Turán's theorem with colors

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Abstract

We consider a generalization of Turán's theorem for edge-colored graphs. Suppose that R (red) and B (blue) are graphs on the same vertex set of size n. We conjecture that if R and B each have more than $(1-1/k)n^2/2$ edges, and K is a $(k+1)$-clique whose edges are arbitrarily colored with red and blue, then $R \cup B$ contains a colored copy of K, for all $k+1 \notin \{4,6,8\}$. If $k+1 \in \{4,6,8\}$, then the same conclusion holds except for certain specified edge-colorings of K_{k+1}.

We prove this conjecture for all 2-edge-colorings of K_{k+1} that contain a monochromatic K_k. We also prove the conjecture for $k+1 \in \{3,4,5\}$.

1 Introduction

Let F be a fixed graph. Classical extremal graph theory asks for the maximum number of edges in an n vertex graph that contains no copy of F as a (not necessarily induced) subgraph. This number is denoted $\mathrm{ex}(n,F)$. The fundamental result is due to Turán, who determined $\mathrm{ex}(n,F)$ precisely when F is a complete graph. The subject has developed primarily via generalizations and extensions of Turán's theorem. These generalizations include proving Turán's theorem for random graphs [9, 11]; replacing density conditions with spectral properties [3]; strengthening the conclusion to an appropriate property about neighborhoods [1, 5]; extending Turán's theorem from graphs to hypergraphs [10], edge-colored graphs [7, 8], weighted graphs or multigraphs [2, 6].

One of these generalizations is due to Füredi and Kündgen [6], who considered the asymptotic maximum number of edges that a multigraph can have without containing v vertices spanning e edges, for all values of v and e. We consider a refinement of this problem where the maximum multiplicity is 2 and $e = \binom{v}{2}$.

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Suppose that we have a red graph R and a blue graph B both on the same vertex set of size n. We can view the union $G = R \cup B$ as a multigraph with maximum multiplicity 2. Pairs of vertices of G come in four types: empty, red, blue, or double. Denote the number of edges in a graph H by $|H|$. The general question one can consider is:

Problem. *Let F be a fixed (simple) graph all of whose edges have been colored either red or blue. What is the minimum m, such that if both $|R|$ and $|B|$ are at least $m+1$, then $G = R \cup B$ contains a colored copy of F?*

In the case that F is monochromatic, we clearly have $m = \mathrm{ex}(n, F)$. The phenomenon we study here is that for certain F, the threshold m is the same *no matter how the edges of F are colored.*

Definition 1 *A simple graph F is* visible *for n if the following holds. Suppose that a red graph R and a blue graph B on the same vertex set of size n satisfy* $\min\{|R|, |B|\} > \mathrm{ex}(n, F)$*. Then no matter how the edges of F are colored red and blue, the union $R \cup B$ contains such a colored copy of F.*

The first author [4] recently showed that a matching of e edges is visible. Note that there are only $e + 1$ possible ways to color such a matching. In general however, the number of nonisomorphic 2-edge-colorings of F can be exponentially large in the order of F, so one might think that dense graphs F are less likely to be visible. In this paper, we consider the case when F is a clique. Our results and conjectures suggest that, apart from sporadic exceptions, this intuition is incorrect, and that cliques are indeed visible. Recall that Turán's theorem states that $\mathrm{ex}(n, K_{k+1}) = |T(n, k)|$, where $T(n, k)$ is the complete k-partite graph on n vertices with almost equal part sizes. When k divides n, we have $|T(n, k)| = (1 - 1/k)n^2/2$.

Our first result, which we believe is of independent interest, determines the correct threshold for m that guarantees several 2-edge-colored copies of K_{k+1}, namely those where there is a vertex whose neighborhood contains a red or blue k-clique The key to the proof is that we prove a stronger statement that facilitates induction.

Theorem 2 *Let $n, k > 0$. Let R (red) and B (blue) be graphs on the same vertex set of size n, with* $\min\{|R|, |B|\} > (1 - 1/k)n^2/2$*. Then $R \cup B$ contains all 2-edge-colored $(k + 1)$-cliques that have a monochromatic k-clique.*

Note that Turán's theorem immediately follows from Theorem 2 (when $k|n$) by letting $R = B$, since $K \cup \{v\}$ is a $(k + 1)$-clique. None of the known proofs of Turán's theorem seem to extend to Theorem 2. In fact, our proof of Theorem 2 provides a new proof of Turán's theorem. It is curious that in order to prove Theorem 2, which probably has only one extremal example, we actually prove a stronger result (Theorem 5 in Section 2) which has exponentially many extremal examples.

There is one obvious example of large graphs R and B whose union does not contain certain 2-edge-colored $(k + 1)$-cliques, namely let $R = B = T(n, k)$. In fact, here $R \cup B$ contains no K_{k+1}

at all. When $k+1$ is even however, there is a more subtle example that avoids only a particular colored K_{k+1}. Let $k+1=2l$. Partition a vertex set V of size n into $V_1 \cup V_2 \cup \ldots \cup V_l$. Let R consist of all edges within V_1 and all edges between two different parts V_p, V_q. Let B consist of all edges within V_i for each $i=2,\ldots,l$, and all edges between two different parts V_p, V_q. Thus the edges of the complete l-partite graph with parts $V_1,\ldots,V_l$, are double, and all other pairs of V are either red or blue. Let $n_i=|V_i|$. Now suppose that $\alpha \in (0,1)$ is fixed, n is large, $n_1 \sim \alpha n$ and $n_i \sim ((1-\alpha)/(l-1))n$ for each $i=2,\ldots,l$. The choice of α that maximizes $\min\{|R|,|B|\}$ is $\alpha=1/2$ for $l=2$, and $\alpha=(\sqrt{l-1}-1)/(l-2)$ for $l>2$. For this choice, we have

$$|R| \sim |B| \sim \begin{cases} \frac{3}{4}\binom{n}{2} & \text{if } l=2 \\ \left(1-\frac{l-2\sqrt{l-1}}{(l-2)^2}\right)\binom{n}{2} & \text{if } l>2. \end{cases} \qquad (*)$$

Now consider a colored K_{2l} whose blues edges from a spanning subgraph such that each component is a complete bipartite graph whose parts have the same size. All other edges are red. Consider a blue component with t vertices in each part. We claim that if this appears in the construction above, then it must use at least t of the parts other than V_1. If there is a vertex in V_1, then none of its t neighbors can be in V_1, and no two of its neighbors can be in the same V_i, so t parts other than V_1 must be used. If no vertex is in V_1, then the t vertices forming one of the independent sets must lie in different V_i's. This proves the claim.

The claim above shows that two vertices in different components must lie in the same V_i for some $i>1$. This is impossible since these two vertices are joined by a red edge in our colored K_{2l}. Consequently, such a colored K_{2l} cannot appear within the construction.

Clearly, a symmetrical construction can be obtained by interchanging the red and blue edges. Let us call any such 2-edge-colored K_{2l} a *$2l$-biclique*.

If we let $R=B=T(n,k)=T(n,2l-1)$, then $|R|=|B| \sim (1-1/(2l-1))\binom{n}{2}$. Asymptotically for fixed k, $(*)$ is larger than this when $l=2,3,4$, equal when $l=5$, and smaller when $l>5$. This leads us to our main conjecture which, if true, is sharp.

Conjecture 3 *Let $k \geq 2$ be fixed and $n \geq k$. Suppose that R (red) and B (blue) are n-vertex graphs on the same vertex set, K is a $(k+1)$-clique whose edges are colored red and blue in some fashion, and $R \cup B$ contains no copy of K. Then*

$$\min\{|R|,|B|\} \leq \begin{cases} (1-1/k)\,\frac{n^2}{2} & \textit{if } k+1 \notin \{4,6,8\} \textit{ or } K \textit{ is not a } (k+1)\textit{-biclique.} \\ \frac{3n^2-2n}{8} & \textit{if } K \textit{ is a 4-biclique.} \\ (2\sqrt{2}-2+o(1))\binom{n}{2} & \textit{if } K \textit{ is a 6-biclique.} \\ (\sqrt{3}/2+o(1))\binom{n}{2} & \textit{if } K \textit{ is an 8-biclique.} \end{cases}$$

Thus in particular, K_{k+1} is visible for n when $k|n$ and $k+1 \notin \{4,6,8\}$.

We verify Conjecture 3 for the first few cases. It is relatively straightforward to do this for $k+1 \in \{3,4\}$. Our main result on Conjecture 3 is

Theorem 4 *Conjecture 3 holds for $k + 1 = 5$.*

We prove Theorem 2 in Section 2, the cases $k + 1 \in \{3, 4\}$ of Conjecture 3 in Section 3, and the case $k + 1 = 5$ in Section 4. In Section 5 we give some open problems and further generalizations.

The degree of a vertex v in a graph G is denoted by $deg_G(v)$. Given a vertex subset S in a graph G, and a vertex $v \notin S$, define $\deg_G(v, S)$ to be the number of edges from v to S. $G[S]$ denotes the subgraph of G induced by the set of vertices S.

2 Red clique in a mixed neighborhood

In this section we prove Theorem 2. In attempting to prove it by induction, we were led to the following result, which is quite a bit stronger, and also needed to make the induction work.

Theorem 5 *Let $n, k > 0$ and $i \in [k]$. Let R (red) and B (blue) be graphs on the same vertex set of size n. Suppose that $|B| + (k-1)|R| > (k-1)n^2/2$. Then there is a red clique K of order k, and a vertex v disjoint from K that is joined to i vertices of K by blue edges, and to the remaining $k - i$ vertices of K by red edges.*

Proof: We proceed by induction on n. Let $G = R \cup B$, and define the weight $w(G)$ of G to be $|B| + (k-1)|R|$.

The result holds vacuously for $n \leq k$, since in this case, $w(G) \leq k\binom{n}{2} \leq (k-1)n^2/2$. So suppose that $n > k$, and the result holds for smaller n.

Define F_t to be a red clique with t vertices together with any blue graph (on the same t vertices) with maximum degree $k - 1$. So F_t has $\binom{t}{2}$ red edges and at most $(k-1)t/2$ blue edges, i.e. there are at most $(k-1)t/2$ double edges in F_t and the rest are red.

Suppose for contradiction, that G does not contain a copy of the required configuration. We will then show that G is an induced F_n. Since $w(G) = w(F_n) = (k-1)n/2 + (k-1)\binom{n}{2} = (k-1)n^2/2$, this gives a contradiction. We will show how to find an F_{t+1} once we have an F_t. Clearly we have F_1. Now suppose we have an F_t with vertex set X. By induction, $w(G - X) \leq (k-1)(n-t)^2/2$. Since $w(F_t) \leq (k-1)t/2 + (k-1)\binom{t}{2}$, we conclude that

$$\sum_{v \in V - X} \deg_B(v, X) + (k-1)\deg_R(v, X) = w(G) - w(G - X) - w(F_t) > (k-1)t(n-t).$$

Hence, there is a vertex $v \in V - X$ with $\deg_B(v, X) + (k-1)\deg_R(v, X) > (k-1)t$. Suppose that v has $t - j$ red edges to F_t for some $j \geq 0$. Then v has at least $(k-1)j + 1$ blue edges to F_t.

Case 1: $j > 0$. The number of double edges from v to F_t is at least

$$(t - j) + ((k-1)j + 1) - t = (k-2)j + 1 \geq k - 1.$$

The number of blue edges from v to F_t is at least k. Consequently, we may choose k edges from v to F_t, of which $k-1$ are double and the last is blue or double. The other endpoints of these k edges contain a red K_k, so we have the desired colored configuration.

Case 2: $j = 0$. In this case v has t (i.e. all) red edges to F_t. Now the induced subgraph $R[X \cup \{v\}]$ is a complete (red) graph. If any vertex in $X \cup \{v\}$ has blue degree at least k in $G[X \cup \{v\}]$, then G contains the required colored configuration. Otherwise, $G[X \cup \{v\}]$ is a copy of F_{t+1}, as desired. $\square$

Theorem 5 is sharp, in fact there are many examples where $|B| + (k-1)|R| = (k-1)n^2/2$ and there is no red k-clique within a blue neighborhood.

- Let $R = K_n$ and B be any $(k-1)$-regular graph; this gives exponentially many (in n) extremal examples.

- When k divides n, set $R = B = T(n,k)$. This also yields Turán's theorem when k divides n.

We note that Theorem 5 proves Conjecture 3 for a non-monochromatic 2-edge-colored K_{k+1} that contains a monochromatic K_k, and Turán's theorem proves it when K_{k+1} is monochromatic.

3 Triangles and Tetrahedra

In this section we prove Conjecture 3 for $k+1 \in \{3,4\}$. The case $k+1 = 3$ follows immediately from Theorem 2 (there are also more direct ways of proving this).

Proof of Conjecture 3 for $k+1 \in \{3,4\}$.

- $k+1 = 3$: Suppose that we are given R and B, each with more than $n^2/4$ edges. Then by Turán's theorem for triangles (i.e., Mantel's theorem), we are guaranteed both a blue triangle and a red triangle. By symmetry, it therefore suffices to find a triangle with one blue and two red edges. This follows directly from Theorem 2.

- $k+1 = 4$ and K is not a 4-biclique: Suppose that we are given graphs R and B, each with more than $n^2/3$ edges. If K is monochromatic, then the result follows from Turán's theorem, considering only the red or blue edges in G. If K contains a monochromatic triangle, then Theorem 5 implies that G contains K.

The only 2-edge-colorings of K_4 that do not contain a monochromatic triangle are a 4-biclique and a 2-edge-coloring in which the red and blue edges both induce a path of length three.

We claim that if $|R| + |B| > (2/3)n^2$, then $G = R \cup B$ contains a 2-edge-colored K_4 in which the red and blue edges both induce paths of length three. We prove this by induction.

The statement is vacuously true for $n < 4$, so suppose $n \geq 4$ and the statement is true for all graphs with fewer vertices.

Since $|R \cap B| > 0$, there is an edge xy in $R \cap B$. We may assume $|G - \{x, y\}| \leq (2/3)(n-2)^2$, otherwise we can apply induction. Consequently, the number of edges joining $\{x, y\}$ to $G - \{x, y\}$ is more than $(8n - 14)/3 > 2(n-2)$. Therefore there is a vertex z in $G - \{x, y\}$ such that $\deg_G(z, \{x, y\}) \geq 3$ and $|G[\{x, y, z\}]| \geq 5$. Again, by induction, $|G - \{x, y, z\}| \leq (2/3)(n-3)^2$, which implies there are more than $4n - 12$ edges joining $\{x, y, z\}$ to $G - \{x, y, z\}$. This implies that there is a vertex w in $G - \{x, y, z\}$ such that $\deg_G(w, \{x, y, z\}) \geq 5$. Then $|G[\{x, y, z, w\}]| \geq 10$ and there is at least one edge between every pair of vertices in $\{x, y, z, w\}$. It is easy to see, by a short case analysis, that $G[\{x, y, z, w\}]$ contains a 2-edge-colored K_4 in which the red and blue edges both induce a path of length three.

• $k + 1 = 4$ and K is a 4-biclique: Suppose that we are given R and B, each with more than $(3n^2 - 2n)/8$ edges and $n \geq 4$. Consider the multigraph $G = R \cup B$ with $|G| > (3n^2 - 2n)/4$. It suffices to show that G contains 4 vertices that induce a subgraph with at least 11 edges. This configuration obviously contains all non-monochromatic K_4's, in particular the 4-biclique, and this will complete the proof.

We will prove this by induction on n. When $n = 4$, both R and B are complete graphs, and when $n = 5$, both miss at most one edge, so we can clearly pick a 4-vertex subgraph with at least 11 edges of G. When $n = 6$, both have at least 13 edges, and a short case analysis shows that again we can pick the required 4-vertex subgraph. We may therefore assume that $n \geq 7$ and the result holds for $n - 3$.

Clearly $|R \cap B| \geq |G| - \binom{n}{2} > n^2/4$, so by Turán's theorem, we conclude that G contains a set T of 3 vertices such that $|G[T]| = 6$. By induction, we may assume that $|G - T| \leq (3(n-3)^2 - 2(n-3))/4$. Consequently, the number of edges of G between T and $G - T$ is more than $(18n - 33)/4$. This is greater than $4(n-3)$, so there is a vertex $v \notin T$ such that $\deg_G(v, T) \geq 5$. Then $G[T \cup \{v\}]$ is the required 4-vertex subgraph. □

4 All five point configurations

In this section, we prove Theorem 4.

We first note that for a monochromatic K_5 the result follows directly from Turán's theorem. Further, if the 2-edge-coloring of K_5 is not monochromatic but contains a monochromatic K_4, the result follows from the general Theorem 5. We only need to consider 2-edge-colorings of K_5 that do not contain a monochromatic K_4. We call such a 2-edge-colored K_5 a *good* K_5.

In order to prove the theorem, we prove a slightly stronger statement.

Theorem 6 *Let R and B be graphs on a set V of vertices of size n. If $|R| + |B| > (3/4)n^2$, then the multigraph $G = R \cup B$ contains every good K_5.*

For convenience, we will consider the graph $G = R \cup B$ to be a simple graph whose edges are colored either red, blue or double (the double edges are precisely the edges in $R \cap B$). Define $w(G) = |R| + |B|$ to be the weight of the graph G. If X and Y are disjoint subsets of vertices, define

$$w(X, Y) = \sum_{v \in X} \deg_G(v, Y)$$

to be the number of edges in G that join a vertex in X to a vertex in Y.

We say that a colored graph G contains a colored graph H, if there is an isomorphism from H to a subgraph H' of G, such that every double edge of H corresponds to a double edge of H', while every red(blue) edge of H corresponds to a red(blue) or double edge of H'.

We prove the theorem by contradiction. Suppose there exists a colored graph G on a set V of n vertices with $w(G) > (3/4)n^2$ that does not contain some good K_5. Choose G such that n is minimum. Since $w(G) \leq n(n-1)$, $n \geq 5$. We will show that G must contain certain subgraphs that together imply that G contains all good K_5, a contradiction.

Lemma 7 *For every nonempty proper subset of vertices $X \subset V$,*

$$w(X, V - X) > (3/4)(2n|X| - |X|^2) - w(G[X]).$$

In particular, for every vertex v, $deg_G(v) > (3/4)(2n - 1)$.

Proof: For every nonempty proper subset of vertices X, we have $w(G[X]) \leq (3/4)|X|^2$, otherwise $G[X]$ is a counterexample with fewer vertices than G. Since $w(X, V - X) = w(G) - w(G[X]) - w(G[V - X])$, the Lemma follows. $\square$

Lemma 8 *G contains a triangle with all edges colored double.*

Proof: Since $|R| + |B| > (3/4)n^2$, $|R \cap B| > n^2/4$. The Lemma follows by Turán's theorem. $\square$

Lemma 9 *G does not contain a 4-clique with all edges colored double.*

Proof: Suppose for contradiction, that X is a set of 4 vertices such that $G[X]$ is a clique with all edges colored double. By Lemma 7,

$$w(X, V - X) > (3/4)(8n - 16) - 12 = 6(n - 4).$$

Therefore there is a vertex $v \in V - X$ such that $\deg_G(v, X) \geq 7$. This implies that $G[X \cup \{v\}]$ is a 5-clique with at least 9 double edges. Such a 5-clique contains all good K_5, contradicting the fact that G is a counterexample. $\square$

Lemma 10 *G contains a 4-clique with 5 edges colored double and one edge that may be red or blue.*

Proof: By Lemma 8, there is a set X of 3 vertices in G such that $G[X]$ is a clique with all edges colored double. By Lemma 7,

$$w(X, V - X) > (3/4)(6n - 9) - 6 = (18n - 51)/4 \geq 4(n - 3).$$

Therefore there exists a vertex $v \in V - X$ such that $\deg_G(v, X) \geq 5$. Hence $G[X \cup \{v\}]$ is a 4-clique with at least 5 double edges. By Lemma 9, there are exactly 5 double edges and the remaining edge may be red or blue. □

By Lemma 10, we may assume, without loss of generality, that G contains a 4-clique with 5 double and 1 red edge. The argument is symmetrical if it contains a 4-clique with 5 double and 1 blue edge.

Lemma 11 *G does not contain a 5-clique with 8 double, 1 red and 1 blue edge.*

Proof: It is easy to see that any good K_5 contains a red and blue edge that are adjacent, and a red and blue edge that are disjoint. Consequently, such a 5-clique contains any good K_5, contradicting the fact that G is a counterexample. □

Lemma 12 *G contains a 5-clique with 8 double edges and 2 disjoint red edges.*

Proof: By Lemma 10 and our assumption, G contains a 4-clique H with 5 double and 1 red edge. Let X be the vertex set of H. By Lemma 7,

$$w(X, V - X) > (3/4)(8n - 16) - 11 > 6(n - 4).$$

Hence, there is a vertex $v \in V - X$ such that $\deg_G(v, X) \geq 7$. Thus $G[X \cup \{v\}]$ is a 5-clique with at least 8 double edges.

If it contains more than 8 double edges, then it contains all good K_5, a contradiction. Similarly, if the two edges that are not colored double are adjacent, then deleting their common endpoint gives a 4-clique with all edges double, contradicting Lemma 9. Since at least one of the edges is red, Lemma 11 implies that both the edges must be red. □

As a consequence of Lemma 12, G contains all good K_5 that contain two disjoint red edges. The only good K_5 that does not contain two disjoint red edges is a 2-edge-coloring of K_5 in which the red edges induce a triangle. We call such a 2-edge-colored K_5 a *special* K_5.

It remains to show that G contains a special K_5.

Lemma 13 *G contains a 6-clique with 11 double edges, 3 pairwise disjoint red edges, and one edge that may be red, blue or double.*

Proof: By Lemma 12, G contains a 5-clique H with 8 double and 2 disjoint red edges. Let $X = \{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of H and v_1v_2, v_3v_4 be the red edges in H. Note that since $w(H) = 18 < 75/4$, $n > 5$ and X is a proper subset of V.

Suppose that there exists a vertex $v \in V - X$ such that $\deg_G(v, X) \geq 9$. If the edge vv_5 is a double edge, then either $G[\{v, v_1, v_3, v_5\}]$ or $G[\{v, v_2, v_4, v_5\}]$ is a 4-clique with all edges double, contradicting Lemma 9. If the edge vv_5 is blue, then $G[\{v, v_1, v_2, v_3, v_5\}]$ is a 5-clique with 8 double, 1 red and 1 blue edge, contradicting Lemma 11. Therefore the edge vv_5 must be red and $G[X \cup \{v\}]$ is a 6-clique with 12 double and 3 disjoint red edges.

Suppose that for every vertex $v \in V - X$, $\deg_G(v, X) \leq 8$. We claim that there exists a vertex $v \in V - X$ such that $\deg_G(v, X) = 8$ and v is adjacent to v_5 in G.

Let $A = \{v \in V - X : \deg_G(v, X) = 8\}$ and let $|A| = a$. By Lemma 7,

$$w(X, V - X) > (3/4)(10n - 25) - 18 = (30n - 147)/4.$$

However, $w(X, V - X) \leq 8a + 7(n - 5 - a) = 7n + a - 35$. This implies $a > (2n - 7)/4$. If no vertex in A is adjacent to v_5, then $\deg_G(v_5) \leq 2(n - 1 - a)$. By Lemma 7, $\deg_G(v_5) > (3/4)(2n - 1)$, which implies $a < (2n - 5)/8$, a contradiction.

Let $v \in V - X$ be a vertex such that $\deg_G(v, X) = 8$ and v is adjacent to v_5.

First, suppose that the edge vv_5 is double. If vv_i, $i \in \{1, 2\}$ and vv_j, $j \in \{3, 4\}$ are double edges, then $G[\{v, v_i, v_j, v_5\}]$ is a 4-clique with all edges double. We may therefore assume, without loss of generality, that the edges vv_3 and vv_4 are not double, which implies the edges vv_1 and vv_2 are double. If any one of the edges vv_3, vv_4 is blue, say vv_3, then $G[\{v, v_1, v_2, v_3, v_5\}]$ is a 5-clique with 8 double, 1 red and 1 blue edge, contradicting Lemma 11. If both the edges vv_3 and vv_4 are red, then $G[\{v, v_1, v_3, v_4, v_5\}]$ is a 5-clique that contains a special K_5. Together with Lemma 12, this implies G contains all good K_5.

Next, suppose that the edge vv_5 is blue. Without loss of generality, we may assume the edge vv_4 is not double. Then $G[\{v, v_1, v_2, v_3, v_5\}]$ is a 5-clique with 8 double, 1 red and 1 blue edge, contradicting Lemma 11.

Finally, if the edge vv_5 is red, then $G[X \cup \{v\}]$ is a 6-clique with 11 double edges, 3 disjoint red edges and one edge that may be red or blue. This proves the Lemma. □

Lemma 14 *G contains a 7-clique with 15 double edges and 6 red edges, such that the subgraph induced by the red edges has two components, a path of length one and a cycle of length 5.*

Proof: By Lemma 13, G contains a 6-clique H with 11 double edges, 3 disjoint red edges and one edge that may be red, blue or double. Let $X = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ be the vertex set of H. Let v_1v_2, v_3v_4, v_5v_6 be the 3 disjoint red edges in H and let v_4v_6 be the edge that may be red, blue or double. Note that since $w(H) \leq 27$, X is a proper subset of V.

By Lemma 7,

$$w(X, V - X) > (3/4)(12n - 36) - 27 = 9n - 54.$$

This implies that there is a vertex $v \in V - X$ such that $\deg_G(v, X) \geq 10$.

If there are 5 double edges joining v to H, then either $G[\{v, v_1, v_3, v_6\}]$ or $G[\{v, v_2, v_4, v_5\}]$ is a 4-clique with all edges double, contradicting Lemma 9. Therefore v is joined to H by 4 double edges and two edges that may be red or blue. Let $A = \{u \in X : uv \text{ is a double edge}\}$. The subgraph $H[A]$ must contain two disjoint red edges, otherwise G contains either a 4-clique with all edges double or a 5-clique with 8 double, one red and one blue edge. This contradicts Lemma 9 or Lemma 11.

Suppose $A = \{v_1, v_2, v_3, v_4\}$. If any of the edges vv_5, vv_6 is blue, say vv_6, then $G[\{v, v_1, v_2, v_3, v_6\}]$ is a 5-clique with 8 double, one red and one blue edge, a contradiction. If both vv_5 and vv_6 are red, then $G[\{v, v_1, v_3, v_5, v_6\}]$ is a 5-clique that contains a special K_5. A symmetrical argument holds if $A = \{v_1, v_2, v_5, v_6\}$ or $A = \{v_3, v_4, v_5, v_6\}$.

The only other possibility is that $A = \{v_1, v_2, v_4, v_6\}$, in which case, the edge v_4v_6 must be red. If any one of the edges vv_3, vv_5 is blue, say vv_3, then $G[\{v, v_1, v_2, v_3, v_6\}]$ is a 5-clique with 8 double, one red and one blue edge. Therefore, both vv_3 and vv_5 must be red and $G[X \cup \{v\}]$ is a 7-clique with 15 double edges and 6 red edges, such that the subgraph induced by the red edges has two components, a path of length one and a cycle of length 5. This completes the proof of the Lemma. $\square$

Lemma 15 *G contains a special K_5.*

Proof: By Lemma 14, G contains a 7-clique H with 15 double edges and 6 red edges such that the subgraph induced by the red edges has two components, a path of length one and a cycle of length 5. Let $X = \{v_1, v_2, \ldots, v_7\}$ be the vertex set of H. Since $w(H) = 36 < (3/4)|X|^2$, X is a proper subset of V. Let v_1v_2 be the path of length one and v_3, v_4, v_5, v_6, v_7 the cycle of length 5 containing the red edges.

Suppose that there exists a vertex $v \in V - X$ and at least 5 double edges joining v to H. If any one of the edges vv_1, vv_2 is double, say vv_1, then at least three of the edges vv_i, $i \in \{3, 4, 5, 6, 7\}$ are double, and we can find two vertices v_j, $v_k \in \{v_3, v_4, v_5, v_6, v_7\}$ such that $G[\{v, v_1, v_j, v_k\}]$ is a 4-clique with all edges double.

We may assume that all the edges vv_i, $i \in \{3, 4, 5, 6, 7\}$ are double. If any of the edges vv_1, vv_2 is blue, say vv_1, then $G[\{v, v_1, v_3, v_4, v_6\}]$ is a 5-clique with 8 double, one red and one blue edge. If both vv_1 and vv_2 are red, then $G[\{v, v_1, v_2, v_3, v_5\}]$ contains a special K_5.

We may therefore assume that for every vertex $v \in V - X$ that is joined to H by 5 double edges, $\deg_G(v, \{v_1, v_2\}) \leq 1$. We claim that there exists a vertex $v \in V - X$ such that $\deg_G(v, X) = 11$ and v is joined to H by four double edges and three edges that may be red or blue.

Let $A = \{u \in V - X : \deg_G(u, X) = 11\}$ and let $|A| = a$. By Lemma 7,

$$w(X, V - X) > (3/4)(14n - 49) - 36 = (42n - 291)/4.$$

However, $w(X, V - X) \leq 11a + 10(n - 7 - a)$, which implies that $a > (2n - 11)/4$. If every vertex in A is joined to H by 5 double edges, then $\deg_G(v, \{v_1, v_2\}) \leq 1$ for all $v \in A$. This implies that $\deg_G(v_1)+\deg_G(v_2) \leq a + 4(n - 2 - a) + 2$. By Lemma 7, $\deg_G(v_1)+\deg_G(v_2) > (3/2)(2n - 1)$. Consequently, $a < (2n - 9)/6$, which is a contradiction.

Hence there is a vertex $v \in A \subset V - X$ that is joined to H by four double edges and three edges that may be red or blue. Let $Y = \{u \in X : uv \text{ is a double edge}\}$. The subgraph $H[Y]$ must contain two disjoint red edges otherwise $H[Y \cup \{v\}]$ contains a 4-clique with all edges double.

Suppose $\{v_1, v_2\} \subset Y$. We may assume, without loss of generality, that v_3 and v_4 are the other vertices in Y. If any one of the edges vv_5, vv_6 is blue, say vv_5, then $G[\{v, v_1, v_2, v_3, v_5\}]$ is a 5-clique with 8 double, one red and one blue edge, a contradiction. If both vv_5 and vv_6 are red, then $G[\{v, v_1, v_3, v_5, v_6\}]$ contains a special K_5.

The only other possibility is that Y contains four vertices from the red five cycle in H. Without loss of generality, $Y = \{v_3, v_4, v_5, v_6\}$. If any of the edges vv_1, vv_2 is blue, say vv_1, then $G[\{v, v_1, v_3, v_4, v_6\}]$ is a 5-clique with 8 double, one red and one blue edge. If both the edges vv_1 and vv_2 are red, then $G[\{v, v_1, v_2, v_3, v_5\}]$ contains a special K_5.

This completes the proof of the Lemma. □

Theorem 6 now follows, since Lemmas 12 and 15 imply that G contains all good K_5, contradicting the fact that G is a counterexample.

5 Open problems

We conclude by mentioning some further possible extensions. The main question to be settled is, of course, Conjecture 3. There is a natural generalization of Theorem 2, that we believe would be necessary for proving Conjecture 3 in general.

Conjecture 16 *Fix numbers $r, b > 0$. Let R (red) and B (blue) be graphs on the same set of vertices of size n. If $b|B| + (r - 1)|R| > (b + r - 2)n^2/2$, then the multigraph $G = R \cup B$ contains a red clique K of order r and a blue clique L of order b disjoint from it, such that every vertex in K is joined to all vertices in L by blue edges.*

Theorem 5 proves Conjecture 16 for the case $b = 1$.

It is of course possible to consider similar questions with more than two colors. The simplest case here is a triangle with edges of distinct colors.

Problem. Let R (red), B (blue) and G (green) be graphs on the same vertex set of size n. How large must $\min\{|R|, |B|, |G|\}$ be to guarantee that $R \cup B \cup G$ contains a multicolored triangle?

In general, we believe it may be possible to generalize many classical extremal graph theory results, including minimum degree conditions, in this way. For example, it would be very interesting if there was some analogue of the Erdős-Simonovits-Stone theorem in this context.

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