On the b-independence number of sparse random graphs

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Abstract

Let graph $G = (V, E)$ and integer $b \geq 1$ be given. A set $S \subseteq V$ is said to be b-independent if $u, v \in S$ implies $d_G(u, v) > b$ where $d_G(u, v)$ is the shortest distance between u and v in G. The b-independence number $\alpha_b(G)$ is the size of the largest b-independent subset of G. When $b = 1$ this reduces to the standard definition of independence number.

We study this parameter in relation to the random graph $G_{n,p}$, $p = d/n$. In particular, when d is a large constant. We show that **whp** that if $d \geq d_{\epsilon,b}$,

$$\left|\alpha_b(G_{n,p}) - \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b}\right)\right| \leq \frac{\epsilon n}{d^b}.$$

1 Introduction

Let graph $G = (V, E)$ and integer $b \geq 1$ be given. A set $S \subseteq V$ is said to be b-independent if $u, v \in S$ implies $d_G(u, v) > b$ where $d_G(u, v)$ is the shortest distance between u and v

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We study this parameter in relation to the random graph $G_{n,p}$, $p = d/n$. In particular, when d is a large constant. Now $\alpha_b(G) = \alpha(G^b)$ where $G^b = (V, E^b)$ and u, v are adjacent in G^b iff $d_G(u, v) \leq b$. If d, b are not too large then the average degree in $G^b_{n,p}$ is approximately d^b and so one might expect that $G^b_{n,d/n}$ and $G_{n,d^b/n}$ are similar. We show in this paper that this is true for the b-independence number.

Theorem 1 *Let b be a positive integer constant and let $\epsilon > 0$ be given. Assume that d is a constant, greater than some fixed $d_{\epsilon,b}$. Then* **whp**

$$\left| \alpha_b(G_{n,p}) - \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b}\right) \right| \leq \frac{\epsilon n}{d^b}.$$

□

The case $b = 1$ was treated in Frieze [4] and so we will assume that $b \geq 2$ from now on. Nierhoff [8] considered this problem and its relation to the k-centre problem of Operations Research. In particular, he proves the upper bound implied in the theorem and is off by a factor of 2 in the lower bound for $d = n^{o(1)}$. This is to be expected as he proves his lower bound via the analysis of a greedy algorithm. Duckworth [2] proved a high probability lower bound of $.2048n$ for the 2-independence number of a random cubic graph.

We will follow the method described in [4]. See also the discussion in Janson, Łuczak and Ruciński [6] where the proof is somewhat simplified by the use of Talagrand's inequality.

2 Proof of Theorem 1

We start with the following lemma, which will prove useful in bounding $\alpha_b(G_{n,p})$ from both above and below.

Lemma 1 *Let $K, L \subseteq [n]$, $|K| = |L| = k$, $|K \cap L| = l$ be given. Then if*

$$p_d = \frac{d(d^b - 1)}{(d-1)n}$$

we have

(a)

$$(1 - p_d)^{\binom{k}{2}} \leq \mathbf{Pr}(K \text{ is } b\text{-independent }) \leq (1 - p_d)^{\binom{k}{2}} \exp\{O(k^3 n^{2b-3} p^{2b-1})\}.$$

(b)

$$\mathbf{Pr}(K, L \text{ are both } b\text{-independent }) \leq (1 - p_d)^{2\binom{k}{2} - \binom{l}{2}} \exp\{O(k^3 n^{2b-3} p^{2b-1})\}.$$

Proof (a) We can use Janson's inequality [1]. To establish notation, let $P_1, P_2, \ldots, P_N$, $N = \binom{k}{2}\sum_{r=1}^{b}(n-2)(n-3)\cdots(n-r)$ enumerate the edge sets of paths of length at most b in K_n which join vertices of K and whose internal vertices are not in K. Let $\mathcal{P}_i$ be the event that the path corresponding to P_i exists in $G_{n,p}$ for $i = 1, 2, \ldots, N$. Then K is b-independent if and only if none of the events $\mathcal{P}_i$ occur.

Let

$$\Delta = \sum_{i \neq j:\ P_i \cap P_j \neq \emptyset} \mathbf{Pr}(\mathcal{P}_i \cap \mathcal{P}_j).$$

Then Janson's inequality states that

$$\prod_{i=1}^{N}(1 - \mathbf{Pr}(\mathcal{P}_i)) \leq \mathbf{Pr}\left(\bigcap_{i=1}^{N} \bar{\mathcal{P}}_i\right) \leq e^{\Delta}\prod_{i=1}^{N}(1 - \mathbf{Pr}(\mathcal{P}_i)). \tag{1}$$

Now

$$\begin{aligned}\prod_{i=1}^{N}(1 - \mathbf{Pr}(\mathcal{P}_i)) &= \left(\prod_{i=1}^{b}(1-p^i)^{(n-2)\cdots(n-i)}\right)^{\binom{k}{2}} \\ &= \left(1 - \sum_{i=1}^{b} p^i n^{i-1}\right)^{\binom{k}{2}} e^{O(1)} \\ &= \left(1 - \frac{d(d^b-1)}{(d-1)n}\right)^{\binom{k}{2}} e^{O(1)}\end{aligned} \tag{2}$$

The $e^{O(1)}$ term in (2) is at least 1 and this gives the lower bound in (a). For the upper bound we need to show that

$$\Delta = O(k^3 n^{2b-3} p^{2b-1}) \tag{3}$$

and apply the upper bound of (1). To obtain this we observe that

$$\Delta \leq \sum_{a_1=2}^{b}\binom{k}{2} n^{a_1-1}p^{a_1} \sum_{a_2=a_1}^{b}\sum_{t=1}^{a_1-1}\binom{a_1}{t} k n^{a_2-t-1}p^{a_2-t}. \tag{4}$$

Explanation: Fix a path length $a_1 \geq 2$ and then the term $\binom{k}{2}n^{a_1-1}p^{a_1}$ bounds the weight of choices for P_i. Then choose another path length $a_2 \geq a_1$ and let t be the size of $|P_i \cap P_j|$. For a given set of t edges in P_i, $kn^{a_2-t-1}p^{a_2-t}$ bounds the weight of the paths P_j which share these edges with P_i.

(3) follows from (4) when d is sufficiently large. This completes the proof of the upper bound in (a). The upper bound in (b) is proved in the same manner. □

We prove a high probability upper bound on $\alpha_b(G_{n,p})$ by the first moment method.

Lemma 2 *Let $\epsilon > 0$ be a constant and*

$$k_1 = \frac{2bn}{d^b}\left(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b} + \frac{\epsilon}{b}\right).$$

Then **whp**

$$\alpha_b(G_{n,p}) \le k_1.$$

Proof Let X_k denote the number of b-independent sets of size k found in $G_{n,p}$. Then, using the upper bound in Lemma 1(a) and putting $k = k_1$, we obtain

$$\begin{aligned}\mathbf{E}(X_k) &\le \binom{n}{k}(1-p_d)^{\binom{k}{2}} \exp\left\{O\left(k^3 n^{2b-3} p^{2b-1}\right)\right\} \\ &\le \exp\left\{k\left[\log\frac{ne}{k} - \frac{k}{2}p_d + O\left(\frac{(\log d)^2}{d}\right)\right]\right\}. \qquad (5)\end{aligned}$$

Now

$$\log\frac{ne}{k} = 1 - \log 2b + b\log d - \log\log d + O\left(\frac{\log\log d}{\log d}\right)$$

and

$$\frac{k}{2}p_d = 1 - \log 2b + b\log d - \log\log d + \epsilon + O\left(\frac{\log d}{d}\right)$$

and so

$$\mathbf{E}(X_k) \le e^{-\epsilon k/2} \qquad (6)$$

for d sufficiently large. □

To prove a lower bound on $\alpha(G_{n,p}^b)$, we partition the vertex set $[n]$ into sets of size $m = \left\lceil \frac{d^b}{b^2(\log d)^2} \right\rceil$ or $m-1$. There will be $n' = \lfloor \frac{nb^2(\log d)^2}{d^b} \rfloor$ sets $P_1 \dots P_{n'}$ in the partition. A set $I \subset [n]$ is said to be P-independent if

(i) I is b-independent.

(ii) $|I \cap P_i| \le 1$ for $1 \le i \le n'$.

Let β_b denote the size of the largest P-independent set. Now let X_k denote the number of P-independent sets of size k found in $G_{n,p}$. It is clear that $\beta_b \le \alpha_b$ and also $X_k > 0$ implies $\beta_b \ge k$.

Following the method of [4], one can put a lower bound on $\alpha_b(G_{n,p})$ by proving two inequalities: Let $\bar{\beta}_b = \mathbf{E}(\beta_b(G_{n,p}))$.

$$\mathbf{Pr}\left(|\beta_b - \bar{\beta}_b| \ge \frac{\epsilon n}{d^b}\right) \le 2\exp\left\{\Omega\Big(-\frac{\epsilon^2 n}{d^b(\log d)^2}\Big)\right\}. \qquad (7)$$

$$\mathbf{Pr}(X_{k_2} > 0) \ge \exp\left\{O\Big(-\frac{n(\log d)^3}{d^{b+1}}\Big)\right\}. \qquad (8)$$

where $k_2 = \frac{2bn}{d^b}(\log d - \frac{\log\log d}{b} - \frac{\log 2b}{b} + \frac{1}{b} - \frac{\epsilon}{b})$.

Then when d is sufficiently large, (7) and (8) imply $|\bar{\beta}_b - k_2| \le \frac{\epsilon n}{d^b}$ and then Theorem 1 follows from (7). □

2.1 Proof of (7)

When $b = 1$ ([4]), one can use Azuma's inequality to prove (7). For $b \ge 2$ we find that the random variable β_b does not have a small worst-case Lipschitz constant. This rules out the use of Talagrand's inequality [9] too. Furthermore, the new inequalities of Kim and Van Vu [7], [10] will not do the job either. The modifications due to Godbole and Hitczenko [5] also fail to help. Instead, we follow the proof idea of the Azuma inequality and patch it up where necessary. Simply put, we estimate the moment generating function without giving too much away and then apply the Markov inequality.

As usual we use the inequalities

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \ge t) \le e^{-\lambda t}\mathbf{E}(e^{\lambda(\beta_b - \bar{\beta}_b)}) \quad (9)$$

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \le -t) \le e^{-\lambda t}\mathbf{E}(e^{\lambda(\bar{\beta}_b - \beta_b)}) \quad (10)$$

$$(11)$$

valid for all $\lambda > 0$.

We will divide the proof into three cases. The general line of attack is the same in all cases, but for the larger values of b we need to solve a couple of extra technical problems.

2.1.1 The Case $b = 2$

We begin with the simplest case, $b = 2$. We will leave b in formulae so that they can be used later for $b \ge 3$.

Now let $Y_i, i = 1, 2, \ldots, n'$, denote the set of edges of $G_{n,p}$ which connect a vertex in P_i to a vertex in $\bigcup_{j \le i} P_j$. Define $Z_i = Z_i(Y_1, Y_2, \ldots, Y_{n'})$ by

$$Z_i = \mathbf{E}(\beta_b \mid Y_1, Y_2, \ldots, Y_i) - \mathbf{E}(\beta_b \mid Y_1, Y_2, \ldots, Y_{i-1}),\ i = 1, 2, \ldots, n'$$

so that

$$\beta_b - \bar{\beta}_b = Z_1 + \cdots + Z_{n'}. \quad (12)$$

Let $\mathbf{Y}_\ell = (Y_1, Y_2, \ldots, Y_\ell)$ for $\ell = 0, 1, \ldots, n'$ and let

$$A = 20b3^{b-2} + 1.$$

We will prove by (backwards) induction on ℓ that for $\ell = n', n'-1, \ldots, 0$,

$$t = \epsilon n/d^b,\ \lambda = \epsilon/(10A^2b^2(\log d)^2) \text{ and } \Theta_\ell = \exp\{-\lambda t + (n' - \ell)(3A^2\lambda^2 + O(\lambda^3))\}, \quad (13)$$

d sufficiently large,

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t) \leq \Theta_\ell \sum_{\mathbf{Y}_\ell} \mathbf{Pr}(\mathbf{Y}_\ell) \prod_{i=1}^{\ell} e^{\lambda Z_i} + (n' - \ell) e^{-n/d^b}. \tag{14}$$

Using (12) we see that when $\ell = n'$, this is simply (9). So assume that (14) is true for some $\ell \leq n'$. Let G_ℓ denote the subgraph of $G_{n,p}$ induced by $[n] \setminus P_\ell$. Define the event:

$$\hat{\mathcal{E}}_\ell = \left\{ \beta_b(G_\ell) \leq I_0 = \frac{4bn \log d}{d^b} \right\}.$$

Going back to (5) with $\epsilon = b \log d + \log\log d + \log 2b - 1$ and $k = I_0$ we see that

$$\begin{aligned} \mathbf{Pr}(\beta_b(G_\ell) > I_0) &\leq \exp\left\{ k \left[\log \frac{ne}{k} - \frac{k}{2} p_d + O\left(p^{2b-1} n^{2b-3} k^2\right) \right] \right\} \\ &\leq \exp\{k[\log\log d - 2b \log d]\} \\ &\leq \exp\left\{ -\frac{5b^2 n (\log d)^2}{d^b} \right\} \end{aligned} \tag{15}$$

for d sufficiently large.

Define $\hat{Y}_\ell, \ldots, \hat{Y}_{n'}$ to be independent copies of $Y_\ell, \ldots, Y_{n'}$. For $i > \ell$ let Y_i' be the subset of edges from Y_i which join P_i and P_ℓ and let $Y_i'' = Y_i - Y_i'$. In the following, let $Y' = \hat{Y}' = \bigcup_{i=\ell+1}^{n'} \hat{Y}_i'$ and let $Y_\ell^* = Y_\ell \cup Y'$, $\hat{Y}_\ell^* = \hat{Y}_\ell \cup Y'$. Then

$$\begin{aligned} Z_\ell &= \sum_{\hat{Y}_\ell, \ldots, \hat{Y}_{n'}} [\beta_b(Y_1, \ldots, Y_\ell, \hat{Y}_{\ell+1}, \ldots, \hat{Y}_{n'}) - \beta_b(Y_1, \ldots, Y_{\ell-1}, \hat{Y}_\ell, \ldots, \hat{Y}_{n'})] \mathbf{Pr}(\hat{Y}_\ell, \ldots, \hat{Y}_{n'}) \\ &= \sum_{\hat{Y}_\ell^*, Y_\ell^*, \hat{\mathbf{Y}}_\ell''} [\beta_b(\mathbf{Y}_{\ell-1}, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') - \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}_\ell^*, \hat{\mathbf{Y}}_\ell'')] \mathbf{Pr}(\hat{Y}_\ell^*, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') \end{aligned}$$

where $\hat{\mathbf{Y}}_\ell'' = (\hat{Y}_{\ell+1}'', \ldots, \hat{Y}_{n'}'')$.

We see that G_ℓ is defined by $\mathbf{Y}_{\ell-1}, \hat{\mathbf{Y}}_\ell''$. So let

$$\mathcal{E}''(\mathbf{Y}_{\ell-1}) = \{\hat{\mathbf{Y}}_\ell'' : \ G_\ell \in \hat{\mathcal{E}}_\ell\}$$

and

$$\mathcal{E}_\ell = \left\{ \mathbf{Y}_{\ell-1} : \ \mathbf{Pr}(G_\ell \in \hat{\mathcal{E}}_\ell \mid \mathbf{Y}_{\ell-1}) \geq 1 - \exp\left\{ -\frac{2b^2 n (\log d)^2}{d^b} \right\} \right\}.$$

It follows from (15) that

$$\mathbf{Pr}(\mathbf{Y}_{\ell-1} \notin \mathcal{E}_\ell) \leq \exp\left\{ -\frac{2b^2 n (\log d)^2}{d^b} \right\}. \tag{16}$$

Using (14) and the independence of $\mathbf{Y}_{\ell-1}, Y_\ell$ write

$$\begin{aligned}
\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t) &\leq \Theta_\ell \sum_{\mathbf{Y}_{\ell-1}} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_{\ell-1}) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) + (n'-\ell)e^{-n/d^b} \\
&\leq \Theta_\ell \sum_{\mathbf{Y}_{\ell-1} \in \mathcal{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_{\ell-1}) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\
&\qquad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_{\ell-1} \notin \mathcal{E}_\ell) + (n'-\ell)e^{-n/d^b}. \qquad (17)
\end{aligned}$$

Now $\mathbf{Y}_{\ell-1} \in \mathcal{E}_\ell$ implies

$$\mathbf{Pr}(\hat{\mathbf{Y}}_\ell'' \notin \mathcal{E}''(\mathbf{Y}_{\ell-1})) \leq \exp\left\{-\frac{b^2 n (\log d)^2}{d^b}\right\}. \qquad (18)$$

So if we fix $\mathbf{Y}_{\ell-1} \in \mathcal{E}_\ell$.

$$\begin{aligned}
Z_\ell = &\sum_{\substack{Y_\ell^*, \hat{Y}_\ell^* \\ \hat{\mathbf{Y}}_\ell'' \in \mathcal{E}_\ell''(\mathbf{Y}_{\ell-1})}} [\beta_b(\mathbf{Y}_{\ell-1}, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') - \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}_\ell^*, \hat{\mathbf{Y}}_\ell'')] \mathbf{Pr}(\hat{\mathbf{Y}}_\ell'', \hat{Y}_\ell^*) \\
&\qquad + O\left(n \exp\left\{-\frac{2b^2 n (\log d)^2}{d^b}\right\}\right). \qquad (19)
\end{aligned}$$

We now estimate in (17), the term

$$\sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) = \mathbf{E}_{Y_\ell}(e^{\lambda Z_\ell})$$

and use (19) to restrict our attention to the case $\hat{\mathbf{Y}}_\ell'' \in \mathcal{E}''(\mathbf{Y}_{\ell-1})$.

Fix $\hat{\mathbf{Y}}_\ell'' \in \mathcal{E}_\ell''(\mathbf{Y}_{\ell-1})$. Let S be a maximum size subset of $[n] \setminus P_\ell$ which is P-independent in G_ℓ and let $I = |S|$. Let S be sub-divided into $S_1 = S \cap \cup_{i=1}^{\ell-1} P_i$ and $S_2 = S \cap \cup_{i=\ell+1}^{n'} P_i$, with I_1 and I_2 denoting $|S_1|$ and $|S_2|$ respectively. For $v \in P_\ell$ let $\delta_1(v,S)$ be the number of edges from Y_ℓ joining v to S_1, and let $\delta_2(v,S)$ be the number of edges from Y' joining v to S_2. Let $\delta(v,S)$ be the total, $\delta_1(v,S) + \delta_2(v,S)$. Let $\hat{\delta}(v,S)$, $\hat{\delta}_1(v,S)$, and $\hat{\delta}_2(v,S)$ be defined similarly, using the edges of $\hat{Y}_\ell$ and $\hat{Y}'$.

Define $W_v = \delta(v,S) 1_{\delta(v,S) \geq 2}$. If we define $W = \sum_{v \in P_\ell} W_v$, we can produce a P-independent set, S^*, in G by removing no more than W vertices from S. If, for every $v \in P_\ell$, we remove W_v neighbors of v from S, then every $v \in P_\ell$ has either one or zero neighbors in S^*. Thus, no path of length 2 exists between any pair of vertices in S^* because there are no paths through P_ℓ and S was 2-independent in G_ℓ. Define $\hat{W}$ in an analogous way using the edge-set $\hat{Y}_\ell$ in place of Y_ℓ. Now observe that

$$\begin{aligned}
I - W &\leq \beta_b(\mathbf{Y}_{\ell-1}, Y_\ell^*, \hat{\mathbf{Y}}_\ell'') \leq I + 1 \\
I - \hat{W} &\leq \beta_b(\mathbf{Y}_{\ell-1}, \hat{Y}_\ell^*, \hat{\mathbf{Y}}_\ell'') \leq I + 1
\end{aligned}$$

and so

$$Z_\ell \le \mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) + \mathbf{E}_{Y'}(W) + 1 \tag{20}$$

In computing $\mathbf{E}_{Y_\ell^*}(W)$ and $\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W})$, it is important to note that $Ip = O(d^{1-b}\log d) \to 0$ as d grows. Therefore, we disregard terms where Ip is raised to a sufficiently high power since those terms will be dominated by Ip and I^2p^2.

$$\begin{aligned}\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) &= Ip(1-(1-p)^{I-1}) \\ &= I^2p^2 - O(I^3p^3) \\ &\le I^2p^2 \end{aligned} \tag{21}$$

So,

$$\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) \le mI^2p^2 \le A. \tag{22}$$

Let $\bar{W}_v = \mathbf{E}_{Y'}(W_v|Y_\ell)$, and let $\bar{W} = \mathbf{E}_{Y'}(W|Y_\ell)$. Note that

$$\mathbf{E}_{Y_\ell}(\bar{W}) = \mathbf{E}_{Y_\ell^*}(W) = \mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}).$$

$$\bar{W}_v = \delta_1(v,S) + I_2p - (1-p)^{I_2}1_{\delta_1(v,S)=1} - I_2p(1-p)^{I_2-1}1_{\delta_1(v,S)=0} \tag{23}$$

For d sufficiently large and $\lambda \le 1/\log d$,

$$\begin{aligned}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v}) &= \mathbf{E}_{Y_\ell}\Bigg(\exp\Big\{\lambda\Big(\delta_1(v,S) + I_2p - (1-p)^{I_2}1_{\delta_1(v,S)=1} \\ &\qquad\qquad - I_2p(1-p)^{I_2-1}1_{\delta_1(v,S)=0}\Big)\Big\}\Bigg) \\ &= e^{\lambda I_2 p}\Bigg[\sum_{t=2}^{I_1}\binom{I_1}{t}e^{\lambda t}p^t(1-p)^{I_1-t} + \exp\big\{\lambda - \lambda(1-p)^{I_2}\big\}(1-p)^{I_1-1}I_1p \\ &\qquad\qquad + \exp\big\{-\lambda I_2p(1-p)^{I_2-1}\big\}(1-p)^{I_1}\Bigg] \\ &= 1 + (I_1^2 + I_2^2 + 2I_1I_2)p^2\lambda + (I_1^2+I_2^2)p^2\lambda^2 + O(I^2p^2\lambda^3) \\ &\le 1 + I^2p^2\lambda + I^2p^2\lambda^2 + O(I^2p^2\lambda^3) \end{aligned} \tag{24}$$

The $\bar{W}_v$ are independent of one another, so

$$\begin{aligned}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}}) &= \mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v})^m \\ &\le \big(1 + \lambda I^2p^2 + \lambda^2I^2p^2 + O(\lambda^3I^2p^2)\big)^m \\ &= 1 + \lambda mI^2p^2 + \lambda^2mI^2p^2 + \lambda^2\frac{m^2}{2}I^4p^4 + O(\lambda^3) \end{aligned} \tag{25}$$

We now turn to $\mathbf{E}_{Y_\ell}\left(e^{\lambda Z_\ell}\right)$.

$$\begin{aligned}
\mathbf{E}_{Y_\ell}(e^{\lambda Z_\ell}) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\mathbf{E}_{Y_\ell}\left(Z_\ell^k\right) \\
&= 1 + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}\mathbf{E}_{Y_\ell}(Z_\ell^k) \qquad\qquad \text{since } \mathbf{E}_{Y_\ell}(Z_\ell) = 0 \\
&\leq 1 + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \sum_{l=0}^{k} \binom{k}{l} (A+1)^{k-l}\mathbf{E}_{Y_\ell}(\bar{W}^l)
\end{aligned}$$

from (20), (22),

$$\begin{aligned}
&= 1 + \sum_{l=0}^{\infty} \frac{\lambda^l}{l!}\mathbf{E}_{Y_\ell}(\bar{W}^l) \sum_{k=l,k\geq 2}^{\infty} \frac{(\lambda(A+1))^{k-l}}{(k-l)!} \\
&= 1 + (e^{\lambda(A+1)} - 1 - \lambda(A+1)) + \lambda\mathbf{E}_{Y_\ell}(\bar{W})(e^{\lambda(A+1)} - 1) \\
&\qquad + e^{\lambda(A+1)} \sum_{l=2}^{\infty} \frac{\lambda^l}{l!}\mathbf{E}_{Y_\ell}(\bar{W}^l) \\
&= 1 + (e^{\lambda(A+1)} - 1 - \lambda(A+1)) + \lambda\mathbf{E}_{Y_\ell^*}(W)(e^{\lambda(A+1)} - 1) \\
&\qquad + e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}} - \lambda\bar{W} - 1) \\
&= e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}}) - \lambda\mathbf{E}_{Y_\ell^*}(W) - \lambda(A+1) \qquad (26) \\
&= e^{\lambda(A+1)}\left(1 + \lambda m I^2 p^2 + \lambda^2 m I^2 p^2 + \lambda^2 \frac{m^2 I^4 p^4}{2} + O(\lambda^3)\right) \\
&\qquad - \lambda(m I^2 p^2 + O(I^3 p^3)) - \lambda(A+1) \\
&= 1 + \lambda^2\left(m I^2 p^2 + \frac{m^2 I^4 p^4}{2} + (A+1) m I^2 p^2 + \frac{(A+1)^2}{2}\right) + O(\lambda^3) \\
&\leq 1 + 3A^2\lambda^2 + O(\lambda^3) \qquad (27)
\end{aligned}$$

Going back to (17) and using (15) we see that

$$\begin{aligned}
\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t) &\leq \Theta_{\ell-1} \sum_{\mathbf{Y}_{\ell-1}} \prod_{i=1}^{\ell-1} e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y}_{\ell-1}) + (n' - \ell + 1)e^{-n/d^b} \\
&= \Theta_{\ell-1} \sum_{\mathbf{Y}_{\ell-1}} \prod_{i=1}^{\ell-1} e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y}_{\ell-1}) + (n' - \ell + 1)e^{-n/d^b},
\end{aligned}$$

completing the induction.

For $\ell = 0$ we read (17) as

$$\begin{aligned}
\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t) &\leq \Theta_0 + n' e^{-n/d^b} \\
&\leq \exp\left\{-\Omega\left(\frac{\epsilon^2 n}{d^b(\log d)^2}\right)\right\}
\end{aligned}$$

if we make the substitutions of (13) for t, λ, Θ_0.

A similar argument handles $\mathbf{Pr}(\beta_b - \bar{\beta}_b \leq -t)$ and (7) follows.

2.1.2 The Case $3 \leq b \leq 5$

We proceed with the case $3 \leq b \leq 5$, letting Y_i, Z_i, t, λ, A and Θ_ℓ be defined as above. We will use a similar strategy of inducting on ℓ to prove,

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t) \leq \Theta_\ell \sum_{\mathbf{Y}} \prod_{i=1}^{\ell} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}) + (n' - \ell) e^{-n/d^b}. \tag{28}$$

To deal with the larger value of b, we need to expand the definition of $\hat{\mathcal{E}}_\ell$ to include several properties of G_ℓ, each of which occurs with high probability. Let $\hat{\mathcal{E}}_\ell$ be the event that all of the following occur:

$\hat{\mathcal{E}}_1 = \{\beta_b(G_\ell) \leq \frac{4bn \log d}{d^b}\}$.

$\hat{\mathcal{E}}_2 = \{$ In G_ℓ, any set of size $k > \frac{n}{d^b}$ has no more than $3dk$ neighbors $\}$.

$\hat{\mathcal{E}}_3 = \{$ G_ℓ contains no more than nd edges $\}$.

$\hat{\mathcal{E}}_1$ is equivalent to the event $\hat{\mathcal{E}}_\ell$ in the $b = 2$ case. The other two events can be analyzed by comparing G_ℓ to $G_{n,p}$. Since G_ℓ is derived from $G_{n,p}$ by removing edges and vertices, if the latter satisfies the criteria for $\hat{\mathcal{E}}_2$ and $\hat{\mathcal{E}}_3$ then the former must also satisfy those same criteria.

Lemma 3
(a)

$$\mathbf{Pr}(G_\ell \notin \hat{\mathcal{E}}_2) \leq \exp\left\{-\frac{n}{13d^{b-1}}\right\}.$$

(b)

$$\mathbf{Pr}(G_\ell \notin \hat{\mathcal{E}}_3) \leq e^{-nd/6}.$$

Proof (a) Given a fixed set K, each vertex is independently a neighbor of K with probability $q = 1 - (1-p)^k < kp$. Applying the Chernoff bound,

$$\mathbf{Pr}(B(n,q) \geq \alpha nq) \leq \left(\frac{e}{\alpha}\right)^{\alpha nq},$$

with $\alpha = \frac{3dk}{nq} \geq 3$,

$$\mathbf{Pr}(|N(K)| > 3dk) \leq \mathbf{Pr}(B(n,q) \geq \alpha nq) \leq (e/\alpha)^{\alpha nq} \leq (e/3)^{3dk} \leq e^{-dk/11}.$$

Therefore, the probability of finding any set of size $k > \frac{n}{d^b}$ with more than $3dk$ neighbors is:

$$\leq \sum_{k=n/d^b}^{n} \binom{n}{k} e^{-dk/11} \leq \sum_{k=n/d^b}^{n} \left(\frac{ne}{k} \cdot e^{-d/11}\right)^k \leq \sum_{k=n/d^b}^{n} e^{-dk/12} \leq \exp\left\{-\frac{n}{13d^{b-1}}\right\}.$$

(b) $\mathbf{Pr}(\hat{\mathcal{E}}_3)$ can also be bounded using the Chernoff bound. □

So, from (15) and Lemma 3,

$$\mathbf{Pr}(G_\ell \notin \hat{\mathcal{E}}_\ell) \leq \exp\left\{-\frac{4b^2 n(\log d)^2}{d^b}\right\} \tag{29}$$

As in the $b = 2$ case, let

$$\mathcal{E}''(\mathbf{Y}_{\ell-1}) = \{\hat{\mathbf{Y}}''_\ell : \ G_\ell \in \hat{\mathcal{E}}_\ell\}$$

and

$$\mathcal{E}_\ell = \left\{\mathbf{Y}_{\ell-1} : \ \mathbf{Pr}(G_\ell \in \hat{\mathcal{E}}_\ell \mid \mathbf{Y}_{\ell-1}) \geq 1 - \exp\left\{-\frac{2b^2 n(\log d)^2}{d^b}\right\}\right\}.$$

Equations (16) and (18) still hold.

As in the $b = 2$ case, we will condition on whether or not $\mathbf{Y}_{\ell-1} \in \mathcal{E}_\ell$ and we see that (17) continues to hold.

We now fix $\mathbf{Y}_\ell \in \mathcal{E}_\ell$ and $\hat{\mathbf{Y}}''_\ell \in \mathcal{E}''_\ell(\mathbf{Y}_\ell)$. Let S be a largest subset of $[n] \setminus P_\ell$ which is P-independent in G_ℓ and let $I = |S|$. Let $S_j = \{v : dist(v, S) = j \text{ in } G_\ell\}$, for $j \geq 0$. Also, let $S_{\leq j} = \cup_{j' \leq j} S_{j'}$. Then

$$\begin{aligned} I &\leq \frac{4bn \log d}{d^b} \\ |S_j| &\leq \frac{4bn \log d}{d^b} (3d)^j = \frac{4(3)^j bn \log d}{d^{b-j}} \\ |S_{\leq j}| &= I + \sum_{k=1}^{j} |S_j| \leq \frac{5(3)^j bn \log d}{d^{b-j}} \end{aligned}$$

Let $\delta_i(v, S_j)$ and $\delta_i(v, S_{\leq j})$ be defined similarly to $\delta_i(v, S)$ for $i = 1, 2$. Also, let $\delta(v, S_j)$, $\delta(v, S_{\leq j})$ be defined similarly to $\delta(v, S)$. Let $W_v = \delta(v, S_{\leq \lfloor b/2-1 \rfloor})\theta_v$ where $\theta_v = 0$ if v has a single neighbor in S_j, for some $j \leq \lfloor b/2 - 1 \rfloor$, and no other neighbors in $S_{\leq b-2-j}$. Otherwise $\theta_v = 1$.

The construction of the P-independent set, S^*, is not quite as simple as in the $b = 2$ case. If, for every $v \in P_\ell$, we remove W_v vertices from S which are distance $\lfloor b/2 \rfloor$ or less from v then we have eliminated all b length connections which pass through a single vertex in P_ℓ. However, there may still be a b length path linking two vertices in S^* if that path passes through $v_1, v_2 \in P_\ell$ such that $W_{v_1} = W_{v_2} = 0$. If such a path exists, it must contain a sub-path linking v_1 and v_2 which lies entirely outside of $S_{\leq \lfloor b/2-1 \rfloor}$. For every such sub-path linking vertices of P_ℓ, there can be at most one $\leq b$ length path connecting vertices of S which is not eliminated by the removal of the W_v elements of S distance $\lfloor b/2 \rfloor$ or less from v. Let T be the number of vertex pairs in P_ℓ connected by a path of length $b-2$ or less, lying entirely outside of $S_{\leq \lfloor b/2-1 \rfloor}$. If we let $W = \sum_{v \in P_\ell} W_v + T$ then we can create a b-independent set, S^*, in G with no fewer than $I - W$ vertices. Define $\hat{W} = \sum_{v \in P_\ell} \hat{W}_v + \hat{T}$ in an analogous way using the edge-set $\hat{Y}_\ell$ in place of Y_ℓ.

Inequality (20) still holds true for Z_ℓ.

$$\begin{aligned}
\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) &= |S_{\leq \lfloor b/2-1 \rfloor}|p - \sum_{j=0}^{\lfloor b/2-1 \rfloor} |S_j| p \,(1-p)^{|S_{\leq b-2-j}|-1} \\
&= p^2 \sum_{j=0}^{\lfloor b/2-1 \rfloor} |S_j||S_{\leq b-2-j}| + o(1) \\
&\leq \frac{20 b^3 3^{b-2} \left(\log d\right)^2}{d^b}
\end{aligned}$$

T can be over-estimated by the total number paths of length $b-2$ or less, which connect a pair of vertices in P_ℓ.

$$\begin{aligned}
\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) &= m\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) + \mathbf{E}_{\hat{Y}_\ell^*}(T) \\
m\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}_v) &\leq \frac{d^b}{b^2 (\log d)^2} \frac{3^{b-2} 20 b^3 \left(\log d\right)^2}{d^b} \\
&\leq 20 b 3^{b-2} \\
&= A - 1 \\
\mathbf{E}_{\hat{Y}_\ell^*}(T) &\leq \binom{m}{2} p + \binom{m}{2} n p^2 + \binom{m}{2} 2 n^2 p^3 = O(n^{-1})
\end{aligned}$$

So,

$$\mathbf{E}_{\hat{Y}_\ell^*}(\hat{W}) \leq A. \tag{30}$$

As in the $b = 2$ case, let $\bar{W}_v = \mathbf{E}_{Y'}(W_v | Y_\ell)$ and $\bar{W} = \mathbf{E}_{Y'}(W | Y_\ell)$.

$$\begin{aligned}
\bar{W}_v = \; & \delta_1(v, S_{\leq \lfloor b/2-1 \rfloor}) + p |S_{\leq \lfloor b/2-1 \rfloor}|_2 \\
& - \sum_{j=0}^{\lfloor b/2-1 \rfloor} 1_{\delta_1(v, S_j)=1, \delta_1(v, S_{\leq b-j-2} \setminus S_j)=0} (1-p)^{|S_{\leq b-j-2}|_2} \\
& - \sum_{j=0}^{\lfloor b/2-1 \rfloor} 1_{\delta_1(v, S_{\leq b-j-2})=0} |S_j|_2 p (1-p)^{|S_{\leq b-j-2}|_2}.
\end{aligned}$$

For d sufficiently large and $\lambda \le 1/\log d$,

$$\begin{aligned}
\mathbf{E}_{Y_\ell}(e^{\lambda \bar{W}_v}) &= e^{\lambda |S_{\le \lfloor b/2-1 \rfloor}|_2 p} \left[\sum_{t=0}^{|S_{\le \lfloor b/2-1\rfloor}|_1} \binom{|S_{\le \lfloor b/2-1\rfloor}|_1}{t} e^{\lambda t} p^t (1-p)^{|S_{\le \lfloor b/2-1\rfloor}|_1 - t} \right. \\
&\quad - \sum_{j=0}^{\lfloor b/2-1\rfloor} \left(e^\lambda - \exp\left\{\lambda - \lambda(1-p)^{|S_{\le b-j-2}|_2}\right\}\right) p|S_j|_1 (1-p)^{|S_{\le b-2-j}|_1 - 1} \\
&\quad \left. - \sum_{j=0}^{\lfloor b/2-1\rfloor} \left(1 - \exp\left\{-\lambda |S_j|_2 p(1-p)^{|S_{\le b-2-j}|_2}\right\}\right)(1-p)^{|S_{\le b-2-j}|_1} \right] \\
&= 1 + \lambda \mathbf{E}_{Y_\ell}(\bar{W}_v) + \frac{\lambda^2}{2} p^2 |S_{\le \lfloor b/2-1\rfloor}|_1^2 - \frac{\lambda^2}{2} p^2 |S_{\le \lfloor b/2-1\rfloor}|_2^2 \\
&\quad + \frac{\lambda^2}{2} p^2 \sum_{j=0}^{\lfloor b/2-1\rfloor} (|S_j|_1 |S_{\le b-2-j}|_1 + |S_j|_2^2) + O(\lambda^3 p^2 |S_{\le \lfloor b/2-1\rfloor}|^2).
\end{aligned}$$

T also depends on edges from both Y_ℓ and Y', and we need to calculate $\bar{T}(Y_\ell) = \mathbf{E}_{Y'}(T|Y_\ell)$ and $\mathbf{E}_{Y_\ell}(e^{\lambda \bar{T}})$. Let T_1 be the number of vertex pairs in P_ℓ connected by a path of length $b-2$ containing at least one edge from Y_ℓ, and let T_2 be the number of vertex pairs from P_ℓ connected by a path of length $b-2$ containing at least one edge from Y'. Some vertex pairs may be counted in both T_1 and T_2, but clearly $T \le T_1 + T_2$ and since T is small relative to $\sum_v W_v$ it is sufficient to approximate T by $T_1 + T_2$.

$$\begin{aligned}
\bar{T} &\le T_1 + \mathbf{E}_{Y'}(T_2) \\
\mathbf{E}_{Y_\ell}(e^{\lambda T}) &\le \mathbf{E}_{Y_\ell}(e^{\lambda T_1}) e^{\lambda \mathbf{E}_{Y'}(T_2)} \\
&= e^{\lambda \mathbf{E}_{Y'}(T_2)} \sum_{t=0}^{m^2/2} e^{\lambda t} \mathbf{Pr}(T_1 = t) \\
&\le e^{\lambda \mathbf{E}_{Y'}(T_2)} \left(1 + \exp\left\{\frac{\lambda m^2}{2}\right\} \mathbf{Pr}(T_1 \ge 1)\right) \\
&\le e^{\lambda \mathbf{E}_{Y'}(T_2)} \left(1 + \exp\left\{\frac{\lambda m^2}{2}\right\} \mathbf{E}(T_1)\right) \\
&= 1 + O(n^{-1}). \qquad (31)
\end{aligned}$$

Since T and the W_v's are independent of one another, we can write:

$$\begin{aligned}
\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}}) &= \mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}_v})^m\mathbf{E}_{Y_\ell}(e^{\lambda\bar{T}}) \\
&= 1+\lambda\mathbf{E}_{Y_\ell}(\bar{W})+\frac{1}{2}\lambda^2\mathbf{E}_{Y_\ell}(\bar{W})^2 \\
&\quad +m\frac{\lambda^2}{2}p^2\left(|S_{\le\lfloor b/2-1\rfloor}|_1^2-|S_{\le\lfloor b/2-1\rfloor}|_2^2+\sum_{j=0}^{\lfloor b/2-1\rfloor}|S_j|_1|S_{\le b-2-j}|_1\right) \\
&\quad +m^2\lambda^2p^4\left(\sum_{j=0}^{\lfloor b/2-1\rfloor}|S_j||S_{\le b/2-1}|\right)^2+O(\lambda^3)
\end{aligned}$$

As in the $b=2$ case, we can derive, in a similar manner to (26),

$$\begin{aligned}
\mathbf{E}\left(e^{\lambda Z_\ell}\right) &\le e^{\lambda(A+1)}\mathbf{E}_{Y_\ell}(e^{\lambda\bar{W}})-\lambda\mathbf{E}_{Y_\ell^*}(W)-\lambda(A+1) \\
&= 1+\frac{1}{2}\lambda^2(A+1)^2+\frac{1}{2}\lambda^2\mathbf{E}_{Y_\ell}(\bar{W})^2+m\lambda^2(A+1)\sum_{j=0}^{b/2-1}\left(p^2|S_j||S_{\le b/2-1}|\right) \\
&\quad +m\frac{\lambda^2}{2}p^2\left(|S_{\le\lfloor b/2-1\rfloor}|_1^2-|S_{\le\lfloor b/2-1\rfloor}|_2^2+\sum_{j=0}^{\lfloor b/2-1\rfloor}|S_j|_1|S_{\le b-2-l}|_1\right) \\
&\quad +m^2\lambda^2p^4\left(\sum_{j=0}^{\lfloor b/2-1\rfloor}|S_j||S_{\le b/2-1}|\right)^2+O(\lambda^3) \\
&\le 1+3A^2\lambda^2
\end{aligned}$$

for d sufficiently large.

Going back to (17) and using (29) we see that

$$\mathbf{Pr}(\beta_b-\bar{\beta}_b\ge t) \;\le\; \Theta_{\ell+1}\sum_{\mathbf{Y}}\prod_{i=1}^{\ell-1}e^{\lambda Z_i}\mathbf{Pr}(\mathbf{Y})+(n'-\ell+1)e^{-n/d^b},$$

completing the induction.

As in the $b=2$ case, letting $\ell=0$, and substituting in t,λ, and Θ_0 yields equation (7).

2.1.3 The Case $b>5$

In the case $b>5$, we will condition on the event that, in G, $\Delta\le\log n$, which we will denote $\mathcal{E}_0$. The probability that G has any vertices of degree greater than $\log n$ is $o(n^{-2})$. Since $\mathbf{Pr}(\mathcal{E}_0)\to 1$ as $n\to\infty$, proving both (7) and (8) conditioned on $\mathcal{E}_0$ is sufficient to prove Theorem 1. The proof follows the same course as the $3 \leq b \leq 5$ case, but the following changes need to be made in order to convert the appropriate probabilities into conditional probabilities.

Inequality (28) becomes:

$$\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t|\mathcal{E}_0) \leq \Theta_\ell \sum_{\mathbf{Y} \in \mathcal{E}_0} \prod_{i=1}^{\ell} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}|\mathcal{E}_0) + (n' - \ell)e^{-n/d^b}. \tag{32}$$

Conditioning on $\mathcal{E}_0$ will change the probability of $Y_\ell \notin \mathcal{E}_1$ and $Y_\ell \notin \mathcal{E}_2$. However, we can over-estimate the new probabilities, using $\mathbf{Pr}(Y_\ell \notin \mathcal{E}_1|\mathcal{E}_0) \leq \frac{\mathbf{Pr}(Y_\ell \notin \mathcal{E}_1)}{\mathbf{Pr}(Y_\ell \in \mathcal{E}_0)}$, and likewise for $\mathcal{E}_2$. This only introduces a constant factor, and the probabilities remain exponentially small.

$\mathcal{E}_3$ is no longer necessary and can be replaced by the claim that G contains no more than $n(\log n)^a$ paths of length a. Given that we are conditioning on $\mathcal{E}_0$, this will always be true.

Inequality (17) becomes:

$$\begin{aligned}\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t|\mathcal{E}_0) &\leq \Theta_\ell \sum_{\mathbf{Y}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell|\mathcal{E}_0) \sum_{Y_\ell s.t. (\mathbf{Y}_\ell, Y_\ell) \in \mathcal{E}_0} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell|\mathcal{E}_0) \\ &\qquad + (n' - \ell)e^{-n/d^b} \\ &\leq \Theta_\ell \sum_{\mathbf{Y}_\ell \in \mathcal{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell|\mathcal{E}_0) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\ &\qquad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_\ell \notin \mathcal{E}_\ell|\mathcal{E}_0) + (n' - \ell)e^{-n/d^b}. \end{aligned} \tag{33}$$

We can overestimate $\mathbf{Pr}(\mathbf{Y}_\ell|\mathcal{E}_0)$ with $\frac{\mathbf{Pr}(\mathbf{Y}_\ell)}{\mathbf{Pr}(\mathcal{E}_0)}$, and substituting this approximation into (32) yields:

$$\begin{aligned}\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t|\mathcal{E}_0) &\leq \frac{\Theta_\ell}{\mathbf{Pr}(\mathcal{E}_0)^{\ell-1}} \sum_{\mathbf{Y}_\ell \in \mathcal{E}_\ell} \prod_{i=1}^{\ell-1} e^{\lambda Z_i} \mathbf{Pr}(\mathbf{Y}_\ell) \sum_{Y_\ell} e^{\lambda Z_\ell} \mathbf{Pr}(Y_\ell) \\ &\quad + e^{\lambda n'} \mathbf{Pr}(\mathbf{Y}_\ell \notin \mathcal{E}_\ell|\mathcal{E}_0) + (n' - \ell)e^{-n/d^b}.\end{aligned}$$

The factor of $\frac{1}{\mathbf{Pr}(\mathcal{E}_0)^{\ell-1}}$, can be not greater than $\frac{1}{\mathbf{Pr}(\mathcal{E}_0)^{n'}}$, which is no more than a constant. A constant factor will not alter the order of magnitude of $\mathbf{Pr}(\beta_b - \bar{\beta}_b \geq t)$ which we are trying to bound in equation (7).

The only remaining change which must be made for the $b > 5$ case is that we now need to account for paths of length greater than 3 when evaluating T in equation (30).

$$\mathbf{E}_{Y_\ell}(\bar{T}) = \mathbf{E}_{\hat{Y}_\ell^*}(T) \leq \binom{m}{2} p + \binom{m}{2} np^2 + \sum_{a=1}^{b-4} \binom{m}{2} 2n(\log n)^a p^2 = O\left(\frac{(\log n)^b}{n}\right).$$

So, following the argument for (31) we obtain

$$\mathbf{E}_{Y_\ell}(e^{\lambda \bar{T}}) = 1 + O\left(\frac{(\log n)^b}{n}\right).$$

2.2 Proof of (8)

We divide the proof of (8) into two cases, rather than three. There is no difference between the $b = 2$ and $b = 3, 4, 5$ cases, but we need to introduce the conditional probabilities in the $b > 5$ case so that (7) and (8) can be properly combined to prove a lower bound on β_b.

2.2.1 The Case $b \leq 5$

One can prove the inequality (8) using the lower bound:

$$\mathbf{Pr}(X_k > 0) \geq \frac{\mathbf{E}(X_k)^2}{\mathbf{E}(X_k^2)}$$

Applying Lemma 1 we obtain

$$\begin{aligned} \mathbf{E}(X_k) &\geq \binom{n'}{k} m^k (1 - p_d)^{\binom{k}{2}} \\ \mathbf{E}(X_k^2) &\leq \binom{n'}{k} \sum_{l=0}^{k} \binom{k}{l} \binom{n'-k}{k-l} m^{k-l} (1 - p_d)^{\left(2\binom{k}{2} - \binom{l}{2}\right)} \exp\left\{O(k^3 n^{2b-3} p^{2b-1})\right\} \end{aligned}$$

$$\frac{\mathbf{E}(X_k^2)}{\mathbf{E}(X_k)^2} \leq \sum_{l=0}^{k} \frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l (1 - p_d)^{\binom{l}{2}}} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \tag{34}$$

Once we have an inequality of this form, we can appeal to [4], which contains the same calculation for the $b = 1$ case. In the $b = 1$ case, with n_1', m_1 to distinguish them from n', m we have

$$\frac{\mathbf{E}(X_k^2)}{\mathbf{E}(X_k)^2} \le \sum_{l=0}^{k} \frac{\binom{k}{l}\binom{n_1'-l}{k-l}}{\binom{n_1'}{k} m_1^l (1-p)^{\binom{l}{2}}} \le \exp\left\{O\left(\frac{(\log d)^{5/2} n}{d^{3/2}}\right)\right\}.$$

To apply this result to the $b > 1$ case, we perform a change of variable.

$$\begin{aligned} p_{new} &= p_d \\ d_{new} &= np_{new} = d^b(1 + O(d^{-1}) \end{aligned}$$

Putting these values into (34), we get:

$$\begin{aligned} \frac{\mathbf{E}(X^2)}{\mathbf{E}(X)^2} &\le \sum_{l=0}^{k} \left[\frac{\binom{k}{l}\binom{n'-l}{k-l}}{\binom{n'}{k} m^l (1-p_{new})^{\binom{l}{2}}}\right] \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \\ &\le \exp\left\{O\left(\frac{(\log d_{new})^{5/2} n}{d_{new}^{3/2}}\right)\right\} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \\ &= \exp\left\{O\left(\frac{b^{5/2}(\log d)^{5/2} n}{d^{3b/2}}\right)\right\} \exp\left\{O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \\ \mathbf{Pr}(X_k > 0) &\ge \exp\left\{-O\left(\frac{(\log d)^3 n}{d^{b+1}}\right)\right\} \end{aligned}$$

2.2.2 The Case $b > 5$

We can introduce the conditional probabilities into the proof of inequality (8) with relative ease. We aim to put a lower bound on $\mathbf{Pr}(X_k > 0|\mathcal{E}_0)$, but we can appeal to the FKG inequality [3] to relate this quantity to $\mathbf{Pr}(X_k > 0)$, which we have already bounded in the $b \le 5$ case. Since both the events, $X_k > 0$ and $\mathcal{E}_0$, are monotone decreasing, increasing in probability with the removal of edges, the FKG inequality shows that:

$$\mathbf{Pr}(X_k > 0|\mathcal{E}_0) \ge \mathbf{Pr}(X_k > 0) \ge \frac{\mathbf{E}(X_k)^2}{\mathbf{E}(X_k^2)}$$

Therefore, our previous calculations are sufficient to prove (8) for the $b > 5$ case as well.

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