Rainbow Matchings and Hamilton Cycles in Random Graphs

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Abstract

Let $HP_{n,m,k}$ be drawn uniformly from all m-edge, k-uniform, k-partite hypergraphs where each part of the partition is a disjoint copy of $[n]$. We let $HP^{(\kappa)}_{n,m,k}$ be an edge colored version, where we color each edge randomly from one of κ colors. We show that if $\kappa = n$ and $m = Kn \log n$ where K is sufficiently large then w.h.p. there is a rainbow colored perfect matching. I.e. a perfect matching in which every edge has a different color. We also show that if n is even and $m = Kn \log n$ where K is sufficiently large then w.h.p. there is a rainbow colored Hamilton cycle in $G^{(n)}_{n,m}$. Here $G^{(n)}_{n,m}$ denotes a random edge coloring of $G_{n,m}$ with n colors. When n is odd, our proof requires $m = \omega(n \log n)$ for there to be a rainbow Hamilton cycle.

1 Introduction

Given an edge-colored hypergraph, a set S of edges is said to be *rainbow* colored if every edge in S has a different color. In this paper we consider the existence of rainbow perfect matchings in k-uniform, k-partite hypergraphs and Hamilton cycles in randomly colored random graphs.

Let $U_1, U_2, \ldots, U_k$ denote k disjoint sets of size n. Let $\mathcal{HP}^{(\kappa)}_{n,m,k}$ denote the set of k-partite, k-uniform hypergraphs with vertex set $V = U_1 \cup U_2 \cup \cdots \cup U_k$ and m edges, each of which has been randomly colored with one of κ colors. The random edge colored graph $HP^{(\kappa)}_{n,m,k}$ is sampled uniformly from $\mathcal{HP}^{(\kappa)}_{n,m,k}$.

In this paper we prove the following result

Theorem 1.1. *There exists a constant $K = K(k)$ such that if $m \geq Kn \log n$ then*

$$\lim_{n\to\infty} \mathbb{P}\left[HP^{(n)}_{n,m,k} \text{ contains a rainbow perfect matching}\right] = 1.$$

This result is best possible in terms of the number of colors n and best possible up to a constant factor in terms of the number of edges.

We get the corresponding result for k-uniform hypergraphs $H^{(n)}_{kn,m,k}$ for free. Here the edge set of $H^{(n)}_{kn,m,k}$ is a random element of $\binom{\binom{[kn]}{k}}{m}$ and each edge is randomly colored from $[n]$.

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Corollary 1.2. *If $m = Ln\log n$ and L is sufficiently large then w.h.p. $H^{(n)}_{kn,m,k}$ contains a rainbow perfect matching.*

Proof. We simply partition $[kn]$ randomly into k sets of size $[n]$. Then we apply Theorem 1.1 with $K = L/k^k$. $\square$

When $k = 2$ the result of Theorem 1.1 can be expressed as follows:

Corollary 1.3. *Let A be an $n \times n$ matrix consructed as follows: Choose $Knlogn$ entries at random and give each a random integer from $[n]$. The remaining entries can be filled with zeroes. Then w.h.p. A contains a latin transversal i.e. an $n \times n$ matrix B with a single non-zero in each row and column, such that each $x \in [n]$ appears as a non-zero of B.*

We can use Theorem 1.1 and a result of Janson and Wormald [5] to prove the following theorem on rainbow Hamilton cycles: Let $G^{(n)}_{n,m}$ denote a copy of $G_{n,m}$ in which each of the m edges has been randomly colored with one of n colors.

Theorem 1.4. *There exists a constant K such that if $m \geq Kn\log n$ then with high probability,*

$$\lim_{\substack{n\to\infty \\ n even}} \mathbb{P}\left[G^{(n)}_{n,m} \text{ contains a rainbow Hamilton cycle}\right] = 1.$$

When n is odd we replace $m = Kn\log n$, by $m = \omega n \log n$, where $\omega \to \infty$ arbitrarily slowly.

The result for n odd is surely an artifact of the proof and we conjecture the same result is true for n odd or even.

Previous results in this area have concentrated on the existence of rainbow Hamilton cycles. For example, Frieze and Loh [4] showed that $G^{(\kappa)}_{n,m}$ contains a rainbow hamilton cycle w.h.p. whenever $m \sim \frac{1}{2}n\log n$[1] and $\kappa \sim n$. This result is asymptotically optimal in number of edges and colors. Theorem 1.4 resolves a question posed at the end of this paper (up to a constant factor) about the number of edges needed when we have a minimum number of colors available. Perarnau and Serra [7] showed that a random coloring of the complete bipartite graph $K_{n,n}$ with n colors contains a rainbow perfect matching. Erdős and Spencer [2] proved the existence of a rainbow perfect matching in the complete bipartite graph $K_{n,n}$ when no color can be used more than $(n-1)/16$ times.

2 Outline of the paper

The proof of Theorem 1.1 is derived directly from the proof in the landmark paper of Johansson, Kahn and Vu [6]. They prove something more general, but one of their main results concerns the "Schmidt-Shamir" problem, viz. how many random (hyper-)edges are needed for a 3-uniform hypergraph to contain a perfect matching. In this context, a perfect matching of a 3-uniform hypergraph H on n vertices V is a set of $n/3$ edges that together partition V.

[1]We write $A_n \sim B_n$ if $A_n = (1 + o(1))B_n$ as $n \to \infty$

There is a fairly natural relationship between rainbow matchings of k-uniform hypergraphs and perfect matchings of $(k+1)$-uniform hypergraphs. This was already exploited in Frieze [3]. The basic idea is to treat an edge $\{u_1, u_2, \ldots, u_k\}$ of color $c \in C$ as an edge $\{u_1, u_2, \ldots, u_k, c\}$ in a $(k+1)$-uniform hypergraph H with vertices $V \cup C$ and edges in $\binom{V}{k} \times C$. Then, assuming that $|V| = k|C|$ we ask for a perfect matching in H. Here we would take $V = [kn]$ and $|C| = n$ and construct H randomly. The "fly in the ointment" so to speak, is that we cannot have two distinct edges $\{u_1, u_2, \ldots, u_k, c_i\}, i = 1, 2$. This seems like a minor technicality and in some sense it is. We have not been able to find a simple way of completely resolving this technicality, other than modifying the proof in [6].

Remark 2.1. *This technicality does not cause a problem for $k \geq 3$. Consider the corresponding independent model in which each possible edge is included with probability p. If $p = O\left(\frac{\log n}{n^{k-1}}\right)$ then the probability that there exists a pair of edges $\{u_1, u_2, \ldots, u_k, c_i\}, i = 1, 2$ is bounded by $O(n^{k+2}p^2n^{-2}) = O(n^{2-k}\log^2 n) = o(1)$. Our choice of p here, is near the end of the process of the edge removal process of [6] and we could start our proof there. So, the only difficulty is with $k = 2$ and this is where we started our research, with rainbow matchings of random graphs. Also, $k = 2$ is the value of k that is needed for Corollary 1.3 and Theorem 1.4. Furthermore, the proof for $k \geq 3$ is no harder than that for $k = 2$.*

We slightly sharpen our focus and consider multi-partite hypergraphs. Let $K_{n,k}$ be the complete k-partite, k-uniform hypergraph where each part has n vertices. Its vertex set V is the union of k disjoint sets $U_1 \cup U_2 \cup \cdots \cup U_k$, each of size n. We let the edge set of $K_{n,k}$ be $\mathcal{V} = \mathcal{V}_k = U_1 \times \cdots \times U_k$. $HP_{n,m,k}$ is obtained by choosing m random edges from $\mathcal{V}$.

Our approach, taken from [6], is to start with a random coloring of the complete k-partite hypergraph $K_{n,k}$. Denote this edge colored graph by $K_{n,k}^{(n)}$. We show in Section 3 that w.h.p. $K_{n,k}^{(n)}$ has a large number of rainbow perfect matchings. We then randomly delete edges one by one showing that w.h.p. the remaining graph H_i, after i steps, still contains many rainbow perfect matchings. Here we need $i \leq N - Kn\log n$ where $N = n^k$ and K is sufficiently large.

We let Φ_i denote the number of rainbow perfect matchings in H_i and consider $\xi_i = 1 - \frac{\Phi_i}{\Phi_{i-1}}$. If we can control the sequence (ξ_i) then we can control the number of rainbow perfect matchings in H_i. It is enough to control $S_i = \sum_i \xi_i$. We will let $\mathbf{w}_i(e)$ denote the number of rainbow perfect matchings that contain a particular edge $e \in E_i$, the edge-set of H_i. S_i will be concentrated around its mean if we show that w.h.p. the maximum value of $\mathbf{w}_i(e)$ is only $O(1)$ times the average value of $\mathbf{w}_i(e)$ over $e \in E_i$. This is the event $\mathcal{B}_i$ defined in (4.7). Proving that $\mathcal{B}_i$ occurs w.h.p. is the heart of the proof.

In Section 4.4 there is a switch from bounding the ratio of max to average to bounding the ratio of max to median. It is then shown that it is unlikely for the maximum to be more than twice the median. Entropy and symmetry play a significant role here and it is perhaps best to leave the reader to enjoy this clever set of ideas from [6] when he/she gets to them.

Once we have Theorem 1.1, it is fairly straightforward to use the result of [5] to obtain Theorem 1.4. This is done in Section 5.

3 The number of rainbow perfect matchings in $K_{n,k}^{(n)}$

To begin, we will show that the number of rainbow perfect matchings in $K_{n,k}^{(n)}$, with its edges randomly colored by n colors is concentrated around its expected value.

Lemma 3.1. *Let $\Phi(K_{n,k}^{(n)})$ represent the number of rainbow matchings of $K_{n,k}^{(n)}$. Then w.h.p.,*

$$\Phi(K_{n,k}^{(n)}) \sim \frac{(n!)^k}{n^n}.$$

Proof. Let X be a random variable representing the number of rainbow matchings in $K_{n,k}^{(n)}$. Then there are $(n!)^{k-1}$ distinct perfect matchings and each has probability $\frac{n!}{n^n}$ of being rainbow colored. Hence,

$$\mathbb{E}\left[X\right] = (n!)^{k-1} \times \frac{n!}{n^n} = \frac{(n!)^k}{n^n}. \tag{3.1}$$

We use Chebyshev's Inequality to show that X is concentrated around this value. It is enough to show that

$$\mathbb{E}\left[X^2\right] \leq (1+o(1))\mathbb{E}\left[X\right]^2.$$

Given a fixed matching M with ℓ edges, let N_ℓ represent the number of matchings covering the same vertex set as M but are edge disjoint from M. Then inclusion-exclusion gives

$$\begin{aligned} N_\ell &= \sum_{i=0}^{\ell}(-1)^i\binom{\ell}{i}((\ell-i)!)^{k-1} \\ &= (\ell!)^{k-1}\sum_{i=0}^{\ell}\frac{(-1)^i}{i!}\left(\frac{(\ell-i)!}{\ell!}\right)^{k-2}. \end{aligned}$$

Now, suppose we have an integer sequence $\lambda = o(\sqrt{\ell})$ and $\lambda \to \infty$ with ℓ. Then the Bonferroni inequalities tell us that

$$(\ell!)^{k-1}\sum_{i=0}^{2\lambda-1}\frac{(-1)^i}{i!\ell^{i(k-2)}}\left(1+o(1)\right) \leq N_\ell \leq (\ell!)^{k-1}\sum_{i=0}^{2\lambda}\frac{(-1)^i}{i!\ell^{i(k-2)}}\left(1+o(1)\right). \tag{3.2}$$

So as long as $\ell \to \infty$,

$$N_\ell = (\ell!)^{k-1}\left(e^{-1}\mathbb{1}_{k=2} + \mathbb{1}_{k\geq 3} + o(1)\right).$$

Then we have

$$\begin{aligned} \mathbb{E}\left[X^2\right] &= \sum_{\ell=0}^{n}(n!)^{k-1}\binom{n}{\ell}N_{n-\ell}\frac{n!}{n^n}\frac{(n-\ell)!}{n^{n-\ell}} \\ &= \mathbb{E}\left[X\right]\sum_{\ell=0}^{n}\frac{n!}{\ell!n^{n-\ell}}N_{n-\ell} \\ &= \mathbb{E}\left[X\right]\sum_{\ell=0}^{\log n}\frac{n!}{\ell!n^{n-\ell}}((n-\ell)!)^{k-1}\left(e^{-1}\mathbb{1}_{k=2}+\mathbb{1}_{k\geq 3}+o(1)\right) \qquad (3.3) \\ &\quad + \mathbb{E}\left[X\right]\sum_{\ell=\log n}^{n}\frac{n!}{\ell!n^{n-\ell}}N_{n-\ell} \qquad (3.4) \end{aligned}$$

We now bound (3.3) and (3.4) in turn. We have that (3.3) is equal to

$$\begin{aligned}
&\mathbb{E}[X]^2 \left(e^{-1}\mathbb{1}_{k=2} + \mathbb{1}_{k\geq 3} + o(1)\right) \sum_{\ell=0}^{\log n} \frac{1+o(1)}{\ell! n^{\ell(k-2)}} \\
&= \mathbb{E}[X]^2 \left(e^{-1}\mathbb{1}_{k=2} + \mathbb{1}_{k\geq 3} + o(1)\right)\left(e\mathbb{1}_{k=2} + \mathbb{1}_{k\geq 3} + o(1)\right) = \mathbb{E}[X]^2 (1+o(1))
\end{aligned}$$

It remains to show that (3.4) is $o(\mathbb{E}[X]^2)$. We split this sum into 2 parts. First, using the trivial bound on $N_{n-\ell} \leq ((n-\ell)!)^{k-1}$, we have

$$\begin{aligned}
\left(\frac{1}{\mathbb{E}[X]^2}\right)\mathbb{E}[X] \sum_{\ell=\log n}^{n-\log n} \frac{n!}{\ell! n^{n-\ell}} N_{n-\ell} &= \sum_{\ell=\log n}^{n-\log n} \frac{n^\ell}{\ell!(n!)^{k-1}} N_{n-\ell} \\
&\leq \sum_{\ell=\log n}^{n-\log n} \frac{n^\ell}{\ell!(n!)^{k-1}} ((n-\ell)!)^{k-1}. \qquad (3.5)
\end{aligned}$$

Since in this range, both ℓ and $n-\ell$ approach infinity with n, we may apply Stirling's approximation to all factorials to get that for some constant c, (3.5) is at most

$$\begin{aligned}
&c \sum_{\ell=\log n}^{n-\log n} n^\ell \cdot \frac{e^\ell}{\ell^{\ell+1/2}} \cdot \frac{e^{(k-1)n}}{n^{(k-1)(n+1/2)}} \cdot \frac{(n-\ell)^{(k-1)(n-\ell+1/2)}}{e^{(k-1)(n-\ell)}} \\
&\leq c \cdot \sum_{\ell=\log n}^{n-\log n} \left(\frac{e^k}{\ell n^{k-2}}\right)^\ell \\
&\leq cn \left(\frac{e^k}{\log n}\right)^{\log n} = o(1).
\end{aligned}$$

For $\ell \geq n - \log n$ we bound $N_{n-\ell} \leq \binom{n}{\log n}((\log n)!)^{k-1}$ and then we have that for some constant c', (3.5) is at most

$$\sum_{\ell=n-\log n}^{n} \frac{n^\ell}{\ell!(n!)^{k-1}} \binom{n}{\log n} ((\log n)!)^{k-1} \leq c' \log n \cdot \frac{e^n \cdot 2^n \cdot (\log n)^{k \log n}}{(n-\log n)!} = o(1).$$

This completes the proof of Lemma 3.1. □

We will need the Chernoff bounds:

Fact 3.2. *Let X be the sum of independent Bernoulli random variables and let $\mathbb{E}[X] = \mu$. Then*

$$\begin{aligned}
\mathbb{P}[|X-\mu| > \epsilon\mu] &\leq 2e^{-\epsilon^2\mu/3} \qquad 0 \leq \epsilon \leq 1. \\
\mathbb{P}[X \geq \alpha\mu] &\leq \left(\frac{e}{\alpha}\right)^{\alpha\mu} \qquad \alpha > e.
\end{aligned}$$

4 Proof of Theorem 1.1

Let the color set be C (so $|C| = n$) and let $\iota : E(K_{n,k}) \to C$ be the random coloring of the edges. Let

$$e_1, \ldots, e_N, \ N = n^k$$

be a random ordering of the edges of $K_{n,k}^{(n)}$, where we have used ι to color the edges of $K_{n,k}$. Let $H_i = K_{n,k}^{(n)} - \{e_1, \ldots, e_i\} = (V, E_i)$. Here if $H = (V, E)$ is a hypergraph and $A \subseteq E, S \subseteq V, D \subseteq C$ then $H - A - S - D$ is the hypergraph on vertex set $V \setminus S$ with those edges in $E \setminus A$ that are disjoint from S and do not use a color from D.

For a color $c \in C$, let $cd_{H_i}(c) = |\{e \in E_i : \iota(e) = c\}|$ be the number of edges of H_i that have color c.

4.1 Tracking the number of rainbow matchings

For an edge-colored hypergraph H, we let $\mathcal{F}(H)$ denote the set of rainbow perfect matchings of H and we let

$$\Phi(H) = |\mathcal{F}(H)|.$$

Let $\mathcal{F}_t = \mathcal{F}(H_t)$ and $\Phi_t = |\mathcal{F}_t|$ and then if $\xi_i = 1 - \frac{\Phi_i}{\Phi_{i-1}}$ then

$$\Phi_t = \Phi_0 \frac{\Phi_1}{\Phi_0} \cdots \frac{\Phi_t}{\Phi_{t-1}} = \Phi_0 (1 - \xi_1) \cdots (1 - \xi_t)$$

or

$$\log \Phi_t = \log \Phi_0 + \sum_{i=1}^{t} \log(1 - \xi_i).$$

where, by Lemma 3.1, we have that w.h.p.

$$\log \Phi_0 = \log \frac{(n!)^k}{n^n} (1 + o(1)) = (k-1) n \log n - c_1 n, \tag{4.1}$$

where

$$0 < c_1 < k + 1. \tag{4.2}$$

Note that for a fixed perfect matching $F \in \mathcal{F}_{i-1}$, we have $\mathbb{P}[e_i \in F] = \frac{n}{N-(i-1)}$. Since a perfect matching is in $\mathcal{F}_{i-1} \setminus \mathcal{F}_i$ if and only if it contains the selected edge e_i, we have

$$\mathbb{E}[\xi_i] = \gamma_i = \frac{n}{N - i + 1} \leq \frac{1}{K \log n}. \tag{4.3}$$

for $i \leq T = N - Kn \log n$.

Equation (4.3) becomes, with

$$p_t = \frac{N-t}{N},$$

$$\sum_{i=1}^{t} \mathbb{E}[\xi_i] = \sum_{i=1}^{t} \gamma_i = n \left(\log \frac{N}{N-t} + O\left(\frac{1}{N-t} \right) \right) = n \left(\log \frac{1}{p_t} + O\left(\frac{1}{N-t} \right) \right) \tag{4.4}$$

using the fact that $\sum_{i=1}^{N} \frac{1}{i} = \log N + (Euler's\ constant) + O(1/N)$.

For $t = T$ this will give

$$p_T = \frac{Kn \log n}{N}$$

and so for $t \le T$ we have

$$\sum_{i=1}^{t} \gamma_i = -n \log p_t + o(n) = (k-1)n \log n - n \log\log n + o(n). \tag{4.5}$$

Our basic goal is to prove that if we define

$$\mathcal{A}_t = \left\{ \log \Phi_t > \log \Phi_0 - \sum_{i=1}^{t} \gamma_i - (c_1+1)n \right\},$$

then

$$\mathbb{P}\left[\bar{\mathcal{A}}_t\right] \le n^{-K^{1/3}/5} \text{ for } t \le T. \tag{4.6}$$

Using (4.1) and (4.5), we see that $\mathcal{A}_t$ implies that

$$\Phi_t = |\mathcal{F}_t| > e^{n \log\log n + O(n)}.$$

Thus taking a union bound over all $t \le T$, we see that (4.6) implies Theorem 1.1 since we may take K as large as we like.

4.2 Important properties

We now define some properties that will be used in the proof.

If $e = (x_1, \ldots, x_k)$ and $c \in C$ then $\mathbf{w}_i(e, c)$ is the number of rainbow matchings of $H_i - \{x_1, \ldots, x_k\}$ that do not use an edge of color c. In particular if e is an edge, then $\mathbf{w}_i(e, \iota(e))$ is the number of rainbow matchings of H_i which use the edge e. We will usually shorten $\mathbf{w}_i(e, \iota(e))$ to $\mathbf{w}_i(e)$ for $e \in E_i$.

In the following we define

$$\mathbf{w}_i(E_i) = \sum_{e \in E_i} \mathbf{w}_i(e) \text{ and } \operatorname{avg}_{e \in E_i} \mathbf{w}_i(e) = \frac{\mathbf{w}_i(E_i)}{|E_i|}.$$

Let

$$\mathcal{B}_i = \left\{ \frac{\max_{e \in E_i} \mathbf{w}_i(e)}{\operatorname{avg}_{e \in E_i} \mathbf{w}_i(e)} \le L = K^{1/2} \right\} \tag{4.7}$$

$$\mathcal{R}_i = \left\{ \begin{array}{c} \forall v \in V,\ \left|d_{H_i}(v) - n^{k-1} p_i\right| \le \epsilon_1 n^{k-1} p_i \\ \text{and} \\ \forall c \in C,\ \left|cd_{H_i}(c) - n^{k-1} p_i\right| \le \epsilon_1 n^{k-1} p_i \end{array} \right\}$$

where $\epsilon_1 = \frac{1}{K^{1/3}}$.

We now consider the first time $t \le T$, if any, where $\mathcal{A}_t$ fails. Then,

$$\bar{\mathcal{A}}_t \cap \bigcap_{i<t} \mathcal{A}_i \subseteq \left[\bigcup_{i<t} \bar{\mathcal{R}}_i \right] \cup \left[\bigcup_{i<t} \mathcal{A}_i \mathcal{R}_i \bar{\mathcal{B}}_i \right] \cup \left[\bar{\mathcal{A}}_t \cap \bigcap_{i<t} (\mathcal{B}_i \mathcal{R}_i) \right]$$

We can therefore write

$$\mathbb{P}\left[\bar{\mathcal{A}}_t \cap \bigcap_{i<t} \mathcal{A}_i\right] < \sum_{i<t} \mathbb{P}\left[\bar{\mathcal{R}}_i\right] + \sum_{i<t} \mathbb{P}\left[\mathcal{A}_i \mathcal{R}_i \bar{\mathcal{B}}_i\right] + \mathbb{P}\left[\bar{\mathcal{A}}_t \cap \bigcap_{i<t} (\mathcal{B}_i \mathcal{R}_i)\right]. \tag{4.8}$$

It will take most of the paper to show that $\mathcal{B}_i$ occurs w.h.p. for all $i \leq T$ thus dealing with the second term of (4.8). However, $\mathcal{R}_i$ (the first term in (4.8)) is easily dealt with.

4.2.1 Dealing with $\mathcal{R}_i$

First, we observe that H_i is distributed as $HP^{(n)}_{n,N-i,k}$ and so for any hypergraph property $\mathcal{P}$ we can write

$$\mathbb{P}\left[H_i \in \mathcal{P}\right] \leq_b m \mathbb{P}_1 \left[HP^{(n)}_{n,p_i,k} \in \mathcal{P}\right], \tag{4.9}$$

where $HP^{(n)}_{n,p_i,k}$ is the corresponding independent model in which each possible edge is included with probability p_i and $m = Np$ and $\mathbb{P}_1$ refers to probabilities computed with respect to $HP^{(n)}_{n,p_i,k}$. This follows from $\mathbb{P}_1 \left[HP^{(n)}_{n,p,k} \in \mathcal{P}\right] \geq \binom{N}{m} p^m (1-p)^{N-m} \mathbb{P}\left[HP^{(n)}_{n,m,k} \in \mathcal{P}\right]$. The notation $A \leq_b B$ is a substitute for $A = O(B)$.

Applying the Chernoff bound and (4.9) we see that for any v, i we have

$$\mathbb{P}\left[|d_{H_i}(v) - n^{k-1}p_i| \geq \epsilon_1 n^{k-1} p_i\right] \leq 2ne^{-\epsilon_1^2 n^{k-1} p_i/3} \leq n^{-K^{1/3}/4}. \tag{4.10}$$

For a fixed color c we see that $cd_{H_i}(c)$ is distributed as the binomial $\text{Bin}(N-i, 1/n)$ which has expectation $n^{k-1}p_i$. Applying the Chernoff bound once more we see then that for a fixed color c we have

$$\mathbb{P}\left[\left|cd_{H_i}(c) - n^{k-1}p_i\right| \geq \epsilon_1 n^{k-1} p_i\right] \leq 2e^{-\epsilon_1^2 n^{k-1} p_i/3} \leq n^{-K^{1/3}/4}. \tag{4.11}$$

Taking the union bound over $v \in [n], i \leq T$ in (4.10) and over $c \in C, i \leq T$ in (4.11) deals with the first term in (4.8).

4.3 Concentration of the number of rainbow matchings

We define

$$\mathcal{E}_i = \{\mathcal{B}_j, \mathcal{R}_j, j < i\}.$$

We will first show that

$$\mathcal{E}_i \implies \xi_i \leq \frac{1}{K^{1/2} \log n}. \tag{4.12}$$

First we have

$$\begin{aligned}\mathbf{w}_{i-1}(E_{i-1}) &= \sum_{e \in E_{i-1}} \sum_{F \in \mathcal{F}_{i-1}} \mathbb{1}_{e \in F} \\ &= \sum_{F \in \mathcal{F}_{i-1}} \sum_{e \in E_{i-1}} \mathbb{1}_{e \in F} = n\Phi_{i-1}.\end{aligned} \tag{4.13}$$

So for any $f \in E_{i-1}$, recalling that $L = K^{1/2}$ from (4.7),

$$\begin{aligned}\Phi_{i-1} &= \frac{1}{n}\mathbf{w}_{i-1}(E_{i-1}) \\ &\geq \frac{1}{Ln}\left|E_{i-1}\right| \max_{e \in E_{i-1}} \mathbf{w}_{i-1}(e) \\ &\geq \frac{N}{Ln} p_{i-1}\mathbf{w}_{i-1}(f).\end{aligned}$$

Hence, if the event $\mathcal{E}_i$ holds then

$$\xi_i \leq \max_{e \in E_{i-1}} \frac{\mathbf{w}_{i-1}(e)}{\Phi_{i-1}} \leq \frac{Ln}{Np_{i-1}} \leq \frac{L}{K \log n} = \frac{1}{K^{1/2}\log n},$$

confirming (4.12).

Now, recalling that $\gamma_i = \mathbb{E}\left[\xi_i\right]$ we define

$$Z_i = \begin{cases} \xi_i - \gamma_i & \text{if } \mathcal{E}_i \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

and let

$$X_t = \sum_{i=1}^{t} Z_i.$$

We will show momentarily that

$$\mathbb{P}\left[X_t \geq n\right] \leq e^{-\Omega(n)}. \tag{4.14}$$

So if we do have $\mathcal{E}_t$ for $t \leq T$ (so that $X_t = \sum_{i=1}^{t}(\xi_i - \gamma_i)$) and $X_t \leq n$ then

$$\sum_{i=1}^{t} \xi_i < \sum_{i=1}^{t} \gamma_i + n \leq (k-1)n\log n - n\log\log n + o(n) \leq (k-1)n\log n$$

and so

$$\sum_{i=1}^{t} \xi_i^2 \leq \frac{1}{K^{1/2}\log n} \cdot \sum_{i=1}^{t} \xi_i \leq kK^{-1/2}n.$$

So, using (4.12) and $\log(1-x) > -x - x^2$ for x small, we have

$$\log \Phi_t > \log \Phi_0 - \sum_{i=1}^{t}(\xi_i + \xi_i^2) > \log \Phi_0 - \sum_{i=1}^{t} \gamma_i - 2n.$$

This deals with the third term in (4.8). (If $\mathcal{E}_t$ holds then $\mathcal{A}_t$ holds with sufficient probability).

Let us now verify (4.14). Note that $|Z_i| \leq \frac{1}{K^{1/2}\log n}$ and that for any $h > 0$

$$\mathbb{P}\left[X_t \geq n\right] = \mathbb{P}\left[e^{h(Z_1+\cdots+Z_t)} \geq e^{hn}\right] \leq \mathbb{E}\left[e^{h(Z_1+\cdots+Z_t)}\right] e^{-hn} \tag{4.15}$$

Now $Z_i = \xi_i - \gamma_i$ (whenever $\mathcal{E}_i$ holds) and $\mathbb{E}\left[\xi_i \mid \mathcal{E}_i\right] = \gamma_i$. The conditioning does not affect the expectation since we have the same expectation given any previous history. Also $0 \leq \xi_i \leq \epsilon = \frac{1}{\log n}$

(whenever $\mathcal{E}_i$ holds). So, with $h \leq 1$, by convexity we have $e^{h\xi_i} \leq 1 - \frac{\xi_i}{\epsilon} + \frac{\xi_i}{\epsilon} e^{h\epsilon}$ for $0 \leq \xi_i \leq \epsilon$ and therefore

$$\mathbb{E}\left[e^{hZ_i}\right] = \mathbb{E}\left[e^{hZ_i} \mid \mathcal{E}_i\right] \mathbb{P}\left[\mathcal{E}_i\right] + \mathbb{E}\left[e^{hZ_i} \mid \neg\mathcal{E}_i\right] \mathbb{P}\left[\neg\mathcal{E}_i\right] \leq$$
$$e^{-h\gamma_i}\mathbb{E}\left[1 - \frac{\xi_i}{\epsilon} + \frac{\xi_i}{\epsilon}e^{h\epsilon}\middle|\mathcal{E}_i\right] \mathbb{P}\left[\mathcal{E}_i\right] + \mathbb{P}\left[\neg\mathcal{E}_i\right] = e^{-h\gamma_i}\left(1 - \frac{\gamma_i}{\epsilon} + \frac{\gamma_i}{\epsilon}e^{h\epsilon}\right) \mathbb{P}\left[\mathcal{E}_i\right] + 1 - \mathbb{P}\left[\mathcal{E}_i\right] \leq$$
$$e^{-h\gamma_i}\left(1 - \frac{\gamma_i}{\epsilon} + \frac{\gamma_i}{\epsilon}(1 + h\epsilon + h^2\epsilon^2)\right) = e^{-h\gamma_i}(1 + \gamma_i h + \gamma_i h^2\epsilon) \leq e^{h^2\epsilon\gamma_i}.$$

So,

$$\mathbb{E}\left[e^{h(Z_1+\cdots+Z_t)}\right] \leq e^{h^2\epsilon\sum_{i=1}^t \gamma_i}$$

and going back to (4.15) we get

$$\mathbb{P}\left[X_t \geq n\right] \leq e^{h^2\epsilon\sum_{i=1}^t \gamma_i - hn}.$$

Now $\sum_{i=1}^t \gamma_i = O(n \log n)$ and so putting h equal to a small enough positive constant makes the RHS of the above less than $e^{-hn/2}$ and (4.14) follows.

4.4 From average to median

If $I \subset [k]$, we write $\mathcal{V}_I$ for the collection of $|I|$-sets of vertices using exactly one vertex from each of $U_i, i \in I$. For $r \leq k$, we let $\mathcal{V}_r = \bigcup_{|I|=r} \mathcal{V}_I$. Given $\mathbf{v} \in \mathcal{V}_r$, we define $I(\mathbf{v})$ by $\mathbf{v} \in \mathcal{V}_{I(\mathbf{v})}$ and $I^c(\mathbf{v}) = [k] \setminus I(\mathbf{v})$.

Now for a multi-set $X \subseteq \mathbf{R}$ we let $\operatorname{med} X$, the median of X, be the largest value $x \in X$ such that there are at least $|X|/2$ elements of X that are at least x. Then define

$$\mathcal{C}_i = \left\{\begin{array}{c} \forall \mathbf{v} \in \mathcal{V}_{k-1}, c \in C, \max_{w \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i((\mathbf{v}, w), c) \leq \max\left\{\frac{\Phi_i}{2^k N}, 2\operatorname{med}_{u \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i((\mathbf{v}, u), c)\right\} \\ \text{and} \\ \forall \mathbf{v} \in \mathcal{V}_k, \max_{c \in C} \mathbf{w}_i(\mathbf{v}, c) \leq \max\left\{\frac{\Phi_i}{2^k N}, 2\operatorname{med}_{c \in C} \mathbf{w}_i(\mathbf{v}, c)\right\}. \end{array}\right\}$$

We will prove

$$\mathbb{P}\left[\mathcal{R}_i\mathcal{C}_i\bar{\mathcal{B}}_i\right] < n^{-K^{1/3}/4} \tag{4.16}$$
$$\mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i\right] < n^{-K^{1/3}/4}. \tag{4.17}$$

Note that (4.16) and (4.17) imply that

$$\mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{B}}_i\right] = \mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{B}}_i\mathcal{C}_i\right] + \mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{B}}_i\bar{\mathcal{C}}_i\right] \leq 2n^{-K^{1/3}/4}.$$

This deals with the middle term in (4.8).

4.5 Proof of (4.16)

First, we suppose that

$$\mathbb{P}\left[\mathcal{R}_i\mathcal{C}_i\right] \geq n^{-K^{1/3}/4}, \tag{4.18}$$

otherwise (4.16) holds trivially.

For $\mathbf{v} \in \mathcal{V}_{k-1}$ and $c \in C$, we let $\psi_V(\mathbf{v}, c) = \max_{w \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i\left((\mathbf{v}, w), c\right)$ and for $\mathbf{v} \in \mathcal{V}_k$, we let $\psi_C(\mathbf{v}) = \max_{c \in C} \mathbf{w}_i(\mathbf{v}, c)$. Let

$$\psi_0 = \mathbf{w}_i(\mathbf{v}', c') = \max_{\mathbf{v} \in \mathcal{V}_k} \max_{c \in C} \mathbf{w}_i(\mathbf{v}, c). \tag{4.19}$$

Lemma 4.1. *Suppose that B is such that $\psi_0 \geq 2^k B$ and that for each $\boldsymbol{v} \in \mathcal{V}_{k-1}, c \in C$ with $\psi_V(\boldsymbol{v}, c) \geq B$, we have*

$$\left| \left\{ w \in \mathcal{V}_{I^c(\boldsymbol{v})} : \mathbf{w}_i((\boldsymbol{v}, w), c) \geq \frac{1}{2} \psi_V(\boldsymbol{v}, c) \right\} \right| \geq \frac{n}{2} \tag{4.20}$$

and for all $\boldsymbol{v} \in V_k$ with $\psi_C(\boldsymbol{v}) \geq B$, we have

$$\left| \left\{ c \in C : \mathbf{w}_i(\boldsymbol{v}, c) \geq \frac{1}{2} \psi_C(\boldsymbol{v}) \right\} \right| \geq \frac{n}{2}. \tag{4.21}$$

Then we have

$$\left| \left\{ (\boldsymbol{v}, c) \in \mathcal{V}_k \times C \,:\, \mathbf{w}_i(\boldsymbol{v}, c) \geq \frac{\psi_0}{2^{k+1}} \right\} \right| \geq \frac{n^{k+1}}{2^{k+1}} \tag{4.22}$$

Proof. Suppose $\mathbf{v}' = (v_1', \dots, v_k'), c'$ are as in (4.19). Then by (4.20), there there is a set $W_1 \subset U_1$ of size at least $n/2$ such that if $w_1 \in W_1$ then $\mathbf{w}_i((w_1, v_2', \dots, v_k'), c') \geq \frac{1}{2} \psi_V((v_2', \dots, v_k'), c') = \frac{1}{2} \psi_0 \geq 2^{k-1} B$. For each $w_1 \in W_1$, since we have $\psi_V((w_1, v_3', \dots, v_k')), c') \geq \frac{1}{2} \psi_0 \geq 2^{k-1} B$, we may apply (4.20) once more to find a set $W_2^{w_1} \subseteq U_2$ of size at least $n/2$ such that if $w_2 \in W_2^{w_1}$ then $\mathbf{w}_i((w_1, w_2, v_3', \dots, v_k'), c') \geq \frac{1}{2} \psi_V((w_1, v_3', \dots, v_k')), c') \geq \frac{1}{4} \psi_0 \geq 2^{k-2} B$.

Continuing in this way, for every choice of $w_1 \in W_1$, $w_2 \in W_2^{w_1}$, $w_3 \in W_3^{w_1, w_2}, \dots, w_k \in W_k^{w_1, \dots w_{k-1}} \subseteq U_k$, we have $\mathbf{w}_i((w_1, \dots, w_k), c') \geq \frac{1}{2^k} \psi_0 \geq B$. Thus every such choice of $w_1, \dots, w_k$, we have $\psi_C((w_1, \dots, w_k)) \geq \frac{1}{2^k} \psi_0 \geq B$, so to finish, we apply (4.21) to find a set $D^{w_1, \dots, w_k} \subseteq C$ of size at least $n/2$ such that if $d \in D^{w_1, \dots, w_k}$ then $\mathbf{w}_i((w_1, \dots, w_k), d) \geq \frac{\psi_0}{2^{k+1}}$. Since there are $n/2$ choices for vertices in each part and $n/2$ choices for colors, we have that the number of choices total is at least $\frac{n^{k+1}}{2^{k+1}}$ as desired. □

For $\mathbf{v} \in \mathcal{V}_k$, let $H_i^{\mathbf{v}c}$ be the hypergraph H_i with vertices in $\mathbf{v}$ removed as well as all edges with color c. Now $\mathbf{w}_i(\mathbf{v}, c)$ equals the number of rainbow matchings in $H_i^{\mathbf{v}c}$. Suppose that $\mathcal{C}_i$ holds and let $B = \frac{\Phi_i}{2^k N}$. Note that $\psi_0 \geq 2^k B$ else we would have $\psi_0 < \frac{\Phi_i}{N} < \operatorname{avg}_{e \in E_i} \mathbf{w}_i(e)$, contradiction.

So for all $\mathbf{v} \in \mathcal{V}_{k-1}$, $c \in C$ with $\psi_V(\mathbf{v}, c) \geq B$, we have

$$\max_{w \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i\left((\mathbf{v}, w), c\right) \leq 2 \operatorname{med}_{w \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i((\mathbf{v}, w), c).$$

This is condition (4.20). Similarly, the second condition of $\mathcal{C}_i$ gives us (4.21). So we may conclude that if

$$W^* = \left\{ (\mathbf{v}, c) \in \mathcal{V}_k \times C \,:\, \mathbf{w}_i(\mathbf{v}, c) \geq \frac{1}{2^{k+1}} \psi_0 \right\}$$

then

$$|W^*| \geq \frac{n^{k+1}}{2^{k+1}}. \tag{4.23}$$

Let

$$E_i^* := \left\{ e \in E_i \,:\, \mathbf{w}_i(e) \geq \frac{1}{2^{k+2}} \max_{e \in E_i} \mathbf{w}_i(e) \right\}. \tag{4.24}$$

We will show that

$$\mathbb{P}\left[|E_i^*| \leq \frac{Np_i}{2^{2k+7}} \middle| \mathcal{R}_i \mathcal{C}_i \right] \leq n^{-K^{1/3}}. \tag{4.25}$$

For $0 < \alpha < \frac{1}{2^{k+1}}$ let

$$X_\alpha = \{x \in U_1 : \exists \text{ at least } \alpha n^k \text{ choices of } (x_2, x_3, \ldots, x_k) \in U_2 \times \cdots \times U_k \text{ and } c \in C \text{ such that } ((x, x_2, \ldots, x_k), c) \in W^*\}.$$

If $|X_\alpha| = \theta_\alpha n$ then (4.23) implies that

$$\theta_\alpha n^{k+1} + \alpha(1 - \theta_a) n^{k+1} \geq \frac{n^{k+1}}{2^{k+1}},$$

which implies that

$$\theta_\alpha \geq \frac{1}{1-\alpha} \left(\frac{1}{2^{k+1}} - \alpha \right). \tag{4.26}$$

Now for $x \in X_\alpha$ and $0 < \beta < \alpha$ let

$$C_\beta(x) = \left\{ c \in C : |\{(x_2, x_3, \ldots, x_k) \in U_2 \times \cdots \times U_k : ((x, x_2, \ldots, x_k), c) \in W^*\}| \geq \beta n^{k-1} \right\}.$$

A similar argument to that for (4.26) shows that if $|C_\beta(x)| = \zeta_\beta n$ then

$$\zeta_\beta \geq \frac{\alpha - \beta}{1 - \beta}. \tag{4.27}$$

Putting $\alpha = \frac{1}{2^{k+2}}$ and $\beta = \frac{1}{2^{k+3}}$, we see by (4.26), (4.27) that there are αn vertices in $X_1 \subseteq U_1$ such that if $x_1 \in X_1$ then there are βn choices for $c_1 \in C_\alpha(x_1) \subseteq C$ such that there are βn^{k-1} choices for $\mathbf{x} = (x_2, \ldots, x_k) \in U_2 \times \cdots \times U_k$, such that if $x_1 \in X_1$, $c_1 \in C_1(x_1)$ then

$$\mathbf{w}_i((x_1, \mathbf{x}), c_1) > \frac{1}{2^{k+1}} \psi_0 \geq \frac{1}{2^{k+1}} \max_{e \in E_i} \mathbf{w}_i(e). \tag{4.28}$$

Now fix $0 \leq \ell \leq 2kn \log n$ and let $\Lambda = 2^\ell$. Fix a vertex $x_1 \in X_1$ and let

$$A_\Lambda(x_1) = \left\{ \mathbf{x} \in \mathcal{V}_{[2,k]}, c_1 \in C : \mathbf{w}_i((x_1, \mathbf{x}), c_1) \geq \Lambda \right\}$$

and let

$$B_\Lambda(x_1) = \left\{ \mathbf{x} \in \mathcal{V}_{[2,k]}, c_1 \in C : (x_1, \mathbf{x}) \in E_i, c_1 = \iota(x_1, \mathbf{x}) \text{ and } \mathbf{w}_i((x_1, \mathbf{x}), c_1) \geq \Lambda \right\}$$

Here Λ will be an approximation to the random variable $\psi_0/2^{k+1}$. Using Λ in place of $\psi_0/2^{k+1}$ reduces the conditioning. There are not too many choices for Λ and so we will be able to use the union bound over Λ. We do not condition on the value ψ_0, but we instead use the fact that $\psi_0/2^{k+1} \in [\Lambda, 2\Lambda]$ for some $\ell \leq 2kn \log n$.

Let S,T denote disjoint subsets of $\{x_1\} \times \mathcal{V}_{[2,k]} \times C$. Note that without the conditioning $\mathcal{R}_i\mathcal{C}_i$ the event $\{S \subseteq A_\Lambda, T \cap A_\Lambda = \emptyset\}$ will be independent of the event

$$\bigcap_{(e,c)\in S} \{e \in E_i, \iota(e) = c\} \cap \bigcap_{(e,c)\in T} \neg \{e \in E_i, \iota(e) = c\}. \tag{4.29}$$

This is because $\mathbf{w}_i((x_1,\mathbf{x}), c_1)$ depends only on the existence and color of edges f where if $\mathbf{x} = (x_2, x_3, \ldots, x_k)$,

$$\{x_1, x_2, \ldots, x_k\} \cap f = \emptyset.$$

If we work with the model HP_{n,k,p_i} in place of H_i, without the conditioning, then $\mathbb{E}[|B_\Lambda(x_1)|] = |A_\Lambda| p_i/n$. Also, we can express $|B_\Lambda(x_1)|$ as the sum of independent Bernoulli random variables, one for each possible value of $\mathbf{x}$. The variable Z corresponding to a fixed $\mathbf{x}$ will be one iff there is a $c_1 \in C$ such that $((x_1,\mathbf{x}), c_1) \in B_\Lambda(x_1)$.

Then equations (4.9) and (4.18) imply that

$$\mathbb{P}[|B_\Lambda(x_1)| \leq \Delta p_i/2n \mid \mathcal{R}_i\mathcal{C}_i] \leq \frac{\mathbb{P}[|B_\Lambda(x_1)| \leq \Delta p_i/2n]}{\mathbb{P}[\mathcal{R}_i\mathcal{C}_i]} \leq n^{k+K^{1/3}/4}\mathbb{P}_1[|B_\Lambda(x_1)| \leq \Delta p_i/2n]. \tag{4.30}$$

If $|A_\Lambda(x_1)| \geq \Delta = \beta^2 N$ then Fact 3.2 and (4.29) imply that

$$\mathbb{P}_1[|B_\Lambda(x_1)| \leq \Delta p_i/2n] \leq e^{-\Delta p_i/12n} \leq n^{-\beta^2 K/12}. \tag{4.31}$$

There are at most n choices for x_1. The number of choices for ℓ is $2kn \log n$ and for one of these we will have $2^\ell \leq \frac{1}{2^{k+1}} \max \mathbf{w}_i(E_i) \leq 2^{\ell+1}$ and so (from (4.30) and (4.31)), with probability $1 - n^{2+o(1)+k+K^{1/3}/4-\beta^2 K/20} \geq 1 - n^{-\beta^2 K/30}$ we have that for each choice of $x_1 \in X_1$ there are $\beta^2 N p_i/2$ choices for $\mathbf{x}, c$ such that $(e = (x_1,\mathbf{x}), c = \iota(x_1,\mathbf{x})) \in B_\Lambda(x_1)$ and $\mathbf{w}_i(e,c) > \frac{1}{2^{k+2}} \max \mathbf{w}_i(E_i)$. Observe that we have 2^{k+2} in place of 2^{k+1}, because we will want the above to hold for a value of Λ where $\Lambda \leq \max \mathbf{w}_i(E_i) \leq 2\Lambda$. This verifies (4.25) and we have

$$\frac{\sum_{e\in E_i} \mathbf{w}_i(e)}{\max \mathbf{w}_i(E_i)} \geq \frac{\sum_{e\in E_i^*} \mathbf{w}_i(e)}{\max \mathbf{w}_i(E_i)} \geq \frac{|E_i^*|}{2^{k+2}} \geq \frac{N p_i}{2^{3k+9}} \geq \frac{|E_i|}{2^{3k+10}}$$

which implies property $\mathcal{B}_i$ if K is sufficiently large.

4.6 Proof of (4.17)

Recall that for a discrete random variable X, the (base e) entropy $H(X)$, is defined by

$$H(X) = \sum_x p_x \log\left(\frac{1}{p_x}\right)$$

where the sum ranges over possible values of X and $p_x = \mathbb{P}[X = x]$. The following lemma is proved in [6].

Lemma 4.2. *Suppose that X is a positive integer valued random variable defined on some finite set S that takes values in an interval $I = \{0, 1, \ldots, \nu\}$ for some positive integer ν. Suppose that*

$H(X) > \log(|S|) - M, M = O(1)$. Suppose that $\mathbb{P}[X = i] = \frac{\mathbf{w}(i)}{\mathbf{w}(S)}$ for some $\mathbf{w} : S \to I$. Then there are $a, b \in I$ with $a \le b \le \rho_M a$ such that for $J = \mathbf{w}^{-1}[a, b]$ we have

$$|J| \ge \sigma_M |S|$$

and

$$\mathbf{w}(J) > 0.7\mathbf{w}(S).$$

Here we can take $\rho_M = 2^{4(M+\log 3)}$ and $\sigma_M = 2^{-2M-2}$.

To prove (4.17), assume that we have $\mathcal{A}_i$ and $\mathcal{R}_i$ and that $\mathcal{C}_i$ fails. Then we have two cases.

Suppose $\mathbf{v} \in \mathcal{V}_{k-1}$, $x \in \mathcal{V}_{I^c(\mathbf{v})}$, and $c \in C$. Let $H_i^{\mathbf{v}xc}$ be the sub-graph of H_i induced by $V \setminus \{\mathbf{v}, x\}$ where all edges of color c have been deleted.

4.6.1 Case 1

Suppose that $\mathcal{C}_i$ fails because there exists $\mathbf{v} \in \mathcal{V}_{k-1}$ and $c \in C$ such that

$$\max_{\xi \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i((\mathbf{v}, \xi), c) > \max\left\{\frac{\Phi_i}{2^k N}, 2\,\mathrm{med}_{\xi \in \mathcal{V}_{I^c(\mathbf{v})}} \mathbf{w}_i((\mathbf{v}, \xi), c)\right\}.$$

Let x be the value of ξ which maximizes $\mathbf{w}_i((\mathbf{v}, \xi), c)$. For ease of notation, let us suppose that $\mathbf{v} = (v_1, \ldots, v_{k-1}) \in U_1 \times \cdots \times U_{k-1}$, so that $I^c(\mathbf{v}) = \{k\}$. Let $X(y, H_i^{\mathbf{v}xc})$ be the (random) edge-color pair containing vertex y in a uniformly random rainbow matching of $H_i^{\mathbf{v}xc}$. Then let $y \in U_k \setminus \{x\}$ be a vertex with

$$\mathbf{w}_i((\mathbf{v}, y), c) \le \mathrm{med}_x\, \mathbf{w}_i\left((\mathbf{v}, x), c\right) \tag{4.32}$$

and

$$h(y, H_i^{\mathbf{v}xc}) := H(X(y, H_i^{\mathbf{v}xc}))$$

maximized subject to (4.32). Then

$$\mathbf{w}_i\left((\mathbf{v}, x), c\right) \ge 2\,\mathrm{med}_u\, \mathbf{w}_i\left((\mathbf{v}, u), c\right) \ge 2\mathbf{w}_i\left((\mathbf{v}, y), c\right).$$

We have, using (4.1) and (4.2) and assuming $\mathcal{A}_i$ that

$$\log \Phi_i > (k-1)n \log n + n \log p_i - (c_1 + 1)n. \tag{4.33}$$

$\Phi(H_i^{\mathbf{v}xc})$ is the number of rainbow matchings of $H_i^{\mathbf{v}xc}$. So,

$$\log \Phi(H_i^{\mathbf{v}xc}) = \log \mathbf{w}_i\left((\mathbf{v}, x), c\right) \ge (k-1)n \log n + n \log p_i - (c_1 + 2)n \tag{4.34}$$

(by the assumption about $\mathbf{v}, x, c$ and the failure of $\mathcal{C}_i$, including $\mathbf{w}_i\left((\mathbf{v}, x), c\right) \ge \Phi_i/((2n)^k)$).

Now a rainbow matching of $H_i^{\mathbf{v}xc}$ is determined by the $\{X(z, H_i^{\mathbf{v}xc}) : z \ne x\}$. Let M denote a uniform random rainbow matching of $H_i^{\mathbf{v}xc}$. Sub-additivity of entropy then implies that

$$H(M) = \log \Phi(H_i^{\mathbf{v}xc}) \le \sum_{z \in U_k \setminus \{x\}} h(z, H_i^{\mathbf{v}xc}). \tag{4.35}$$

By our choice of y, we have $h(z, H_i^{\mathbf{v}xc}) \leq h(y, H_i^{\mathbf{v}xc})$ for at least half the z's in $U_k \setminus \{x\}$. Also, for all $z \in U_k \setminus \{x\}$, we have

$$h(z, H_i^{\mathbf{v}xc}) \leq \log d_{H_i^{\mathbf{v}xc}}(z) \leq \log\left((1+\epsilon_1)n^{k-1}p_i\right).$$

Here we use the fact that $\mathcal{R}_i$ holds.

So,

$$\log \Phi(H_i^{\mathbf{v}xc}) \leq \frac{n}{2}\left(h(y, H_i^{\mathbf{v}xc}) + \log((1+\epsilon_1)n^{k-1}p_i)\right) \tag{4.36}$$

and hence by combining (4.34) and (4.36) we get

$$\begin{aligned} h(y, H_i^{\mathbf{v}xc}) &\geq \frac{2}{n}\log \Phi(H_i^{\mathbf{v}xc}) - \log\left((1+\epsilon_1)n^{k-1}p_i\right) \\ &\geq \frac{2}{n}\left((k-1)n\log n + n\log p_i - (c_1+2)n\right) - (k-1)\log n - \log p_i - \epsilon_1 \\ &= 2(k-1)\log n + 2\log p_i - (c_1+2) - (k-1)\log n - \log p_i - \epsilon_1 \\ &\geq \log\left(d_{H_i^{\mathbf{v}xc}}(y)\right) - (c_1+3). \end{aligned} \tag{4.37}$$

To summarise what we have proved so far: If we have $\mathcal{A}_i, \mathcal{R}_i$ but not $\mathcal{C}_i$ then (4.37) holds.

Now for $i = 1, \ldots, k-1$, let $W_i = U_i \setminus \{v_i\}$ and $W = W_1 \times \cdots \times W_{k-1}$. Let $L = C \setminus \{c\}$ and for $(\mathbf{z}, c') \in W \times L$, let $\mathbf{w}_i'(\mathbf{z}, c')$ be the number of rainbow matchings of $H_i - \{\mathbf{v}, \mathbf{z}, x, y\} - \{c, c'\}$. We define $\mathbf{w}_y((\mathbf{z}, y), c')$ on

$$W_y := \left\{((\mathbf{z}, y), c') \,:\, \mathbf{z} \in W,\ c' \in L,\ (\mathbf{z}, y) \in E_i,\ \iota((\mathbf{z}, y)) = c'\right\}$$

as $\mathbf{w}_i'(\mathbf{z}, c')$ and define $\mathbf{w}_x((\mathbf{z}, x), c')$ on

$$W_x := \left\{((\mathbf{z}, x), c') \,:\, \mathbf{z} \in W,\ c' \in L,\ (\mathbf{z}, x) \in E_i,\ \iota((\mathbf{z}, x)) = c'\right\}$$

as $\mathbf{w}_i'(\mathbf{z}, c')$. Then the random variable $X(y, H_i^{\mathbf{v}xc})$, which is the edge-color pair containing y in a random rainbow matching of $H_i^{\mathbf{v}xc}$, is chosen according to $\mathbf{w}_y$ and $X(x, H_i^{\mathbf{v}yc})$ which is the edge-color pair containing x in a random rainbow matching of $H_i^{\mathbf{v}yc}$, is chosen according to $\mathbf{w}_x$.

Equation (4.37) tells us that $H(X(y, H_i^{\mathbf{v}xc})) = h(y, H_i^{\mathbf{v}xc}) \geq \log|W_y| - (c_1+3)$. We may therefore apply Lemma 4.2 to conclude that there exist $a \leq b \leq \rho a$, $\rho = \rho_{c_1+3}$ and a set $J \subseteq W_y$ with $|J| \geq \sigma|W_y| \geq (1-\epsilon_1)\sigma n^{k-1}p_i$, $\sigma = \sigma_{c_1+3}$ such that $\mathbf{w}_y(J) \geq 0.7\mathbf{w}_y(W_y)$ and $J = \mathbf{w}_y^{-1}([a, b])$.

We also let $J' := \mathbf{w}_x^{-1}([a, b])$ and note that

$$\mathbf{w}_x(J') \leq \mathbf{w}_x(W_x) = \mathbf{w}_i\left((\mathbf{v}, y), c\right) \leq .5\mathbf{w}_i\left((\mathbf{v}, x), c\right)$$

while on the other hand

$$\mathbf{w}_y(J) \geq 0.7\mathbf{w}_i\left((\mathbf{v}, x), c\right) \geq 1.4\mathbf{w}_x(J'). \tag{4.38}$$

We will condition on $H_i[V \setminus \{\mathbf{v}, x, y\}]$ and denote the conditioning by $\mathcal{E}_1$ i.e. we will fix the edges and edge colors of this subgraph of H_i.

Next enumerate

$$\left\{((\mathbf{z}, y), c') : \Phi(H_i - \{\mathbf{v}, \mathbf{z}, x, y\} - \left\{c, c'\right\}) \in [a, b]\right\} = \left\{((\mathbf{z}_j, y), c_j),\ j = 1, 2, \ldots, \Lambda\right\}.$$

Remark 4.3. *At this point we have a small technical problem. To estimate a probability below, we need to drop the conditioning $\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i$ and then later compensate by inflating our estimates by $1/\mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i\right]$. The existence of a, b depends on this conditioning and we need to deal with this fact. We tackle this as we did in Section 4.5 with respect to ℓ and Λ. So we will consider pairs of integers $1 \le \lambda \le \mu \le \lambda + \log_2 \rho \le 2n^2$. Then for some pair λ, μ we will find $2^\lambda \le a \le b \le 2^\mu$. It is legitimate in the argument to replace a by 2^λ and b by 2^μ and in the analysis below consider a, b as fixed, independent of H_i. We can then inflate our estimates of probabilities by $O(n^2)$ to account for the number of possible choices for λ, μ.*

We define the events

$$\mathcal{D}_{e,\delta} = \{e \in E_i,\, \iota(e) = \delta\}.$$

For the moment replace H_i by HP_{n,k,p_i}. We note that the event $\Phi(H_i - \{\mathbf{v}, \mathbf{z}_j, x, y\} - \{c, c_j\}) \in [a, b]$ does not depend on the occurrence or otherwise of $\mathcal{D}_{(\mathbf{z}_j,y),c_j}$ for any k. Hence, given $\{((\mathbf{z}_j, y), c_j),\ j = 1, 2, \ldots, \Lambda\}$ we find that without conditioning on $\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i$, $|J|$ is distributed as the sum of independent Bernoulli random variables, as in (4.29). Note also that $\mathcal{R}_i$ implies that $|W_y| \ge (1-\epsilon_1)n^{k-1}p_i$. We can assume that $\mathbb{P}\left[\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i\right] \ge n^{-K^{1/3}/4}$, else we have proved (4.17) by default. (We have extra conditioning $\mathcal{E}_1$, but this is independent of the $\mathcal{D}_{e,\delta}$). Therefore, using Fact 3.2,

$$1 = \mathbb{P}\left[|J| \ge (1-\epsilon_1)\sigma n^{k-1}p_i \mid \mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i\mathcal{E}_1\right] \le n^{K^{1/3}/4}\left(\frac{2e\Lambda}{(1-\epsilon_1)\sigma N}\right)^{(1-\epsilon_1)\sigma n^{k-1}p_i}.$$

It follows that for K sufficiently large, we have

$$\Lambda \ge \frac{\sigma N}{10}. \tag{4.39}$$

Then let

$$\Gamma_j = H_i - \{\mathbf{v}, \mathbf{z}_j, x, y\} - \{c, c_j\}.$$

Note that the $\Phi(\Gamma_j) = \mathbf{w}_i'(\mathbf{z}_j, c_j)$ are completely determined by the conditioning $\mathcal{E}_1$.

Then let

$$\mathbf{w}_y(J) = \sum_{\mathbf{z}\in W} \mathbb{1}_{\{\mathbf{z},y\}\in E_i} \sum_{j:\mathbf{z}_j=\mathbf{z}} \Phi(\Gamma_j)\cdot \mathbb{1}_{\iota((\mathbf{z}_j,y))=c_j} \tag{4.40}$$

$$\mathbf{w}_x(J') = \sum_{\mathbf{z}\in W} \mathbb{1}_{\{\mathbf{z},x\}\in E_i} \sum_{j:\mathbf{z}_j=\mathbf{z}} \Phi(\Gamma_j)\cdot \mathbb{1}_{\iota(\{\mathbf{z}_j,x\})=c_j} \tag{4.41}$$

Let

$$X_{\mathbf{z}} = \sum_{j:\mathbf{z}_j=\mathbf{z}} \Phi(\Gamma_j)\cdot \mathbb{1}_{\iota(\{\mathbf{z}_j,x\}=c_j}.$$

$$Y_{\mathbf{z}} = \sum_{j:\mathbf{z}_j=\mathbf{z}} \Phi(\Gamma_j)\cdot \mathbb{1}_{\iota((\mathbf{z}_j,y)=c_j}.$$

Note that $X_{\mathbf{z}}, Y_{\mathbf{z}} \le b$.

We have

$$Z_y = \frac{\mathbf{w}_y(J)}{b} = \sum_{\mathbf{z}\in W} \mathbb{1}_{\{\mathbf{z},y\}\in E_i} \frac{Y_{\mathbf{z}}}{b}$$

$$Z_x = \frac{\mathbf{w}_x(J')}{b} = \sum_{\mathbf{z}\in W} \mathbb{1}_{\{\mathbf{z},x\}\in E_i} \frac{X_{\mathbf{z}}}{b}$$

It follows directly from the expressions (4.40), (4.41) that Z_y and Z_x are both equal to the sum of (conditionally) independent random variables, each bounded between 0 and 1. Furthermore, we see from (4.40), (4.41) that

$$\mathbb{E}\left[Z_y \mid \mathcal{E}_1\right] = \mathbb{E}\left[Z_x \mid \mathcal{E}_1\right]. \tag{4.42}$$

What we have to show now is that we can assume that this (conditional) expectation is large.

Let

$$L_{\mathbf{z}} = \{j : \mathbf{z}_j = \mathbf{z}\}$$

and

$$W' = \{\mathbf{z} \in W : |L_{\mathbf{z}}| \geq \gamma n\}$$

where $\gamma = \sigma/20$.

Note that

$$\mathbf{z} \in W' \text{ implies that } \mathbb{E}\left[Y_{\mathbf{z}} \mid \mathcal{E}_1\right] \geq a|L_{\mathbf{z}}|n^{-1} \geq a\gamma.$$

We have

$$|L_{\mathbf{z}}| \leq n \text{ and } \sum_{\mathbf{z}} |L_{\mathbf{z}}| = \Lambda.$$

We deduce that

$$|W'|n + \gamma n(n^{k-1} - |W'|) \geq \Lambda \geq \frac{\sigma N}{10}.$$

Therefore

$$|W'| \geq \frac{\sigma - 10\gamma}{10(1-\gamma)} n^{k-1} \geq \frac{\sigma n^{k-1}}{20}.$$

Hence,

$$\mathbb{E}\left[Z_y \mid \mathcal{E}_1\right] \geq |W'| p_i \times \frac{a\gamma}{b} \geq \frac{K\sigma \log n}{20\rho}.$$

Now, Hoeffding's theorem implies concentration of Z_y around its (conditional) mean i.e. for arbitrarily small constant ϵ and for large enough K,

$$\mathbb{P}\left[\exists \mathbf{v} \in \mathcal{V}_{k-1}, c \in C : |Z_y - \mathbb{E}\left[Z_y \mid \mathcal{E}_1\right]| \geq \epsilon \mathbb{E}\left[Z_y \mid \mathcal{E}_1\right] \mid \mathcal{E}_1\right] \leq n^{k-dK},$$

for some $d = d(k)$.

The same holds for Z_x. But this together with (4.42) contradicts (4.38). This completes the proof of Case 1 of (4.17). We should of course multiply all probability upper by bounds by $O(n^2)$ to account for Remark 4.3, and there is ample room for this. We can also multiply by $O(n^2)$ to account for the number of choices for x, y.

4.6.2 Case 2

Suppose that $\mathcal{C}_i$ fails because there are vertices $\mathbf{v} = (v_1, \ldots, v_k) \in \mathcal{V}_k$ such that

$$\max_{d \in C} \mathbf{w}_i(\mathbf{v}, d) > \max\left\{\frac{\Phi_i}{(2n)^k}, 2\,\mathrm{med}_{d \in C}\, \mathbf{w}_i(\mathbf{v}, d)\right\}.$$

Let c be the color that maximizes $\mathbf{w}_i(\mathbf{v}, d)$. Let $c^* \in C \setminus \{c\}$ be a color with $\mathbf{w}_i(\mathbf{v}, c^*) \leq \mathrm{med}_c\, \mathbf{w}_i(\mathbf{v}, c)$ and

$$h(c^*, H_i^{\mathbf{v}c}) := H(X(c^*, H_i^{\mathbf{v}c}))$$

maximized subject to this constraint. Similarly to Case 1, $X(c^*, H_i^{\mathbf{v}c})$ denotes the edge-color pair using the color c^* in a uniformly random rainbow matching of $H_i^{\mathbf{v}c}$. Then we can show as before that

$$h(c^*, H_i^{\mathbf{v}c}) \geq \log\left(cd_{H_i^{\mathbf{v}c}}(c^*)\right) - (c_1 + 3). \tag{4.43}$$

Indeed, we have

$$\mathbf{w}_i(\mathbf{v}, c) \geq 2\,\mathrm{med}_d\, \mathbf{w}_i(\mathbf{v}, d) \geq 2\mathbf{w}_i(\mathbf{v}, c^*). \tag{4.44}$$

We have (4.33) and so if $\Phi(H_i^{\mathbf{v}c})$ is the number of rainbow matchings of $H_i^{\mathbf{v}c}$,

$$\log \Phi(H_i^{\mathbf{v}c}) = \log \mathbf{w}_i(\mathbf{v}, c) \geq (k-1)n \log n + n \log p_i - (c_1 + 2)n \tag{4.45}$$

(by the assumption about v, w, c and the failure of $\mathcal{C}_i$, including $\mathbf{w}_i(\{v, w\}, c) \geq \Phi_i/((2n)^k)$).

Now, as in (4.35),

$$\log \Phi(H_i^{\mathbf{v}c}) \leq \sum_{d \in C \setminus \{c\}} h(z, H_i^{\mathbf{v}d}).$$

By our choice of c^*, we have $h(d, H_i^{\mathbf{v}c}) \leq h(c^*, H_i^{\mathbf{v}c})$ for at least half the d's in $C \setminus \{c\}$. Also, for all $d \in C \setminus \{c\}$, we have

$$h(d, H_i^{\mathbf{v}c}) \leq \log cd_{H_i^{\mathbf{v}c}}(d) \leq \log\Big((1+\epsilon_1)n^{k-1}p_i\Big).$$

So

$$\log \Phi(H_i^{\mathbf{v}c}) \leq \frac{n}{2} h(y, H_i^{\mathbf{v}c}) + \frac{n}{2}\log((1+\epsilon_1))n^{k-1}p_i) \tag{4.46}$$

and hence by combining (4.45) and (4.46) we get (4.43), just as we obtained (4.37) from (4.34) and (4.36).

Now for $i = 1, \ldots, k$, we let $W_i = U_i \setminus \{v_i\}$ and $W = W_1 \times \cdots \times W_k$. We let $L = C \setminus \{c, c^*\}$ and for $\mathbf{z} = (z_1, \ldots, z_k) \in W$, let $\mathbf{w}_i'(\mathbf{z})$ be the number of rainbow matchings of $H_i - \{\mathbf{v}, \mathbf{z}\}$ which do not use c^* or c. Then define $\mathbf{w}_{c^*}(\mathbf{z})$ on

$$W_{c^*} := \{\mathbf{z} \in W :\ \mathbf{z} \in E_i,\ \iota(\mathbf{z}) = c^*\}$$

as $\mathbf{w}_i'(\mathbf{z})$ and define $\mathbf{w}_c(\mathbf{z})$ on

$$W_c := \{\mathbf{z} \in W :\ \mathbf{z} \in E_i,\ \iota(\mathbf{z}) = c\}$$

as $\mathbf{w}_i'(\mathbf{z})$. Then the random variable $X_{c^*} = X(c^*, H_i^{\mathbf{v}c})$, which is the edge of color c^* in a random rainbow matching of $H_i^{\mathbf{v}c}$, is chosen according to $\mathbf{w}_{c^*}$ and $X_c = X(c, H_i^{\mathbf{v}c^*})$ which is the edge of color c in a random rainbow matching of $H_i^{\mathbf{v}c^*}$ is chosen according to $\mathbf{w}_c$.

Equation (4.43) tells us that $H(X_{c^*}) \geq \log |W_{c^*}| - (c_1+3)$. Therefore we may apply Lemma 4.2 to conclude that there exist $\alpha \leq \beta \leq \rho\alpha$ and a set $J \subseteq W_{c^*}$ with $|J| \geq \sigma |W_{c^*}| \geq (1-\epsilon_1)\sigma n^{k-1} p_i$ such that $\mathbf{w}_{c^*}(J) \geq 0.7\mathbf{w}_{c^*}(W_{c^*}) = 0.7\mathbf{w}_i(\mathbf{v}, c)$ and $J = \mathbf{w}_{c^*}^{-1}([\alpha, \beta])$. We also let $J' := \mathbf{w}_c^{-1}([\alpha, \beta])$ and note that

$$\mathbf{w}_c(J') \leq \mathbf{w}_c(W_c) = \mathbf{w}_i(\mathbf{v}, c^*) \leq .5\mathbf{w}_i(\mathbf{v}, c)$$

while on the other hand

$$\mathbf{w}_{c^*}(J) \geq 0.7\mathbf{w}_i(\mathbf{v}, c) \geq 1.4\mathbf{w}_c(J'). \tag{4.47}$$

Now let H_i denote the graph induced by the edges $\mathbf{e} \in W$ for which $\iota(\mathbf{e}) \neq c^*, c$. Fix H_i and let $F_i = W \setminus E(H_i)$.

Next enumerate

$$\Psi = \{\mathbf{z} \in F_i : \Phi(H_i - \{\mathbf{v}, \mathbf{z}\} - \{c^*, c\}) \in [\alpha, \beta]\} = \{\mathbf{z}_j,\ j = 1, 2, \ldots, \Lambda\}.$$

Here we can proceed as indicated in Remark 4.3 and treat α, β as constants.

Suppose that we replace H_i by HP_{n,k,p_i}. In this case, Ψ is determined by H_i and is independent of the events $\mathbf{z}_j \in E_i, \iota(\mathbf{z}_j) \in \{c, c^*\}$. It follows that if we omit the conditioning $\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i$ then $|W_{c^*}|$ is distributed as $Bin(\Lambda, p_i/n)$. We still have the conditioning $\mathcal{A}_i\mathcal{R}_i\bar{\mathcal{C}}_i$ but we can argue as before that (4.39) holds.

Then with

$$\Gamma_j = H_i - \{v, w, x_j, y_j\} - \{c, c^*\}$$

(i.e. the graph induced by vertices $V \setminus \{\mathbf{v}, \mathbf{z}_j\}$, not including edges of color c, c^*), we have

$$\mathbf{w}_{c^*}(J) = \sum_{j=1}^{\Lambda} \Phi(\Gamma_j) \mathbb{1}_{\mathbf{z}_j \in E_i, \iota(\mathbf{z}_j) = c^*} \tag{4.48}$$

$$\mathbf{w}_c(J') = \sum_{j=1}^{\Lambda} \Phi(\Gamma_j) \mathbb{1}_{\mathbf{z}_j \in E_i, \iota(\mathbf{z}_j) = c} \tag{4.49}$$

We have already observed the conditioning on H_i means that the $\Phi(\Gamma_j)$ are independent of the $\mathbb{1}_{\mathbf{z}_j \in E_i}, \mathbb{1}_{\iota(\mathbf{z}_j)=c^*}, \mathbb{1}_{\iota(\mathbf{z}_j)=c}$. Thus we may condition on the values of the $\Phi(\Gamma_j)$.

It follows directly from the expressions (4.48), (4.49) that $Z_{c^*} = \mathbf{w}_{c^*}(J)/\beta$ and $Z_c = \mathbf{w}_c(J')/\beta$ are both equal to the sum of independent random variables, each bounded between α/β and 1. Furthermore, we see from (4.48), (4.49) that

$$\mathbb{E}[Z_c \mid \mathcal{E}_1] = \mathbb{E}[Z_{c^*} \mid \mathcal{E}_1]. \tag{4.50}$$

We can argue as before that $\Lambda \geq \sigma N/10$. Then note that

$$\mathbb{E}[Z_{c^*} \mid \mathcal{E}_1] \geq \frac{\alpha\Lambda p_i}{n\beta} \geq \frac{K\sigma \log n}{10\rho}.$$

Now, Hoeffding's theorem implies concentration of Z_{c^*} around its (conditional) mean i.e. for arbitrarily small constant ϵ and for large enough K,

$$\mathbb{P}[\exists \mathbf{v} \in \mathcal{V}_k : |Z_{c^*} - \mathbb{E}[Z_{c^*} \mid \mathcal{E}_1]| \geq \epsilon\mathbb{E}[Z_{c^*} \mid \mathcal{E}_1] \mid \mathcal{E}_1] \leq n^{k-d'K},$$

for some $d' = d'(k)$.

The same holds for Z_c. There is room to inflate all probability bounds by $O(n^4)$ as in Case 1. But this together with (4.50) contradicts (4.38). This completes the proof of Case 2 of (4.17), as well the proof of Theorem 1.1.

5 Proof of Theorem 1.4

Janson and Wormald [5] proved the following theorem.

Theorem 5.1. *Let $G = G_{n,2r}$, $4 \leq r = O(1)$ be a random $2r$-regular graph with vertex set $[n]$. Suppose that the edges of G are randomly colored with n colors so that each color appears exactly r times. Then w.h.p. G contains a rainbow Hamilton cycle.*

Suppose then that we have $G = G_{n,m}^{(n)}$ where $n = 2\nu$ is even and $m = Kn \log n$ where K is sufficiently large. We randomly assign an integer $\ell(e) \in \{1, 2, 3, 4\}$ to each edge. We then randomly partition the set $[n] \times [4]$ into 8 sets $C_1, C_2, \ldots, C_8$ of size ν. We then partition the edges of G into 8 sets $E_1, E_2, \ldots, E_8$. We place an edge e into E_i if $(c(e), \ell(e)) \in C_i$ where $c(e)$ is the color of e. An edge goes into each E_i with the same probability, 1/8, and so w.h.p. we find that $|E_i| \geq m/10$ for $i = 1, 2, \ldots, 8$. If $|E_i| = m_i$ then the subgraph H_i induced by E_i is distributed as $G_{n,m_i}^{(\nu)}$ and so we can apply Theorem 1.1 to argue that w.h.p. each H_i contains a rainbow perfect matching M_i. If we let $\Gamma = \bigcup_{i=1}^{8} M_i$ and drop the $\ell(e)$ part of the coloring, then it almost fits the hypothesis of Theorem 5.1. It is 8-regular and each color appears exactly 4 times. Now M_1 is a uniform random perfect matching of K_n and in general M_i is a uniform random matching, disjoint from $M_1, \ldots, M_{i-1}$. If we take 8 independent random perfect matchings $M_1', M_2', \ldots, M_8'$ then the probability that they are disjoint is bounded below by an absolute constant. We omit the proof. It mirrors the proof that the configuration model (Bollob'as [1]) of 8-regular (multi)graphs is simple with probability bounded below. So, if $\Gamma' = \bigcup_{i=1}^{8} M_i'$ has a rainbow Hamilton cycle w.h.p. when each color appears exactly 4 times, then so does Γ. It is however well-known, see for example Wormald [8] that Γ' is contiguous to the random 8-regular graph $G_{n,8}$ and this implies Theorem 1.4 for the case where n is even.

When $n = 2\nu + 1$ is odd, and $m = \omega n \log n$ where $\omega \to \infty$ then we proceed as follows. Let $p = m/N$ and for convenience, we work with $G = G_{n,p}^{(n)}$, an edge colored copy of $G_{n,p}$, in place of $G_{n,m}^{(n)}$. We decompose $G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_{\omega/K}$ where each Γ_i is an almost independent copy of $G_{n,p'}^{(n)}$ where $1 - p = (1 - p')^{\omega/K}$. The dependence will come when we insist that if an edge appears in Γ_i and $\Gamma_{i'}$ then it has the same color in both. We fix an i and we choose some edge $e = \{x, y\}$ and contract it to a vertex ξ. We also delete all edges of Γ_i that have color $c(e)$ to obtain Γ_i'. Edges in Γ_i' between vertices not including ξ now occur independently with probability $p'' = (n-1)p'/n$. Edges involving ξ appear with about twice this probability. Now $n - 1$ is even and by making K large enough, we can make the probability that any Γ_i' fails to contain a rainbow Hamilton cycle H_i less than $1/n$. Let $e_j = \{\xi, z_j\}, j = 1, 2$ be the edges of H_i that are incident with ξ. Now replace ξ with x, y. If the edges e_1, e_2 are disjoint in Γ_i then H_i can be lifted to a rainbow Hamilton cycle in Γ_i. This happens with probability 1/2 and the lift successes are independent. So the probability that none of the Γ_i contain a rainbow Hamilton cycle is at most $2^{-\omega/K} \to 0$. This completes the proof of Theorem 1.4.

References

[1] B. Bollobás, A probabilistic proof of an asymptotic formula for the number of labelled regular graphs, *European Journal of CVombinatorics* 1 (1980) 311-316.

[2] P. Erdős and J. Spencer, Lopsided Lovsz Local Lemma and Latin transversals, *Discrete Applied Mathematics* 30 (1991) 151154.

[3] A.M. Frieze, Loose Hamilton Cycles in Random 3-Uniform Hypergraphs, *Electronic Journal of Combinatorics* 17, N28.

[4] A.M. Frieze and P. Loh, Rainbow Hamilton cycles in random graphs, to appear.

[5] S. Janson and N. Wormald, Rainbow Hamilton cycles in random regular graphs, *Random Structures Algorithms* 30 (2007) 35-49.

[6] A. Johansson, J. Kahn and V. Vu, Factors in Random Graphs, *Random Structures and Algorithms* 33 (2008) 1-28.

[7] G. Perarnau and O. Serra, Rainbow Matchings: existence and counting, *Combinatorics, Probability and Computing* 22, (2013) 783-799.

[8] N. Wormald, Models of random regular graphs, in *Surveys in Combinatorics, London Mathematical Society Lecture Note Series 267*, Cambridge University Press, Cambridge, 1999 (J.D.Lamb and D.A.Preece, Eds.), Proceedings of the 1999 British Combinatorial Conference, Cambridge University Press, 239-298.