{a} \in \mathcal{W}$ was arbitrary, it follows from **Thm. 7101** that $\mathbf{H}$ is skew.

Since $\mathbf{H}$ is skew, we may consider $\mathbf{H}$ in light of the results in §**7.1**. Assume that $\mathbf{H} \neq \mathbf{0}$. Since $\dim \mathcal{W} = 3$, we see from **Prop. 7108** that $\dim \operatorname{Null} \mathbf{H} = 1$. We say that the one-dimensional subspace of $\mathcal{V}$, $\operatorname{Null} \mathbf{H}$, is the **axis** of the magnetic part. The number α (as given in **Prop. 7108**) is called the **intensity** of the magnetic part.

Our next task is to describe $\mathbf{F}$ in greater detail. With **Prop. D19** of Appendix D in mind, we offer the following.

7201 Definition: *We say that* $\mathbf{F}$ *is* **regular** *if there is a regular two-dimensional* $\mathbf{F}$*-space, and* **singular** *if no two-dimensional* $\mathbf{F}$*-space is regular (i.e., all two-dimensional* $\mathbf{F}$*-spaces are singular). (See* **Def. 5104** *for definitions of regular and singular subspaces.)*

As is happens, being able to distinguish whether $\mathbf{F}$ is regular or singular is as fine a distinction as is necessary in order to obtain a useful description of $\mathbf{F}$. We assume that $\mathbf{F}$ is not zero.

7202 Theorem: *Assume that* $\mathbf{F}$ *is regular. Then there is exactly one two-dimensional positive-regular* $\mathbf{F}$*-space* $\mathcal{U} \subset \mathcal{V}$*. Given a world-direction* $\mathbf{d}$ *belonging to* $\mathcal{U}^{\perp}$*, we may determine an orthonormal list-basis* $(\mathbf{d}, \widehat{\mathsf{E}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}})$ *of* $\mathcal{V}$ *so that the matrix of* $\mathbf{F}$ *relative to this basis is given by*

$$[\mathbf{F}] = \begin{bmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{bmatrix},$$

where ε *is the intensity of the electric part of the field relative to* $\mathbf{d}$ *and* η *is the intensity of the magnetic part of* $\mathbf{F}$ *relative to* $\mathbf{d}$*. We have* $\mathbf{H}_{\mathbf{d}} = \eta(\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}})$ *(in* $\mathcal{W} := \{\mathbf{d}\}^{\perp}$*), and hence* E *belongs to the axis of the magnetic part.* ε *and* η *do not depend on the choice of* $\mathbf{d} \in \mathcal{U}^{\perp} \cap \mathcal{F}_1$*.*

Remark: In this Theorem, and wherever appropriate, we interpret the exterior product "$\wedge$" (see **Def. 7105**) as relative to $\mathcal{W} := \{\mathbf{d}\}^\perp$. Strictly speaking, both $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{f}}$ belong to $\mathcal{V}$, and hence $\eta(\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}})$ belongs to $\mathsf{Skew}\ \mathcal{V}$, while $\mathbf{H_d}$ belongs to $\mathsf{Skew}\ \mathcal{W}$. In such instances, we consider $\widehat{\mathbf{g}}$ and $\widehat{\mathbf{f}}$ to be members of $\mathcal{W}$ (which they are), and therefore consider the exterior product "$\eta(\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}})$" to be a member of $\mathsf{Skew}\ \mathcal{W}$.

Proof: We proceed with a proof of this Theorem in some detail. We first choose a two-dimensional regular $\mathbf{F}$-space, say $\mathcal{U}$. By **Cor. 7103**, $\mathcal{U}^\perp$ is also a two-dimensional $\mathbf{F}$-space. We leave as an Exercise to show that one of $\mathcal{U}$ and $\mathcal{U}^\perp$ must be positive-regular and the other must have signature $(1,1)$. We assume without loss, then, that $\mathcal{U}$ is positive-regular and that $\mathcal{U}^\perp$ has signature $(1,1)$. We also leave as an Exercise that with these restrictions, $\mathcal{U}$ is uniquely determined by $\mathbf{F}$.

Since $\mathcal{U}$ is an $\mathbf{F}$-space, we may consider the linear mapping $\mathbf{F}_{|\mathcal{U}} : \mathcal{U} \to \mathcal{U}$ as given in **Def. D18** of Appendix D. Now $\mathsf{sig}\,\mathcal{U} = (2,0)$, and evidently $\mathbf{F}_{|\mathcal{U}}$ is skew since $\mathbf{F}$ is skew. Hence by **Thm. 7107**, we may determine $\eta \in \mathbb{P}$ such that $\mathbf{F}^2_{|\mathcal{U}} = -\eta^2 \mathbf{1}_{\mathcal{U}}$. We apply a similar analysis to $\mathcal{U}^\perp$, yielding $\varepsilon \in \mathbb{P}$ such that $\mathbf{F}^2_{|\mathcal{U}^\perp} = \varepsilon^2 \mathbf{1}_{\mathcal{U}^\perp}$.

We denote the electric and magnetic parts of $\mathbf{F}$ relative to $\mathbf{d}$ by E and $\mathbf{H}$, respectively. Since $\mathsf{E} = \mathbf{Fd}$ (see **Def. 7200**) and $\mathbf{F}^2_{|\mathcal{U}^\perp} = \varepsilon^2 \mathbf{1}_{\mathcal{U}^\perp}$, we have that $\mathsf{E} \in \mathcal{U}^\perp$ (since $\mathbf{d} \in \mathcal{U}^\perp$ and $\mathcal{U}^\perp$ is an $\mathbf{F}$-space), and

$$|\mathsf{E}|^2 = \mathbf{Fd} \cdot \mathbf{Fd} = -\mathbf{d} \cdot \mathbf{F}^2\mathbf{d} = -\mathbf{d} \cdot \varepsilon^2 \mathbf{d} = \varepsilon^2,$$

and hence $|\mathsf{E}| = \varepsilon$ is the intensity of the electric part. As a result, we may determine $\widehat{\mathsf{E}} \in \mathcal{U}^\perp$ such that $\mathsf{E} = \varepsilon\widehat{\mathsf{E}}$ and $(\mathbf{d}, \widehat{\mathsf{E}})$ is an orthonormal list-basis of $\mathcal{U}^\perp$. (Note that when $\varepsilon \neq 0$, $\widehat{\mathsf{E}}$ is uniquely determined. When $\varepsilon = 0$, $\widehat{\mathsf{E}}$ may be replaced by $-\widehat{\mathsf{E}}$.) An application of **Thm. 7107** yields that the matrix of $\mathbf{F}_{|\mathcal{U}^\perp}$ relative to $(\mathbf{d}, \widehat{\mathsf{E}})$ is given by

$$[\mathbf{F}_{|\mathcal{U}^\perp}] = \begin{bmatrix} 0 & \varepsilon \\ \varepsilon & 0 \end{bmatrix}.$$

By applying a similar analysis to $\mathbf{F}_{|\mathcal{U}}$, we may determine an orthonormal list-basis $(\widehat{\mathbf{f}}, \widehat{\mathbf{g}})$ of $\mathcal{U}$ such that the matrix of $\mathbf{F}_{|\mathcal{U}}$ relative to this list-basis is given by

$$[\mathbf{F}_{|\mathcal{U}}] = \begin{bmatrix} 0 & \eta \\ -\eta & 0 \end{bmatrix}.$$

It is readily seen that $(\mathbf{d}, \widehat{\mathsf{E}}, \widehat{\mathbf{f}}, \widehat{\mathbf{g}})$ is an orthonormal list-basis of $\mathcal{V}$. The above calculations reveal that the matrix of $\mathbf{F}$ with respect to this list-basis is given by

$$[\mathbf{F}] = \begin{bmatrix} 0 & \varepsilon & 0 & 0 \\ \varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{bmatrix}.$$

It may be shown that $\mathbf{H} = \eta(\widehat{\mathbf{f}} \wedge \widehat{\mathbf{g}})$, and hence that η is the intensity of the magnetic part. Moreover, we have $\mathsf{E} \in \mathsf{Null}\, \mathbf{H}$; *i.e.,* E belongs to the axis of $\mathbf{H}$. Note that the intensities of the electric and magnetic parts do not depend on $\mathbf{d}$ as long as $\mathbf{d} \in \mathcal{U}^{\perp}$. ◊

Remark: If $\mathbf{d} \in \mathcal{F}_1$ does not belong to $\mathcal{U}^{\perp}$, then the electric and magnetic parts of $\mathbf{F}$ have a more complicated form and E does not necessarily belong to the axis of the magnetic part.

The following Theorem deals with the case when $\mathbf{F}$ is singular. Although we will not apply this Theorem in the present book, it is very important when analyzing electromagnetic waves because they are given by non-constant fields whose values are singular skew lineons.

7203 Theorem: *Assume that* $\mathbf{F}$ *is singular. Then* $\mathsf{Rng}\, \mathbf{F}$ *and* $\mathsf{Null}\, \mathbf{F}$ *are singular two-dimensional subspaces of* $\mathcal{V}$ *and* $\mathcal{S} := \mathsf{Rng}\, \mathbf{F} \cap \mathsf{Null}\, \mathbf{F}$ *is a one-dimensional totally singular subspace of* $\mathcal{V}$. *For every* $\mathbf{e} \in \mathsf{Rng}\, \mathcal{F}$ *with* $\mathbf{e} \cdot \mathbf{e} = 1$, *we have*

$$\mathbf{n} := \mathbf{F}\mathbf{e} \in \mathcal{S}, \quad \mathbf{n} \cdot \mathbf{n} = 0, \tag{72.1}$$

and

$$\mathbf{F} = \mathbf{n} \wedge \mathbf{e}. \tag{72.2}$$

If $\mathbf{d}$ *is any given world-direction, we can determine* $\alpha \in \mathbb{P}^{\times}$ *and an orthonormal list-basis* $(\mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g})$ *of* $\mathcal{V}$ *such that the matrix of* $\mathbf{F}$ *relative to it is*

$$[\mathbf{F}] = \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \alpha & 0 & -\alpha & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{72.3}$$

The intensity of the electric part and the intensity of the magnetic part of $\mathbf{F}$ *relative to* $\mathbf{d}$ *are both equal to* α *and the electric part* $\mathsf{E}_{\mathbf{d}}$ *is perpendicular to the axis of the magnetic part* $\mathbf{H}_{\mathbf{d}}$*; i.e.,* $\mathsf{E}_{\mathbf{d}} \in (\mathsf{Null}\ \mathbf{H})^{\perp} = \mathsf{Rng}\ \mathbf{H}_{\mathbf{d}}$.

7204 Lemma: *Let a two-dimensional* $\mathbf{F}$*-space* $\mathcal{U} \subset \mathcal{V}$ *be given. Then* $\mathcal{T} := \mathcal{U} \cap \mathcal{U}^{\perp}$ *is the only totally singular subspace of* $\mathcal{U}$*, and we have* $\dim \mathcal{T} = 1$, $\mathbf{F}_{>}(\mathcal{U}) \subset \mathcal{T}$, $\mathbf{F}_{>}(\mathcal{U}^{\perp}) \subset \mathcal{T}$*. Also, we have* $\dim (\mathcal{U} + \mathcal{U}^{\perp}) = 3$. $\mathcal{U} + \mathcal{U}^{\perp}$ *is an* $\mathbf{F}$*-space and* $\mathbf{F}_{>}(\mathcal{U} + \mathcal{U}^{\perp}) \subset \mathcal{T}$.

Proof: The fact that $\mathcal{T}$ is totally singular and $\dim \mathcal{T} = 1$ follows from **Prop. 5209**. It follows from **Prop. 5102** that $\dim \mathcal{U}^{\perp} = 4 - 2 = 2$ and from **Cor. 7103** that $\mathcal{U}^{\perp}$ is also an $\mathbf{F}$-space.

Now let $\mathbf{u} \in \mathcal{U}^{\times}$ be given. Since $\mathbf{F}$ is skew, we have $\mathbf{u} \cdot \mathbf{F}\mathbf{u} = 0$ and hence $\mathbf{F}\mathbf{u} \in \{\mathbf{u}\}^{\perp}$. Since $\mathcal{U}$ is an $\mathbf{F}$-space, we also have $\mathbf{F}\mathbf{u} \in \mathcal{U}$ and hence $\mathbf{F}\mathbf{u} \in \{\mathbf{u}\}^{\perp} \cap \mathcal{U}$. In the case when $\mathbf{u} \in \mathcal{T}$, we have $\mathbf{F}\mathbf{u} \in \mathcal{T}$ because $\mathcal{U}$ and $\mathcal{U}^{\perp}$ and hence $\mathcal{T} = \mathcal{U} \cap \mathcal{U}^{\perp}$ are $\mathbf{F}$-spaces. In the case when $\mathbf{u} \in \mathcal{U} \setminus \mathcal{T}$, we can apply **Prop. 5209** to conclude that $\mathbf{F}\mathbf{u} \in \mathcal{T}$ also. Since $\mathbf{u} \in \mathcal{U}$ was arbitrary, it follows that $\mathbf{F}_{>}(\mathcal{U}) \subset \mathcal{T}$. Applying the same argument to $\mathcal{U}^{\perp}$ instead of $\mathcal{U}$, we also obtain $\mathbf{F}_{>}(\mathcal{U}^{\perp}) \subset \mathcal{T}$. The rest of the Lemma is an easy consequence. $\Diamond$

Proof of Theorem: In view of **Prop. D19** of Appendix D, we may choose a two-dimensional $\mathbf{F}$-space $\mathcal{U}$. Since $\mathcal{U}$ is singular, the Lemma can be applied to it. Put $\mathcal{W} := \mathcal{U} + \mathcal{U}^{\perp}$. Then $\mathbf{F}_{>}(\mathcal{W}) \subset \mathcal{T}$ is equivalent to $\mathsf{Rng}\ \mathbf{F}_{|\mathcal{W}} \subset \mathcal{T}$. Since $\dim \mathcal{T} = 1$, it follows from **Prop. D14** that $1 \geq \dim \mathsf{Rng}\ \mathbf{F}_{|\mathcal{W}} = 3 - \dim \mathsf{Null}\ \mathbf{F}_{|\mathcal{W}}$ and hence $\dim \mathsf{Null}\ \mathbf{F}_{|\mathcal{W}} \geq 2$. Since $\mathsf{Null}\ \mathbf{F}_{|\mathcal{W}} \subset \mathsf{Null}\ \mathbf{F}$, we also have $\dim \mathsf{Null}\ \mathbf{F} \geq 2$. Since *every* subspace of $\mathsf{Null}\ \mathbf{F}$ is an $\mathbf{F}$-space and since all two-dimensional $\mathbf{F}$-spaces must be singular, it follows from **Cor. 5210** that $\dim \mathsf{Null}\ \mathbf{F} = 2$.

We now apply the Lemma to $\mathsf{Null}\ \mathbf{F}$. Since $\mathsf{Rng}\ \mathbf{F} = (\mathsf{Null}\ \mathbf{F})^{\perp}$ by **Prop. 7104**, the first statement of the Theorem is proved.

Now let $\mathbf{e} \in \mathsf{Rng}\ \mathbf{F}$ with $\mathbf{e} \cdot \mathbf{e} = 1$ be given and put $\mathbf{n} := \mathbf{F}\mathbf{e}$. Since $\mathbf{e} \in \mathsf{Rng}\ \mathbf{F}$, it follows from the Lemma that $\mathbf{n} \in \mathbf{F}_{>}(\mathsf{Rng}\ \mathcal{F}) \subset \mathcal{S}$ and hence $\mathbf{n} \cdot \mathbf{n} = 0$. We cannot have $\mathbf{n} = \mathbf{0}$, because otherwise we would have $\mathbf{e} \in \mathsf{Rng}\ \mathbf{F} \cap \mathsf{Null}\ \mathbf{F} = \mathcal{S}$, which is incompatible with $\mathbf{e} \cdot \mathbf{e} = 1$. Since $\dim \mathsf{Rng}\ \mathbf{F} = 2$, $\{\mathbf{e}, \mathbf{n}\}$ must be a basis of $\mathsf{Rng}\ \mathbf{F}$. We now choose

$\mathbf{w} \in \mathcal{V}$ such that $\mathbf{e} = \mathbf{Fw}$. We have $\mathbf{e} \cdot \mathbf{w} = \mathbf{e} \cdot \mathbf{n} = 0$ because $\mathbf{F}$ is skew.

Now let $\mathbf{v} \in \mathcal{V}$ be given. Since $\mathbf{Fv} \in \mathsf{Rng}\,\mathbf{F}$, we can determine $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbf{Fv} = \lambda\mathbf{n} + \mu\mathbf{e}. \tag{72.4}$$

Taking the inner product with $\mathbf{e}$ gives

$$\mu = \mathbf{e} \cdot \mathbf{Fv} = -\mathbf{v} \cdot \mathbf{Fe} = -\mathbf{v} \cdot \mathbf{n}.$$

Taking the inner product of (72.4) with $\mathbf{w}$ gives

$$\lambda(\mathbf{n} \cdot \mathbf{w}) = \mathbf{w} \cdot \mathbf{Fv} = -\mathbf{v} \cdot \mathbf{Fw} = -\mathbf{v} \cdot \mathbf{e}.$$

Since

$$\mathbf{n} \cdot \mathbf{w} = \mathbf{Fe} \cdot \mathbf{w} = -\mathbf{e} \cdot \mathbf{Fw} = -\mathbf{e} \cdot \mathbf{e} = -1,$$

we conclude that $\lambda = \mathbf{v} \cdot \mathbf{e}$. Therefore, (72.4) gives

$$\mathbf{Fv} = (\mathbf{v} \cdot \mathbf{e})\mathbf{n} - (\mathbf{v} \cdot \mathbf{n})\mathbf{e} = (\mathbf{n} \otimes \mathbf{e} - \mathbf{e} \otimes \mathbf{n})\mathbf{v}.$$

Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, we conclude, using **Def. 7105**, that (72.1) and (72.2) are valid.

Now let a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given. Since $\mathsf{Null}\,\mathbf{F}$ is singular, it follows from **Prop. 5209** that $\mathcal{F}_1 \cap \mathsf{Null}\,\mathbf{F} = \emptyset$, and hence $\mathbf{Fd} \neq \mathbf{0}$. Therefore we may determine $\alpha \in \mathbb{P}^\times$ and $\mathbf{e} \in \mathsf{Rng}\,\mathbf{F}$ with $\mathbf{e} \cdot \mathbf{e} = 1$ such that

$$\mathbf{Fd} = \alpha\mathbf{e}. \tag{72.5}$$

We have $\mathbf{d} \cdot \mathbf{e} = 0$ and that α is the intensity of the electric field $\alpha\mathbf{e}$ relative to $\mathbf{d}$. Applying the part of the Theorem already proved, we find that (72.1) and (72.2) are valid for this choice of $\mathbf{e}$.

Put $\mathbf{f} := \frac{1}{\alpha}\mathbf{n} - \mathbf{d}$. Since $\mathbf{n} \cdot \mathbf{e} = 0$, we have $\mathbf{f} \cdot \mathbf{e} = 0$. Since

$$\mathbf{n} \cdot \mathbf{d} = \mathbf{Fe} \cdot \mathbf{d} = -\mathbf{e} \cdot \mathbf{Fd} = -\mathbf{e} \cdot (\alpha\mathbf{e}) = -\alpha,$$

we have

$$\mathbf{f} \cdot \mathbf{d} = -\frac{1}{\alpha}\alpha - \mathbf{d} \cdot \mathbf{d} = -1 + 1 = 0$$

and hence,

$$\mathbf{f} \cdot \mathbf{f} = \mathbf{f} \cdot \left(\frac{1}{\alpha}\mathbf{n} - \mathbf{d}\right) = \frac{1}{\alpha}(\mathbf{f} \cdot \mathbf{n}) = -\frac{1}{\alpha}(\mathbf{d} \cdot \mathbf{n}) = 1.$$

Substituting $\mathbf{n} = \alpha(\mathbf{d} + \mathbf{f})$ into (72.2) yields

$$\mathbf{F} = \alpha(\mathbf{d} \wedge \mathbf{e}) + \alpha(\mathbf{f} \wedge \mathbf{e}). \tag{72.6}$$

We may choose $\mathbf{g} \in \{\mathbf{d}, \mathbf{e}, \mathbf{f}\}^{\perp}$ such that $|\mathbf{g}| = 1$. Then $(\mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g})$ is an orthonormal list-basis of $\mathcal{V}$. It easily follows from (72.6) that the matrix of $\mathbf{F}$ relative to this basis is given by (72.3). Also, $(\mathbf{e}, \mathbf{f}, \mathbf{g})$ is an orthonormal list-basis of $\mathcal{W} := \{\mathbf{d}\}^{\perp}$ and $\mathbf{H_d} = \alpha(\mathbf{f} \wedge \mathbf{e}) \in \mathsf{Skew}\,\mathcal{W}$ is the magnetic part of $\mathbf{F}$ relative to $\mathbf{d}$. The intensity of the magnetic part of $\mathbf{F}$ relative to $\mathbf{d}$ is α (dependent on the choice of $\mathbf{d}$), and the axis of $\mathbf{H_d}$ is $\mathsf{Null}\,\mathbf{H_d} = \mathbb{R}\mathbf{g}$. The electric part $\mathsf{E}_\mathbf{d} = \alpha\mathbf{e}$ belongs to $(\mathsf{Null}\,\mathbf{H_d})^{\perp} = \mathsf{Rng}\,\mathbf{H_d}$. $\Diamond$

7.3 Particles in Constant Electromagnetic Fields

We now begin an investigation of particles in a constant electromagnetic field. In other words, we endeavor to describe the worldpaths of charged particles which are injected into constant electromagnetic fields. Although we do not offer a formal definition of a "charged particle", suffice it to say that charged particles are those particles whose paths can be altered by the presence of an electromagnetic field.

We first offer an informal justification for using skew lineons to model electromagnetic fields. Given $x \in \mathcal{E}$ and a world-direction $\mathbf{d} \in \mathcal{F}_1$, an electromagnetic field should assign a force per unit mass which would be the force on each charged particle whose worldpath passes through x and whose world-direction at x is $\mathbf{d}$. Moreover, this force per unit mass should be proportional to the charge of the particle.

In light of the discussion in §**5.6**, this force must also belong to $\{\mathbf{d}\}^{\perp}$. As a result, the electromagnetic field at x should be a mapping which assigns to each world-direction a corresponding force per unit mass; *i.e.,* a mapping $\varphi : \mathcal{F}_1 \to \mathcal{V}$ such that $\varphi(\mathbf{d}) \in \{\mathbf{d}\}^{\perp} \subset \mathcal{V}^{+} \cup \{\mathbf{0}\}$ for all $\mathbf{d} \in \mathcal{F}_1$. We see that $\mathbf{d} \cdot \varphi(\mathbf{d}) = 0$ for all $\mathbf{d} \in \mathcal{F}_1$.

Now assume that there is a lineon $\mathbf{F} \in \mathsf{Lin}\,\mathcal{V}$ whose restriction $\mathbf{F}|_{\mathcal{F}_1}$ is φ; that is, $\mathbf{F}\mathbf{d} = \varphi(\mathbf{d})$ for all $\mathbf{d} \in \mathcal{F}_1$. One can show that this is impossible unless $\mathbf{v} \cdot \mathbf{F}\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{V}$; *i.e.,* unless $\mathbf{F}$ is skew (see **Thm. 7101**). Hence, rather than specifying a mapping such as φ, we might as well consider the electromagnetic field at x as a skew lineon. This motivates the following definition.

7300 Definition: *An* **electromagnetic field on** $\mathcal{E}$ *is given by a mapping* $\widehat{\mathbf{F}} : \mathcal{E} \to \mathsf{Skew}\ \mathcal{V}$*; i.e., a skew lineon field.*

Let an electromagnetic field $\widehat{\mathbf{F}}$ be given. For simplicity, we assume that $\widehat{\mathbf{F}}$ is constant, and determine $\mathbf{F} \in \mathsf{Skew}\ \mathcal{V}$ such that $\mathbf{F} = \widehat{\mathbf{F}}(x)$ for all $x \in \mathcal{E}$. Although this restriction may seem severe, many interesting problems may still be discussed.

Now let a charged particle be given whose worldpath $\mathcal{L}$ is described by a smooth time-parameterization $p : I \to \mathcal{E}$ (where I is a genuine interval in $\mathbb{R}$), and whose world-momentum is given by a smooth mapping $\mathbf{p} : I \to \mathcal{V}$ so that the world-momentum of the particle at $p(t)$ is $\mathbf{p}(t) \in \mathbb{P}^{\times} p^{\bullet}(t)$ for all $t \in I$. We put $\mathbf{u} := p^{\bullet}$, and recall that $\mathsf{Rng}\ \mathbf{u} \subset \mathcal{F}_1$ (see **Prop. 3409**). Given the above remarks and considering the disucssion in §**5.6**, it seems reasonable to require that

$$\mathbf{p}^{\bullet} = e\mathbf{F}\mathbf{u},$$

where e is the charge of the particle. This relationship is often referred to as the *Lorentz law.*

Remark: We emphasize that this is only an approximation, as the charged particle itself generates its own electromagnetic field, the effects of which are too complicated to be included in our discussion.

We now assume that the particle's charge and mass are constant throughout the life of the particle, and we denote them by e and m, respectively. Since we assume that the mass of the particle is constant, we see that $\mathbf{p} = m\mathbf{u}$ and hence $\mathbf{p}^{\bullet} = m\mathbf{u}^{\bullet}$. Hence, we may rewrite the Lorentz law as

$$p^{\bullet\bullet} = \mathbf{u}^{\bullet} = \gamma\mathbf{F}\mathbf{u}, \quad \text{where} \quad \gamma := e/m. \tag{73.1}$$

For this discussion, we assume that $\mathbf{F}$ is regular and non-zero. Recall from the previous section that we may determine a two-dimensional positive-regular subspace $\mathcal{U}$ of $\mathcal{V}$ and $\varepsilon, \eta \in \mathbb{P}$ such that $\mathbf{F}_{>}(\mathcal{U}) \subset \mathcal{U}$, $\mathbf{F}_{>}(\mathcal{U}^{\perp}) \subset \mathcal{U}^{\perp}$, and

$$\mathbf{F}^2_{|\mathcal{U}} = -\eta^2 \mathbf{1}_{\mathcal{U}} \quad \text{and} \quad \mathbf{F}^2_{|\mathcal{U}^{\perp}} = \varepsilon^2 \mathbf{1}_{\mathcal{U}^{\perp}}. \tag{73.2}$$

Note that ε and η are not both zero; otherwise, (73.2) would imply that $\mathbf{F}$ is the zero mapping.

We begin with a few observations. Since $\mathcal{U}$ is regular, then $(\mathcal{U},\mathcal{U}^\perp)$ is a decomposition of $\mathcal{V}$ (see **Def. D07** of Appendix D). We may therefore determine $\mathbf{v},\mathbf{w} : I \to \mathcal{V}$ such that $\mathbf{u} = \mathbf{v} + \mathbf{w}$, $\mathsf{Rng}\ \mathbf{v} \subset \mathcal{U}$, and $\mathsf{Rng}\ \mathbf{w} \subset \mathcal{U}^\perp$, and hence $\mathsf{Rng}\ \mathbf{v}^\bullet \subset \mathcal{U}$ and $\mathsf{Rng}\ \mathbf{w}^\bullet \subset \mathcal{U}^\perp$.

Now $\mathbf{u}^\bullet = \gamma\mathbf{F}\mathbf{u} = \gamma\mathbf{F}\mathbf{v} + \gamma\mathbf{F}\mathbf{w}$. Since $\mathsf{Rng}\ \mathbf{F}\mathbf{v} \subset \mathcal{U}$ and $\mathsf{Rng}\ \mathbf{F}\mathbf{w} \in \mathcal{U}^\perp$, it follows from the fact that $(\mathcal{U},\mathcal{U}^\perp)$ is a decomposition of $\mathcal{V}$ that

$$\mathbf{u}^\bullet = \mathbf{v}^\bullet + \mathbf{w}^\bullet, \quad \mathbf{v}^\bullet = \gamma\mathbf{F}\mathbf{v}, \quad \text{and} \quad \mathbf{w}^\bullet = \gamma\mathbf{F}\mathbf{w}. \tag{73.3}$$

If it were the case that $\gamma = 0$, then it is easily seen that $\mathbf{u}$ would be constant, and hence the particle would be a free particle. But $\gamma = 0$ means that the particle has zero charge, and thus an electromagnetic field would have no effect on its path. So we assume, from now on, that $\gamma \neq 0$.

We have from (73.2) that $\mathbf{F}^2\mathbf{v} = -\eta^2\mathbf{v}$ and $\mathbf{F}^2\mathbf{w} = \varepsilon^2\mathbf{w}$. By differentiating (73.3) and using (73.2) and (73.3), we see that

$$\mathbf{v}^{\bullet\bullet} = \gamma\mathbf{F}\mathbf{v}^\bullet = \gamma^2\mathbf{F}^2\mathbf{v} = -\gamma^2\eta^2\mathbf{v}, \tag{73.4}$$

$$\mathbf{w}^{\bullet\bullet} = \gamma\mathbf{F}\mathbf{w}^\bullet = \gamma^2\mathbf{F}^2\mathbf{w} = \gamma^2\varepsilon^2\mathbf{w}. \tag{73.5}$$

Assume for convenience that $0 \in I$, and suppose that the values of $\mathbf{u}$ and $\mathbf{u}^\bullet$ at $0 \in I$ are prescribed; that is, given $\mathbf{u}_0 \in \mathcal{F}_1$ and $\mathbf{u}_0' \in \mathcal{V}^+ \cup \{\mathbf{0}\}$, we put

$$\mathbf{u}(0) := \mathbf{u}_0, \quad \mathbf{u}^\bullet(0) := \mathbf{u}_0'. \tag{73.6}$$

Note that $\mathbf{u}_0$ is the world-direction of the charged particle at $0 \in I$.

Since $(\mathcal{U},\mathcal{U}^\perp)$ is a decomposition of $\mathcal{V}$, we may determine $\mathbf{u}_m, \mathbf{u}_m' \in \mathcal{U}$ and $\mathbf{u}_e, \mathbf{u}_e' \in \mathcal{U}^\perp$ such that

$$\mathbf{u}_0 = \mathbf{u}_m + \mathbf{u}_e, \quad \mathbf{u}_0' = \mathbf{u}_m' + \mathbf{u}_e'. \tag{73.7}$$

Here, the subscripts "m" and "e" denote the relationships to the magnetic and electric parts of $\mathbf{F}$, respectively, as indicated in (73.2) with the relationships between η and $\mathcal{U}$, and ε and $\mathcal{U}^\perp$.

Since $\mathbf{u}_0 \in \mathcal{F}_1$, we see that

$$-1 = \mathbf{u}_m \cdot \mathbf{u}_m + \mathbf{u}_e \cdot \mathbf{u}_e.$$

Since $\mathbf{u}_m \in \mathcal{U}$ and $\mathcal{U}$ is positive-regular, we must have $\mathbf{u}_m \cdot \mathbf{u}_m \geq 0$, and hence $\mathbf{u}_e \cdot \mathbf{u}_e \leq -1$. Put $\mu := \tau(\mathbf{u}_e) = \sqrt{-\mathbf{u}_e \cdot \mathbf{u}_e}$ and $\nu := \sqrt{1 - \mu^{-2}}$. Then we may deduce from the remarks above that

$$\mu \geq 1, \quad \nu \in [0, 1], \quad \text{and} \quad |\mathbf{u}_m| = \mu\nu. \tag{73.8}$$

Since $\mu = \tau(\mathbf{u}_e)$, we may choose $\hat{\mathbf{u}}_e \in \mathcal{U}^\perp$ with $\tau(\hat{\mathbf{u}}_e) = 1$ such that

$$\mathbf{u}_e = \mu\hat{\mathbf{u}}_e. \tag{73.9}$$

Assume now that $\nu \neq 0$. Then $|\mathbf{u}_m| \neq 0$ (see (73.8)), and thus we may determine $\hat{\mathbf{u}}_m \in \mathcal{U}$ with $|\hat{\mathbf{u}}_m| = 1$ such that

$$\mathbf{u}_m = \mu\nu\hat{\mathbf{u}}_m. \tag{73.10}$$

Since $\mathbf{u}_0 = \mathbf{v}(0) + \mathbf{w}(0)$ and $\mathbf{u}_0' = \mathbf{v}^\bullet(0) + \mathbf{w}^\bullet(0)$, it follows from (73.7) that

$$\mathbf{v}(0) = \mathbf{u}_m, \qquad \mathbf{v}^\bullet(0) = \mathbf{u}_m', \tag{73.11}$$

$$\mathbf{w}(0) = \mathbf{u}_e, \qquad \mathbf{w}^\bullet(0) = \mathbf{u}_e'. \tag{73.12}$$

Evaluating the middle equation in (73.3) at 0 and using (73.11) yields

$$\mathbf{u}_m' = \gamma\mathbf{F}\mathbf{u}_m. \tag{73.13}$$

We use this and (73.4) to see that

$$\begin{aligned} \mathbf{u}_m' \cdot \mathbf{u}_m' &= \gamma^2\mathbf{F}\mathbf{u}_m \cdot \mathbf{F}\mathbf{u}_m \\ &= -\gamma^2\mathbf{F}^2\mathbf{u}_m \cdot \mathbf{u}_m \\ &= \gamma^2\eta^2\mathbf{u}_m \cdot \mathbf{u}_m, \end{aligned}$$

and hence (see (73.8)) that

$$|\mathbf{u}_m'| = \gamma\eta|\mathbf{u}_m| = \gamma\eta\mu\nu. \tag{73.14}$$

Since $\mathbf{F}$ is skew, it follows from (73.13) that

$$\mathbf{u}_m' \cdot \mathbf{u}_m = \gamma\mathbf{F}\mathbf{u}_m \cdot \mathbf{u}_m = 0.$$

We may similarly consider (73.12) to conclude that

$$|\mathbf{u}_e'| = \gamma\mu\varepsilon, \quad \mathbf{u}_e' \cdot \mathbf{u}_e = 0. \tag{73.15}$$

From now on, whenever we refer to the electric or magnetic part of $\mathbf{F}$, we will be referring to the magnetic or electric part of $\mathbf{F}$ relative to the world-direction $\hat{\mathbf{u}}_e$. We put $\mathcal{W} := \{\hat{\mathbf{u}}_e\}^\perp$. Recall that $\mathbf{H} = \mathbf{PF}|_\mathcal{W}$, where $\mathbf{P}$ is the orthogonal projection of $\mathcal{V}$ onto $\mathcal{W}$ (see **Def. 7200**).

Now that some preliminary observations have been made, we proceed to an explicit description of the worldpath of the charged particle. We first consider the case when $\eta \neq 0$ and $\varepsilon \neq 0$. Since $\eta \neq 0$, then we may, as a result of (73.14), choose $\hat{\mathbf{u}}'_m \in \mathcal{U}$ with $|\hat{\mathbf{u}}'_m| = 1$ such that

$$\mathbf{u}'_m = \gamma\eta\mu\nu\hat{\mathbf{u}}'_m.$$

It may easily be shown that both $\mathbf{H}\hat{\mathbf{u}}_m = \eta\hat{\mathbf{u}}'_m$ and $\mathbf{H}\hat{\mathbf{u}}'_m = -\eta\hat{\mathbf{u}}_m$, and hence η is the intensity of the magnetic part of $\mathbf{F}$ relative to $\hat{\mathbf{u}}_e$.

Solving (73.4) with the data given in (73.11) yields

$$\mathbf{v}(s) = \mu\nu(\cos(\gamma\eta s)\hat{\mathbf{u}}_m + \sin(\gamma\eta s)\hat{\mathbf{u}}'_m)$$

for all $s \in I$.

Since $\varepsilon \neq 0$, we see from (73.15) that we may choose $\hat{\mathbf{u}}'_e \in \mathcal{U}^\perp$ with $|\hat{\mathbf{u}}'_e| = 1$ such that

$$\mathbf{u}'_e = \gamma\mu\varepsilon\hat{\mathbf{u}}'_e. \tag{73.16}$$

Evaluating the right equation of (73.3) at 0 and using (73.12) results in

$$\mathbf{u}'_e = \gamma\mathbf{F}\mathbf{u}_e.$$

Recall that the electric part (*i.e.*, the electric part of $\mathbf{F}$ relative to $\hat{\mathbf{u}}_e$) is given by $\mathbf{F}\hat{\mathbf{u}}_e$. Thus, it follows from (73.9) and (73.16) that

$$\mathsf{E} = \mathbf{F}\hat{\mathbf{u}}_e = \frac{1}{\mu}\mathbf{F}\mathbf{u}_e = \frac{1}{\gamma\mu}\mathbf{u}'_e = \varepsilon\hat{\mathbf{u}}'_e.$$

Recall that ε is the intensity of the electric part.

It is clear that $(\hat{\mathbf{u}}_e, \hat{\mathbf{u}}'_e, \hat{\mathbf{u}}_m, \hat{\mathbf{u}}'_m)$ is an orthonormal list-basis for $\mathcal{V}$. We see by comparing the above analysis with the case when $\mathbf{F}$ is regular in **Thm. 7202** that the matrix of $\mathbf{F}$ relative to this basis is given by

$$[\mathbf{F}] = \begin{bmatrix} 0 & \epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta \\ 0 & 0 & -\eta & 0 \end{bmatrix}.$$

Solving (73.5) with the information provided in (73.12) results in

$$\mathbf{w}(s) = \mu(\cosh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \sinh(\gamma\varepsilon s)\hat{\mathbf{u}}'_e)$$

for all $s \in I$.

We combine the above descriptions of $\mathbf{v}$ and $\mathbf{w}$ to yield

$$\begin{aligned}\mathbf{u}(s) &= \mathbf{v}(s) + \mathbf{w}(s)\\ &= \mu(\cosh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \sinh(\gamma\varepsilon s)\hat{\mathbf{u}}'_e + \nu(\cos(\gamma\eta s)\hat{\mathbf{u}}_m + \sin(\gamma\eta s)\hat{\mathbf{u}}'_m))\end{aligned}$$

for all $s \in I$.

We may proceed to solve for p (recall that $\mathbf{u} = p^{\bullet}$) and obtain

$$p(s) = q + \frac{\mu}{\gamma}\left(\frac{1}{\varepsilon}\left(\sinh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \cosh(\gamma\varepsilon s)\hat{\mathbf{u}}'_e\right)\right.$$
$$\left. + \frac{\nu}{\eta}\left(\sin(\gamma\eta s)\hat{\mathbf{u}}_m - \cos(\gamma\eta s)\hat{\mathbf{u}}'_m\right)\right)$$

for all $s \in I$, where

$$q = p(0) - \frac{\mu}{\gamma\varepsilon}\hat{\mathbf{u}}'_e + \frac{\mu\nu}{\gamma\eta}\hat{\mathbf{u}}'_m.$$

We now describe geometrically the worldpath of the particle given by p in $q + \mathcal{W}$. We proceed as in §**5.7**, using the notation explained there. We put

$$J := \mathsf{Rng}\ \bar{\mathsf{t}}^q_{\hat{\mathbf{u}}_e} = \left\{\frac{\mu}{\gamma\varepsilon}\sinh(\gamma\varepsilon s) \mid s \in I\right\}.$$

Then $\alpha : I \to J$ and $p_\perp : J \to q + \mathcal{W}$ are given by

$$\alpha(s) = \frac{\mu}{\gamma\varepsilon}\sinh(\gamma\varepsilon s) \tag{73.17}$$

for all $s \in I$ and

$$p_\perp(t) = q + \frac{\mu}{\gamma\varepsilon}\cosh(\gamma\varepsilon\alpha^{\leftarrow}(t))\hat{\mathbf{u}}'_e + \frac{\mu\nu}{\gamma\eta}(\sin(\gamma\eta\alpha^{\leftarrow}(t))\hat{\mathbf{u}}_m - \cos(\gamma\eta\alpha^{\leftarrow}(t))\hat{\mathbf{u}}'_m)$$

for all $t \in J$. We first note that

$$t \mapsto q + \frac{\mu\nu}{\gamma\eta}(\sin(\gamma\eta\alpha^{\leftarrow}(t))\hat{\mathbf{u}}_m - \cos(\gamma\eta\alpha^{\leftarrow}(t))\hat{\mathbf{u}}'_m)$$

describes a circle with center q and radius $\rho := \mu\nu/\gamma\eta$. Thus, the path described by $p_\perp$ lies on the surface of a cylinder in $q + \mathcal{W}$ of radius ρ.

If we define $\theta : J \to \mathbb{R}$ by

$$\theta(t) := \gamma\eta\alpha^{\leftarrow}(t)$$

for all $t \in J$, we see that

$$p_\perp(t) = q + \frac{\mu}{\gamma\varepsilon}\cosh\left(\frac{\varepsilon}{\eta}\theta(t)\right)\hat{\mathbf{u}}'_e + \frac{\mu\nu}{\gamma\eta}(\sin(\theta(t))\hat{\mathbf{u}}_m - \cos(\theta(t))\hat{\mathbf{u}}'_m)$$

for all $t \in J$, and hence the particle describes a path which can be likened to wrapping a catenary around a cylinder of radius ρ.

Assuming that $[0, 50] \subset J$, and with mappings $x, y, z : J \to \mathbb{R}$ given by

$$\begin{aligned} x(t) &:= -\rho\cos(\theta(t)), \\ y(t) &:= \rho\sin(\theta(t)), \text{ and} \\ z(t) &:= \frac{\mu}{\gamma\epsilon}\cosh\left(\frac{\varepsilon}{\eta}\theta(t)\right) \end{aligned}$$

for all $t \in J$, we see that

$$p_\perp(t) = q + z(t)\hat{\mathbf{u}}'_e + y(t)\hat{\mathbf{u}}_m + x(t)\hat{\mathbf{u}}'_m$$

for all $t \in J$. We may graphically describe $p_\perp$ as in the following figure, where values of $\nu := 1/2$, $\gamma := 1$, $\varepsilon := 0.03$, and $\eta := 1$ have been assigned, and where t ranges over the interval $[0, 50]$.

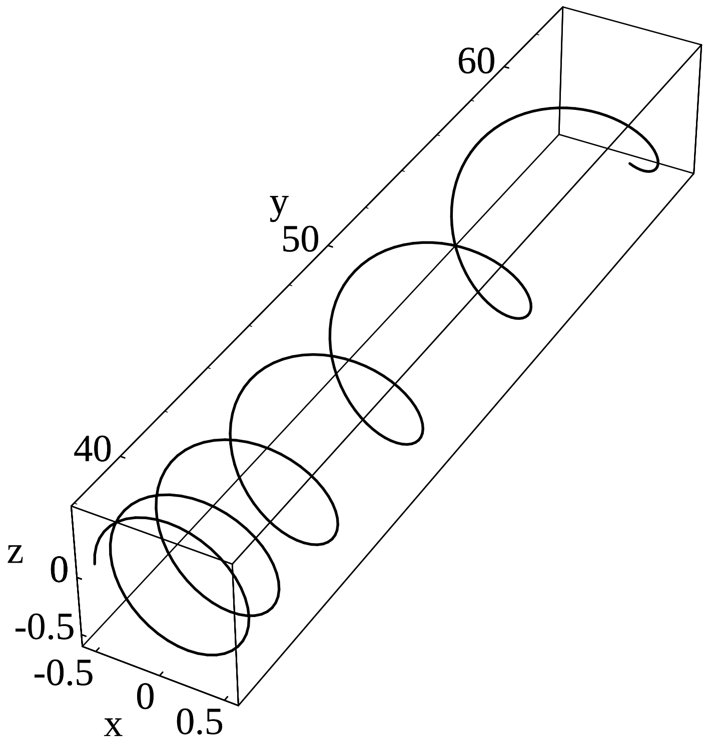

Figure 73a

If we define $r, \omega : J \to \mathbb{R}$ by

$$r(t) := \left(\frac{\mu}{\gamma\varepsilon}\cosh\left(\frac{\varepsilon}{\eta}\theta\right)\right)^{\bullet}(t)$$

and

$$\omega(t) := \theta^{\bullet}(t)$$

for all $t \in J$, where r can be interpreted as the rate that the particle travels along the direction of the axis of the cylinder and ω can be interpreted as the angular velocity with which the particle moves "around" the cylinder, it can be shown that

$$\frac{r(t)}{\omega(t)} = \frac{\varepsilon t}{\eta}$$

for all $t \in J$, and hence this ratio is independent of the mass and the charge of the particle.

In the case where $\eta \neq 0$ and $\varepsilon = 0$, we find that

$$p(s) = q + \mu s \hat{\mathbf{u}}_e + \frac{\mu\nu}{\gamma\eta}\left(\sin(\gamma\eta s)\hat{\mathbf{u}}_m - \cos(\gamma\eta s)\hat{\mathbf{u}}'_m\right)$$

for all $s \in I$, where

$$q = p(0) + \frac{\mu\nu}{\gamma\eta}\hat{\mathbf{u}}'_m.$$

The details of the calculations are left to the Exercises.

The path that the particle makes in $q + \mathcal{W}$ is described by the parameterization

$$p_\perp(t) = q + \frac{\mu\nu}{\gamma\eta}\left(\sin\left(\frac{\gamma\eta}{\mu}t\right)\hat{\mathbf{u}}_m - \cos\left(\frac{\gamma\eta}{\mu}t\right)\hat{\mathbf{u}}'_m\right)$$

for all $t \in J$ (where J and $p_\perp$ are as in §**5.7**), which is easily seen to describe a circle centered at q with radius $\rho := \mu\nu/\gamma\eta$. This relationship is sometimes written

$$\frac{e}{m} = \frac{1}{\eta\rho}\frac{\nu}{\sqrt{1-\nu^2}},$$

or equivalently,

$$e\eta\rho = m\mu\nu,$$

and is referred to in this form as the *cyclotron formula* because it is useful in the design of cyclotrons. One may easily verify that $|p^\bullet_\perp(t)| = \nu$ for all $t \in J$, so that the speed of the charged particle relative to $\hat{\mathbf{u}}_e$ is ν.

In the event that $\eta = 0$ and $\varepsilon \neq 0$, we see that

$$p(s) = q + \frac{\mu}{\gamma\varepsilon}(\sinh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \cosh(\gamma\varepsilon s)\hat{\mathbf{u}}'_e) + \mu\nu s\hat{\mathbf{u}}_m$$

for all $s \in I$, where

$$q = p(0) - \frac{\mu}{\gamma\varepsilon}\hat{\mathbf{u}}'_e.$$

The details of the calculations are left to the Exercises.

The path of this particle in $q + \mathcal{W}$ is a catenary; this is useful in the design of cathode ray tubes like the ones used in television sets. Note that α is again as in (73.17), and thus

$$p_\perp(t) = q + \frac{\mu}{\gamma\varepsilon}\cosh(\gamma\varepsilon\alpha^{\leftarrow}(t))\hat{\mathbf{u}}'_e + \mu\nu\alpha^{\leftarrow}(t)\hat{\mathbf{u}}_m$$

for all $t \in J$. When $x, y : J \to \mathbb{R}$ satisfy $p_\perp(t) = q + y(t)\hat{\mathbf{u}}'_e + x(t)\hat{\mathbf{u}}_m$ for all $t \in J$, we see that

$$y(t) = \frac{\mu}{\gamma\varepsilon}\cosh\left(\frac{\gamma\varepsilon}{\mu\nu}x(t)\right)$$

for all $t \in J$. Hence $p_\perp$ may be graphically described by a catenary as in the following figure.

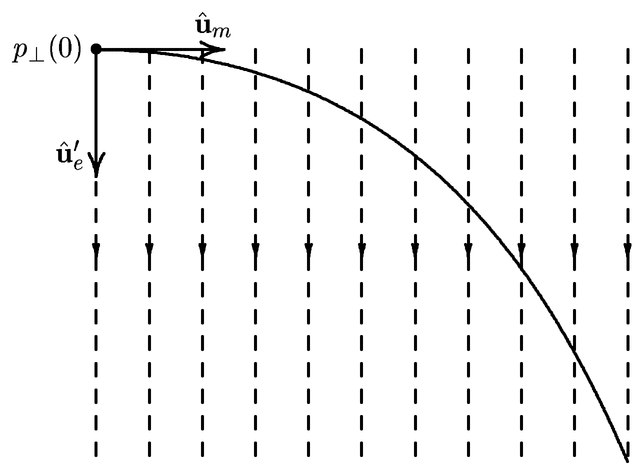

Figure 73b

Finally, we examine the case when $\nu = 0$. We have from (73.8) that $|\mathbf{u}_m| = 0$, and from (73.14) that $|\mathbf{u}'_m| = 0$. Since $\mathbf{u}_m, \mathbf{u}'_m \in \mathcal{U}$ and $\mathcal{U}$ is positive-regular, it follows that $\mathbf{u}_m = \mathbf{u}'_m = \mathbf{0}$. Thus, we see that $\mathbf{u}_0 = \mathbf{u}_e$ and $\mathbf{u}'_0 = \mathbf{u}'_e$. We also see that (73.4) and (73.11) reduce to

$$\mathbf{v}^{\bullet\bullet} = -\gamma^2\eta^2\mathbf{v}, \quad \mathbf{v}(0) = \mathbf{v}^{\bullet}(0) = \mathbf{0}.$$

The solution to this differential equation is simply $\mathbf{v}(s) = \mathbf{0}$ for all $s \in I$, and hence $\mathsf{Rng}\ \mathbf{u} = \mathsf{Rng}\ \mathbf{w} \subset \mathcal{U}^\perp$. Since $\mathsf{Rng}\ \mathbf{H} \subset \mathcal{U}$, we see that the magnetic part of $\mathbf{F}$ in this case has no effect on the path of the particle.

We proceed to examine the two possible cases.

1. $\varepsilon = 0$.

   We see from (73.15) that $|\mathbf{u}_e'| = 0$, and since $\mathbf{u}_e' = \mathbf{u}_0' \in \mathcal{V}^+ \cup \{\mathbf{0}\}$, that $\mathbf{u}_e' = \mathbf{0}$. Then (73.5) and (73.12) become (since $\nu = 0$ yields $\mu = 1$)

   $$\mathbf{w}^{\bullet\bullet} = \mathbf{0}, \quad \mathbf{w}(0) = \hat{\mathbf{u}}_e, \quad \mathbf{w}^\bullet(0) = \mathbf{0}.$$

   Solving for $\mathbf{w}$ and using the fact that $\mathbf{v} = \mathbf{0}$ yields that

   $$\mathbf{u}(s) = \hat{\mathbf{u}}_e$$

   for all $s \in I$. We may solve for p to get

   $$p(s) = p(0) + s\hat{\mathbf{u}}_e$$

   for all $s \in I$. In this case, we see that the particle behaves as a free particle (with zero charge).

2. $\varepsilon \neq 0$.

   In this case, we solve (73.5) with (73.12) to yield

   $$\mathbf{u}(s) = \cosh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \sinh(\gamma\varepsilon s)\hat{\mathbf{u}}_e'$$

   for all $s \in I$. Solving for p yields

   $$p(s) = q + \frac{1}{\gamma\varepsilon}(\sinh(\gamma\varepsilon s)\hat{\mathbf{u}}_e + \cosh(\gamma\varepsilon s)\hat{\mathbf{u}}_e')$$

   for all $s \in I$, where

   $$q = p(0) - \frac{1}{\gamma\varepsilon}\hat{\mathbf{u}}_e'.$$

   Again using the notation of §**5.7**, we find that

   $$p_\perp(t) = q + \frac{1}{\gamma\varepsilon}\sqrt{1 + (\gamma\varepsilon t)^2}\,\hat{\mathbf{u}}_e'$$

for all $t \in J$. Thus, we see that the particle moves along a straight line with direction $\hat{\mathbf{u}}'_e$. However, the speed relative to $\hat{\mathbf{u}}_e$ varies; in fact, we have

$$|p^\bullet_\perp(t)| = \frac{\gamma\varepsilon|t|}{\sqrt{1+(\gamma\varepsilon t)^2}}$$

for all $t \in J$. Note that as $|t|$ gets larger, $|p^\bullet_\perp(t)|$ approaches 1. Also, recall that $\hat{\mathbf{u}}'_e$ is the direction of the electric part, so that the charged particle always moves in the direction of the electric part.

Exercises

Let a Minkowskian spacetime $\mathcal{E}$ be given. Assume that $\mathsf{sig}\,\mathcal{V} = (3,1)$.

EXERCISES, I

1. Complete the proof of **Thm. 7107**.
2. Prove **Prop. 7108**.
3. Suppose that $\mathbf{F} \in \mathsf{Skew}\,\mathcal{V}$ is regular, and that $\mathcal{U}$ is a two-dimensional $\mathbf{F}$-space. Show that of $\mathcal{U}$ and $\mathcal{U}^\perp$, one is positive-regular and the other has signature $(1,1)$ (see §**7.2**).
4. Show that when $\mathbf{F}$ is regular, the $\mathbf{F}$-space $\mathcal{U}$ is uniquely determined by $\mathbf{F}$ (see §**7.2**).
5. Show that $\mathcal{T} := \mathcal{U} \cap \mathcal{U}^\perp$ is a one-dimensional totally singular subspace of $\mathcal{V}$ and that $\mathcal{T} = \mathcal{U} \cap \mathcal{V}^0 = \mathcal{U}^\perp \cap \mathcal{V}^0$ (see §**7.2**).
6. Show that $\mathsf{dim}\,\mathcal{Y} \neq 1$, where $\mathcal{Y}$ is given as in §**7.2**.
7. Complete the calculations in the case when $\eta \neq 0$ and $\varepsilon = 0$ (see §**7.3**).
8. Complete the calculations in the case when $\eta = 0$ and $\varepsilon \neq 0$ (see §**7.3**).

EXERCISES, II

1. Show that $\mathbf{H} = (\mathbf{F} - \mathbf{d} \wedge \mathbf{F}\mathbf{d})_{|\mathcal{W}}$ (see **Def. 7200**).

2. Show that $\mathbf{H} = \eta(\widehat{\mathbf{g}} \wedge \widehat{\mathbf{f}})$, where η, $\widehat{\mathbf{f}}$, and $\widehat{\mathbf{g}}$ are as described in §**7.2**.

3. Show that $\mathbf{H} = \alpha(\widehat{\mathbf{f}} \wedge \widehat{\mathsf{E}})$, where α, $\widehat{\mathbf{f}}$, and $\widehat{\mathsf{E}}$ are given as in §**7.2**.

4. Let $p : I \to \mathcal{E}$ be a smooth time-parameterization of the worldpath of a particle with constant mass $m \in \mathbb{P}^\times$ and constant charge $e \in \mathbb{R}$ in a constant electromagnetic field with value $\mathbf{F} \in \mathsf{Skew}\,\mathcal{V}$.

   Let a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given. Determine $\mu : I \to \mathbb{P}^\times$, $\nu : I \to \mathbb{P}^\times$, and $\mathbf{f} : I \to \{\mathbf{d}\}^\perp$ such that

   $$p^\bullet(s) = \mu(s)(\mathbf{d} + \nu(s)\mathbf{f}(s)) \quad \text{and} \quad |\mathbf{f}(s)| = 1$$

   for all $s \in I$. Prove that

   $$\mu^\bullet(s) = \frac{e}{m}\mu(s)\nu(s)(\mathsf{E}_{\mathbf{d}} \cdot \mathbf{f}(s))$$

   for all $s \in I$.