Vertex covers by edge disjoint cliques

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Abstract

Let H be a simple graph having no isolated vertices. An (H,k)-*vertex-cover* of a simple graph $G=(V,E)$ is a collection $H_1,\ldots,H_r$ of subgraphs of G satisfying

1. $H_i \cong H$,   for all $i=1,\ldots,r$,

2. $\cup_{i=1}^r V(H_i)=V$,

3. $E(H_i)\cap E(H_j)=\emptyset$,   for all $i\neq j$, and

4. each $v\in V$ is in at most k of the H_i.

We consider the existence of such vertex covers when H is a complete graph, $K_t, t\geq 3$, in the context of extremal and random graphs.

1 Introduction

Let H be a simple graph having no isolated vertices. For the purposes of this discussion we say that the simple graph $G=(V,E)$ has property $\mathcal{C}_{H,k}$ if there is a collection $H_1,\ldots,H_r$ of subgraphs of G satisfying

P1. $H_i \cong H$,   for all $i=1,\ldots,r$,

P2. $\cup_{i=1}^r V(H_i)=V$,

P3. $E(H_i)\cap E(H_j)=\emptyset$,   for all $i\neq j$, and

P4. each $v\in V$ is in at most k of the H_i.

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We call the family $\{H_1, \ldots, H_r\}$ an (H, k)-vertex-cover of G. Thus when $k = 1$ we ask for the existence of a partition of V into *vertex disjoint* copies of H i.e. the existence of an *H-factor.* In this case we assume the necessary divisibility condition, i.e. that $|V(H)|$ divides $|V|$. We study this property when G is a random graph and also when G is extremal w.r.t. minimum degree. In the main we will focus on the case where H is a complete graph K_t and denote our property by $\mathcal{C}_{t,k}$.

Random Graphs. The precise threshold for the occurrence of $\mathcal{C}_{2,1}$ i.e. the existence of a perfect matching was found by Erdős and Rényi [7] as part of a series of papers which laid the foundations of the theory of random graphs. The precise threshold for the occurrence of $\mathcal{C}_{3,1}$ i.e the existence of a vertex partition into triangles remains as one of the most challenging problems in this area (see, for example, the Appendix by Erdős to the monograph by Alon and Spencer [1]).

The thresholds for H-factors have been studied for example by Ruciński [15] and by Alon and Yuster [3]. For a graph H, let

$$m_1(H) = \max\Bigl(\frac{|E(H')|}{|V(H')| - 1}\Bigr)$$

where the maximum is taken over all subgraphs H' of the graph H with at least two vertices. In [15], Ruciński showed that the probability $p(n) = O(n^{-1/m_1(H)})$ is a sharp threshold for the property $\mathcal{C}_{H,1}$ for any graph H such that $m_1(H) > \delta(H)$ where $\delta(H)$ stands, as usual, for the minimum degree of the graph H. Note that, for example, H being a complete graph is excluded. Hence, the first interesting open case is $H = K_3$. In [11], Krivelevich showed that the probability $p(n) = O(n^{-3/5})$ is enough for the random graph to have a K_3-factor **whp**[1] and, in general, if $p(n) = O(n^{-2t/(t-1)(t+2)})$ then the random graph $G_{n,p}$ contains a K_t-factor **whp** (provided t divides n).

An obvious necessary condition for the existence of a (K_t, k)-vertex-cover is that every vertex be incident with at least one copy of K_t.

Theorem 1. *Let $m = \binom{n}{2}((t-1)!(\log n + c_n))^{1/\binom{t}{2}} n^{-2/t}$. Then*

$$\lim_{n\to\infty} \mathbf{Pr}(G_{n,m} \text{ contains a } (K_t, 2)\text{-vertex-cover}) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases}$$

(Here, $G_{n,m}$ stands for the probability space over the set of all graphs on n vertices and with m edges endowed with the uniform probability measure.) We will prove this as a consequence of the slightly stronger hitting time version. We consider the graph process $G_m = ([n], E_m), m = 0, 1, \ldots, \binom{n}{2}$, where $E_0 = \emptyset$ and G_m is obtained from G_{m-1} by choosing e_m randomly from $\binom{[n]}{2} \setminus E_{m-1}$ and putting

[1] A sequence of events $\mathcal{E}_n$ occurs *with high probability,* **whp**, if $\mathbf{Pr}(\mathcal{E}_n) = 1 - o(1)$.

$E_m = E_{m-1} \cup \{e_m\}$. We define two *hitting times*:

$$\begin{aligned} \tau_1 = \tau_1(t) &= \min\{m : \text{ Every } v \in [n] \text{ is contained in a copy of } K_t \text{ in } G_m\} , \\ \tau_2 = \tau_2(t) &= \min\{m : G_m \text{ contains a } (K_t, 2)\text{-vertex-cover}\}. \end{aligned}$$

Theorem 2. *For every fixed $t \geq 3$,*

$$\lim_{n\to\infty} \mathbf{Pr}(\tau_1 = \tau_2) = 1.$$

Moreover, there exists **whp** *a $(K_t, 2)$-vertex-cover of G_{τ_2} containing $(1 + o(1))\frac{n}{t}$ copies of K_t.*

Remark 1. In fact, our proof of Theorem 2 implies that G_{τ_2} possesses **whp** a $(K_t, 2)$-vertex-cover containing at most $\left(\frac{1}{t} + \frac{1}{(\log n)^{1/t}}\right) n$ copies of K_t.

Remark 2. Theorem 2 lends weight to the common conjecture that the threshold for a K_t-factor is m of Theorem 1.

We prove Theorem 2 in Section 2 and show how Theorem 1 follows from Theorem 2 in Section 3.

Extremal Graphs. For a graph G on n vertices what is the smallest minimum degree that insures G has $\mathcal{C}_{t,k}$? For $t \geq 3$ and $k \geq 2$ let

$$f(n, t, k) = \max\{d : \exists G \text{ such that } \delta(G) = d, |V(G)| = n \text{ and } G \notin \mathcal{C}_{t,k}\}.$$

We will assume that n is large with respect to t, but k can be arbitrarily large. The smallest minimum degree that guarantees a K_t-factor (this would be, up to divisibility considerations, $f(n, t, 1) + 1$) was established in the following deep theorem of Hajnal and Szemerédi [9].

Theorem 3 (Hajnal, Szemerédi). *If $|V(G)| = n$ and $\delta(G) \geq (1 - \frac{1}{t})n$ then G contains $\lfloor n/t \rfloor$ vertex-disjoint copies of K_t.*

Our central result in this section is the following:

Theorem 4. *Let $t \geq 3$, $k \geq 2$, $n \geq 6t^2 - 4t$ and*

$$n = q[(t-1)k + 1] + r \text{ where } 1 \leq r \leq (t-1)k + 1.$$

Then

$$n - qk - \left\lceil \frac{r}{t-1} \right\rceil \leq f(n, t, k) \leq n - qk - - \left\lceil \frac{r}{t-1} \right\rceil + 1 \ .$$

Note that it follows from Theorem 4 that

$$f(n, t, k) = \left\lfloor \frac{[(t-2)k + 1]n}{(t-1)k + 1} \right\rfloor + c \tag{1}$$

where $c \in \{0, 1, 2\}$. It is tempting to believe that $f(n, t, k)$ equals the lower bound given in Theorem 4. This is not the case in general.

Theorem 5. *Let $n \geq 6$ and $k \geq (n-1)/2$.*

$$f(n, 3, k) = \left\lceil \frac{n}{2} \right\rceil.$$

Note that the value of $f(n,3,k)$ given in Theorem 5 equals the lower bound in Theorem 4 for n even, but equals the upper bound for n odd. (Here $q = 0$ and $r = n$).

For H a simple graph with no isolated vertices and G an arbitrary graph an (H, ∞)-*vertex-cover* of G is a collection $H_1, \ldots, H_r$ of subgraphs of G satisfying P1, P2 and P3. Thus, G has an (H, ∞)-vertex-cover if and only if there exists a k such that G has a (H, k)-vertex-cover. To motivate our results on (H, ∞)-vertex-covers, we recall the following well-known extension of Theorem 3. Given an arbitrary graph H, Komlós, Sárközy and Szemerédi [13] showed that there is a constant c (depending only on the graph H) such that if $\delta(G) \geq \left(1 - \frac{1}{\chi(H)}\right) n$ for a graph G on n vertices, then there is a union of vertex-disjoint copies of H covering all but at most c vertices of G. Weakening the condition on $\delta(G)$ we show in the following theorem the existence of (H, ∞)-vertex-covers for graphs H having the property that there is a vertex u of H such that $\chi(H \setminus \{u\}) = \chi(H) - 1 \geq 3$.

Theorem 6. *Let H be a graph such that $\chi(H) \geq 4$ and such that there is a vertex u of H with the property that $\chi(H \setminus \{u\}) = \chi(H) - 1$. Then for every $\epsilon > 0$ and every graph G on n vertices, if $\delta(G) \geq \left(1 - \frac{1}{\chi(H)-1} + \epsilon\right)n$, then G has an (H, ∞)-vertex-cover provided n is large enough.*

Theorems 4, 5 and 6 are proved in Section 4.

2 Proof of Theorem 2

In this section we will use the following Chernoff bounds on the tails of the binomial random variable $B(n, p)$. For $0 \leq \epsilon \leq 1$ and $\theta > 0$

$$\mathbf{Pr}(B(n,p) \leq (1-\epsilon)np) \leq e^{-\epsilon^2 np/2} \quad (2)$$

$$\mathbf{Pr}(B(n,p) \geq (1+\epsilon)np) \leq e^{-\epsilon^2 np/3} \quad (3)$$

$$\mathbf{Pr}(B(n,p) \geq \theta np) \leq (e/\theta)^{\theta np} \quad (4)$$

All Lemmas introduced in this section will be proven in the subsections that follow.

Let $t \geq 3$ be fixed. We construct a $(K_t, 2)$-vertex-cover in G_m by dividing our graph process into 3 phases and using edges from different phases for different purposes. Before describing the phases, we make some preliminary definitions and the observation that we may restrict our attention to G_m where m lies in a small interval. Let $\alpha, \beta > 0$ be constants such that

$$\beta^{\binom{t}{2}} > 19/20 \text{ and } \alpha + \beta < 1,$$

and let

$$m_a = \alpha\binom{n}{2}\left((t-1)!\log n\right)^{1/\binom{t}{2}} n^{-2/t}, \text{ and}$$
$$m_b = \beta\binom{n}{2}\left((t-1)!\log n\right)^{1/\binom{t}{2}} n^{-2/t}.$$

Furthermore, for $i = 0, 1$ let

$$m_i = \binom{n}{2}\left((t-1)!(\log n - (1-2i)\log\log n)\right)^{1/\binom{t}{2}} n^{-2/t}.$$

Lemma 1.
$$\mathbf{Pr}(\tau_1 \notin [m_0, m_1]) = o(1)\ .$$

We will use the term 'a collection of K_t's' in the graph G, for a family $\mathcal{A} \subseteq \binom{V(G)}{t}$ such that $G[S]$ is complete for all $S \in \mathcal{A}$. For such a collection $\mathcal{A}$ we set

$$V(\mathcal{A}) = \bigcup_{S\in\mathcal{A}} S \quad \text{ and } \quad E(\mathcal{A}) = \bigcup_{S\in\mathcal{A}} \binom{S}{2},$$

say $\mathcal{A}$ 'covers' a vertex v if $v \in V(\mathcal{A})$, and say $\mathcal{A}$ 'covers' a set of vertices T if $T \subseteq V(\mathcal{A})$.

We are now ready to describe the 3 phases. In the first phase we simply choose m_a edges uniformly at random, producing the graph $G^1 = ([n], E^1)$. Thus,

$$G^1 = G_{n,m_a}.$$

In the second phase we form the graph $G^2 = ([n], E^2)$ by choosing m_b edges uniformly at random. This is done independently of phase 1 and without knowledge of which edges were placed in phase 1. Thus,

$$G^2 = G_{n,m_b},$$

and a particular edge may appear in both G^1 and G^2. Let $F = E^1 \cup E^2$ and $m_{-1} = |F|$. The third phase is the graph process $H_i = ([n], F_i), i = m_{-1}, \ldots, m_1$ where $F_{m_{-1}} = F$ and F_{i+1} is the union of F_i and the set containing a single edge chosen uniformly at random from $\binom{n}{2}\backslash F_i$. In other words, in the third phase we start with the collection of edges generated in phases 1 and 2 and then add new edges one at time until m_1 edges have been placed. Note that for $m_a+m_b \le i \le m_1$ the graphs G_i and H_i are identically distributed.

We henceforth assume that

$$m_a + m_b \le m \le m_1$$

and that every vertex in $H_m = G_m$ lies in at least one copy of K_t. We will show that

$$\textbf{whp } G_m \text{ has a } (K_t, 2)\text{-vertex-cover.} \tag{5}$$

Theorem 2 follows from (5) and Lemma 1.

How do we construct the $(K_t, 2)$-vertex-cover? We first use the phase one edges to greedily cover as many vertices as possible with vertex disjoint K_t's. Let Ξ be an arbitrary maximal collection of vertex disjoint K_t's in G^1, $X \subseteq [n]$ be the set of vertices not covered by Ξ, and

$$r = \left\lceil \frac{n}{(\log n)^{1/t}} \right\rceil.$$

We can easily randomise this choice of K_t's so that X is a random $|X|$-subset of $[n]$. This will be used in the proof of Lemma 4.

Lemma 2. *Let $G = G_{n,m_a}$.*

$$\mathbf{Pr}\left(\exists R \subset [n] \text{ such that } |R| = r \text{ and } G[R] \text{ contains no } K_t\text{'s}\right) = o(1).$$

It follows from Lemma 2 that **whp**

$$|X| \leq r. \tag{6}$$

In other words, after using only a small fraction of the edges in G_m, only $o(n)$ vertices remain to be covered. We will use the phase 2 edges (as well as a handful of the phase 1 and phase 3 edges) to form a vertex disjoint collection of K_t's that covers X but does not use any edge in $E(\Xi)$.

Before describing the vertex disjoint collection of K_t's that covers X, we make further definitions and preliminary observations. Our first observation concerns the random graph process G_{m_1} alone. Let $\nu_3 = 4$, $\nu_4 = 3$ and $\nu_i = 2$ for $i = 5, 6, \ldots$. We define a *cluster* to be a collection $\mathcal{C} = \{S_1, \ldots, S_l\}$ of K_t's in G_{m_1} such that $l \leq 2\nu_t$

$$\begin{array}{c} \kappa_i \geq 1 \quad \text{for} \quad i = 2, \ldots, l \\ \kappa_i = t \quad \Rightarrow \quad \kappa_{i-1} = 1 \quad \wedge \quad |S_i \cap S_{i-1}| \geq 2 \\ \text{and} \quad |\{i : \kappa_i \neq 1\}| = \nu_t \end{array}$$

where

$$\kappa_i = \left| S_i \cap \left(\bigcup_{j=1}^{i-1} S_j \right) \right| \quad \text{for} \quad i = 2, \ldots, l.$$

Roughly speaking, a cluster is a small collection of K_t's that have many or large pairwise intersections.

Lemma 3.

$$\mathbf{Pr}(G_{m_1} \text{ contains a cluster}) = o(1).$$

We now turn our attention to the graph G^2. For $v \in [n]$ let Υ_v be the collection of K_t's in G^2 that contain v; to be precise,

$$\Upsilon_v = \left\{ S \in \binom{[n]}{t} : v \in S \text{ and } \binom{S}{2} \subseteq E^2 \right\}.$$

Since Υ_v depends only on the graph G^2 while X is small and depends only on the graph G^1, it is usually the case that no $V(\Upsilon_v)$ contains many members of X. To make this statement precise, we let

$$q = \left\lceil \frac{\log n}{\log \log \log n} \right\rceil.$$

Lemma 4.

$$\mathbf{Pr}(\exists v \in [n] \textit{ such that } |V(\Upsilon_v) \cap X| > q) = o(1).$$

We say that

$$\begin{array}{l} v \in [n] \text{ is } \textit{large} \text{ if } \quad |\Upsilon_v| \geq \frac{\log n}{20}, \text{ and} \\ v \in [n] \text{ is } \textit{small} \text{ if } \quad |\Upsilon_v| < \frac{\log n}{20}. \end{array}$$

With high probability the small vertices are, with respect to connections via K_t's, far apart. To make this statement precise, we define a *chain* to be a pair u, v of distinct small vertices and a collection $S_1, S_2, S_3, S_4 \in \binom{[n]}{t}$ of (not necessarily distinct) sets such that $u \in S_1$, $v \in S_4$,

$$S_1 \cap S_2, S_2 \cap S_3, S_3 \cap S_4 \neq \emptyset, \qquad \text{and} \qquad \binom{S_i}{2} \subseteq E(G_{m_1}) \text{ for } i = 1, 2, 3, 4.$$

Lemma 5.

$$\mathbf{Pr}(G_{m_1} \textit{ contains a chain}) = o(1).$$

We also note that no K_t containing a small vertex intersects any other K_t in more than a single vertex. A *link* is a small vertex $u \in [n]$ and distinct $S_1, S_2 \in \binom{[n]}{t}$ such that $u \in S_1$, $|S_1 \cap S_2| \geq 2$, and $\binom{S_1}{2}, \binom{S_2}{2} \subseteq E(G_{m_1})$.

Lemma 6.

$$\mathbf{Pr}(G_{m_1} \textit{ contains a link}) = o(1).$$

Finally, let

$$X_1 = \{v \in X : v \text{ is small}\},$$
$$X_2 = \{v \in X : v \text{ is large}\}, \text{ and}$$
$$\Phi = \left\{ S \in \binom{[n]}{t} : \binom{S}{2} \subseteq E(G_{m_1}) \text{ and } S \cap X_1 \neq \emptyset \right\}.$$

We are now prepared to describe the remainder of the $(K_t, 2)$-cover.

We henceforth assume (6),

$$\begin{aligned} G_{m_1} &\text{ does not contain a cluster,} && (7)\\ \forall v \in [n] \quad &|V(\Upsilon_v) \cap X| \leq q, && (8)\\ G_{m_1} &\text{ does not contain a chain,} && (9)\\ G_{m_1} &\text{ does not contain a link,} && (10) \end{aligned}$$

and that n is sufficiently large (in a sense that is made clear below). We will show that there exist collections Ξ_1 and Ξ_2 of vertex disjoint K_t's in G_m such that $\Xi_1 \cup \Xi_2$ covers $X_1 \cup X_2$ and

$$V(\Xi_1) \cap V(\Xi_2) = \emptyset \text{ and } E(\Xi) \cap E(\Xi_1 \cup \Xi_2) = \emptyset. \tag{11}$$

If follows from Lemmas 1, 2, 3, 4, 5 and 6 that (11) implies Theorem 2.

We cover X_1 in a rather crude way. Let Ξ_1 be an arbitrary collection of K_t's in G_m that covers X_1. Note that the collection Ξ_1 uses edges from all 3 phases and that we make use of the fact that every vertex is contained in some K_t in G_m when forming Ξ_1. By (9), Ξ_1 is vertex disjoint.

We cover

$$X_2' := X_2 \backslash V(\Xi_1)$$

in a more sophisticated way: we apply the Lovász Local Lemma. We first 'trim' the Υ_v's. For $v \in X_2'$ let Υ_v' be the collection of sets in $S \in \Upsilon_v$ such that

$$\begin{gathered} S \cap X = \{v\} \\ T \in \binom{[n]}{t} \wedge \binom{T}{2} \subseteq E(G_{m_1}) \Rightarrow |S \cap T| \leq 1, \qquad \text{and} \\ S \cap V(\Phi) \subseteq \{v\}. \end{gathered} \tag{12}$$

In words, we get Υ_v' from Υ_v by throwing away those sets in Υ_v that contain an element of X other than v, intersect another K_t in more than one vertex, or contain a vertex of a K_t that contains a small vertex. By (8) there are at most q sets in Υ_v that contain an element of X other than v. We will show

$$\text{there are } \leq \tbinom{2\nu_t t}{t} \text{ sets in } \Upsilon_v \text{ that intersect another } K_t \text{ in} \geq 2 \text{ vertices.} \tag{13}$$

By (9) at most 1 set in Υ_v intersects $V(\Phi)$. Therefore, we may choose $\Theta_v \subseteq \Upsilon_v'$ such that

$$|\Theta_v| = \left\lceil \frac{\log n}{21} \right\rceil \qquad \text{for all} \qquad v \in X_2'. \tag{14}$$

Proof of (13) Let $\hat{\Upsilon}_v$ denote the collection of K_t's in Υ_v which intersect another K_t in more than one vertex. Let $B = V(\hat{\Upsilon}_v)$. We construct copies $X_1, X_2, \ldots, X_l$

of K_t in G_{m_1} as follows: Suppose we have constructed $X_1, X_2, \dots, X_k$. Either (i) $B \subseteq V_k = V(X_1 \cup X_2 \cup \dots \cup X_k)$ or (ii) $B \not\subseteq V_k$. In case (ii) choose $X_{k+1} \in \Upsilon_v$ which is not contained in V_k. If $|X_{k+1} \cap V_k| = 1$ then choose X_{k+2} where $|X_{k+2} \cap X_{k+1}| \geq 2$. If this process continues for ν_t iterations we will have produced a cluster. Thus $l \leq 2\nu_t$ and $|B| \leq 2t\nu_t$, which implies (13).

Now, consider the probability space in which each $v \in X_2'$ chooses $S_v \in \Theta_v$ uniformly at random and independently of the other vertices. For $u \neq v \in X_2'$, $S \in \Theta_u$ and $T \in \Theta_v$ such that $S \cap T \neq \emptyset$ let $A_{u,v,S,T}$ be the event that $S_u = S$ and $S_v = T$. These are the 'bad' events in our application of the Lovász Local Lemma. Clearly,

$$\mathbf{Pr}(A_{u,v,S,T}) = \frac{1}{|\Theta_v||\Theta_u|} \leq \left(\frac{21}{\log n}\right)^2 =: p. \tag{15}$$

Events A_{u_1,u_2,S_1,S_2} and A_{v_1,v_2,T_1,T_2} are dependent if and only if

$$\{u_1, u_2\} \cap \{v_1, v_2\} \neq \emptyset.$$

Thus, the degree in the dependency graph is bounded above by

$$\begin{aligned} d := 2 \max_{u \in X_2'} \sum_{S \in \Theta_u} \sum_{v \in X_2'} |\{T \in \Theta_v : S \cap T \neq \emptyset\}| \\ \leq 2 \max_{u \in X_2'} \sum_{w \in V(\Theta_u)} |\Upsilon_w \cap X| \\ \leq 2tq \left\lceil \frac{\log n}{21} \right\rceil \qquad\qquad \text{by (8)} \\ \leq \frac{t(\log n)^2}{10 \log\log\log n}. \end{aligned} \tag{16}$$

It follows from (15) and (16) that

$$pd \leq \frac{45}{\log\log\log n} = o(1).$$

Thus, for n sufficiently large, it follows from the Lovász Local Lemma that there exists a vertex disjoint collection Ξ_2 of K_t's in G^2 that covers X_2' but covers no vertex in $V(\Xi_1)$.

It remains to show that

$$E(\Xi) \cap E(\Xi_1 \cup \Xi_2) = \emptyset.$$

This is an immediate consequence of (10) and (12). We have established (11) and completed the proof. $\square$

2.1 Proof of Lemma 1

Let $p_i = m_i / \binom{n}{2}$ for $i = 0, 1$.

We first apply Janson's inequality to show that **whp** every vertex in G_{n,p_1} is contained in a copy of K_t (we follow the notation of [1, pages 95 and 96]). Let v be a fixed vertex and let and Z denote the number of copies of K_t in G which are incident with v. Next let $S_1, S_2, \dots, S_{\binom{n-1}{t-1}}$ be an enumeration of the copies of K_t in K_n which contain v. Letting B_j be the event $\binom{S_j}{2} \subseteq E(G_{n,p_i})$, we have

$$\mu = \sum_{j=1}^{\binom{n-1}{t-1}} \mathbf{Pr}(B_j) = \binom{n-1}{t-1} p_1^{\binom{t}{2}} = (\log n + \log\log n)(1 + O(1/n)) \tag{17}$$

and

$$\begin{aligned} \Delta &= \sum_{|S_j \cap S_k| \geq 2} \mathbf{Pr}(B_j \cap B_k) \\ &= \binom{n-1}{t-1} \sum_{r=2}^{t-1} \binom{t-1}{r-1}\binom{n-t}{t-r} p_i^{2\binom{t}{2}-\binom{r}{2}} \\ &= O\left(\sum_{r=2}^{t-1} n^{2t-r-1-\frac{2}{t}(2\binom{t}{2}-\binom{r}{2})+o(1)} \right) \\ &= O(n^{2t^{-1}-1+o(1)}) \ . \end{aligned} \tag{18}$$

Then, by Janson's inequality, we have

$$\begin{aligned} \mathbf{Pr}(Z=0) &\leq \exp\left\{-\mu + \frac{1}{1-\epsilon}\frac{\Delta}{2}\right\} \\ &= \frac{1}{n\log n}\exp\left\{O(n^{-1+o(1)}) + O(n^{2t^{-1}-1+o(1)})\right\} \\ &= o(1/n). \end{aligned} \tag{19}$$

It follows that

$$\mathbf{Pr}(\exists u \in [n] : \ u \text{ is not contained in a copy of } K_t \text{ in } G_{n,p_1}) = o(1). \tag{20}$$

The event $\{\exists u \in [n] : \ u$ is not contained in a copy of $K_t\}$ is monotone decreasing and so (20) implies that **whp** every vertex in $[n]$ is contained in a copy of K_t in G_{n,m_1}. In other words, $\tau_1 \leq m_1$ **whp**.

We now turn to the random graph G_{n,p_0} in order to establish our almost sure lower bound on τ_1. For $v \in [n]$ let Z_v be the number of K_t's in G_{n,p_0} that contain v, and let Y denote the number of vertices v such that $Z_v = 0$. Since

$$M = (1-p_0)^{\binom{n-1}{t-1}} = (1+o(1))\frac{\log n}{n} \tag{21}$$

is a lower bound on $\mathbf{Pr}(Z_v = 0)$ for each $v \in [n]$, we have

$$\mathbf{E}(Y) \geq (1+o(1))\log n. \tag{22}$$

We now show that $\mathbf{Var}(Y)$ is small. Indeed,

$$\mathbf{Pr}(Z_1 = Z_2 = 0) \le \mathbf{Pr}(\mathcal{E}_1) + \mathbf{Pr}(\bar{\mathcal{E}}_2\bar{\mathcal{E}}_3 \mid \bar{\mathcal{E}}_1) \tag{23}$$

where, if N_i is the set of neighbors of i in G_{n,p_0},

$$\begin{aligned}
\mathcal{E}_1 &= \left\{\left((1-n^{-\frac{1}{4t}})np_0 \le |N_1|, |N_2| \le 2np_0\right)\right. \\
&\qquad\qquad \left.\vee \left(|N_1 \cap N_2| \ \ge n^{-\frac{1}{4t}}np_0\}\right)\right\} \\
\mathcal{E}_2 &= \{G_{n,p_0} \text{ contains a copy } H \text{ of } K_{t-1} \text{ such that } H \subseteq N_1\} \\
\mathcal{E}_3 &= \{G_{n,p_0} \text{ contains a copy } H \text{ of } K_{t-1} \text{ such that } H \subseteq N_2 \setminus N_1\}.
\end{aligned}$$

Applying (2)–(4) we get,

$$\mathbf{Pr}(\mathcal{E}_1) \le 5\exp\left\{-n^{1-\frac{5}{2t}+o(1)}\right\}.$$

Note that

$$\mathbf{Pr}(\overline{\mathcal{E}_2} \wedge \overline{\mathcal{E}_3}|N_1,N_2) = \mathbf{Pr}(\overline{\mathcal{E}_2}|N_1,N_2)\mathbf{Pr}(\overline{\mathcal{E}_3}|N_1,N_2)$$

because, conditioning on N_1 and N_2, these events depend on disjoint sets of edges. Let W_1 and W_2 be fixed sets that satisfy

$$\left(1-\frac{1}{n^{\frac{1}{4t}}}\right)np_0 \le |W_1| \le 2np_0 \text{ and } \left(1-\frac{2}{n^{\frac{1}{4t}}}\right) \le |W_2 \setminus W_1| \le 2np_0.$$

It follows from another application of Janson's inequality that

$$\begin{aligned}
&\mathbf{Pr}(\overline{\mathcal{E}_2}|N_1 = W_1, N_2 \setminus N_1 = W_2), \mathbf{Pr}(\overline{\mathcal{E}_3}|N_1 = W_1, N_2 \setminus N_1 = W_2) \\
&\qquad\qquad \le \exp\left\{-\log n + \log\log n + O(n^{-\frac{1}{4t}+o(1)}) + O(n^{-1+\frac{2}{t}+o(1)})\right\}.
\end{aligned}$$

Therefore,

$$\mathbf{Pr}(Z_1 = Z_2 = 0) = \frac{\log^2 n}{n^2}(1+o(1)),$$

and it follows from (21) that

$$\mathbf{Var}(Y) = o(\log^2 n).$$

It then follows from Chebyshev's inequality that

$$\mathbf{Pr}(Y = 0) = o(1). \tag{24}$$

Since the event $\{Y = 0\}$ is monotone increasing, it follows from (24) that

$$\mathbf{Pr}(\text{every vertex in } G_{n,m_0} \text{ is contained in a copy of } K_t) = o(1).$$

In other words, we have shown that **whp** $\tau_1 > m_0$.

2.2 Proof of Lemma 2

Let $p_a = m_a/\binom{n}{2}$ and consider the random graph $G = G_{n,p_a}$. For $S \in \binom{[n]}{t}$ let B_S be the event that the induced graph $G[S]$ is complete. For R a fixed subset of $[n]$ such that

$$|R| = r = \left\lceil \frac{n}{(\log n)^{1/t}} \right\rceil$$

let the random variable X_R be the number of copies of K_t contained in R. We clearly have

$$\begin{aligned}
\mu &:= \mathbf{E}[X_R] \\
&= \sum_{S \in \binom{R}{t}} \mathbf{Pr}(B_S) \\
&= \binom{r}{t} p_a^{\binom{t}{2}} \\
&= \binom{r}{t} \frac{\alpha^{\binom{t}{2}} (t-1)! \log n}{n^{t-1}} \\
&= \frac{r^t}{t!} (1 + O(1/r)) \frac{\alpha^{\binom{t}{2}} (t-1)!}{n^{t-1}} \log n \\
&= \Omega(n)
\end{aligned}$$

We apply Janson's inequality (again, we follow the notation of [1]) to show that $\mathbf{Pr}(X_R = 0)$ is small. In order to do so, we must bound the parameter Δ.

$$\begin{aligned}
\Delta &= \sum_{S,T \in \binom{R}{t} : 2 \le |S \cap T| \le t-1} \mathbf{Pr}(B_S \wedge B_T) \\
&= \binom{r}{t} \sum_{i=2}^{t-1} \binom{t}{i} \binom{r-t}{t-i} p_a^{2\binom{t}{2} - \binom{i}{2}} \\
&= \sum_{i=2}^{t-1} O\left(n^{2t - i - \frac{2}{t}\left(2\binom{t}{2} - \binom{i}{2}\right) + o(1)} \right) \\
&= \sum_{i=2}^{t-1} O\left(n^{2 + \frac{i(i-1)}{t} - i + o(1)} \right) \\
&= O\left(n^{2/t + o(1)} \right).
\end{aligned}$$

Thus, Janson's inequality gives

$$\mathbf{Pr}(X_R = 0) \le e^{-c_1 n}$$

where c_1 is a positive constant. Applying the first moment method, we have

$$\begin{aligned}
\mathbf{Pr}\left(\bigvee_{R\in\binom{[n]}{r}}\{X_R=0\}\right) &\le \binom{n}{r}e^{-c_1 n} \\
&\le \left(\frac{ne}{r}\right)^r e^{-c_1 n} \\
&= \exp\left\{r\left(1+\frac{\log\log n}{t}\right)-c_1 n\right\} \\
&= o(1)
\end{aligned}$$

Since this event is monotone, the same holds for G_{n,m_a}.

2.3 Proof of Lemma 3

Let $\mathcal{C}=\{S_1,\ldots,S_l\}$ be a fixed collection of K_t's in K_n such that $l\le 2\nu_t$

$$\begin{aligned}
\kappa_i \ge 1 \quad &\text{for} \quad i=2,\ldots,l \\
\kappa_i = t \quad \Rightarrow \quad &\kappa_{i-1}=1 \quad \wedge \quad |S_i\cap S_{i-1}|\ge 2 && (25)\\
\text{and} \quad &|\{i:\kappa_i\ne 1\}|=\nu_t && (26)
\end{aligned}$$

where

$$\kappa_i=\left|S_i\cap\left(\bigcup_{j=1}^{i-1}S_j\right)\right| \qquad \text{for} \qquad i=2,\ldots,l.$$

Let $a=|V(\mathcal{C})|$ and $b=|E(\mathcal{C})|$.

Claim 7.

$$a-\frac{2b}{t}<-\frac{1}{t}$$

Proof. We observe this difference as we 'build' the collection $\mathcal{C}$ one K_t at a time. For $j=1,\ldots,l$ let $\mathcal{C}_j=\{S_1,\ldots,S_j\}$, $a_j=|V(\mathcal{C}_j)|$, $b_j=|E(\mathcal{C}_j)|$ and $d_j=a_j-2b_j/t$. Note that

$$d_1=1,$$

and

$$d_{i+1}-d_i\le(t-\kappa_{i+1})-\frac{2}{t}\left(\binom{t}{2}-\binom{\kappa_{i+1}}{2}\right)=(\kappa_{i+1}-1)\left(\frac{\kappa_{i+1}}{t}-1\right). \qquad (27)$$

Thus

$$\begin{aligned}
\kappa_{i+1}=1 \quad &\Rightarrow \quad d_{i+1}-d_i=0 \\
\text{and} \quad 2\le\kappa_{i+1}\le t-1 \quad &\Rightarrow \quad d_{i+1}-d_i\le\frac{2}{t}-1. && (28)
\end{aligned}$$

Furthermore, it follows from (25) that

$$\kappa_{i+1} = t \quad \Rightarrow \quad b_{i+1} \geq b_i + t - 2 \quad \Rightarrow \quad d_{i+1} - d_i \leq -\frac{2(t-2)}{t}. \tag{29}$$

Since (by (28) and (29)) the difference $a_i - 2b_i/t$ decreases by at least $1 - 2/t$ whenever $\kappa_{i+1} \neq 1$, it follows from (26) that $a - 2b/t = d_l < -1/t$. □

Let $\mathcal{E}_i$ be the event that there exists a cluster in G_{m_1} with a vertex set of cardinality i, and let b_i be the minimum number of edges in a cluster on i vertices. With $p_{m_1} = m_1/\binom{n}{2}$ we have

$$\begin{aligned}\mathbf{Pr}(\mathcal{E}_i) &\leq \binom{n}{i} 2^{\binom{i}{t}} p_{m_1}^{b_i} \\ &= O\left(n^{i - \frac{2b_i}{t} + o(1)}\right) \\ &= O(n^{-\frac{1}{t} + o(1)}).\end{aligned}$$

The lemma then follows from the fact that the cardinality of the vertex set of a cluster is at most $2\nu_t t$, a constant depending only on t.

2.4 Proof of Lemma 4

We first argue that **whp**

$$|\Upsilon_v| \leq 4 \log n \qquad \text{for all } v \in [n]. \tag{30}$$

We can calculate in G_{n,p_b} where $p_b = m_b/N$, $N = \binom{n}{2}$ and then use monotonicity to translate the result to G^2. It follows from Lemma 3 and (13) that **whp** after removing $O(1)$ K_t's from Υ_v we have a collection $\tilde{\Upsilon}_v$ of K_t's which are disjoint except for there containing v. So in G_{n,p_b}

$$\mathbf{Pr}(|\tilde{\Upsilon}_v| \geq \kappa = 3.9 \log n) \leq \frac{\binom{n-1}{t-1}^{\kappa}}{\kappa!} p_b^{\kappa\binom{t}{2}} \leq \frac{(\log n)^{\kappa}}{\kappa!} \leq (e/3.9)^{3.9 \log n} = o(n^{-3/2}).$$

This verifies (30).

Now fix a vertex v. Then $|V(\Upsilon_v)| < 4t \log n$ and $|X| \leq r$. Also, X and $V(\Upsilon_v)$ are chosen independently. It follows that

$$\begin{aligned}\mathbf{Pr}(|V(\Upsilon_v) \cap X| \geq q) &\leq \frac{\binom{4t \log n}{q}\binom{n-q}{r-q}}{\binom{n}{r}} \\ &\leq \left(\frac{4ter \log n}{qn}\right)^q \\ &\leq \left(\frac{4te \log\log\log n \log n}{(\log n)^{(t+1)/t}}\right)^{\log n / \log\log\log n} \\ &= O(n^{-A})\end{aligned}$$

for any constant $A > 0$.

There are n choices for v and the lemma follows.

2.5 Proof of Lemmas 5 and 6

Let

$$p = ((t-1)!\log n)^{1/\binom{t}{2}}\, n^{-2/t} \quad \text{and} \quad p_{m_1} = \frac{m_1}{\binom{n}{2}}.$$

The main work of this section is the following claim.

Claim 8. *Let $H = (A, B)$ be a fixed graph whose vertex set A is a subset of $[n]$, and let $x, y \in A$ be distinct fixed vertices. If $b := |B|$ and $a := |A| \leq 4t$ then*

1. $\mathbf{Pr}\left((x \textit{ is small }) \wedge (H \subseteq G_{m_1})\right) = O(p_{m_1}^b n^{-3/4})$
2. $\mathbf{Pr}\left((x \textit{ and } y \textit{ are small}) \wedge (H \subseteq G_{m_1})\right) = O(p_{m_1}^b n^{-3/2})$

Proof. We only prove 2; the proof of 1 is both similar and easier. Let $\mathcal{R}_x$ be the event that x is small, $\mathcal{R}_y$ be the event that y is small, and let $\mathcal{R}_H$ be the event $B \subseteq E(G_{m_1})$. Furthermore, let

$$N_x = \{v \in [n] : x \sim_{G^2} v\} \backslash A \quad \text{and} \quad N_y = \{v \in [n] : y \sim_{G^2} v\} \backslash (A \cup N_x),$$

G_x be the induced graph $G^2[N_x]$, and $G_y = G^2[N_y]$. Finally, let $\epsilon > 0$ be a constant such that

$$\beta + \epsilon < 1 \text{ and } (\beta - \epsilon)^{\binom{t}{2}} \geq \frac{3}{4} + \frac{1}{20}(1 + \log 20). \tag{31}$$

Case 1. $t = 3$

We condition on the event that N_x and N_y are of nearly the expected size. Let $\mathcal{R}_1$ be the event that

$$(\beta - \epsilon)np \leq |N_x|, |N_y| \leq (\beta + \epsilon)np, \tag{32}$$

and $\mathcal{R}_2$ be the event that

$$|E(G_x)|, |E(G_y)| \leq \frac{\log n}{20}. \tag{33}$$

We have

$$\mathbf{Pr}(\mathcal{R}_H \wedge \mathcal{R}_x \wedge \mathcal{R}_y) \leq \mathbf{Pr}(\mathcal{R}_2 | \mathcal{R}_1 \wedge \mathcal{R}_H)\mathbf{Pr}(\mathcal{R}_H) + \mathbf{Pr}(\overline{\mathcal{R}_1}). \tag{34}$$

Now the Chernoff bounds show that in $G_{n,p_{m_1}}$ we have

$$\mathbf{Pr}(\bar{\mathcal{R}}_1) = O(\exp\{-n^{1-\frac{2}{t}+o(1)}\}), \tag{35}$$

and we can inflate this by $O(n)$ to show the same for G_{m_1}.

Then, where $N = \binom{n}{2}$

$$\begin{aligned} \mathbf{Pr}(\mathcal{R}_H) &\leq \binom{\binom{a}{2}}{b}\binom{N-b}{m_1-b}\Big/\binom{N}{m_1} \\ &= O(p_{m_1}^b). \end{aligned} \tag{36}$$

To bound $\mathbf{Pr}(\mathcal{R}_2)$ we condition on $N_x = S, N_y = T$ satisfying (32), where S, T are fixed subsets of $[n]$. Now let $\hat{\mathcal{R}}_2$ denote the event

$$|E(S)|, |E(T)| \leq \frac{\log n}{20}.$$

We show that for $\gamma \geq \beta - \epsilon$, in $G_{n,\gamma p}$ we have

$$\mathbf{Pr}_{\gamma p}(\hat{\mathcal{R}}_2) = O(n^{-3/2}). \tag{37}$$

The monotonicity of $\hat{\mathcal{R}}_2$ plus the concentration of the number of edges of $G_{n,\gamma p}$ around $\gamma N p$ then allows us to assert (37) for G^2. Indeed, then

$$O(n^{-3/2}) = \mathbf{Pr}_{\gamma p}(\hat{\mathcal{R}}_2) = \sum_m \binom{N}{m}(\gamma p)^m(1-\gamma p)^{N-m}\mathbf{Pr}_m(\hat{\mathcal{R}}_2)$$

and so taking $\beta - \epsilon \leq \gamma$ we see that if $\mathbf{Pr}_{m_1}(\hat{\mathcal{R}}_2) \geq An^{-3/2}$ then $\mathbf{Pr}_{\gamma p}(\hat{\mathcal{R}}_2) \geq An^{-3/2}/2$.

The random variable $X = |E(G_x)|$ (in $G_{n,\gamma p}$) is a binomial random variable $B(s,p)$ where $s = \binom{|S|}{2}$, having mean μ where

$$(\beta - \epsilon)^3 \log n < \mu < (\beta + \epsilon)^3 \log n.$$

So,

$$\begin{aligned} \mathbf{Pr}_{\gamma p}\left(X \leq \frac{\log n}{20}\right) &\leq \sum_{l=0}^{\lfloor \frac{\log n}{20} \rfloor} \binom{s}{l}(\gamma p)^l(1-\gamma p)^{s-l} \\ &\leq (1+o(1)) \sum_{l=0}^{\lfloor \frac{\log n}{20} \rfloor} e^{-\mu}\frac{\mu^l}{l!} \\ &\leq 2e^{-\mu}\frac{\mu^{\lfloor \frac{\log n}{20} \rfloor}}{\lfloor \frac{\log n}{20} \rfloor!} \\ &\leq 3\exp\left\{-\log n\left((\beta-\epsilon)^3 - \frac{1}{20}(1+\log 20)\right)\right\} \\ &\leq 3n^{-3/4} \end{aligned}$$

We apply the same argument to $|E(G_y)|$ (adding the appropriate conditioning on the number of edges within N_y). The proof now follows from (34) – (37).

Case 2. $t \geq 4$

We bound $\mathbf{Pr}(\mathcal{R}_x \wedge \mathcal{R}_y \wedge \mathcal{R}_H)$ by conditioning on the event that the neighborhoods of x and y are of nearly the expected size and have nearly the expected number of edges. Let $\mathcal{R}_3$ is the event that

$$\begin{gathered}(\beta-\epsilon)pn \leq |N_x|, |N_y| \leq (\beta+\epsilon)pn, \\ (\beta-\epsilon)p\binom{|N_x|}{2} \leq |E(G_x)| \leq (\beta+\epsilon)p\binom{|N_x|}{2}, \text{ and} \\ (\beta-\epsilon)p\binom{|N_y|}{2} \leq |E(G_y)| \leq (\beta+\epsilon)p\binom{|N_y|}{2}.\end{gathered}$$

Let $\mathcal{R}_4$ be the event that both G_x and G_y contain fewer than $\frac{\log n}{20}$ copies of K_t. We now bound the probability of $\mathcal{R}_x \wedge \mathcal{R}_y \wedge \mathcal{R}_H$ as follows:

$$\begin{aligned}\mathbf{Pr}(\mathcal{R}_x \wedge \mathcal{R}_y \wedge \mathcal{R}_H) &\leq \mathbf{Pr}(\mathcal{R}_4|\mathcal{R}_H \wedge \mathcal{R}_3)\mathbf{Pr}(\mathcal{R}_H) + \mathbf{Pr}(\overline{\mathcal{R}_3}) \\ &\leq \mathbf{Pr}(\mathcal{R}_4|\mathcal{R}_H \wedge \mathcal{R}_3)O(p_{m_1}^b) + O(\exp\{-n^{1-\frac{2}{t}+o(1)}\}).\end{aligned} \tag{38}$$

We bound $\mathbf{Pr}(\mathcal{R}_4|\mathcal{R}_H \wedge \mathcal{R}_3)$ by an application of the Poisson approximation on the number of K_t's in the random graph $G_{n,m}$ given by Theorem 6.1 of [8, page 68]. We let n' and m' be integers satisfying

$$(\beta-\epsilon)pn \leq n' \leq (\beta+\epsilon)pn, \text{ and} \tag{39}$$

$$(\beta-\epsilon)p\binom{n'}{2} \leq m' \leq (\beta+\epsilon)p\binom{n'}{2}, \tag{40}$$

and condition on the event that $|N_x| = n'$ and $|E(G_x)| = m'$. Note that under this conditioning G_x can be viewed as the random graph $G_{n',m'}$. Following the notation of [8], we have

$$\frac{1}{2}(n')^{2-\frac{2}{t-2}}\omega_1 \leq m' \leq \frac{1}{2}(n')^{2-\frac{2}{t-2}}\omega_2$$

where

$$\omega_1 = (\beta-\epsilon)^{\frac{t}{t-2}}\left((t-1)!\log n\right)^{1/\binom{t-1}{2}}$$

and

$$\omega_2 = (\beta+\epsilon)^{\frac{t}{t-2}}\left((t-1)!\log n\right)^{1/\binom{t-1}{2}}.$$

Let $X = X_{K_t}$ be the number of copies of K_t in $G_{n',m'}$. The expected number of such K_t's, $\lambda := \mathbf{E}[X]$, is then bounded as follows:

$$(\beta-\epsilon)^{\binom{t}{2}}\log n \leq \lambda \leq (\beta+\epsilon)^{\binom{t}{2}}\log n.$$

It then follows from Theorem 6.1 of [8] that

$$\begin{aligned}
\mathbf{Pr}\left(X \le \frac{\log n}{20}\right) &\le (1+o(1)) \sum_{k=0}^{\lfloor \frac{\log n}{20} \rfloor} e^{-\lambda}\frac{\lambda^k}{k!} \\
&\le 2e^{-\lambda}\frac{\lambda^{\lfloor \frac{\log n}{20} \rfloor}}{\lfloor \frac{\log n}{20} \rfloor !} \\
&\le 2e^{-\lambda}\left(\frac{20e\lambda}{\log n}\right)^{\frac{\log n}{20}} \\
&\le 2\exp\left\{-(\beta-\epsilon)^{\binom{t}{2}}\log n\right\}(20e)^{\frac{\log n}{20}} \\
&= 2\exp\left\{-\log n\left((\beta-\epsilon)^{\binom{t}{2}} - \frac{1}{20}(1+\log 20)\right)\right\} \\
&\le 2n^{-3/4}
\end{aligned}$$

With (38) this completes the proof. □

Proof of Lemma 5. Let $\mathcal{S}_1$ be the event that there is a chain in G_{m_1}. For a fixed collection $\mathcal{A}$ of K_t's in K_n and distinct $u, v \in [n]$ which define a possible chain, it follows from an argument along the line of the proof of Claim 7 that

$$|V(\mathcal{A})| \le 1 + \frac{2|E(\mathcal{A})|}{t}$$

and it follows from Claim 8 that

$$\mathbf{Pr}\left((u \text{ and } v \text{ are small }) \wedge E(\mathcal{A}) \subseteq E(G_{m_1})\right) \le O(p_{m_1}^{|E(\mathcal{A})|}n^{-3/2}).$$

Applying the first moment method we have

$$\begin{aligned}
\mathbf{Pr}(\mathcal{S}_1) &\le \binom{n}{2}\sum_{i=t}^{4t-3}\binom{n-2}{i-2}2^{\binom{i}{t}}O(p_{m_1}^{\frac{(i-1)t}{2}}n^{-3/2}) \\
&\le \sum_{i=t}^{4t-3} O(n^{i-\frac{2}{t}\frac{(i-1)t}{2}-\frac{3}{2}+o(1)}) \\
&\le \sum_{i=t}^{4t-3} O(n^{-\frac{1}{2}+o(1)}) \\
&= o(1)
\end{aligned}$$

□

Proof of Lemma 6. Let $\mathcal{S}_2$ be the event that there is a link in G_{m_1}. For fixed $S, T \in \binom{[n]}{t}$ such that $|S \cap T| \ge 2$ and $x \in S \cup T$ it follows from Claim 8 that

$$\mathbf{Pr}\left((x \text{ is small }) \wedge \binom{S}{2} \cup \binom{T}{2} \subseteq E(G_{m_1})\right) = O(p_{m_1}^{\binom{t}{2}-\binom{|S\cap T|}{2}}n^{-3/4}).$$

Applying the first moment method we have

$$\begin{aligned}
\mathbf{Pr}(\mathcal{S}_2) &\leq n\binom{n-1}{t-1}\sum_{i=2}^{t-1}\binom{t}{i}\binom{n-t}{t-i}O(p_{m_1}^{2\binom{t}{2}-\binom{i}{2}}n^{-3/4}) \\
&\leq \sum_{i=2}^{t-1} O(n^{2t-i-2(t-1)+\frac{2}{t}\binom{i}{2}-\frac{3}{4}+o(1)}) \\
&\leq \sum_{i=2}^{t-1} O(n^{\frac{5}{4}-i+\frac{i(i-1)}{t}+o(1)}) \\
&= o(1)
\end{aligned}$$

□

3 Proof of Theorem 1.

For a graph G and a vertex v, we defined prior to (21) $Z_v(G) = Z_v$ to be the number of K_t's in G that contain v and $Y(G) = Y$ to be the number of vertices u with $Z_u = 0$.

In view of Theorem 2 we need only prove that

$$\lim_{n\to\infty} \mathbf{Pr}(Y(G_{n,m}) = 0) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases} \tag{41}$$

Using Theorem 2 of Łuczak [14] we can derive (41) from the more easily obtained

$$\lim_{n\to\infty} \mathbf{Pr}(Y(G_{n,p}) = 0) = \begin{cases} 0 & c_n \to -\infty \\ e^{-e^{-c}} & c_n \to c \\ 1 & c_n \to \infty \end{cases} \tag{42}$$

where $p = m/\binom{n}{2}$. Furthermore we need only consider the case $c_n \to c$ as the others follow by monotonicity. Equation (42) can be proved by showing that $Y(G_{n,p})$ is asymptotically Poisson. In particular we need only show that for $k = O(1)$,

$$\lim_{n\to\infty} n^k \mathbf{Pr}(Z_i(G_{n,p}) = 0,\ 1 \leq i \leq k) = e^{-ck} \tag{43}$$

and then apply e.g. Theorem 20 of Bollobás [5].

Equation (43) follows from

$$\mathbf{Pr}(Z_i(G_{n,p}) = 0 \mid Z_j(G_{n,p}) = 0,\ 1 \leq j < i) \sim \frac{e^{-c}}{n} \tag{44}$$

for $1 \leq i \leq k$.

Using N_j to denote the neighbourhood of j in $G_{n,p}$ we let

- ν_1 denote the number of K_{t-1} in $N_i \setminus \bigcup_{j=1}^{i-1} N_j$.
- ν_2 denote the number of K_{t-1} in N_i which use a vertex of $\bigcup_{j=1}^{i-1} N_j$.

We then let $\mathcal{C}_i = \{Z_j(G_{n,p}) = 0,\ 1 \le j < i\}$ and write

$$\mathbf{Pr}(Z_i(G_{n,p}) = 0 \mid \mathcal{C}_i) = \mathbf{Pr}(\nu_1 = 0 \mid \mathcal{C}_i)(1 - \mathbf{Pr}(\nu_2 \neq 0 \mid \nu_1 = 0, \mathcal{C}_i)). \tag{45}$$

Then $\mathbf{Pr}(\nu_1 = 0 \mid \mathcal{C}_i) \sim e^{-c}/n$ follows from Janson's inequality and $\mathbf{Pr}(\nu_2 \neq 0 \mid \nu_1 = 0, \mathcal{C}_i) \le \mathbf{Pr}(\nu_2 \neq 0) = o(1/n)$ follows from the FKG inequality and a first moment calculation. □

4 Proofs of Theorems 4 – 6

We prove Theorem 4 via an application of the following theorem of Hajnal and Szemerédi. For $k \le n$ the *Turán graph* $T_k(n)$ is the complete k-partite graph on n vertices where the parts in the vertex partition have cardinalities

$$\left\lfloor \frac{n}{k} \right\rfloor, \left\lfloor \frac{n+1}{k} \right\rfloor, \ldots, \left\lfloor \frac{n+k-1}{k} \right\rfloor.$$

In other words, the parts in the partition are as near as possible to being equal (i.e. the partition is a so-called *equipartition*). Below we use the following theorem proved by Hajnal and Szemerédi (cf. Theorem 3).

Theorem 7 (Hajnal, Szemerédi). *If G is a graph on n vertices having maximum degree $\Delta(G) = \Delta$ then*

$$G \subseteq T_{\Delta+1}(n).$$

For a graph G, let $\overline{G}$ be the complement of G. It is easy to see that Theorem 7 is equivalent to

Theorem 8. *If G is a graph on n vertices having minimum degree $\delta(G) = \delta$ then*

$$\overline{T_{n-\delta}(n)} \subseteq G.$$

Let a (K_t, l)-vertex-cover be a K_t-vertex-cover in which each vertex appears in at most l copies of K_t.

Proof of Theorem 4. We establish the lower bound by example. Consider the complete t-partite graph on n vertices having parts $V_1, \ldots, V_t$ such that $|V_1| = q$ and

$$|V_2|, \ldots, |V_t| \in \left\{ lq + \left\lceil \frac{r}{t-1} \right\rceil, lq + \left\lfloor \frac{r}{t-1} \right\rfloor \right\}.$$

If $q = 0$ then G contains no t-clique and therefore has no (K_t, l)-vertex–cover. If $q > 0$ then, by the definition of r, there exists V_i such that $|V_i| > ql$, and G has no (K_t, l)-vertex–cover.

Suppose G is a graph on n vertices having

$$\delta(G) \geq n - ql - \left\lceil \frac{r}{t-1} \right\rceil + 2.$$

Let

$$s = ql + \left\lceil \frac{r}{t-1} \right\rceil - 2.$$

It follows from Theorem 8 that $\overline{T_s(n)} \subseteq G$. In words, there exists an equipartition $V(G) = V_1 \cup \cdots \cup V_s$ such that the induced graph $G[V_i]$ is complete for $i = 1, \ldots s$. We will show that the collection of cliques $G[V_1], \ldots, G[V_s]$ can be transformed into a (K_t, l)-vertex-cover.

Claim 9.

$$t - 1 \leq |V_i| \leq t \textit{ for } i = 1, \ldots, s.$$

Proof. We merely observe that $s(t-1) < n$ while $st \geq n$.

$$\begin{aligned}\left[ql + \left\lceil \frac{r}{t-1} \right\rceil - 2\right](t-1) &\leq ql(t-1) + \left(\frac{r}{t-1} + 1\right)(t-1) - 2(t-1) \\ &\leq ql(t-1) + r - (t-1) \\ &< n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\left[ql + \left\lceil \frac{r}{t-1} \right\rceil - 2\right] t &\geq \left[ql + \frac{r}{t-1} - 2\right] t \\ &= n + q(l-1) + \frac{r}{t-1} - 2t.\end{aligned} \tag{46}$$

Now, since $n \geq 6t^2 - 4t$, at least one of the following holds:

- $r \geq 2t(t-1)$
- $q \geq 2t$
- $q(t-1)l \geq 4t(t-1)$.

In any of these situations, the expression in (46) is greater than or equal to n. □

If follows from Claim 9 that we may assume that for some m we have $|V_1| = \cdots = |V_m| = t - 1$ and $|V_{m+1}| = \cdots = |V_s| = t$.

Claim 10.

$$m < (l-1)(q+1).$$

Proof. Since $V_1, \dots, V_s$ is a partition, we must have $(t-1)m + t(s-m) = n$. However,

$$\begin{aligned}(t-1)(l-1)(q+1)+t\left[ql + \left\lceil \frac{r}{t-1} \right\rceil - 2 - (l-1)(q+1)\right] \\ &= q[(t-1)l+1] + t\left\lceil \frac{r}{t-1} \right\rceil + 1 - l - 2t \\ &\leq q[(t-1)l+1] + t\left(\frac{r}{t-1} + \frac{t-2}{t-1}\right) + 1 - l - 2t \\ &\leq n + \frac{1}{t-1} + t\frac{t-2}{t-1} + 1 - 2t \\ &= n - t \\ &< n\end{aligned}$$

□

We transform $G[V_1], \dots, G[V_s]$ into a (K_t, l)-vertex-cover by expanding the clique V_i by one vertex for $i = 1, \dots, m$. To be precise, we will show that there exist $x_1, \dots, x_m \in V(G)$ such that

1. $x_i \sim v \quad \forall v \in V_i$,
2. $|\{x_i : x_i = v\}| \leq l - 1 \quad \forall v \in V(G)$,
3. $x_i \in V_j \Rightarrow x_j \notin V_i$,
4. $x_i \notin V_i$.

Note that the third condition must be included to prevent two of the expanded cliques from containing a common edge. For $i = 1, \dots, m$ let

$$A_i = \{v \in V(G) \setminus V_i : v \sim u \quad \forall u \in V_i\}$$

Claim 11. $|A_i| \geq q + t$ *for* $i = 1, \dots m$.

Proof. Since, for $v \in V_i$,

$$\begin{aligned}|\{x \in V(G) \backslash V_i : x \not\sim v\}| &\leq n - 1 - \delta(G) \\ &\leq ql + \left\lceil \frac{r}{t-1} \right\rceil - 3,\end{aligned}$$

we have

$$\begin{aligned}
&|\{x \in V(G)\backslash V_i : \exists v \in V_i \text{ such that } x \not\sim v\}| \\
&\qquad\qquad\qquad\qquad \le (t-1)\left[ql + \left\lceil \frac{r}{t-1} \right\rceil - 3\right] \\
&\qquad\qquad\qquad\qquad \le ql(t-1) + (t-1)\left(\frac{r}{t-1} + \frac{t-2}{t-1}\right) - 3(t-1) \\
&\qquad\qquad\qquad\qquad = ql(t-1) + r - 2t + 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
|A_i| &= |V(G)\backslash V_i| - |\{x \in V(G)\backslash V_i : \exists v \in V_i \text{ such that } x \not\sim v\}| \\
&\ge n - (t-1) - [ql(t-1) + r - 2t + 1] \\
&= q + t
\end{aligned}$$

□

Now, we choose the x_i's one at a time in an order $x_1 = x_{i_1}, x_{i_2}, \dots x_{i_m}$ as follows. Suppose $x_{i_1}, \dots, x_{i_k}$ have been chosen.

$$\text{If } x_{i_k} \in V_j \text{ and } j \notin \{i_1, \dots, i_k\} \text{ then } j = i_{k+1}. \tag{47}$$

Otherwise i_{k+1} is chosen arbitrarily from $\{j : 1 \le j \le m\}\backslash\{i_1, \dots, i_k\}$. In other words, we chose the x_i's in an order such that at most one x_i falls in V_j before x_j is chosen. For $k = 1, \dots, m$ let

$$U_k = \{v \in V(G) : |\{1 \le j < k : x_{i_j} = v\}| = l - 1\}.$$

In words, U_k is the set of vertices that satisfy 2. with equality after $x_{i_1}, \dots, x_{i_{k-1}}$ have been determined. Thus, we must have $x_{i_k} \notin U_k$. By Claim 10

$$|U_k| \le \left\lfloor \frac{m-1}{l-1} \right\rfloor < q + 1. \tag{48}$$

For $k = 1, \dots, m$ let

$$R_k = \bigcup_{1 \le j < k : x_{i_j} \in V_k} V_{i_j}.$$

(Note that the union here is over zero or one set only). By condition 3. we must have $x_{i_k} \notin R_k$. By the construction of the ordering given in (47),

$$|R_k| \le t - 1. \tag{49}$$

An arbitrary $x_{i_k} \in (A_{i_k}\backslash U_k)\backslash R_k$ satisfies 1, 2, and 3. By (48), (49) and Claim 11 such an element exists. □

Proof of Theorem 6. Let $\epsilon > 0$ and let G be a graph on n vertices with $\delta(G) = \delta \geq (1 - \frac{1}{\chi(H)-1} + \epsilon)n$. We show that any collection of edge disjoint copies of H that does not cover $V(G)$ can be extended to cover at least one new vertex. To be precise, we show that if a family $\mathcal{F} = \{\Gamma_1, \ldots, \Gamma_m\}$ of copies of H in G and a vertex $v \in V(G)$ satisfy

$$\begin{aligned} & m < n, \\ & \Gamma_i = (V(\Gamma_i), E(\Gamma_i)) \text{ are copies of } H \text{ in } G \text{ for all } i, \\ & E(\Gamma_i) \cap E(\Gamma_j) = \emptyset \text{ for all } i \neq j, \end{aligned} \tag{50}$$

and

$$v \notin \cup_{i=1}^{m} V(\Gamma_i) ,$$

then there exists a family $\mathcal{F}' = \{\Upsilon_1, \ldots, \Upsilon_l\}$ such that for all i $\Upsilon_i = (V(\Upsilon_i), E(\Upsilon_i))$ are copies of H in G

$$E(\Upsilon_i) \cap E(\Upsilon_j) = \emptyset \text{ for all } i \neq j \tag{51}$$

and

$$\cup_{i=1}^{l} V(\Upsilon_i) \supseteq \Big(\bigcup_{i=1}^{m} V(\Gamma_i)\Big) \cup \{v\} .$$

Note that we include the possibility of $m = 0$. Clearly, an inductive argument based on (50) and (51) above implies the theorem. Further, we may assume $m < n$ in (50). Suppose, on the contrary, that we have a family $\mathcal{F}^* = \{\Gamma_1, \ldots, \Gamma_m\}, m \geq n$, constructed inductively by (50) and (51) such that it does not cover all vertices. However, by the inductive construction of $\mathcal{F}^*$ every vertex is already in some copy of H included in the family $\mathcal{F}^*$. A contradiction.

To proceed with the proof we need to establish some notational conventions. Let u be the vertex of H such that $\chi(H \setminus \{u\}) = \chi(H) - 1$. Set $H' = H \setminus \{u\}, h = |V(H)|$, and $e_H = |E(H)|$. For $\mathcal{F}$ and a vertex v as in (50), let N_v be the set of neighbors of v, $d_v = |N_v|$ and $F = \cup_{i=1}^{m} E(\Gamma_i)$. Our analysis will focus on the consideration of the subgraphs $L = G[N_v]$ and $L' = (N_v, E(L) \setminus F)$. We extend $\mathcal{F}$ to $\mathcal{F}'$ by simply finding a copy of H which contains v but no edges in F. Clearly, if there exists a copy of H' in L', then this H' together with v gives a copy of H that extends $\mathcal{F}$. (Note H' is a subgraph of $L = G[N_v]$).

We have for $|E(L)| \geq \frac{d_v}{2}\Big(\delta - (n - d_v)\Big)$. Since $\delta \geq \Big(\frac{\chi-2}{\chi-1} + \epsilon\Big)n$ is equivalent to $\delta - n \geq -\frac{1}{\chi-2}\delta + \epsilon n \frac{\chi-1}{\chi-2}$, we get

$$\begin{aligned} |E(L)| &\geq \frac{d_v}{2}\Big(\delta - (n - d_v)\Big) \\ &\geq \frac{d_v}{2}\Big(d_v - \frac{1}{\chi-2}\delta + \epsilon n \frac{\chi-1}{\chi-2}\Big) \\ &\geq \frac{d_v^2}{2} \cdot \frac{\chi-3}{\chi-2} + \epsilon n \frac{d_v}{2} \cdot \frac{\chi-1}{\chi-2} . \end{aligned}$$

Since we are assuming that $|\mathcal{F}| < n$, we have

$$|F \cap E(L)| \le |F| \le e_H n \ ,$$

and it follows

$$\begin{aligned}
|E(L')| &= |E(L)| - |F \cap E(L)| \\
&\ge \frac{d_v^2}{2} \cdot \frac{\chi - 3}{\chi - 2} + \epsilon n \cdot \frac{d_v}{2} \cdot \frac{\chi - 1}{\chi - 2} - e_H n \\
&\ge \binom{d_v}{2} \cdot \frac{\chi - 3}{\chi - 2} + \frac{1}{2}\epsilon \binom{d_v}{2} \frac{\chi - 1}{\chi - 2} \\
&\quad + \left(\frac{1}{2}\epsilon \binom{d_v}{2} \frac{\chi - 1}{\chi - 2} + \frac{d_v}{2} \cdot \frac{\chi - 3}{\chi - 2} + \epsilon \frac{d_v}{2} \cdot \frac{\chi - 1}{\chi - 2} - e_H n\right) .
\end{aligned}$$

Letting $\epsilon' = \frac{1}{2} \cdot \frac{\chi - 1}{\chi - 2} \cdot \epsilon$ and d_v be large enough (i.e. n large enough), we conclude that

$$\frac{1}{2}\epsilon \binom{d_v}{2} \frac{\chi - 1}{\chi - 2} + \frac{d_v}{2} \cdot \frac{\chi - 3}{\chi - 2} + \epsilon \frac{d_v}{2} \cdot \frac{\chi - 1}{\chi - 2} - e_H n \ge 0$$

and thus, $|E(L')| \ge \left(\frac{\chi-3}{\chi-2} + \epsilon'\right)\binom{d_v}{2}$. By the Erdős - Stone theorem there exists a copy of H' in L'. Taking this copy of H' together with v and edges needed gives us a new copy of H by which we extend $\mathcal{F}$ to $\mathcal{F}'$.

Proof of Theorem 5. We are going to determine the exact value of $f(n, 3, k), k \ge \frac{n-1}{2}$ and $n \ge 6$. First, note that in any (K_3, ∞)-vertex-cover of a graph G on n vertices no vertex lies in more than $\frac{n-1}{2}$ copies of K_3. In order to get a tight result we assume G is a graph on n vertices with $\delta(G) \ge \lceil n/2 \rceil + 1$. Let $\mathcal{F} = \{\Gamma_1, \dots, \Gamma_m\}$ and v be as in (50) with $H = K_3$. We use the notation introduced in the proof of Theorem 6. Unlike in the proof of Theorem 6, in order to get a tight result it does not suffice to simply add a new K_3 to $\mathcal{F}$. Our argument includes consideration of several different kinds of modifications of $\mathcal{F}$.

It follows from our minimal degree condition that

$$d_L(x) \ge 2, \quad \text{for all } \ x \in N_v. \tag{52}$$

If there is an edge in L not contained in $F = \cup_{i=1}^m E(\Gamma_i)$ then this edge together with v gives an extension of $\mathcal{F}$ that contains v, and therefore we can assume

$$E(L) \subset F. \tag{53}$$

It follows from (52) and (53) that $|F \cap E(L)| \ge d_v = |N_v|$, and therefore

$$3|\mathcal{F}_3| + |\mathcal{F}_2| \ge d_v \ge \frac{n}{2} + 1 \ , \tag{54}$$

where $\mathcal{F}_j = \{\Gamma \in \mathcal{F} : |V(\Gamma) \cap V(L)| = j\}, j = 2, 3$. Since $H = K_3$, to simplify the description we identify $\Gamma \in \mathcal{F}$ with its vertex set, i.e. $\Gamma = \{x_1, x_2, x_3\}$. Consider

$\Gamma_A = \{x_1, x_2, y\} \in \mathcal{F}_2$ with $x_1, x_2 \in N_v$ and $y \in V(G) \setminus (N_v \cup \{v\})$. If there exists $\Gamma_B \in \mathcal{F}, \Gamma_B \neq \Gamma_A$, such that $y \in \Gamma_B$ then $(\mathcal{F}\backslash\{\Gamma_A\}) \cup \{\{x_1, x_2, v\}\}$ is an extension of $\mathcal{F}$ containing v. Therefore, we can assume

$$|\mathcal{F}_2| \leq |V(G) \setminus (N_v \cup \{v\})| \leq \frac{n}{2} - 2\,, \tag{55}$$

because otherwise there exists a pair $\Gamma_A, \Gamma_B \in \mathcal{F}, \Gamma_A = \{x_1, x_2, y\}, \Gamma_B = \{z_1, z_2, y\}$ as above. It follows from (54) and (55) that $|\mathcal{F}_3| \geq 1$. Now, consider $\Gamma_A \in \mathcal{F}_3$. If there exists $\Gamma_B \in \mathcal{F}$ such that $\Gamma_A \cap \Gamma_B = \{x\}$ then $(\mathcal{F} \cup \{\Gamma_A\backslash\{x\} \cup \{v\}\})\backslash\{\Gamma_A\}$ is an extension of $\mathcal{F}$ containing v. So, we can henceforth assume

$$\Gamma_A \in \mathcal{F}_3, \Gamma_B \in \mathcal{F} \implies \Gamma_A \cap \Gamma_B = \emptyset. \tag{56}$$

Once again, we consider $\Gamma_A = \{x_1, x_2, x_3\} \in \mathcal{F}_3$. Since $d_G(x_i) \geq n/2 + 1 > 3$ (here we use our assumption on n) there exists $u \in V\backslash\{v, x_1, x_2, x_3\}$ and $a \neq b \in \{1, 2, 3\}$ such that u is adjacent to both x_a and x_b. Let $c = \{1, 2, 3\}\backslash\{a, b\}$ and set

$$\mathcal{F}' = \mathcal{F}\backslash\{\Gamma_A\} \cup \{\{x_a, x_b, u\}, \{x_a, x_c, v\}\}.$$

By (56) the family $\mathcal{F}'$ is edge-disjoint and covers v.

In order to prove the lower bound on $f(n, 3, k)$ we consider the following two graphs. If $n = 2m$, H_n^e is the complete bipartite graph on the vertex set $Z_1 \cup Z_2, |Z_1| = |Z_2| = m$. In the case $n = 2m + 1$, H_n^o consists of the edges of the complete bipartite graph on the vertex set $Z_1 \cup Z_2, |Z_1| = m + 1, |Z_2| = m$. Moreover, if $|Z_1|$ is even, H_n^o contains edges of a perfect matching of Z_1 and in the case $|Z_1|$ is odd, H_n^o contains edges of a maximal matching, say M, of Z_1 together with a single edge $\{x, y\}$ where x is the vertex of Z_1 which does not belong to M and y is any vertex of $Z_1 \setminus \{x\}$. Clearly, $\delta(H_n^e) = \lceil n/2 \rceil$ and $\delta(H_n^o) = \lceil n/2 \rceil$. Further, neither of H_n^e and H_n^o contains a (K_3, ∞)-vertex-cover because H_n^e does not contain any copy of K_3 and H_n^o contains only at most $\lceil (n+1)/4 \rceil$ copies of K_3. □

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