Ramsey games with giants

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Abstract

The classical result in the theory of random graphs, proved by Erdős and Rényi in 1960, concerns the threshold for the appearance of the giant component in the random graph process. We consider a variant of this problem, with a Ramsey flavor. Now, each random edge that arrives in a sequence of rounds must be colored with one of r colors. The goal can be either to create a giant component in every color class, or alternatively, to avoid it in every color. One can analyze the offline or online setting for this problem. In this paper, we consider all these variants and provide nontrivial upper and lower bounds; in certain cases (like online avoidance) the obtained bounds are asymptotically tight.

1 Introduction

Let $G_{n,m}$ be the Erdős-Rényi random graph with n labeled vertices and m randomly chosen edges. A celebrated result of Erdős and Rényi, probably the single most important result in the theory of random graphs, discovered a threshold for the appearance of the giant component in this random model. Erdős and Rényi proved that if $m \leq (1-\epsilon)\frac{n}{2}$ for a constant $\epsilon > 0$, then **whp**[1] the random graph $G_{n,m}$ has all of its connected components of order at most logarithmic in n; on the other hand, if $m \geq (1+\epsilon)\frac{n}{2}$ then **whp** $G_{n,m}$ has a unique connected component of linear size, the so called giant component, while all other components are at most logarithmic in size. This result can be formulated equivalently in terms of the random graph process: if the process starts with the empty graph G_0 on n vertices, and at stage $i \geq 1$ a random missing edge is added to G_{i-1} to form G_i, then after the first $(1-\epsilon)\frac{n}{2}$ rounds the resulting graph typically has all connected components of at most logarithmic size, while after $(1+\epsilon)\frac{n}{2}$ rounds **whp** the unique giant component is born, while all other components

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[1]As customary, we write that a graph property $\mathcal{P}$ holds with high probability, or **whp** for brevity, if the probability of $\mathcal{P}$ tends to 1 as the number of vertices n tends to infinity.

are of size $O(\log n)$. Since then, there have been numerous extensions to this fundamental result. One further ramification is considered in this paper.

Recently, quite a lot of attention and research effort has been devoted to controlled random graph processes. In processes of this type, an input graph or a graph process is usually generated fully randomly, but then an algorithm has access to this random input and can manipulate it in some well defined way (say, by dropping some of the input edges, or by coloring them), aiming to achieve some preset goal. There is usually the so called *online* version where the algorithm must decide on its course of action based only on the history of the process so far and without assuming any familiarity with future random edges, and the *offline* version, where the algorithm has access to the whole history of the process and makes its decisions based on the full knowledge of the process. We will give corresponding accurate definitions for our setting later.

Applied to the question about the appearance of the giant component, the first such version chronologically is probably the so-called *Achlioptas process*. This process is named after Dimitris Achlioptas, who posed the following question about 10 years ago. Suppose random edges arrive in pairs, and an online algorithm can choose one of them, put it into the graph, and return the other edge to the pool. Is it possible to design an algorithm that **whp** delays the appearance of the giant components for noticeably longer than the Erdős-Rényi $0.5n$ steps? This question was answered affirmatively in [6] by the first two authors of the present paper, who exhibited an algorithm that **whp** survives for at least $0.535n$ rounds without creating the giant component. Since then, there has been a series of papers about the giant component in Achlioptas processes, where a variety of scenarios and goals (online and offline algorithms, delaying or accelerating the appearance of the giant component) have been considered. See, e.g., any of [3, 4, 7, 8, 9, 13, 28].

Here we consider a Ramsey-type version of controlled random processes. In this version, incoming random edges are colored by an algorithm in one of r colors, for a fixed $r \geq 2$. The goal of the algorithm is to achieve or maintain a certain monotone graph property in all of the colors. This setting originates in the papers of Rödl and Ruciński [26, 27], who determined when $G_{n,m}$ satisfies the Ramsey property of having a monochromatic copy of a fixed graph H in any r-coloring of the edges. In our terminology they considered the offline version of the problem, and the property $\mathcal{P}$ to avoid in each color was the appearance of a copy of a fixed graph H. The online version of the problem for the case of two colors and $H = K_3$ was treated by Friedgut, Kohayakawa, Rödl, Ruciński and Tetali in [14], and extended to a wider variety of graphs by Marciniszyn, Spöhel and Steger in [20, 21]. The online setting of achieving Hamiltonicity in each of r colors has been addressed in [17].

In the present paper, we investigate several Ramsey-type problems involving the giant component. We consider whether or not it is possible to color the edges of $G_{n,m}$ in r colors with the objective of creating a giant component in every color class, or of avoiding a giant component in every color. We study both the offline and online settings. In the offline setting, an algorithm gets access to the entire graph, generated according to the probability distribution $G_{n,m}$; in the online setting the edges of $G_{n,m}$ are first ordered in a random order and then revealed to the algorithm one by one (i.e., the algorithm observes the random graph process and colors each new edge as it arrives).

The main objective of this paper is to show new interesting questions, and not necessarily to get precise answers to all of them. We do determine the offline thresholds for these problems for all values of r, but the online setting remains open. There, we show that for two colors, there is always a separation phenomenon away from the trivial bounds, and then calculate asymptotic bounds for large numbers of colors.

As a warm-up, consider the offline threshold for creating a giant in every color. Recall that if $m < (1-\epsilon)\frac{n}{2}$ for any fixed $\epsilon > 0$, then **whp** $G_{n,m}$ itself has all components of size $O(\log n)$. On the other hand, one can show that for $m > (1+\epsilon)\frac{n}{2}$, **whp** it is possible to color the edges of $G_{n,m}$ with any fixed number of colors $r \geq 2$, so that every color class contains a component of order $\Omega(n)$. Indeed, Ajtai, Komlós, and Szemerédi proved in [2] that **whp** $G_{n,(1+\epsilon)\frac{n}{2}}$ contains a path of length $c_\epsilon n$. (Here and later in the paper, we will write c_ϵ to specify a positive constant determined only by ϵ.) By splitting this path into r paths of length $c_\epsilon n/r$, the result follows.

The question of avoiding giants in all colors offline is not so simple. It turns out that the threshold for avoiding giants in r colors is precisely the same as that of r*-orientability*, which says that it is possible to direct all of the edges of the graph so that the resulting digraph has maximum in-degree at most r. Cain, Sanders and Wormald [11], and Fernholz and Ramachandran [12] recently discovered that this threshold coincides with the number of edges needed to make the $(r+1)$-core have average degree above $2r$. More precisely, they showed that for any integer $r \geq 2$, there is an explicit threshold ψ_r such that the following holds. For any $\epsilon > 0$, if $m > (\psi_r + \epsilon)n$, then **whp** $G_{n,m}$ contains a subgraph with average degree at least $2r + c_\epsilon$, where $c_\epsilon > 0$. On the other hand, if $m < (\psi_r - \epsilon)n$, then $G_{n,m}$ is r-orientable **whp**. As the $r = 2$ case is often of particular interest, we note that (as calculated in [11]) $\psi_2 \approx 1.794$, and the asymptotic dependence of ψ_r on r is $\psi_r = r - \frac{1}{2}\left(\frac{2}{e} + o(1)\right)^r$. We now state our first main theorem in terms of this threshold.

Theorem 1.1. *Given any fixed r, let ψ_r be the threshold referenced above. For any $\epsilon > 0$, if $m < (\psi_r - \epsilon)n$, then **whp** it is possible to color the edges of $G_{n,m}$ with r colors such that each color class contains components of order only $o(n)$. On the other hand, if $m > (\psi_r + \epsilon)n$, then **whp** every r-edge-coloring of $G_{n,m}$ has a color class with a component of order at least $c_\epsilon n$.*

Remark. This was also recently and independently discovered by Spöhel, Steger, and Thomas [29].

We also consider online versions of these problems, in which the m edges come sequentially, and each must be colored as soon as it appears. Precisely, we consider the process to be a sequence of m rounds. In each round, a random edge arrives, independently and uniformly distributed over all pairs of vertices. If it repeats an existing edge, then we do not force ourselves to recolor it. This is not an important issue, because we will never consider more than $O(n)$ rounds, but it is more convenient to use this product probability space with full independence between the rounds.

Here, we have several results. First we state them for avoiding giants in all colors. The offline upper bound of course supplies an upper bound for the online case as well. Indeed, a standard coupling argument (Fact 2.3 in the next section) translates the offline upper bound to the case where the rounds have independent edges (possibly with repetitions). So, after $(\psi_r + \epsilon)n$ rounds, **whp** every possible coloring of them contains a giant component, where the dependence of ψ_r on r is $\psi_r = r - \frac{1}{2}\left(\frac{2}{e} + o(1)\right)^r$.

On the other hand, by taking the natural online adaptation of the offline avoidance strategy, which was based on edge orientation, we found a randomized online algorithm which matches the first-order asymptotic of $\psi_r = (1 - o(1))r$.

Theorem 1.2. *For any $\epsilon > 0$, the following holds for all sufficiently large r. There is an online randomized algorithm which can last for $(1-\epsilon)rn$ rounds, while keeping all connected components in each of r color classes smaller than $o(n)$ **whp**.*

For large r, this is asymptotically a factor of 2 better than the trivial bound of $\frac{rn}{2}$ rounds, obtained by coloring each edge independently at random. However, the above theorem only beats the trivial bound after $r > 50$, at which point the resulting ϵ falls below $\frac{1}{2}$. For the extreme case of small r, we have the following result using an entirely different strategy, which improves upon the trivial bound for all r by a factor of approximately 1.06.

Theorem 1.3. *There is an online algorithm which can 2-color edges for* $1.06n$ *rounds, while keeping all connected components in both color classes of size at most* $O(\log n)$ ***whp***.

Remark. Although the theorem is stated only for $r = 2$, it immediately gives a strategy for all even r, by splitting the colors into $\frac{r}{2}$ pairs. At each round, one of the color pairs is randomly chosen, and the above algorithm is used to decide which of the two colors in the pair to use. Then, this will avoid giants in all colors for $1.06n \cdot \frac{r}{2}$ rounds **whp**. For odd r, one can run the above modification for $1.06n \cdot \frac{r-1}{2}$ rounds using only the first $r-1$ colors, and then an additional $(1-\epsilon)\frac{n}{2}$ rounds using only the r-th color. This beats the trivial bound of $\frac{rn}{2}$ by a factor which approaches 1.06 as r grows.

When the objective is to create giants in every color class, the trivial bounds are as follows. Certainly, if fewer than $(1-\epsilon)\frac{n}{2}$ edges are observed, then **whp** there will be no giant in the uncolored graph, so one cannot hope to create r monochromatic giants any faster. Note that this trivial lower bound turned out to be the truth in the offline setting, even though it does not grow with r. We will show that in the online case, there is a lower bound which does.

Theorem 1.4. *There is a constant* $c \approx 0.043$ *such that after* $(c \log_2 r)n$ *edges are* r*-colored by any online algorithm,* ***whp*** *some color class still has all components of order only* $O(\log n)$*. For* $r = 2$*, the same result holds for* $c'n$ *edges for any* $c' < 2 - \sqrt{2} \approx 0.586$*.*

On the other hand, the trivial strategy of randomly coloring each edge succeeds when the number of edges surpasses $rn/2$. We are able to give an online algorithm which asymptotically performs far better than the trivial one.

Theorem 1.5. *There is an online algorithm such that for any* $\epsilon > 0$*, after* $(c_r + \epsilon)n$ *edges every color class contains a connected component of order at least* $c_\epsilon n$ ***whp****, where the dependence of* c_r *on* r *is* $c_r = (1 + o(1))\frac{\sqrt{r}}{2}$*.*

For the specific case of 2 colors, one can adapt the argument and obtain a value of $c_2 = \frac{3}{4}$, but we give a slightly more sophisticated strategy which creates giants even faster.

Theorem 1.6. *There is an online algorithm such that for any* $\epsilon > 0$*, after* $0.733n$ *rounds both color classes contain connected components of order at least* $c_\epsilon n$ ***whp***.

This paper is organized as follows. The next section reviews some standard probabilistic facts, and then develops a general tool which extends a recent result of Spencer and Wormald from [28]. This allows us to control the evolution of the susceptibility of a graph under the addition of random edges. Section 3 completely resolves the offline case, by proving Theorem 1.1. For the online setting, Sections 4 and 5 consider the respective problems of avoiding and creating giants. The final section contains some concluding remarks.

Throughout our paper, we will omit floor and ceiling signs whenever they are not essential, to improve clarity of presentation. All logarithms are in base $e \approx 2.718$ unless otherwise specified. The following asymptotic notation will be utilized extensively. For two functions $f(n)$ and $g(n)$, we write $f(n) \ll g(n)$, $f(n) = o(g(n))$, or $g(n) = \omega(f(n))$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and $f(n) = O(g(n))$ or $g(n) = \Omega(f(n))$ if there exists a constant M such that $|f(n)| \leq M|g(n)|$ for all sufficiently large n. The number of vertices n is assumed to be sufficiently large where necessary.

2 Preliminaries

In this section, we review some standard facts commonly used in Probabilistic Combinatorics. Then, we use them to prove a useful result (Theorem 2.6) which shows that a certain graph parameter, the *susceptibility*, tracks a natural differential equation. This extends a result of Spencer and Wormald, and we state it in a general-purpose form for the convenience of possible future citations.

2.1 Probabilistic tools

We recall the Chernoff bound for exponential concentration of the binomial distribution. The following formulation appears in, e.g., [1].

Fact 2.1. *For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that any binomial random variable X with mean μ satisfies $\mathbb{P}\left[|X-\mu| > \epsilon\mu\right] < e^{-c_\epsilon \mu}$.*

A binomial random variable is the sum of independent indicator variables. We also need concentration in settings without complete independence. Recall that a martingale is a sequence $X_0, X_1, \ldots$ of random variables such that each conditional expectation $\mathbb{E}\left[X_{t+1} \mid X_0, \ldots, X_t\right]$ is precisely X_t. The Hoeffding-Azuma inequality (see, e.g., [1]) provides concentration for martingales with bounded stepwise increments $|X_{t+1} - X_t|$, and this has been widely used in probabilistic combinatorics.

When only one-sided concentration is needed, it can be convenient to consider instead a *supermartingale*, which only requires $\mathbb{E}\left[X_{t+1} \mid X_0, \ldots, X_t\right] \le X_t$ for all t. We will use the analogue of Hoeffding-Azuma for supermartingales, which follows from exactly the same proof as for martingales (see, e.g., [19] or [30]).

Fact 2.2. *Let $X_0, \ldots, X_n$ be a supermartingale, with bounded differences $|X_{i+1} - X_i| \le C$. Then for any $\lambda \ge 0$,*

$$\mathbb{P}\left[X_n \ge X_0 + \lambda\right] \;\le\; \exp\left\{-\frac{\lambda^2}{2C^2 n}\right\}.$$

We can also define *submartingales* via the requirements $\mathbb{E}\left[X_{t+1} \mid X_0, \ldots, X_t\right] \ge X_t$; estimates on their lower tails, similar to the above fact, follow by symmetry.

Finally, we will frequently switch between the models $G_{n,p}$, $G_{n,m}$, and the product space of m independent uniform random edges, depending on which one is the most convenient. Adding more edges makes it harder to avoid giants, but easier to create them, so all properties we consider are monotone. Hence the following fact allows us to translate results between the models, while still keeping everything sharp to first-order.

Fact 2.3. *Fix any constant $\epsilon > 0$, and suppose that $m = m(n)$ tends to infinity with n, but $m = o(n^2)$. Then there are couplings of the corresponding probability spaces such that the following hold.*

- **(i)** *$G_{n,m} \subset G_{n,p}$ **whp** for $p = (1+\epsilon)\frac{2m}{n}$, and $G_{n,m} \supset G_{n,p}$ **whp** for $p = (1-\epsilon)\frac{2m}{n}$.*
- **(ii)** *The graph formed by generating m random edges (possibly with repetition) is always contained in $G_{n,m}$, and **whp** contains $G_{n,m'}$ with $m' = (1-\epsilon)m$.*

Proof sketch. By the standard coupling of $G_{n,m}$ and $G_{n,p}$ via the random graph process, part (i) follows from the Chernoff bound on $\mathrm{Bin}\left[\binom{n}{2}, p\right]$. For part (ii), one can similarly couple $G_{n,m'}$ with the product space of m edges by considering an infinite sequence of independent random edges. Then, the m-edge product space is the projection onto the first m choices, and $G_{n,m'}$ is the graph consisting of the first m' distinct edges. So, it suffices to show that **whp**, there are at least $(1-\epsilon)m$ distinct edges among the first m sampled with replacement. Observe that when the k-th edge is sampled, the probability that it is a repetition of a previously sampled edge is always less than $k/\binom{n}{2} < \frac{\epsilon}{2}$ since $m = o(n^2)$. Therefore, the number of samples which are repetitions is stochastically dominated by $\text{Bin}\left[m, \frac{\epsilon}{2}\right]$, which is at most ϵm **whp** by the Chernoff bound. Then, the number of distinct edges is at least $(1-\epsilon)m$, as desired. $\square$

2.2 Evolution of susceptibility

One of the most useful parameters for studying the giant component of a graph is the *susceptibility*. For a graph G, this is defined as $S(G) = \frac{1}{n}\sum_v C_v$, where C_v is the size of the connected component in G containing v. Note that this also equals $\frac{1}{n}$ times the sum of the squares of the component sizes. Many researchers have investigated the evolution of the susceptibility under random edge addition, starting with Bohman and Kravitz, who used this to analyze the Achlioptas process in [9].

More recently, Spencer and Wormald proved in [28] that for m up to $(1-\epsilon)\frac{n}{2}$, the susceptibility of the m-edge random graph evolves like the solution $\phi(m)$ of the differential equation $\phi' = \frac{2}{n}\phi^2$ with initial condition $\phi(0) = 1$. The heuristic for this differential equation is quite natural, although the formal proof is nontrivial. Indeed, when a random edge is added to some intermediate (and subcritical) G, its endpoints typically lie in different components, each of which has expected size $S(G)$. If both component sizes are close to $S(G)$, then the increment to $S(G)$ after adding the edge is roughly $\frac{1}{n}\left[(S(G)+S(G))^2 - 2S(G)^2\right] = \frac{2}{n}S(G)^2$. Thus, one might expect the evolution of $S(G)$ to follow $\phi' = \frac{2}{n}\phi^2$. The solution of this differential equation is $\phi(m) = \left(1 - \frac{2}{n}m\right)^{-1}$, so it only "blows up" when m reaches $\frac{n}{2}$. This matches the classical threshold of the giant component, because the result of Spencer and Wormald concentrates $S(G_{n,m})$ around $\phi(m)$ for m up to $(1-\epsilon)\frac{n}{2}$. In this range, $S(G_{n,m})$ is then bounded by a constant, and we can always trivially bound the size of the largest component by $\sqrt{nS(G)}$, so the largest component is $o(n)$ **whp**.

However, once we start to color edges, the color classes are no longer Erdős-Rényi random graphs. It is then crucial to control the evolution of susceptibility from initial graphs which are non-empty. One of the main contributions of [28] was a result of this nature, but it only controlled one phase of evolution. In order to formulate it, we need the following definition.

Definition 2.4. *A graph has a* ***K, c component tail*** *if for all positive integers s, at most Ke^{-cs}-fraction of its vertices lie in components of order at least s.*

Note that a K, c component tail immediately implies that all components have order $O(\log n)$. Now we restate a key result of Spencer and Wormald (Theorem 3.1 of [28]), translated into an equivalent form via Fact 2.3.

Fact 2.5. *Let L, K, c, γ be positive real numbers. Let G be a graph on n vertices with a K, c component tail and $S(G) \le L$. Add $(1-\gamma)\frac{n}{2L}$ independent random edges to G, ignoring repeated edges, and let the result be G'. Then there exist K', c' such that G' has a K', c' component tail* ***whp****.*

The K', c' component tail is very useful, because it bounds the entire distribution of the component sizes. However, our arguments also need control of the new value of the susceptibility after random edge addition, so we prove the following extension of the above result. This can be done using the methods used in [28], but we include here an alternate (and simpler) proof, following ideas from [5].

Theorem 2.6. *Let L, K, c, γ be positive real numbers. Let G be a graph on n vertices with a K, c component tail and $S(G) \le L$. Add $(1-\gamma)\frac{n}{2L}$ independent random edges to G, ignoring repeated edges, and let the result be G'. Then there exist K', c' such that* ***whp*** *G' has a K', c' component tail,* ***and*** $\boldsymbol{S(G') \le \frac{L}{\gamma} + o(1)}$.

Remark. The bound $\frac{L}{\gamma}$ arises from the following heuristic. Suppose that the initial susceptibility is L. We will show that its evolution is dictated by the differential equation $\phi' = \frac{2}{n}\phi^2$ with initial condition $\phi(0) = L$, whose solution is $\phi(t) = \left(\frac{1}{L} - \frac{2}{n}t\right)^{-1}$. Substituting $t = (1-\gamma)\frac{n}{2L}$ gives $\frac{L}{\gamma}$.

Proof. Note that by definition, the susceptibility is always at least 1, so we will implicitly use $L \ge 1$ throughout the proof. Let $T = (1-\gamma)\frac{n}{2L}$. Let $e_1, \dots, e_T$ denote the incoming edges, and let G_t be the graph after the addition of the first t of them. Fact 2.5 gives constants K', c' such that G_T has a K', c' component tail **whp**.

Let $\phi(t) = \left(\frac{1}{L} - \frac{2}{n}t\right)^{-1}$. We now formalize our heuristic argument which suggests that $S(G_t)$ evolves like $\phi(t)$. For each t, let $\mathcal{E}_t$ be the event that G_t has a K', c' component tail and $S(G_t) \le \phi(t) + e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}}$. Note that we will only run t up to $T \le n$, so the exponential factor is only at most a constant, and hence the error term tends to zero as n grows. Now, consider the sequence of random variables:

$$X_t \;=\; \begin{cases} S(G_t) - \phi(t) - e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}} & \text{if } \mathcal{E}_{t-1} \text{ holds,} \\ X_{t-1} & \text{otherwise.} \end{cases}$$

We claim that X_t is a supermartingale. Indeed, suppose that G_t has components of order $C_1, C_2, \dots$ If the incoming edge $v_1 v_2$ has v_1 in the i-th component and v_2 in the j-th component, then the susceptibility increases by exactly $\frac{1}{n}[(C_i + C_j)^2 - C_i^2 - C_j^2] = \frac{2}{n} C_i C_j$ when $i \ne j$, and zero otherwise. Therefore,

$$\begin{aligned} \mathbb{E}\left[S(G_{t+1}) \mid e_1, \dots, e_t\right] &= S(G_t) + \sum_{i \ne j} \frac{2}{n} C_i C_j \cdot \frac{C_i}{n} \frac{C_j}{n-1} \\ &\le S(G_t) + \frac{2}{n-1}\left(\frac{1}{n}\sum_i C_i^2\right)^2 \\ &= S(G_t) + \frac{2}{n-1} S(G_t)^2. \end{aligned}$$

We use this to bound the expected conditional increment in X_t. Note that for the purposes of bounding $\mathbb{E}\left[X_{t+1} \mid e_1, \dots, e_t\right]$ we may assume that $\mathcal{E}_t$ holds (otherwise this conditional expectation is trivially equal to X_t). Using the above, and the convexity of ϕ and the exponential, we have:

$$\begin{aligned}
&\mathbb{E}\left[X_{t+1} - X_t \mid e_1, \ldots, e_t, \mathcal{E}_t\right] \\
&\quad\leq \frac{2}{n-1} S(G_t)^2 - (\phi(t+1) - \phi(t)) - \left(e^{\frac{5L}{\gamma}\frac{t+1}{n}} - e^{\frac{5L}{\gamma}\frac{t}{n}}\right) n^{-\frac{1}{3}} \\
&\quad\leq \frac{2}{n-1} S(G_t)^2 - \phi'(t) - \frac{5L}{\gamma}\frac{1}{n} e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}} \\
&\quad= \frac{2}{n-1} S(G_t)^2 - \frac{2}{n}\phi(t)^2 - \frac{5L}{\gamma}\frac{1}{n^{4/3}} e^{\frac{5L}{\gamma}\frac{t}{n}} \\
&\quad\leq \frac{2}{n-1}\left(\phi(t) + e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}}\right)^2 - \frac{2}{n}\phi(t)^2 - \frac{5L}{\gamma}\frac{1}{n^{4/3}} e^{\frac{5L}{\gamma}\frac{t}{n}} \\
&\quad= \frac{2}{n(n-1)}\phi(t)^2 + \frac{4}{(n-1)n^{1/3}}\phi(t) e^{\frac{5L}{\gamma}\frac{t}{n}} + \frac{2}{(n-1)n^{2/3}} e^{\frac{10L}{\gamma}\frac{t}{n}} - \frac{5L}{\gamma}\frac{1}{n^{4/3}} e^{\frac{5L}{\gamma}\frac{t}{n}}.
\end{aligned}$$

We will only run t up to $T = (1-\gamma)\frac{n}{2L}$, so we always have $\frac{t}{n} < 1$, as well as $\phi(t) \leq \frac{L}{\gamma}$ because ϕ is increasing. Plugging in these bounds, the $\phi(t)$ and exponential factors are replaced by constants, so the asymptotic behavior of each term is determined by the power of n in the denominator. Hence the second and fourth terms dominate, giving

$$\begin{aligned}
\mathbb{E}\left[X_{t+1} - X_t \mid e_1, \ldots, e_t, \mathcal{E}_t\right] &\leq (1+o(1))\left(\frac{4}{n^{4/3}}\frac{L}{\gamma} e^{\frac{5L}{\gamma}} - \frac{5L}{\gamma}\frac{1}{n^{4/3}} e^{\frac{5L}{\gamma}}\right) \\
&= -(1+o(1))\frac{L}{\gamma n^{4/3}} e^{\frac{5L}{\gamma}},
\end{aligned}$$

which is negative for sufficiently large n. Therefore, X_t is indeed a supermartingale. Observe that $X_0 \leq -n^{-1/3}$. We will use the Hoeffding-Azuma inequality (Fact 2.2) to prove that **whp**, $X_t < 0$ for every $t \leq T$. For this, note that the one-step change in X_t is zero if G_t does not have a K', c' component tail. Otherwise, as previously remarked, all components of G_t are bounded by some $C \log n$, so the maximum change in the susceptibility is $\frac{2}{n}(C \log n)^2$. To bound the one-step change in the error term $\phi(t) + e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}}$, which is an increasing convex function, it suffices to use the first derivative at $t = T$. Recalling that $T = (1-\gamma)\frac{n}{2L}$, this turns out to be precisely

$$\left.\frac{d}{dt}\right|_{t=T} = \left[\left(\frac{1}{L} - \frac{2}{n}T\right)^{-2} \cdot \frac{2}{n}\right] + \left[e^{\frac{5L}{\gamma}\frac{T}{n}} n^{-1/3} \cdot \frac{5L}{\gamma n}\right] = \left(\frac{\gamma}{L}\right)^{-2}\frac{2}{n} + e^{\frac{5L}{\gamma}\frac{T}{n}}\frac{5L}{\gamma n^{4/3}},$$

which is clearly $O(n^{-1})$ because γ and L are constants, and $T \leq n$. Applying the Hoeffding-Azuma inequality with $\lambda = n^{-1/3}$, we find that for each $t \leq T \leq n$,

$$\mathbb{P}\left[X_t \geq 0\right] \leq \exp\left\{-\frac{n^{-2/3}}{2 \cdot \left(\frac{2}{n}(C \log n)^2\right)^2 t}\right\} \leq \exp\left\{-\frac{n^{1/3}}{8C^4 \log^4 n}\right\}.$$

A union bound over all $t \leq T$ shows that **whp**, all $X_t < 0$. Furthermore, Fact 2.5 implies that **whp**, G_T has a K', c' component tail.

To complete our argument, we claim that whenever all of these high-probability events happen, then all $\mathcal{E}_t$ occur for $0 \leq t \leq T$. We prove this by induction on t. Each $\mathcal{E}_t$ has two parts: a component tail and an upper bound on $S(G_t)$. The K', c' component tail property is automatically satisfied for all t because $G_t \subset G_T$, and we are assuming that G_T already has this (monotone) property. We concentrate on the upper bounds for $S(G_t)$ in the remainder of this proof. For the base case $t = 0$, the susceptibility part of $\mathcal{E}_0$ is immediate by definition since $S(G_0) = \phi(0) < \phi(0) + e^{\frac{5L}{\gamma}\frac{0}{n}} n^{-\frac{1}{3}}$. For our induction step, given that $\mathcal{E}_{t-1}$ occurs, the definition of X_t is then $S(G_t) - \phi(t) - e^{\frac{5L}{\gamma}\frac{t}{n}} n^{-\frac{1}{3}}$ instead of the alternative X_{t-1}. Yet we assumed that $X_t < 0$, so that gives the susceptibility part of $\mathcal{E}_t$, and completes the induction.

Therefore, we conclude that $\mathcal{E}_T$ occurs **whp**, which in particular means that $S(G_T) \le \phi(T) + e^{\frac{5L}{\gamma}\frac{T}{n}} n^{-\frac{1}{3}} = \frac{L}{\gamma} + o(1)$, as desired. □

3 Offline avoidance of giants

In this section, we prove Theorem 1.1, which has two parts, a lower and an upper bound. The lower bound relies on the following relationship between orientability and decomposition. Recall that we call a graph r-orientable if it is possible to orient all edges such that all in-degrees are at most r.

Lemma 3.1. *The edges of any r-orientable graph G can be colored with r colors such that for every pair of distinct vertices u, v, there are at most 2 monochromatic paths in each color connecting u and v.*

Proof. Fix an orientation of G with all in-degrees at most r, and greedily color the edges by r colors so that at each vertex, all incoming edges are differently colored. Consider a particular color class. By construction, it is a directed graph with all in-degrees at most 1, so it is a disjoint union of unicyclic components. Then, every pair of vertices is linked by at most two paths in that color, as desired. □

The previous lemma produces a coloring whose connectivity is very fragile. Our next lemma quantifies this, showing that the (*a priori*, possibly large) monochromatic components shatter easily.

Lemma 3.2. *For any $\epsilon > 0$, there is $c > 0$ such that the following holds. Let G be a graph on n vertices with maximum degree $\log n$, where every pair of distinct vertices is connected by at most 2 distinct paths. Independently delete each edge of G with probability ϵ. Then, **whp** all connected components of the resulting graph have order at most $ne^{-c\frac{\log n}{\log\log n}} = o(n)$.*

Proof. Define c such that $1 - \epsilon = e^{-8c}$, and recall from Section 2.2 that the susceptibility of a graph is $\frac{1}{n}\sum_v C_v$, where C_v is the size of the connected component containing v. Let the random variable S be the susceptibility of the graph G' which remains after the edge deletions. Since $\sum_v C_v$ equals the sum of the squares of the component sizes, all components of G' have order at most $\sqrt{nS}$. Thus, it suffices to show that $S \le ne^{-2c\frac{\log n}{\log\log n}}$ **whp**.

Fix an arbitrary vertex v. Since G has maximum degree $\log n$, the total number of vertices within distance $D = \frac{1}{2}\frac{\log n}{\log\log n}$ of v is at most $(\log n)^D = \sqrt{n}$. Any other vertex u has probability at most $2(1-\epsilon)^D = 2e^{-8cD}$ of being connected to v after the deletion. This is because there are at most 2 paths between u and v, and each path has length at least D. Therefore, by linearity of expectation, the expected size of the component containing v is $\mathbb{E}[C_v] \le \sqrt{n} + n \cdot 2e^{-4c\frac{\log n}{\log\log n}} \le ne^{-3c\frac{\log n}{\log\log n}}$. Another application of linearity of expectation gives $\mathbb{E}[S] \le ne^{-3c\frac{\log n}{\log\log n}}$. So, by Markov's inequality, S exceeds $ne^{-2c\frac{\log n}{\log\log n}}$ with probability at most $e^{-c\frac{\log n}{\log\log n}} = o(1)$, completing the proof. □

Remark. The self-contained argument above only requires a relatively weak maximum degree condition, and is sufficient for our purposes. It is worth mentioning that under the stronger assumption that G is a random graph, one can use the substantially less trivial Lemma 11 of [8] to sharpen the eventual bound to n^{1-c_ϵ}, as Spöhel, Steger and Thomas do in [29]. Indeed, that lemma claims that if $m = cn$, then there is a large constant K such that in $G_{n,m}$, the number of vertices within distance $(\log n)/K$ of any vertex v is at most $n^{\frac{\log 2K}{K}}$ **whp**. Using this fact above instead of our exploration to depth $\frac{1}{2}\frac{\log n}{\log \log n}$ bounds all connected components below n^{1-c_ϵ}.

Let us now state the result of Cain, Sanders and Wormald [11], and Fernholz and Ramachandran [12], on the matching thresholds for orientability and average degree.

Fact 3.3. *For any integer $r \geq 2$, there is an explicit threshold ψ_r such that the following holds. For any $\epsilon > 0$, if $m < (\psi_r - \epsilon)n$, then $G_{n,m}$ is r-orientable **whp**. On the other hand, if $m > (\psi_r + \epsilon)n$, then **whp** $G_{n,m}$ contains a subgraph with average degree at least $2r + c_\epsilon$, where $c_\epsilon > 0$.*

We are now ready to prove the lower bound, which we first translate to $G_{n,p}$ for convenience. By applying Fact 2.3 and rescaling ϵ, it suffices to show that if $p = 2(1-\epsilon)(\psi_r - \epsilon)/n$, then **whp** there is a coloring of $G_{n,p}$ where every color class has all components of order $o(n)$.

Proof of lower bound of Theorem 1.1. Let $p' = 2(\psi_r - \epsilon)/n$, and observe that $G_{n,p}$ can be obtained from $G' = G_{n,p'}$ by independently deleting each edge with probability ϵ. First, consider the graph G' before deletions. By Fact 3.3, G' is r-orientable **whp**. Also, it is easy to see that since np is at most the constant $2\psi_r$, G' has maximum degree at most $\log n$ **whp**. Indeed, each individual degree is distributed as $\text{Bin}(n-1, p)$, and $\mathbb{P}\left[\text{Bin}(n-1,p) > \log n\right] \leq \binom{n}{\log n} p^{\log n} \leq \left(\frac{enp}{\log n}\right)^{\log n}$. Since np is bounded by a constant, this is $o(n^{-1})$, so a union bound over all n vertices implies that the maximum degree is at most $\log n$ **whp**.

Thus, by Lemma 3.1, we can color the edges of G' so that every pair of distinct vertices is connected by at most two paths in each color. This, together with our degree bound and Lemma 3.2, shows that after deleting each edge of G' independently with probability ϵ (to obtain $G_{n,p}$), **whp** all color classes have connected components of order only $o(n)$. □

For the upper bound, we use the second half of Fact 3.3, which gives a subgraph of high average degree. It turns out that this is already enough to ensure a giant. To see this, we first show that small sets of vertices typically induce low average degree in the random graph.

Lemma 3.4. *For any $\lambda, \epsilon > 0$, there is a constant $c > 0$ such that in $G_{n,p}$ with $p = \frac{\lambda}{n}$, **whp** every set of at most cn vertices induces a subgraph with average degree less than $2 + \epsilon$.*

Proof. Without loss of generality, assume that $\epsilon < 1$ and $\lambda \geq 1$. Let $c = (e^3\lambda^2)^{-\frac{2}{\epsilon}}$. We will take a union bound over all subsets of $t \leq cn$ vertices. For a fixed value of t, the probability that some t-set of vertices induces at least $\left(1+\frac{\epsilon}{2}\right)t$ edges is at most

$$\begin{aligned}
\binom{n}{t} \cdot \mathbb{P}\left[\text{Bin}\left[\binom{t}{2}, \frac{\lambda}{n}\right] \geq \left(1+\frac{\epsilon}{2}\right)t\right] &\leq \binom{n}{t} \cdot \binom{t^2/2}{(1+\frac{\epsilon}{2})t} \left(\frac{\lambda}{n}\right)^{(1+\frac{\epsilon}{2})t} \\
&\leq \left(\frac{en}{t}\right)^t \cdot \left(\frac{et^2/2}{(1+\frac{\epsilon}{2})t} \cdot \frac{\lambda}{n}\right)^{(1+\frac{\epsilon}{2})t} \\
&= \left[\left(\frac{en}{t}\right) \cdot \left(\frac{e\lambda}{2+\epsilon} \cdot \frac{t}{n}\right)^{1+\frac{\epsilon}{2}}\right]^t \\
&= \left[e\left(\frac{e\lambda}{2+\epsilon}\right)^{1+\frac{\epsilon}{2}} \cdot \left(\frac{t}{n}\right)^{\frac{\epsilon}{2}}\right]^t \\
&\leq \left[\frac{e^3\lambda^2}{2} \cdot \left(\frac{t}{n}\right)^{\frac{\epsilon}{2}}\right]^t.
\end{aligned}$$

To complete our union bound, we sum the final expression over the range $1 \leq t \leq cn$. We split this into two intervals, separating at $t = \log n$. Observe that the quantity in the square brackets increases in t, and reaches $\frac{1}{2}$ when $t = cn$. So, the sum over the interval $\log n \leq t \leq cn$ is at most $\sum_{\log n}^{cn} 2^{-t} = o(1)$. For the other interval $t < \log n$, the square bracket is still at most $\frac{1}{2} \leq 1$, so we can ignore the outer exponentiation and conclude that the final expression is at most $\frac{e^3\lambda^2}{2} \cdot \left(\frac{\log n}{n}\right)^{\frac{\epsilon}{2}}$. Multiplying this by the number of values of t in this interval ($\log n$), we see that the final sum is still $o(1)$. Therefore, the property holds **whp**, as claimed. ☐

From this, we immediately derive the following useful corollary, which ensures a giant in any subgraph of average degree at least $2+\epsilon$.

Corollary 3.5. *For any $\lambda, \epsilon > 0$, there is a constant $c > 0$ such that in $G_{n,m}$ with $m = \lambda n$,* ***whp*** *every subgraph with average degree at least $2+\epsilon$ contains a connected component of order at least cn.*

Proof. By the previous lemma and Fact 2.3, **whp** $G_{n,m}$ has the property that every set of at most cn vertices induces a subgraph with average degree less than $2+\epsilon$. Then, consider any subgraph H with average degree at least $2+\epsilon$. Separating H into its connected components, we find that some component must have average degree at least $2+\epsilon$. Therefore, that component must have order at least cn, as desired. ☐

Proof of upper bound of Theorem 1.1. By Fact 3.3, if $m > (\psi_r + \epsilon)n$, **whp** $G_{n,m}$ contains a subgraph H with average degree at least $2r + c_\epsilon$. No matter which colors appear on the edges of H, some color class will have average degree at least $2 + c_\epsilon/r$, and therefore contain a giant **whp** by Corollary 3.5. ☐

4 Online avoidance of giants

In this section, we consider the online case of the avoidance problem. We first show that a natural adaptation of the offline algorithm gives an asymptotically sharp result for large numbers of colors. Then, we consider the other extreme with 2 colors, and show that the trivial bound (surviving for $(1-\epsilon)n$ rounds by randomly coloring each incoming edge) is not tight.

4.1 Many colors

Our offline algorithm avoided giant components by orienting edges to minimize in-degrees. By replacing the offline orientation procedure with an online one, this strategy naturally extends to the online setting. Online edge orientation has been extensively studied, in the famous equivalent formulation known as the "power of two random choices" with balls and bins (see [22] for a survey of results). Indeed, that setting had n bins, with kn balls coming sequentially, each with two independent random choices for a destination bin. The objective was to control the maximum load across all of the bins. This can be interpreted as a graph orientation problem, where each pair of bin choices corresponds to an incoming edge with the two choices as endpoints. The edge's orientation records which bin the ball is sent to, and the goal of controlling the maximum in-degree is precisely the same as that of controlling the maximum load in the balls-and-bins problem.

It is now well-known that when the objective is to minimize the maximum in-degree, the stochastically optimal online orientation strategy is to always orient each incoming edge towards the endpoint which currently has lower in-degree. However, it turns out that for the purpose of proving Theorem 1.2, one can use a random orientation strategy, which is easier to analyze. Our coloring algorithm, which we call ORIENT, internally maintains a set of orientations for all edges it has seen. To color a new edge e, it randomly orients it with equal probability toward one of its endpoints. Let the new in-degree of that endpoint be d. If $d < r$, then color d is used for the edge e. Otherwise, color r is used. Observe that just as in Lemma 3.1, each of the first $r - 1$ color classes is a disjoint union of unicyclic components. Therefore, each of these color classes has every pair of vertices connected by at most two paths, so it will shatter by the same argument as in the proof of Theorem 1.1.

The new challenge in this section is to control the r-th color class. Fortunately, it turns out that it is extremely sparse. To prove this, it is more convenient to work in the random directed graph $\overrightarrow{G}_{n,p}$, in which each of the $n(n-1)$ possible directed edges appears independently with probability $p/2$. Note that in this model, it is possible for both $\overrightarrow{uv}$ and $\overleftarrow{uv}$ to be present simultaneously. Our first claim is that $\overrightarrow{G}_{n,p}$ typically has no long cycles containing many vertices of high in-degree. This is relevant because every edge in color r has an endpoint with in-degree at least r.

Lemma 4.1. *For any $\epsilon > 0$, the following holds for every sufficiently large constant r. Let $\overrightarrow{G} = \overrightarrow{G}_{n,p}$ be a random directed graph with $p = (1-\epsilon)\frac{2r}{n}$, and G be the undirected graph on the same vertex set obtained by collapsing all edges between each vertex pair into a single undirected edge. Then,* ***whp*** *G does not contain any cycles of length at least $\sqrt[4]{\log n}$ for which at least half of the vertices on the cycle had in-degree at least r in $\overrightarrow{G}$.*

Proof. We will use a union bound to show that a large family of objects do not appear in the random directed graph. Let us define an *isomorphism type* to be a directed simple graph whose underlying undirected graph is a cycle, say with vertices $v_1, \ldots, v_t$, along with a subset of at least $t/2$ of its vertices which have been designated as "high-in-degree vertices." Note that we do not require the edges of the cycle to be oriented in a consistent direction. The number of distinct t-vertex isomorphism types is at most $2^t \cdot 2^t$, because each of the t edges can be oriented in 2 ways, and the number of different subsets of vertices that can be designated as high-in-degree is at most 2^t.

We say that $\overrightarrow{G}_{n,p}$ contains a copy of this isomorphism type if there is an embedding of the vertices v_i such that all consecutive edges $v_i v_{i+1}$ are present in the correct direction, and all designated high-in-degree vertices v_i already have in-degree at least $r - 2$ from vertices other than v_{i-1}, v_{i+1}. We do not restrict our attention to induced copies, so other edges may also be present. If we can show that over all isomorphism types with $t \geq \sqrt[4]{\log n}$, the expected total number of copies in $\overrightarrow{G}_{n,p}$ is $o(1)$, then we will be done by Markov's inequality.

So, let us focus on a particular isomorphism type with t vertices. There are at most n^t ways to embed the t vertices of the cycle. Each edge $v_i v_{i+1}$ independently appears with its correct orientation with probability exactly $p/2$. Next, consider a designated high-in-degree vertex v_i. Crucially, we only require in-degree at least $r-2$ from vertices *other than* v_{i-1} and v_{i+1}. The reason for this exclusion is that the previous step may already have exposed the edges $\overrightarrow{v_{i-1}v_i}$ and $\overrightarrow{v_{i+1}v_i}$. But now, since our model even allows edges in both directions between vertex pairs, the probability that each designated vertex indeed has high in-degree is independently $\mathbb{P}\left[\mathrm{Bin}\left[n-3, \frac{p}{2}\right] \geq r-2\right]$. Since $p = (1-\epsilon)\frac{2r}{n}$ and r is large, each of these individual probabilities is bounded by the probability that $\mathrm{Bin}\left[n-3, (1-\epsilon)\frac{r}{n}\right]$ exceeds its mean by at least an $\frac{\epsilon}{2}$-fraction. By the Chernoff bound, this happens with probability at most $e^{-c_\epsilon r}$ for some constant c_ϵ. By choosing large enough r, we may assume that this is below $\frac{1}{64r^2}$. Putting everything together, we find that the expected number of copies of a fixed isomorphism type in $\overrightarrow{G}_{n,p}$ is at most

$$n^t \left(\frac{p}{2}\right)^t \left(\frac{1}{64r^2}\right)^{t/2} \leq \left(\frac{1}{8}\right)^t.$$

We initially showed that the number of distinct t-vertex isomorphism types is at most 4^t, so the expected total number of copies of all t-vertex isomorphism types is at most 2^{-t}. This is a geometric series, so its sum over all $t \geq \sqrt[4]{\log n}$ is still $o(1)$, as desired. □

Remark 1. Since we had a convergent geometric series at the end of the proof, the $\sqrt[4]{\log n}$ bound is not tight. In fact, any function which grows with n is sufficient.

Remark 2. If one is interested in beating the trivial bound, which corresponds to $p \approx \frac{r}{n}$, one can choose ϵ to be extremely close to, but just below, $\frac{1}{2}$. One can numerically check that if $\epsilon = 0.4999$ and $r \geq 51$, then the probability that $\mathrm{Bin}\left[n, (1-\epsilon)\frac{r}{n}\right]$ exceeds $r-2$ is at most $\frac{1}{16.1r^2}$ for large n, because the Binomial converges to a Poisson variable with mean $0.5001r$. Continuing the argument, this will show that the expected number of appearances of all t-vertex isomorphism types is at most $\left(\frac{4}{\sqrt{16.1}}\right)^t$, which is still a convergent geometric series, so the same result will follow.

Next, we establish an easy bound which holds for ordinary random graphs.

Lemma 4.2. *For every constant c, **whp** in $G_{n,p}$ with $p = \frac{c}{n}$, every set of $t \leq \sqrt[3]{\log n}$ vertices induces at most t edges.*

Proof. The expected number of sets with $t \leq \sqrt[3]{\log n}$ and at least $t+1$ edges can be bounded by

$$\sum_{t=4}^{\sqrt[3]{\log n}} \binom{n}{t} \binom{\binom{t}{2}}{t+1} p^{t+1} \leq \sum_{t=4}^{\sqrt[3]{\log n}} \frac{t}{en} \left(\frac{ne}{t} \cdot \frac{tec}{2n}\right)^{t+1} = o(1).$$

□

We now combine the previous two lemmas to show that the r-th color class shatters easily. In the proof of Lemma 3.2, the control of connectivity was done by bounding the number of distinct paths between every pair of vertices. This time, we use the notion of an *essential edge*. We say that an edge e on a path is *essential* if every other path connecting the same endpoints also contains e. It turns out that in the r-th color class, every long path contains a huge number of essential edges.

Lemma 4.3. *For any $\epsilon > 0$, the following holds **whp** for every sufficiently large constant r. Let G be the graph formed by the r-th color class after $(1-\epsilon)rn$ independent random edges have been colored by* ORIENT*. Then every path in G of length at least $\sqrt[3]{\log n}$ has the property that more than half of its edges are essential.*

Proof. Since each (random) incoming edge is randomly directed by ORIENT, one can think of the input as a sequence of random directed edges, which is then deterministically colored using the rule in ORIENT. By a similar argument to Fact 2.3, it suffices to consider the more convenient model where the input sequence is a random permutation of the edges of a random directed graph $G = \overrightarrow{G}_{n,p}$ with $p = (1-\epsilon)\frac{2r}{n}$. Throughout this proof, although G is a directed graph, whenever we speak of cycles or paths, we are referring to undirected cycles and paths in the underlying undirected graph. In other words, we are ignoring the edge orientations when seeking these structures.

Note that if an edge of G is oriented toward a vertex with in-degree less than r, then regardless of the permutation, it will never be colored r. So, let $H \subset G$ be obtained by deleting all edges oriented into vertices of in-degree less than r. Then H entirely contains the r-th color class. Let A be the set of vertices whose in-degrees were less than r, and let B be those that had in-degree at least r. Observe that we deleted all edges oriented toward vertices in A, so A spans no edges in H. In particular, any cycle in H has at least half of its vertices in B, i.e., with in-degree at least r.

Therefore, by Lemma 4.1, **whp** all cycles in H have length at most $\sqrt[4]{\log n}$. Also, condition on the result of Lemma 4.2, which shows that in G (and hence also H), every set of $t \le \sqrt[3]{\log n}$ vertices induces at most t edges. These two graph properties will be enough to show that long paths in H contain many essential edges.

Let $P = v_1, \ldots, v_t$ be a path in H with length at least $\sqrt[3]{\log n}$. Suppose for contradiction that at least half of its edges are non-essential. We claim that since $\sqrt[4]{\log n} \ll \sqrt[3]{\log n}$, there must be non-essential edges $v_i v_{i+1}$ and $v_j v_{j+1}$ such that $i < j$ and $3\sqrt[4]{\log n} < j - i < 7\sqrt[4]{\log n}$. Indeed, if this were false, then out of the $7\sqrt[4]{\log n}$ edges immediately following each non-essential edge in P, at least $\frac{4}{7}$-fraction of them would be essential. Then an averaging argument would contradict the fact that at least half of the edges were non-essential.

Now, since $v_i v_{i+1}$ is non-essential, there is another path $P' = w_1, \ldots, w_s$ with $w_1 = v_1$ and $w_s = v_t$ which avoids the edge $v_i v_{i+1}$. Let a be the largest index such that $w_a \in \{v_1, \ldots, v_i\}$, and let b be the next index after a such that $w_b \in P$. These exist because P and P' both contain v_1 and v_t. Note that by definition, w_b is actually in $\{v_{i+1}, \ldots, v_t\}$, and the segment of P' from w_a to w_b intersects P only at w_a and w_b. So, there is a cycle C_1 formed by going from w_a to w_b along P', and then back to w_a along P. Importantly, the common edges between C_1 and P are a contiguous interval containing the edge $v_i v_{i+1}$.

Similarly, we can find a cycle C_2 containing the edge $v_j v_{j+1}$. Crucially, C_1 and C_2 are distinct (although not necessarily disjoint) because $j - i > 3\sqrt[4]{\log n}$ and we conditioned on all cycles being shorter than $\sqrt[4]{\log n}$. Yet $j - i < 7\sqrt[4]{\log n}$, so the union of C_1, C_2, and the path $v_i v_{i+1} \ldots v_j$ forms a subgraph of order $k \le 9\sqrt[4]{\log n}$ which spans at least $k+1$ edges. Since we also conditioned on all such subgraphs having order at least $\sqrt[3]{\log n}$, this is a contradiction. Therefore, the path P must have had at least half of its edges essential. $\square$

The previous lemma shows that the r-th color class is typically quite fragile as well. We now combine this with an adaptation of our offline argument, and prove that it is possible to avoid giants in all colors for nearly rn rounds **whp**.

Proof of Theorem 1.2. By rescaling ϵ, it suffices to give a randomized coloring algorithm that avoids giants in all colors **whp**, for a sequence of $m = (1-\epsilon)^3 rn$ independent random edges (possibly with repetitions). As in our proof of Theorem 1.1, it is convenient to color a slightly denser random graph, because the deletion of fictitious edges shatters all large components.

Strictly speaking, we cannot simply apply ORIENT to a larger sequence of edges, because for this problem the input is a sequence of m edges, which must be processed online. We will therefore take some care in specifying how we randomly interleave the input into a longer sequence of edges, so that all operations are clearly online. Let us denote the final sequence of real and fictitious edges by $e_1, \ldots, e_{m'}$, where $m' = (1-\epsilon)^2 rn$. Initially, we select a random subset of m of the m' indices to correspond to the positions of the real edges. We then generate independent random edges for all other e_i, and pass the resulting sequence to ORIENT. Note that since the input distribution is uniform over all sequences of m edges, the augmented sequence of edges consists of m' independent random edges.

Let σ denote the colored sequence of m' edges produced by ORIENT. The graph formed by σ has maximum degree at most $\log n$ **whp** by the same argument as in the offline case. We also know by construction that there are at most 2 paths between every pair of vertices in each of the first $r-1$ color classes. For the r-th color class, Lemma 4.3 ensures that **whp**, all paths longer than $\sqrt[3]{\log n}$ have at least half of their edges essential. Let $\mathcal{P}$ denote the collection of all of these properties. We will write $\sigma \in \mathcal{P}$ when all of them hold.

Now, we delete the $\epsilon m'$ fictitious edges to recover the coloring of the original edges. Note that since the algorithm knows which m edges are real (that was the input), the edges to delete are completely determined. But crucially, it used an independent source of randomness to interleave the original m edges into the full sequence of m' edges. Therefore, if we only condition on σ (and not on the input), then the distribution of which m edges were original is uniform over all possible subsets of m positions. Formally, we are calculating the probability of success by summing over all colored sequences σ of m' edges. We have

$$\mathbb{P}\left[\text{success}\right] \;=\; \sum_{\sigma} \mathbb{P}\left[\text{success} \mid \sigma\right] \mathbb{P}\left[\sigma\right] \;\geq\; \sum_{\sigma \in \mathcal{P}} \mathbb{P}\left[\text{success} \mid \sigma\right] \mathbb{P}\left[\sigma\right]$$

Since we showed that $\sigma \in \mathcal{P}$ **whp**, it suffices to show that $\mathbb{P}\left[\text{success} \mid \sigma\right] \geq 1 - o(1)$ for all $\sigma \in \mathcal{P}$. We noted above that conditioned on σ, the $\epsilon m'$ edges to delete were uniformly distributed over all subsets. Therefore, it remains to show that given any coloring with property $\mathcal{P}$, the deletion of a random ϵ-fraction of its edges **whp** shatters all large connected components. We accomplish this by deleting every edge independently with probability $\frac{\epsilon}{2}$, which will imply the result by a similar coupling argument to Fact 2.3, since $\text{Bin}\big(m', \frac{\epsilon}{2}\big) \leq \epsilon m'$ **whp**.

For each of the first $r-1$ color classes, Lemma 3.2 shows that all components shatter to $o(n)$ **whp**, as in the offline proof. For the r-th color class, we now adapt the proof of Lemma 3.2 to use essential edges. Indeed, let us bound the expected size of the component C_v containing a particular vertex v after the deletions. Since the maximum degree in $G_{n,rn}$ is $\log n$, the total number of vertices within distance $D = \frac{1}{2}\frac{\log n}{\log\log n}$ of v is at most $(\log n)^D = \sqrt{n}$. Any other vertex u is at distance at least $D \gg \sqrt[3]{\log n}$ away from v, so a shortest path from v to u contains at least $D/2$ essential edges. The deletion of any essential edge disconnects u, v, so if edges are deleted with probability $\frac{\epsilon}{2}$, then u and v remain connected only with probability at most $(1-\frac{\epsilon}{2})^{D/2} = e^{-c\frac{\log n}{\log\log n}}$ for some constant c. Hence the expected size of C_v is at most $\sqrt{n} + ne^{-c\frac{\log n}{\log\log n}} = o(n)$, and by linearity of expectation, the expected susceptibility $\mathbb{E}[S]$ of the graph after deletions is $o(n)$. Since the size of the largest component is at most $\sqrt{nS}$, Markov's inequality implies that the r-th color class also has all components smaller than $o(n)$, completing the proof. $\square$

4.2 Two colors

The trivial algorithm, which randomly colors each edge blue or red, clearly lasts for $(1-\epsilon)n$ rounds **whp**. We now present a better algorithm which lasts for $1.06n$ rounds **whp**. To color a new edge e, it considers the set of colors C that appear on isolated edges which are incident with any of its endpoints. (If e is not incident to any isolated edges, then C is empty.) When C contains exactly one color, the algorithm colors the edge e with the other color. Otherwise, it randomly colors e either blue or red with equal probability.

We analyze this by tracking a certain partition of the vertex set. Split the set of isolated edges into two groups based on their color, and call them the *red matching* and the *blue matching*, respectively. After the k-th round, let:

$$\begin{aligned} I_k &= \text{number of isolated vertices,} \\ B_k &= \text{number of vertices in the blue matching,} \\ R_k &= \text{number of vertices in the red matching,} \end{aligned}$$

and let $J_k = n - I_k - B_k - R_k$ be the number of remaining vertices. These parameters correspond to the decomposition of the graph into its isolated vertices, the blue matching, the red matching, and the remainder.

Lemma 4.4. *With probability $1-o(1)$, the following hold for all $t \le 1.1$:*

$$\begin{aligned} \left|\tfrac{1}{n} I_{tn} - e^{-2t}\right| &\le n^{-1/3}, \\ \left|\tfrac{1}{n} B_{tn} - te^{-4t}\right| &\le e^{8} n^{-1/3}, \\ \left|\tfrac{1}{n} R_{tn} - te^{-4t}\right| &\le e^{8} n^{-1/3}. \end{aligned}$$

Proof. The probability that a particular vertex is not incident to any of the first tn edges is exactly $\left(\frac{n-1}{n} \cdot \frac{n-2}{n-1}\right)^{tn} = \left(1-\frac{2}{n}\right)^{tn}$, which tends to e^{-2t} from below as n grows. Routine calculus easily bounds the convergence rate by $O(n^{-1})$, so $\mathbb{E}\left[\frac{1}{n} I_{tn}\right] = e^{-2t} + O(n^{-1})$. Now consider the edge-exposure martingale where Y_k is the conditional expectation of I_{tn} given the first k rounds. Changing the outcome of any particular round can only affect I_{tn} by at most 2, and there are tn rounds to determine I_{tn}, so by the Hoeffding-Azuma inequality (see Theorem 7.4.1 of [1]) I_{tn} is within (say) $\frac{1}{2} n^{2/3}$ of its expectation with probability $e^{-\Omega(n^{1/3})}$. This gives the desired asymptotic for I_{tn}.

We estimate B_{tn} next. We claim that conditioned on the first k incoming edges $e_1, \ldots, e_k$, the expected change $B_{k+1} - B_k$ is

$$\mathbb{E}\left[B_{k+1} - B_k \mid e_1, \ldots, e_k\right] \;=\; 2 \cdot \left(\frac{I_k}{n}\right)^2 \frac{1}{2} - \frac{4B_k}{n} + O(n^{-1}). \tag{1}$$

The first summand comes from the creation of a blue isolated edge from 2 isolated vertices, which contributes 2 to B_k. The probability that both endpoints are isolated vertices is $\frac{I_k}{n} \cdot \frac{I_k - 1}{n-1}$. Since

$\frac{1}{n(n-1)} - \frac{1}{n^2} = O(n^{-3})$ and $I_k \le n$, this is $\left(\frac{I_k}{n}\right)^2 - O(n^{-1})$. The $\frac{1}{2}$ factor comes from the fact that the edge is randomly colored blue or red.

For the second summand, the only way we lose blue isolated edges is when an endpoint of the incoming edge is incident to a blue isolated edge. The probability that the two endpoints hit two different blue isolated edges (hence contributing -4) is $\frac{B_k}{n} \cdot \frac{B_k - 2}{n-1}$. On the other hand, the probability that they hit exactly one isolated edge (hence contributing -2) is $2 \cdot \frac{B_k}{n}\left(1 - \frac{B_k - 1}{n-1}\right)$. Thus the expected contribution from these losses is

$$(-4) \cdot \frac{B_k}{n} \cdot \frac{B_k - 2}{n-1} + (-2) \cdot 2 \cdot \frac{B_k}{n}\left(1 - \frac{B_k - 1}{n-1}\right) \;=\; -\frac{4B_k}{n} + O(n^{-1}),$$

matching the second summand.

Since we showed that $\frac{1}{n} I_{tn} = (1 - o(1))e^{-2t}$ **whp**, equation (1) suggests that $b(t) = \frac{1}{n} B_{tn}$ should satisfy the differential equation

$$\frac{db}{dt} \;=\; (e^{-2t})^2 - 4b, \qquad b(0) \;=\; 0,$$

whose solution is $b(t) = te^{-4t}$.

We now verify this formally, using the same method as for the proof of Theorem 2.6. For each k, let $\mathcal{E}_k$ be the event that $\left|\frac{1}{n} I_k - e^{-\frac{2k}{n}}\right| \le n^{-\frac{1}{3}}$ and $\left|\frac{1}{n} B_k - b\left(\frac{k}{n}\right)\right| \le e^{\frac{7k}{n}} n^{-\frac{1}{3}}$. Now, consider the sequence of random variables

$$W_k \;=\; \begin{cases} B_k - nb\left(\frac{k}{n}\right) - e^{\frac{7k}{n}} n^{\frac{2}{3}} & \text{if } \mathcal{E}_{k-1} \text{ occurs,} \\ W_{k-1} & \text{otherwise.} \end{cases}$$

We claim that W_k is a supermartingale. Assume that $\mathcal{E}_k$ occurs. Then, using (1) we obtain

$$\begin{aligned} &\mathbb{E}\left[W_{k+1} - W_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \\ &\quad\le \left(\frac{I_k}{n}\right)^2 - \frac{4B_k}{n} + O(n^{-1}) - n\left[b\left(\frac{k+1}{n}\right) - b\left(\frac{k}{n}\right)\right] - \left[e^{\frac{7(k+1)}{n}} - e^{\frac{7k}{n}}\right] n^{2/3}. \end{aligned}$$

Since $\frac{I_k}{n} \le e^{-\frac{2k}{n}} + n^{-\frac{1}{3}}$ and $e^{-\frac{2k}{n}} \le 1$, we have $\left(\frac{I_k}{n}\right)^2 \le e^{-\frac{4k}{n}} + 2n^{-\frac{1}{3}} + O(n^{-\frac{2}{3}})$. Similarly, $-\frac{4B_k}{n} \le -4b\left(\frac{k}{n}\right) + 4e^{\frac{7k}{n}} n^{-\frac{1}{3}}$. Recall that for any twice-differentiable function f, Taylor's formula ensures that for any t, h, there is some $0 \le \xi \le 1$ such that $f(t+h) - f(t) = f'(t)h + \frac{1}{2} f''(t + \xi h)h^2$. Since the second derivative of our function $b(t)$ is bounded on the interval $0 \le t \le 1.1$, Taylor's formula gives $b\left(\frac{k+1}{n}\right) - b\left(\frac{k}{n}\right) = \frac{1}{n} b'\left(\frac{k}{n}\right) + O(n^{-2})$. By a similar argument, $e^{\frac{7(k+1)}{n}} - e^{\frac{7k}{n}} = \frac{7}{n} e^{\frac{7k}{n}} + O(n^{-2})$. Combining all of these estimates and using $b' = e^{-4t} - 4b$, we obtain

$$\begin{aligned} &\mathbb{E}\left[W_{k+1} - W_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \\ &\quad\le e^{-\frac{4k}{n}} + 2n^{-\frac{1}{3}} + O(n^{-\frac{2}{3}}) - 4b\left(\frac{k}{n}\right) + 4e^{\frac{7k}{n}} n^{-\frac{1}{3}} - b'\left(\frac{k}{n}\right) - 7e^{\frac{7k}{n}} n^{-\frac{1}{3}} \\ &\quad= \left(2 - 3e^{\frac{7k}{n}}\right) n^{-\frac{1}{3}} + O(n^{-\frac{2}{3}}). \\ &\quad< 0, \end{aligned}$$

so W_k is indeed a supermartingale. Next, we bound the stepwise differences $W_{k+1} - W_k$. The change in B_k is at most 4, and our Taylor estimates show that the error term $nb\left(\frac{k}{n}\right) - e^{\frac{7k}{n}} n^{\frac{2}{3}}$ changes by at most an absolute constant because $b'(\frac{k}{n})$ is bounded on $k \le 1.1n$. Therefore, the Hoeffding-Azuma inequality implies that since $W_0 = -n^{\frac{2}{3}}$,

$$\mathbb{P}\left[\exists k \le 1.1n : W_k \ge 0\right] \le e^{-\Omega(n^{1/3})}. \qquad (2)$$

Similarly, if

$$\hat{W}_k \;=\; \begin{cases} B_k - nb(\frac{k}{n}) + e^{\frac{7k}{n}} n^{\frac{2}{3}} & \text{if } \mathcal{E}_{k-1} \text{ occurs,} \\ \hat{W}_{k-1} & \text{otherwise.} \end{cases}$$

then

$$\begin{aligned} &\mathbb{E}\left[\hat{W}_{k+1} - \hat{W}_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \\ &\qquad \ge \left(\frac{I_k}{n}\right)^2 - \frac{4B_k}{n} + O(n^{-1}) - n\left[b\left(\frac{k+1}{n}\right) - b\left(\frac{k}{n}\right)\right] + \left[e^{\frac{7(k+1)}{n}} - e^{\frac{7k}{n}}\right] n^{2/3}. \end{aligned}$$

Since $\frac{I_k}{n} \ge e^{-\frac{2k}{n}} - n^{-\frac{1}{3}}$ and $e^{-\frac{2k}{n}} \le 1$, we have $\left(\frac{I_k}{n}\right)^2 \ge e^{-\frac{4k}{n}} - 2n^{-\frac{1}{3}}$. Also, $-\frac{4B_k}{n} \ge -4b(\frac{k}{n}) - 4e^{\frac{7k}{n}} n^{-\frac{1}{3}}$. Using the same estimates as before for $b(\frac{k+1}{n}) - b(\frac{k}{n})$ and $e^{\frac{7(k+1)}{n}} - e^{\frac{7k}{n}}$, we obtain

$$\begin{aligned} &\mathbb{E}\left[\hat{W}_{k+1} - \hat{W}_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \\ &\qquad \ge\; e^{-\frac{4k}{n}} - 2n^{-\frac{1}{3}} - 4b\left(\frac{k}{n}\right) - 4e^{\frac{7k}{n}} n^{-\frac{1}{3}} + O(n^{-1}) - b'\left(\frac{k}{n}\right) + 7e^{\frac{7k}{n}} n^{-\frac{1}{3}} \\ &\qquad =\; \left(-2 + 3e^{\frac{7k}{n}}\right) n^{-\frac{1}{3}} + O(n^{-1}). \\ &\qquad >\; 0, \end{aligned}$$

so $\hat{W}_k$ is a submartingale. Applying the Hoeffding-Azuma inequality once again we see that

$$\mathbb{P}\left[\exists k \le 1.1n : \hat{W}_k \le 0\right] \le e^{-\Omega(n^{1/3})}. \qquad (3)$$

We have now shown that **whp**, $W_k < 0$, $\hat{W}_k > 0$, and $\left|\frac{1}{n} I_k - e^{-\frac{2k}{n}}\right| \le n^{-1/3}$ for every $k \le 1.1n$. Whenever these all happen, the same induction argument as in the conclusion of the proof of Theorem 2.6 shows that every $\mathcal{E}_k$ necessarily holds as well. In particular,

$$\left|B_k - nb\left(\frac{k}{n}\right)\right| \;\le\; e^{\frac{7k}{n}} n^{\frac{2}{3}} \;<\; e^8 n^{\frac{2}{3}},$$

for all $k \le 1.1n$. This completes the proof for B_{tn}, and the result for R_{tn} follows by symmetry. □

Now that we have control of the vertex partition, we study the evolution of the susceptibility. We have symmetry between blue and red, so it suffices to show that the susceptibility of the blue color class does not "blow up" before $1.06n$ rounds. Let X_k be the sum of the squares of the component sizes in the blue color class after the i-th round. Note that this is precisely n times the susceptibility of the blue color class. In the remainder of this proof, we will show that $\frac{1}{n} X_{tn}$ tracks $x(t)$, which is the solution of the differential equation

$$\frac{dx}{dt} \;=\; x^2 + 3b^2 - 2bx, \qquad x(0) \;=\; 1, \qquad (4)$$

where $b(t) = te^{-4t}$. (The precise form of the differential equation will be derived in what follows.) Numerical methods confirm that this differential equation "blows up" only at $t \approx 1.065$, and $x(t) \le 209$ for all $t \le 1.06$.

Lemma 4.5. *Suppose that $\frac{1}{n}X_k < 210$. Then the expected change in X_k is:*

$$\mathbb{E}\left[X_{k+1} - X_k \mid e_1, \dots, e_k; \tfrac{1}{n}X_k < 210\right] \;=\; \left(\frac{X_k}{n}\right)^2 + \frac{1}{n^2}\left[4B_k^2 - 4B_kX_k - R_k^2 + 2R_kX_k\right] + O(n^{-1}).$$

Proof. Let the connected components in the blue color class be $C_1, C_2, \dots$. Suppose that the $(k+1)$-st edge has endpoints in C_i, C_j. If $i = j$, or if the edge is colored red, then the sum of the squares of the blue components does not change. Otherwise, it increases by exactly $(|C_i|+|C_j|)^2 - |C_i|^2 - |C_j|^2 = 2|C_i||C_j|$. Therefore,

$$\mathbb{E}\left[X_{k+1} - X_k \mid e_1, \dots, e_k; \tfrac{1}{n}X_k < 210\right] \;=\; \sum_{i \neq j} 2|C_i||C_j| \cdot \frac{|C_i|}{n}\frac{|C_j|}{n-1} \cdot p_{ij}$$

where p_{ij} is the probability that an edge with endpoints in C_i and C_j is colored blue. Note that p_{ij} is usually $\frac{1}{2}$, but is sometimes 0 or 1 when the endpoints hit isolated edges. The factor of $n-1$ in the denominator is cumbersome, so we will replace it with an n. To do this, note that $\sum_{i \neq j} 2|C_i|^2|C_j|^2 \cdot p_{ij} \leq 2(\sum_i |C_i|^2)^2 = 2X_k^2 \leq 2(210n)^2 = O(n^2)$. Since $\frac{1}{n(n-1)} - \frac{1}{n^2} = O(n^{-3})$, the total additive error we will make by replacing the $n-1$ with an n is $O(n^{-1})$. Therefore,

$$\mathbb{E}\left[X_{k+1} - X_k \mid e_1, \dots, e_k; \tfrac{1}{n}X_k < 210\right] \;=\; \frac{2}{n^2}\sum_{i \neq j} |C_i|^2|C_j|^2 \cdot p_{ij} + O(n^{-1}).$$

Let S be the right hand side of this equality, and let S' be what it would be if all p_{ij} were equal to $\frac{1}{2}$. Then

$$S' \;=\; \frac{1}{n^2}\sum_{i \neq j} |C_i|^2|C_j|^2 + O(n^{-1}) \;\leq\; \left(\frac{X_k}{n}\right)^2 + O(n^{-1}). \tag{5}$$

Now we estimate the total error we made in S' by replacing all p_{ij} with $\frac{1}{2}$. Whenever $p_{ij} = 0$, we overestimated by $\frac{1}{n^2}|C_i|^2|C_j|^2$, and when $p_{ij} = 1$, we underestimated by that same amount. To systematically examine all of the cases when $p_{ij} \neq \frac{1}{2}$, we classify the components C_i of the blue color class into *types*, which we represent with the letters B, R, I, and J. We say that C_i has type B if it is part of the blue matching (hence a single edge), type R if it is part of the red matching (hence a single vertex), type I if it is an isolated vertex, and type J otherwise. Now we break into cases depending on the types of C_i and C_j. In each case, we calculate the sum of all $|C_i|^2|C_j|^2$ of that type.

Case BB. In this case, both C_i and C_j have type B, meaning that they are isolated edges from the blue matching. If the incoming edge has one endpoint in C_i and one endpoint in C_j, our algorithm will definitely color it red, so $p_{ij} = 0$. Any $|C_i|^2|C_j|^2$ of this type is precisely $2^2 \cdot 2^2 = 16$. The number of C_i of type B is $\frac{B_k}{2}$, because the blue matching consists of $\frac{B_k}{2}$ isolated blue edges. So, the number of pairs C_i, C_j of type BB with $i \neq j$ is $\frac{B_k}{2} \cdot \left(\frac{B_k}{2} - 1\right) = \frac{B_k^2}{4} - O(n)$. Therefore, the sum of all $|C_i|^2|C_j|^2$ of this type is $4B_k^2 - O(n)$.

Cases BI, IB. Again $p_{ij} = 0$. Any $|C_i|^2|C_j|^2$ of this type is precisely $2^2 \cdot 1^2 = 4$. There are $\frac{B_k}{2} \cdot I_k$ pairs C_i, C_j of type BI, and the same number of type IB, so the sum is $4B_kI_k$.

Cases BJ, JB. Again $p_{ij} = 0$. Let Z be the set of indices j such that C_j has type J. Since there are $\frac{B_k}{2}$ components C_i of type B, the sum of $|C_i|^2|C_j|^2$ over all pairs of type BJ alone is

$$\begin{aligned}\frac{B_k}{2}\sum_{j\in Z} 2^2 \cdot |C_j|^2 &= 2B_k \sum_{j\in Z} |C_j|^2 \\ &= 2B_k(X_k - \sum_{j\notin Z} |C_j|^2) \\ &= 2B_k\left(X_k - I_k - R_k - \frac{B_k}{2}\cdot 2^2\right) \\ &= 2B_k(X_k - I_k - R_k - 2B_k).\end{aligned}$$

The explanation is as follows. X_k is the sum of all $|C_j|^2$. Then, we break the sum over $j \notin Z$ of $|C_j|^2$ into the cases when C_j has type I, R, or B, in which $|C_j|$ is always 1, 1, and 2, respectively.

The total contribution from pairs of type BJ and JB is twice that from BJ alone, so it is $4B_k(X_k - I_k - R_k - 2B_k)$.

Case RR. Now $p_{ij} = 1$. Any $|C_i|^2|C_j|^2$ of this type is precisely $1^2 \cdot 1^2 = 1$. The number of C_i of type R is R_k, because the red matching consists of $\frac{R_k}{2}$ isolated red edges, which give R_k isolated vertices in the blue color class. So, the number of pairs C_i, C_j of type RR with $i \neq j$ is $R_k \cdot (R_k - 1) = R_k^2 - O(n)$. Thus the sum of $|C_i|^2|C_j|^2$ is $R_k^2 - O(n)$.

Cases RI, IR. Again $p_{ij} = 1$. Any $|C_i|^2|C_j|^2$ of this type is precisely $1^2 \cdot 1^2 = 1$. There are $R_k \cdot I_k$ pairs C_i, C_j of type RI, and the same number of type IR, so the sum is $2R_kI_k$.

Cases RJ, JR. Again $p_{ij} = 1$. Let Z be the set of indices j such that C_j has type J. Since there are R_k components C_i of type R, the sum of $|C_i|^2|C_j|^2$ over all pairs of type RJ is

$$R_k \sum_{j\in Z} 1^2 \cdot |C_j|^2 = R_k(X_k - I_k - R_k - 2B_k),$$

where we used the exact same calculation as in the case BJ for $\sum_{j\in Z} |C_j|^2$. We double this to include the contribution from JR, and obtain a total sum of $2R_k(X_k - I_k - R_k - 2B_k)$.

All other cases. For all other pairs of types, our algorithm chooses a random color, so $p_{ij} = \frac{1}{2}$, and there is no difference between S and S'.

Combining all of the above calculations, we express $\mathbb{E}\left[X_{k+1} - X_k \mid e_1, \ldots, e_k; \frac{1}{n}X_k < 210\right] = S$ in terms of $S' \leq \left(\frac{X_k}{n}\right)^2 + O(n^{-1})$.

$$\begin{aligned}S &= S' - \tfrac{1}{n^2}\left[(4B_k^2 - O(n)) + 4B_kI_k + 4B_k(X_k - I_k - R_k - 2B_k)\right] \\ &\qquad + \tfrac{1}{n^2}\left[(R_k^2 - O(n)) + 2R_kI_k + 2R_k(X_k - I_k - R_k - 2B_k)\right]. \\ &= S' + \frac{1}{n^2}\left[4B_k^2 - 4B_kX_k - R_k^2 + 2R_kX_k\right] + O(n^{-1}) \\ &\leq \left(\frac{X_k}{n}\right)^2 + \frac{1}{n^2}\left[4B_k^2 - 4B_kX_k - R_k^2 + 2R_kX_k\right] + O(n^{-1}),\end{aligned}$$

as desired. □

By Lemma 4.4, $\frac{1}{n}B_k$ and $\frac{1}{n}R_k$ track $b(t) = te^{-4t}$, so Lemma 4.5 indeed indicates that the differential equation (4) estimates $\frac{1}{n}X_{tn}$. We now prove this formally. Our method uses Hoeffding-Azuma, so we need bounded differences. In our proof of Theorem 2.6, we achieved this by controlling the distribution of the component sizes with the result of Spencer and Wormald (Fact 2.5).

Recall that a graph has a K, c component tail if for all positive integers s, at most Ke^{-cs}-fraction of vertices lie in components of order at least s. In particular, the empty graph has a K, c component tail with $K = e$ and $c = 1$. Fact 2.5 then ensures that after a period of random edge addition, the resulting graph still has a K', c' component tail. However, the period only lasts for about $0.5n$ edges when starting with the empty graph, and our process needs to run for $1.06n$ rounds. To work around this issue, we use several iterations.

Define the sequence $t_0, \ldots, t_{19}$, by letting $t_0 = 0$, and $t_{i+1} = t_i + \frac{1}{4x(t_i)}$, where $x(t)$ is the solution of the differential equation (4). The motivation for this sequence is as follows. Suppose we have already established that the blue graph after $t_i n$ rounds has a K_i, c_i component tail, and its susceptibility L is approximately $x(t_i)$, specifically, that $L < 1.5x(t_i)$. Then, we could apply Fact 2.5 with $L = 1.5x(t_i)$, $K = K_i$, $c = c_i$, and $\gamma = \frac{1}{4}$, to conclude that after $t_{i+1}n$ rounds, even if all new edges were colored blue, the blue graph would still have a K_{i+1}, c_{i+1} component tail **whp**. This allows us to define sequences $K_0 \le \cdots \le K_{19} = K'$ and $c_1 \ge \cdots \ge c_{19} = c'$. We confirmed numerically that $t_{19} > 1.06$, so this would allow us to maintain a K', c' component tail for $1.06n$ rounds. Now we formalize this heuristic, and prove our two-color avoidance theorem.

Proof of Theorem 1.3. For each $0 \le k \le 1.06n$, let $\mathcal{E}_k$ be the event that all of the following hold:

$$\mathcal{E}_k \;=\; \begin{cases} \left|\frac{1}{n}B_k - b\left(\frac{k}{n}\right)\right| \;\le\; e^8 n^{-\frac{1}{3}}, \\ \left|\frac{1}{n}R_k - b\left(\frac{k}{n}\right)\right| \;\le\; e^8 n^{-\frac{1}{3}}, \\ \frac{1}{n}X_k \;\le\; x\left(\frac{k}{n}\right) + e^{\frac{500k}{n}} n^{-\frac{1}{4}}, \\ \text{and the blue graph has a } K', c' \text{ component tail.} \end{cases}$$

We define a supermartingale. Let

$$Z_k \;=\; \begin{cases} X_k - nx\left(\frac{k}{n}\right) - e^{\frac{500k}{n}} n^{\frac{3}{4}} & \text{if } \mathcal{E}_{k-1} \text{ occurs,} \\ Z_{k-1} & \text{otherwise.} \end{cases}$$

We only consider $k \le 1.06n$, and $x(t) \le 209$ for all $t \le 1.06$, so if $\mathcal{E}_k$ holds, we have $\frac{1}{n}X_k < 210$. Then Lemma 4.5 gives

$$\begin{aligned} &\mathbb{E}\left[Z_{k+1} - Z_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \\ &\quad\le\; \left(\frac{X_k}{n}\right)^2 + \frac{1}{n^2}\left[4B_k^2 - 4B_k X_k - R_k^2 + 2R_k X_k\right] + O(n^{-1}) \\ &\qquad\qquad - n\left[x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right)\right] - \left[e^{\frac{500(k+1)}{n}} - e^{\frac{500k}{n}}\right] n^{\frac{3}{4}}. \end{aligned}$$

Now we estimate each term. Since $\frac{X_k}{n} \le x\left(\frac{k}{n}\right) + e^{\frac{500k}{n}} n^{-\frac{1}{4}}$ and $k \le 1.06n$, we have $\left(\frac{X_k}{n}\right)^2 \le x^2\left(\frac{k}{n}\right) + 2x\left(\frac{k}{n}\right)e^{\frac{500k}{n}} n^{-\frac{1}{4}} + O(n^{-\frac{1}{2}})$. Similarly, $\frac{B_k^2}{n^2} = b^2\left(\frac{k}{n}\right) + O(n^{-\frac{1}{3}})$, and the same estimate holds for $\frac{R_k^2}{n^2}$. Also, $\frac{1}{n}(2R_k - 4B_k) = -2b\left(\frac{k}{n}\right) + O(n^{-\frac{1}{3}})$, so

$$\frac{1}{n}(2R_k - 4B_k) \cdot \frac{X_k}{n} \;\le\; -2b\left(\frac{k}{n}\right)\left[x\left(\frac{k}{n}\right) - e^{\frac{500k}{n}} n^{-\frac{1}{4}}\right] + O(n^{-\frac{1}{3}}).$$

From Taylor bounds similar to those in the proof of Lemma 4.4, we have $x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right) = \frac{1}{n}x'\left(\frac{k}{n}\right) + O(n^{-2})$ and $e^{\frac{500(k+1)}{n}} - e^{\frac{500k}{n}} = \frac{500}{n}e^{\frac{500k}{n}} + O(n^{-2})$. Combining all of these bounds, and using $x' = x^2 + 3b^2 - 2bx$, the entire estimate simplifies to

$$\mathbb{E}\left[Z_{k+1} - Z_k \mid e_1, \ldots, e_k, \mathcal{E}_k\right] \;\le\; \left[2x\left(\frac{k}{n}\right) + 2b\left(\frac{k}{n}\right) - 500\right] e^{\frac{500k}{n}} n^{-\frac{1}{4}} + O(n^{-\frac{1}{3}}),$$

which is indeed less than zero for large n because $b(t) = te^{-4t}$ is always less than 1, and $x(t) \le 209$ for all $t \le 1.06$. Therefore $Z_0, \ldots, Z_{1.06n}$ is a supermartingale. Note that $Z_0 = -n^{\frac{3}{4}}$. Now because we are dealing with a graph with a K', c' tail, just as in the proof of Theorem 2.6 we have $|Z_{k+1} - Z_k| = O(\log^2 n)$ and then the Hoeffding-Azuma inequality implies that for each $k \le 1.06n$,

$$\mathbb{P}\left[Z_k \ge 0\right] \le e^{-\Omega(n^{1/2}/\log^4 n)}.$$

Therefore, by a union bound, **whp** $Z_k < 0$ for all $k \le 1.06n$. Also, Lemma 4.4 implies that **whp**, $\left|\frac{B_k}{n} - b\left(\frac{k}{n}\right)\right| \le e^8 n^{-\frac{1}{3}}$ and $\left|\frac{R_k}{n} - b\left(\frac{k}{n}\right)\right| \le e^8 n^{-\frac{1}{3}}$ for every $k \le 1.06n$. Let $\mathcal{E}$ be the conjunction of all of these high-probability events.

To complete our argument, we show by induction that **whp**, for each $0 \le i \le 19$, the blue graph after $t_i n$ rounds has a K_i, c_i component tail. The base case $i = 0$ is trivial. For the induction step, suppose that it is true for i. Condition on the blue graph after $t_i n$ rounds having a K_i, c_i component tail, as well as on the event $\mathcal{E}$ that all $Z_k < 0$ and all B_k, R_k are concentrated. Then, the same argument as in the conclusion of the proof of Theorem 2.6 forces all $\mathcal{E}_k$ to occur for $k \le t_i n$, since $K_i \le K'$ and $c_i \ge c'$. In particular, $\mathcal{E}_{t_i n}$ already implies that after $t_i n$ rounds, the blue graph has susceptibility $\frac{1}{n}X_{t_i n} \le x(t_i) + o(1) < 1.5x(t_i)$. Applying Fact 2.5 with $L = 1.5x(t_i)$, $K = K_i$, $c = c_i$, and $\gamma = \frac{1}{4}$, we see that **whp**, even if all new edges were colored blue, the blue graph after $t_i n + \left(1 - \frac{1}{4}\right)\frac{n}{2\cdot 1.5x(t_i)} = t_{i+1}n$ rounds would have a K_{i+1}, c_{i+1} component tail. This finishes the induction, so **whp** the blue graph after $t_{19}n > 1.06n$ rounds has a K', c' component tail. In particular, all connected components are of order $O(\log n)$, so there is no giant in the blue color class. The same result follows for the red color class by symmetry. □

5 Online creation of giants

Recall that the trivial bounds for the online creation of giants are as follows. No algorithm can create giants in all colors in fewer than $(1 - \epsilon)\frac{n}{2}$ total edges, because that is not even enough to make a giant in the uncolored graph. On the other hand, if one randomly colors each incoming edge, then monochromatic giants will appear after $(r + \epsilon)\frac{n}{2}$ total edges. In this section, we prove Theorems 1.4, 1.5, and 1.6, which improve the above trivial lower and upper bounds for the online creation of giants.

5.1 Lower bound

The previous argument iterated Fact 2.5 to maintain the component tail property, using a customized argument to control the susceptibility for a specific algorithm. In this section, we need to consider an arbitrary coloring strategy, so we use our general-purpose tool (Theorem 2.6) to control the susceptibility. This will establish a lower bound of $\Omega(n \log r)$ for the number of edges required to create giants online in each of r color classes. We need the following simple bound for random graphs.

Lemma 5.1. *Let λ be a constant. The random graph $G_{n,p}$ with $p = \frac{\lambda}{n}$ contains at most $o\big(\frac{n}{\log n}\big)$ cycles of length at most $\sqrt{\log n}$,* ***whp****.*

Proof. The expected number of cycles of length k in $G_{n,p}$ is at most $\frac{n^k}{2k}p^k = \frac{\lambda^k}{2k}$, so the expected number of cycles of length at most $\sqrt{\log n}$ is below $\sum_{k=3}^{\sqrt{\log n}} \frac{\lambda^k}{2k}$. If $\lambda \leq 1$, this is below $\sqrt{\log n}$. Otherwise, it is below $\sqrt{\log n} \cdot \lambda^{\sqrt{\log n}}$. In both cases, the conclusion follows from Markov's inequality. □

Next, we need a worst-case bound on how large the susceptibilities of different color classes can be when a graph is colored.

Lemma 5.2. *Let K, c be positive real constants. Let G be an n-vertex graph with a K, c component tail. Also assume that G contains $o\big(\frac{n}{\log n}\big)$ cycles of length at most $\sqrt{\log n}$. Consider any 2-coloring of the edges of G, and let $G^{(1)}$ and $G^{(2)}$ be the n-vertex subgraphs of G obtained by keeping only edges in the first or second color, respectively. Then $S(G^{(1)}) + S(G^{(2)}) \leq S(G) + 1 + o(1)$.*

Proof. Each component of $G^{(i)}$ is entirely contained within a component of G, so we may break down the left hand side by components of G. Consider first the components of G which are larger than $\sqrt{\log n}$. Since G has a K, c component tail, the number of vertices in such components is at most $Ke^{-c\sqrt{\log n}}n$. The component tail also implies that there is some constant C such that all components of G are bounded by $C \log n$. Since $S(G^{(1)}) + S(G^{(2)}) = \frac{1}{n}\sum_v (C_v^{(1)} + C_v^{(2)})$, where $C_v^{(i)}$ is number of vertices in the component of $G^{(i)}$ containing v, the total contribution from vertices in components of G with order at least $\sqrt{\log n}$ is only $\frac{1}{n} \cdot Ke^{-c\sqrt{\log n}}n \cdot 2C \log n = o(1)$.

Next, consider the components of order at most $\sqrt{\log n}$ which contain cycles. Since the susceptibility is $\frac{1}{n}$ times the sum of squares of component sizes, each component of this type contributes at most $\frac{1}{n} \cdot 2(\sqrt{\log n})^2$ to $S(G^{(1)}) + S(G^{(2)})$. By assumption, G only has $o\big(\frac{n}{\log n}\big)$ cycles small enough to fit into these components, so the number of such components is at most $o\big(\frac{n}{\log n}\big)$. Therefore, their total contribution to $S(G^{(1)}) + S(G^{(2)})$ is at most $\frac{1}{n} \cdot 2(\sqrt{\log n})^2 \cdot o\big(\frac{n}{\log n}\big) = o(1)$.

The main contribution comes from the remaining components, which are all trees. Any tree T in G contributes $\frac{1}{n}\sum_{v\in T} |T|$ to $S(G)$. We claim that it contributes at most $\frac{1}{n}\sum_{v\in T}(|T| + 1)$ to $S(G^{(1)}) + S(G^{(2)})$, i.e., the additional amount is at most $\frac{1}{n}|T|$. Indeed, T's contribution to $S(G^{(i)})$ is precisely $\frac{1}{n}$ times the sum of the sizes of the $G^{(i)}$-components that contain each vertex $v \in T$. Trees have the property that each pair of vertices is connected by a unique path, so we can express the size of the $G^{(i)}$-component containing v as $\sum_{w\in T} I_{v,w}^{(i)}$, where the indicator $I_{v,w}^{(i)}$ is 1 if the unique path between v and w is monochromatic in color i, and 0 otherwise. Hence, the total contribution of T to $S(G^{(1)}) + S(G^{(2)})$ is $\frac{1}{n}\sum_{v,w\in T}(I_{v,w}^{(1)} + I_{v,w}^{(2)})$. Since T is a tree, the only time both indicators $I_{v,w}^{(i)}$ can be 1 is when $w = v$. So for each v, the sum $\sum_{w\in T}(I_{v,w}^{(1)} + I_{v,w}^{(2)})$ is at most $|T| + 1$, as claimed. Summing over all tree components, we see that their total contribution to $S(G^{(1)}) + S(G^{(2)})$ exceeds $S(G)$ by at most $\frac{1}{n}$ times the sum of the sizes of tree components, which is at most 1. Combining this with the contributions from non-tree components above, we obtain $S(G^{(1)}) + S(G^{(2)}) \leq S(G) + 1 + o(1)$, as desired. □

Now we proceed to prove Theorem 1.4, using the previous two lemmas, and Theorem 2.6 to control the evolution of susceptibility. We will show that for any r which is a power of two, **whp** no online algorithm can create giants in all r colors within $(c \log_2 r)n$ edges, where $c \approx 0.043$. This clearly implies the desired asymptotic bound. Our calculated bound for $r = 2$ will follow as a special case.

Proof of Theorem 1.4. Let C_0 be the set of all $r = 2^t$ colors. Let γ be a constant parameter which we will specify later. The graph is initially empty, with susceptibility $L_0 = 1$. By Theorem 2.6, after $(1-\gamma)\frac{n}{2}L_0^{-1}$ edges, the graph formed by the union of those edges has a K_1, c_1 component tail and susceptibility at most $\frac{L_0}{\gamma} + o(1)$ **whp**. Arbitrarily divide the colors into two groups of size 2^{t-1} each. Lemmas 5.1 and 5.2 ensure that no matter how the edges were colored, one of the two color groups determines a graph G_1 with susceptibility at most $L_1 + o(1)$, where $L_1 = \frac{1}{2}\left(1 + \frac{L_0}{\gamma}\right)$. Note that G_1 still has a K_1, c_1-component tail, and let C_1 be the set of 2^{t-1} colors we picked.

We iterate this procedure a total of t times. For example, in the next step, we advance by $(1-\gamma)\frac{n}{2}L_1^{-1}$ more edges. Even if all of them received colors in C_1 (i.e., were added to G_1), the susceptibility of the graph determined by C_1-colors is at most $\frac{L_1}{\gamma} + o(1)$ **whp**, by Theorem 2.6. Arbitrarily divide the colors of C_1 into two groups of size 2^{t-2} each. Again by Lemmas 5.1 and 5.2, one of the two color groups, say C_2, determines a graph G_2 with susceptibility at most $L_2 + o(1)$, where $L_2 = \frac{1}{2}\left(1 + \frac{L_1}{\gamma}\right)$.

After t iterations, we conclude that there is some single color c such that the graph G_t determined by all edges of color c has a K_t, c_t component tail and susceptibility at most L_t. A final application of Theorem 2.6 implies that we can add $\frac{n}{2}(L_t^{-1} - \epsilon)$ more random edges and still have all components in color c of order $O(\log n)$ **whp**.

It remains to count the total number of edges which we have accumulated. The relationship between the L_i's is $L_{i+1} = \frac{1}{2}\left(1 + \frac{L_i}{\gamma}\right) = \frac{1}{2} + \frac{L_i}{2\gamma}$, so

$$\begin{aligned}
L_0 &= 1, \\
L_1 &= \frac{1}{2} + \frac{1}{2\gamma}, \\
L_2 &= \frac{1}{2} + \frac{1}{4\gamma} + \frac{1}{4\gamma^2}, \\
L_3 &= \frac{1}{2} + \frac{1}{4\gamma} + \frac{1}{8\gamma^2} + \frac{1}{8\gamma^3},
\end{aligned}$$

and in general,

$$\begin{aligned}
L_t &= \frac{1}{2} + \frac{1}{2(2\gamma)} + \frac{1}{2(2\gamma)^2} + \cdots + \frac{1}{2(2\gamma)^{t-1}} + \frac{1}{(2\gamma)^t} \\
&< 1 + \frac{1}{2\gamma} + \cdots + \frac{1}{(2\gamma)^t} \\
&< \left(1 - \frac{1}{2\gamma}\right)^{-1}.
\end{aligned}$$

Thus, the total number of edges added (not even counting the final step) is at least

$$(1-\gamma)\frac{n}{2}\sum_{i=0}^{t-1} L_i^{-1} \;>\; (1-\gamma)\frac{n}{2} \cdot t\left(1 - \frac{1}{2\gamma}\right).$$

By routine calculus, the optimal choice for γ is $\frac{1}{\sqrt{2}}$, giving $(1-\gamma)\left(1 - \frac{1}{2\gamma}\right) = \frac{3}{2} - \sqrt{2} \approx 0.086$. Since $t = \log_2 r$, we indeed see that **whp**, no online algorithm can create giants in all colors within $0.043n\log_2 r$ edges. This completes the proof of the asymptotic bound.

For the specific case of $r = 2$ colors, we can add the final batch of $\frac{n}{2}(L_t^{-1} - \epsilon)$ random edges (here $t = 1$) to get a specific bound which beats the trivial bound of $n/2$ edges. Since $L_1 = \frac{1}{2}(1 + \frac{1}{\gamma})$, this gives a total edge count of

$$\begin{aligned}(1-\gamma)\frac{n}{2} + \frac{n}{2}(L_1^{-1} - \epsilon) &= \frac{n}{2}\left[(1-\gamma) + \left(\frac{1}{2}\left(1 + \frac{1}{\gamma}\right)\right)^{-1} - \epsilon\right] \\ &= \frac{n}{2}\left[(1-\gamma) + \frac{2\gamma}{\gamma+1} - \epsilon\right].\end{aligned}$$

By routine calculus, the optimal choice for γ is $\sqrt{2} - 1$. Therefore, **whp**, no online algorithm can create giants in both colors within $(2 - \sqrt{2} - \epsilon)n$ edges, as claimed. □

5.2 Upper bound for many colors

In this section, we present an online coloring algorithm which creates giants in all r color classes within roughly $\frac{n}{2}\sqrt{r}$ edges. The strategy is based on the classical fact that there are infinitely many values of r such that the edges of K_r can be perfectly partitioned into cliques of order roughly $\sqrt{r}$.

Fact 5.3. *Let $r = q^2 + q + 1$ for some prime power q. The edges of K_r can be partitioned into disjoint sets $E_1, \ldots, E_r$ such that each E_i is precisely the edge set of some clique of order $q + 1$.*

Proof. The projective plane of order $r = q^2 + q + 1$ is the finite geometry where points and lines correspond to dimension-1 and dimension-2 subspaces of $\mathbb{F}_q^3$, respectively. This object contains exactly $\frac{q^3-1}{q-1} = q^2 + q + 1$ points and the same number of lines, and has the property that every pair of distinct points determines a unique line.

Identify the vertices of K_r with the points of the projective plane. Let the $q + 1$ vertices of the clique corresponding to E_i be the points contained in the i-th line of the projective plane. The edge partition property is then equivalent to the incidence property of the projective plane. □

We also need the giant component threshold in certain inhomogeneous random graph models, where the edge probability is not uniformly p at all $\binom{n}{2}$ possible sites. Instead, the probability of each edge depends on the locations of its endpoints. Bollobás, Janson, and Riordan recently completed a far-reaching study of phase transitions in these types of inhomogeneous models in [10]. We use a special case of their work, regarding the specific model below.

Fix a symmetric $k \times k$ matrix $A = (a_{ij})$. Let $G_{n,A}$ be the n-vertex random graph defined as follows. Split the n vertices into k groups of size n/k. Between each pair of distinct vertices, say from the i-th and j-th groups (where i may equal j), place an independent random edge with probability $\frac{a_{ij}}{n}$. Note that when $A = cJ_k$, where J_k is the $k \times k$ all-ones matrix, $G_{n,A}$ is the Erdős-Rényi random graph $G_{n,p}$ with $p = \frac{c}{n}$.

The following result was proved as Theorem 3.1 of [10]. Here, the L_2 operator norm $\|B\|_2$ of a $k \times k$ matrix B is $\sup\{\|Bx\|_2 : \|x\|_2 = 1\}$, and the 2-norm of a vector $(x_1, \ldots, x_k)$ is $\sqrt{\sum x_i^2}$.

Fact 5.4. *Let $A = (a_{ij})$ be a symmetric $k \times k$ matrix, and let $\overline{A}$ be its normalization $\left(\frac{a_{ij}}{k}\right)$. If $\|\overline{A}\|_2 > 1$, then $G_{n,A}$ contains a giant component **whp**.*

Remark 1. In the same theorem, Bollobás, Janson, and Riordan also proved the complementary result that when $\|\overline{A}\|_2 \leq 1$, the largest component of $G_{n,A}$ is $o(n)$ **whp**. However, we do not need this part for our analysis.

Remark 2. The L_2 operator norm of a real symmetric matrix A always equals its *spectral radius* $\rho(A)$, which is the maximum $|\lambda_i|$ over all eigenvalues λ_i. Indeed, A is diagonalizable with an orthonormal basis of real eigenvectors, so let the eigenvalues and eigenvectors be $\lambda_1, \ldots, \lambda_k$ and $v_1, \ldots, v_k$, respectively. Expressing any vector x in this basis as $\sum c_i v_i$, we have that the condition $\|x\|_2 = 1$ is precisely $\sum c_i^2 = 1$, and $\|Ax\|_2 = \sqrt{\sum \lambda_i^2 c_i^2}$. Therefore, $\|Ax\|_2$ has maximum value equal to the largest absolute value of an eigenvalue.

Remark 3. As we mentioned above, the Erdős-Rényi model $G_{n,p}$ with $p = \frac{c}{n}$ corresponds to $G_{n,A}$ with $A = cJ_k$. The normalized matrix $\overline{A} = \frac{c}{k} J_k$ has eigenvalues c and 0, so Fact 5.4 implies the classical result that the giant component appears after $p = \frac{1}{n}$.

We use this to study the k-partite random graph $G_{n,p}^{(k)}$, which has n vertices split into equal groups of size $\frac{n}{k}$, and independent random edges with probability $p = \frac{c}{n}$ between pairs of vertices from distinct groups. In the above framework, this is $G_{n,A}$ with $A = c(J_k - I_k)$.

Corollary 5.5. *Let $k \geq 2$ be a positive integer, and let $c > \frac{k}{k-1}$ be a real number. Then the k-partite random graph $G_{n,p}^{(k)}$ with $p = \frac{c}{n}$ contains a giant component **whp**.*

Proof. By Fact 5.4 and our second remark, the problem reduces to determining the eigenvalues of $\overline{A} = \frac{c}{k}(J_k - I_k)$. These are precisely $\frac{c}{k}(k-1)$ and $\frac{c}{k}(0-1)$, so since $k \geq 2$, the giant component appears once $c > \frac{k}{k-1}$. □

We are now ready to state our algorithm and prove its effectiveness. Note that a coloring algorithm that produces giants in r colors trivially gives coloring algorithms for any $r' < r$ as well, simply by using the first color whenever any color beyond r' was to be used. So, Theorem 1.5 is a consequence of the following more precise formulation, combined with the Prime Number Theorem and Fact 2.3.

Theorem. *Let $r = q^2 + q + 1$ for some prime power q. There is an online algorithm such that for any $\epsilon > 0$, **whp** all r color classes contain giant components within $\left(\frac{r}{q} + \epsilon\right)\frac{n}{2}$ edges.*

Proof. Arbitrarily partition the n vertices into r sets $V_1, \ldots, V_r$, each of size $\frac{n}{r}$. By Fact 5.3, there is a partition $E_1 \cup \ldots \cup E_r$ of the edges of K_r, such that each E_t is precisely the edge set of some clique of order $q + 1$. Our online coloring algorithm is then as follows. Usually, the incoming edge will have endpoints in distinct parts V_i and V_j. In that case, color the edge with the index t of the E_t which contains the edge ij in the partitioned graph K_r. Otherwise, if the incoming edge is spanned by a single V_i, then discard the edge entirely. Note that this is even stronger than coloring it, because we will now find giants without using those edges at all.

Our algorithm disregards the entire history of the process, since the color of each edge is a function of the locations of its endpoints. In particular, the order of the edges is irrelevant, so the performance only depends on the final edge set. Thus, by Fact 2.3, it suffices to show that if this strategy is applied to $G_{n,p}$ with $p = \left(\frac{r}{q} + \frac{\epsilon}{2}\right)\frac{1}{n}$, then it creates giants in all colors **whp**. By passing to this independent model, each color class itself becomes a $(q+1)$-partite random graph $G_{n',p}^{(q+1)}$, on only $n' = \frac{n}{r}(q+1) \approx \frac{n}{\sqrt{r}}$ vertices. Indeed, E_t is the edge set of a clique on some set S of $q + 1$ vertices of K_r, so the edges that receive color t are precisely those with endpoints in some V_i and V_j with $i \neq j$ and $i, j \in S$.

Finally, we can apply Corollary 5.5 with $k = q+1$, since $p = \frac{c'}{n'}$ with $c' = \left(\frac{r}{q} + \frac{\epsilon}{2}\right)\frac{1}{n} \cdot \frac{n}{r}(q+1) > \frac{q+1}{q}$. Therefore, each individual color class contains a giant component **whp**. Taking a union bound over all r (finitely many) color classes finishes the proof. □

5.3 Upper bound for 2 colors

To adapt our strategy from the previous section to the case $r = 2$, we must specify symmetric 0-1 matrices A_1 and A_2 which sum to the $k \times k$ all-ones matrix J_k. We then split the vertices into k equal parts $V_1, \dots, V_k$, and color an edge with endpoints in some V_i, V_j with color 1 if the ij-entry of A_1 is 1, and color 2 otherwise.

Then, after applying this strategy to the edges of $G_{n,p}$ with $p = \frac{c}{n}$, the i-th color class is a copy of G_{n,cA_i}. By the second remark after Fact 5.4, this contains a giant component when the spectral radius $\rho(\frac{c}{k}A_i)$ exceeds 1. Since our objective is to create giants in both colors as rapidly as possible, we want to select A_1 and A_2 such that $A_1 + A_2 = J_k$, but $\min\{\rho(A_1), \rho(A_2)\}$ is as large as possible. This appears to be a nontrivial problem, but one simple way to choose the matrices is to let A_1 have 1's in the top-left $t \times t$ submatrix, and 0's everywhere else. This leads to the following bound.

Proposition 5.6. *For every $\epsilon > 0$, it is possible to create giants in two colors online within $\left(\frac{3}{4} + \epsilon\right)n$ rounds **whp**.*

Proof sketch. Since A_1 is just J_t embedded in an all-zeros matrix, its spectral radius is precisely t. Next, note that $A_2 = J_k - A_1$ has rank 2, so it has at most 2 nonzero eigenvalues λ_1, λ_2. The trace of A_2 is $k - t$, so $\lambda_1 + \lambda_2 = k - t$. Also, the main diagonal of A_2^2 has its first t entries equal to $k - t$, and the remaining $k - t$ entries equal to k, giving $\text{tr}(A_2^2) = t(k - t) + (k - t)k = k^2 - t^2$. This trace also equals $\lambda_1^2 + \lambda_2^2$, because the nonzero eigenvalues of A_2^2 are λ_1^2 and λ_2^2. Solving this system of equations, one finds that the largest eigenvalue of A_2 is $\frac{1}{2}(k - t + \sqrt{k^2 + 2kt - 3t^2})$. Recall that the largest eigenvalue of A_1 is t, and we wanted the largest possible $\min\{\rho(A_1), \rho(A_2)\}$. Routine calculus shows that the optimal choice of t is $\frac{2}{3}k$, giving both $\rho(A_i) = \frac{2}{3}k$. So, we choose the particular 3×3 matrices

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Therefore, as we remarked at the beginning, Fact 5.4 shows that when this strategy is applied to $G_{n,p}$ with $p = \frac{c}{n}$, both colors will contain giant components if their spectral radii $\rho(\frac{c}{k}A_i)$ exceed 1, i.e., once $c > \frac{3}{2}$. By Fact 2.3, this happens after $\left(\frac{3}{2} + \epsilon\right)\frac{n}{2}$ rounds, so we are done. □

Remark. Although the partition we chose may appear naïve, there is evidence to suggest that it may be optimal. Note that if we ignore the main diagonal (an effect that can be made negligible by choosing large k) and seek $A_1 + A_2 = J_k - I_k$, then A_1 and A_2 are the adjacency matrices of a graph and its complement.

Several researchers have studied the question of bounding the sum of the spectral radii of the adjacency matrices of complementary graphs (see [15, 18, 23, 24, 25, 31]). In particular, Nikiforov recently conjectured in [23] that the sum of these two spectral radii is always at most $\frac{4}{3}k + O(1)$, where k is the number of vertices. If true, this would imply that $\min\{\rho(A_1), \rho(A_2)\} \le \frac{2}{3}k + O(1)$, which our construction achieved. In fact, in his extremal example, one graph was a clique on a subset of the vertices, which is essentially the same as our construction. So, perhaps $\frac{3}{4}n$ is the limit of what can be achieved by any strategy as above.

Next, we prove Theorem 1.6, which shows that by making the strategy more adaptive, one can create giants even faster. The algorithm in the proof of Proposition 5.6 fixed a subset R of vertices in advance, and used the first color whenever an edge was spanned by R. The key idea is to let the subset R depend on the outcomes of the first few rounds. To analyze this strategy, we will need two results from the literature. The first is a folklore result on the susceptibility of the Erdős-Rényi random graph.

Fact 5.7. *Let $0 < t < \frac{1}{2}$ be a fixed parameter. Then there exist constants K, c such that* ***whp****, the graph on n vertices formed by tn independent random edges has susceptibility $\frac{1}{1-2t} + o(1)$, and a K, c component tail.*

Justification. This result is well-known. Nevertheless, for completeness, we will show how a formal proof can be derived as a consequence of Theorem 1.1 of Spencer and Wormald in [28]. We do not state their full theorem here, as it is much broader in scope, and hence necessarily more technical. Instead, we provide some pointers for the interested reader to check this conclusion. Page 591 of their paper specifies the *bounded size algorithm* which corresponds to the Erdős-Rényi evolution. In terms of these parameters, their target susceptibility function $S(t)$ for the $tn/2$-edge random graph is the solution of their differential equation (37), where their subscript $\overrightarrow{j}$ only takes the single value $(\omega, \omega, \omega, \omega)$. In their notation, this is simply $S'(t) = I((\omega, \omega, \omega, \omega), t)$. The right hand side evaluates to $S(t)^2$ because all $x_\omega(t) = 1$ (pointed out on page 597) and their equation (6) implies that $S_\omega = S$ for the Erdős-Rényi evolution. The solution of $S'(t) = S(t)^2$ with initial condition $S(0) = 1$ is $S(t) = \frac{1}{1-t}$. This indeed matches Fact 5.7 because Spencer and Wormald parameterize their susceptibility $S(t)$ with respect to the random graph with $tn/2$ edges, whereas we consider tn edges. □

The second result we need is Theorem 3.1 of [28], again translated to account for the fact that their parameterization is for $tn/2$ edges, instead of tn edges.

Fact 5.8. *Let L, K, c, ϵ be positive real numbers. Let G be a graph on n vertices with a K, c component tail and $S(G) = L$. Then, after adding $(1+\epsilon)\frac{n}{2L}$ more independent random edges, the resulting graph contains a giant component* ***whp****.*

Proof of Theorem 1.6. Let the colors be red and blue. We state the coloring strategy in terms of a constant parameter t, which we can optimize at the end. (The best choice turns out to be $t \approx 0.189$.) For the first tn rounds, color all edges red. Then, permanently fix R to be the set of all vertices incident to a red edge at that time. Color each future edge red whenever both endpoints lie in R, and blue otherwise.

Let $\alpha = \frac{|R|}{n}$. Lemma 4.4 shows that $\alpha = (1 - e^{-2t} + o(1))$ **whp**. Let us analyze how many rounds are required for a red giant to appear. By Fact 5.7, the (completely red) graph G at time tn has susceptibility $S(G) = \frac{1}{1-2t} + o(1)$ **whp**, so the sum of the squares of its component sizes is $\left(\frac{1}{1-2t} + o(1)\right)n$. Let G_R be the subgraph of G induced by R. The sum of the squares of the components in G_R is precisely $S(G)n - (1-\alpha)n$, because all components of G outside R are singletons. Therefore, since G_R has αn vertices, its susceptibility L is:

$$L \;=\; \frac{1}{\alpha n}[S(G)n - (1-\alpha)n] \;=\; (1+o(1))\frac{1}{\alpha}\left[\frac{1}{1-2t} - e^{-2t}\right].$$

Then, by Fact 5.8, **whp** the red graph will contain a giant component after $(1+\epsilon)\frac{|R|}{2L}$ more random edges are added with both endpoints in R. By a standard coupling as in Fact 2.3, this happens after $(1+\epsilon)\frac{|R|}{2L} \cdot \alpha^{-2}$ more rounds **whp**, since each incoming edge falls within R with probability α^2. Substituting $|R| = \alpha n$, we find that a red giant appears after a grand total of $tn + (\frac{1}{2\alpha L} + \epsilon)n = \left[t + \frac{1}{2}\left(\frac{1}{1-2t} - e^{-2t}\right)^{-1} + \epsilon\right]n$ rounds **whp**.

To analyze the blue graph, observe that by a similar coupling to Fact 2.3, after $tn+(1+\epsilon)\frac{cn}{2}$ rounds the blue graph contains $G_{n,cA}$ **whp**, where A is the $n \times n$ matrix with 0's in the top-left $|R| \times |R|$ submatrix, and 1's everywhere else. Plugging $|R| = \alpha n$ into the eigenvalue calculation from the proof of Proposition 5.6, we see that the largest eigenvalue of A is $\frac{n}{2}\left(1-\alpha+\sqrt{1+2\alpha-3\alpha^2}\right)$. Thus, Fact 5.4 implies that **whp**, the giant component appears in the blue graph once c surpasses $\frac{2}{1-\alpha+\sqrt{1+2\alpha-3\alpha^2}}+\epsilon$, i.e., when the total number of rounds exceeds $tn + \frac{1+\epsilon}{1-\alpha+\sqrt{1+2\alpha-3\alpha^2}}n$.

Since $\alpha = 1 - e^{-2t} + o(1)$, it is now routine to numerically optimize t. It turns out that the best choice is $t \approx 0.189$, which gives $\alpha \approx 0.314$. Then, both of the bounds at the ends of the previous two paragraphs are satisfied after $0.733n$ rounds, completing the proof. □

6 Concluding remarks

In this paper we have introduced several rather natural algorithmic variants of the classical problem of the appearance of the giant component in a random graph/process. As expected, the offline cases of these problems appear to be much more accessible, and indeed we managed to solve both the avoidance and the embracing versions asymptotically for any fixed r. The online case seems to be more challenging; there we showed that in all cases one can do better than the trivial algorithms that randomly color each incoming edge, but for creating giants, rather sizable gaps remain.

It would certainly be nice to settle the case of two colors for creating and avoiding giants online in both color classes, but that could be difficult. A more approachable problem might be to close the asymptotic gap between the lower bound of $\Omega(\log r) \cdot n$ and the upper bound of $O(\sqrt{r}) \cdot n$ for the question of creating giants in r colors. In particular, can one show a lower bound of the form $r^a n$ for some positive constant a?

Another, perhaps more technical, issue that we would like to see settled is the nature of an algorithm for avoiding giants online. Our online avoidance algorithm is randomized. Is there a deterministic strategy that matches its performance in the online setting?

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