Rainbow Hamilton cycles in uniform hypergraphs

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Abstract

Let $K_n^{(k)}$ be the complete k-uniform hypergraph, $k \geq 3$, and let ℓ be an integer such that $1 \leq \ell \leq k-1$ and $k-\ell$ divides n. An ℓ-overlapping Hamilton cycle in $K_n^{(k)}$ is a spanning subhypergraph C of $K_n^{(k)}$ with $n/(k-\ell)$ edges and such that for some cyclic ordering of the vertices each edge of C consists of k consecutive vertices and every pair of adjacent edges in C intersects in precisely ℓ vertices.

We show that, for some constant $c = c(k,\ell)$ and sufficiently large n, for every coloring (partition) of the edges of $K_n^{(k)}$ which uses arbitrarily many colors but no color appears more than $cn^{k-\ell}$ times, there exists a rainbow ℓ-overlapping Hamilton cycle C, that is every edge of C receives a different color. We also prove that, for some constant $c' = c'(k,\ell)$ and sufficiently large n , for every coloring of the edges of $K_n^{(k)}$ in which the maximum degree of the subhypergraph induced by any single color is bounded by $c'n^{k-\ell}$, there exists a properly colored ℓ-overlapping Hamilton cycle C, that is every two adjacent edges receive different colors. For $\ell = 1$, both results are (trivially) best possible up to the constants. It is an open question if our results are also optimal for $2 \leq \ell \leq k-1$.

The proofs rely on a version of the Lovász Local Lemma and incorporate some ideas from Albert, Frieze, and Reed.

1 Introduction

By a *coloring* of a hypergraph H we mean any function $\phi : H \to \mathbb{N}$ assigning natural numbers (colors) to the edges of H. (In this paper we do not consider vertex colorings.)

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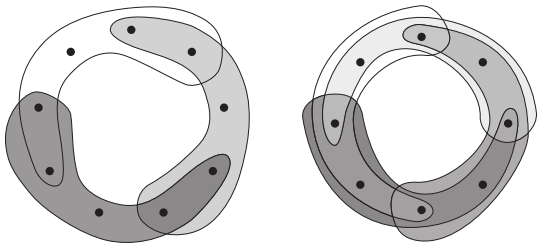

Figure 1: A 2-overlapping and a 3-overlapping 5-uniform cycle.

A hypergraph H together with a given coloring ϕ will be dubbed *a colored hypergraph*. A subhypergraph F of a colored hypergraph H is said to be *properly colored* if every two adjacent edges of F receive different colors. (Two different edges are *adjacent* if they share at least one vertex.) We say that a subhypergraph F of a colored hypergraph H is *rainbow* if every edge of F receives a different color, that is, when ϕ is injective on F.

In order to force the presence of properly colored or rainbow subhypergraphs one has to restrict the colorings ϕ, either globally or locally. A coloring ϕ is *r-bounded* if every color is used at most r times, that is, $|\phi^{-1}(i)| \leq r$ for all $i \in \mathbb{N}$. A coloring ϕ is *r-degree bounded* if the hypergraph induced by any single color has maximum degree bounded by r, that is, $\Delta(H[\phi^{-1}(i)]) \leq r$ for all $i \in \mathbb{N}$.

In this paper we study the existence of properly colored and rainbow Hamilton cycles in colored k-uniform complete hypergraphs, $k \geq 3$. (A hypergraph is *k-uniform* if every edge has size k; it is *complete* if all k-element subsets of the vertices form edges.) There is a broad literature on this subject for $k = 2$, that is, for graphs. Indeed, setting $r = cn$, Alon and Gutin proved in [2], improving upon earlier results from [5, 6, 16] that if $c < 1 - 1/\sqrt{2}$ then any r-degree bounded coloring of the edges of the complete graph K_n yields a properly colored Hamilton cycle (for the history of the problem, see [3]). It had been conjectured in [5] that the constant $1 - 1/\sqrt{2}$ can be replaced by $\frac{1}{2}$ which is the best possible.

Rainbow Hamilton cycles in r-bounded colorings of the complete graph have been studied in [1, 8, 10, 12]. Hahn and Thomassen conjectured that their existence is guaranteed if $r = cn$ for some $c > 0$. This was confirmed by Albert, Frieze, and Reed in [1] with $c = \frac{1}{64}$. Again, $c = 1/2$ seems to be a critical value here, since one can use each of $n - 1$ colors exactly $n/2$ times, making the presence of rainbow Hamilton cycles impossible. In striking contrast, there is literally nothing known on properly colored or rainbow Hamilton cycles in colored k-uniform hypergraphs for $k \geq 3$.

The notion of a hypergraph cycle can be ambiguous. In this paper we are *not* concerned with the Berge cycles as defined by Berge in [4] (see also [11]). Instead, following a recent trend in the literature ([7, 13, 15]), given an integer $1 \leq \ell < k$, we define an *ℓ-overlapping cycle* as a k-uniform hypergraph in which, for some cyclic ordering of its vertices, every edge consists of k consecutive vertices, and every two consecutive edges (in the natural ordering of the edges induced by the ordering of the vertices) share exactly ℓ vertices. (See Fig. 1 for an example of a 2-overlapping and a 3-overlapping 5-uniform cycle.)

The two extreme cases of $\ell = 1$ and $\ell = k - 1$ are referred to as, respectively, *loose* and *tight* cycles. Note that the number of edges of an ℓ-overlapping cycle with s vertices is $s/(k - \ell)$. Note also that when $k - \ell$ divides s, every tight cycle on s vertices contains an

ℓ-overlapping cycle on the same vertex set (with the same cyclic ordering).

Given a k-uniform hypergraph H on n vertices, where $k-\ell$ divides n, an ℓ-overlapping cycle contained in H is called *Hamilton* if it goes through every vertex of H, that is, if $s = n$. We denote such a Hamilton cycle by $C_n^{(k)}(\ell)$.

In this paper we prove the following two results. Let $K_n^{(k)}$ be the complete k-uniform hypergraph of order n.

Theorem 1.1 *For every $1 \le \ell < k$ there is a constant $c = c(k,\ell)$ such that if n is sufficiently large and $k-\ell$ divides n then any $cn^{k-\ell}$-bounded coloring of $K_n^{(k)}$ yields a rainbow copy of $C_n^{(k)}(\ell)$.*

Theorem 1.2 *For every $1 \le \ell < k$ there is a constant $c' = c'(k,\ell)$ such that if n is sufficiently large and $k-\ell$ divides n then any $c'n^{k-\ell}$-degree bounded coloring of $K_n^{(k)}$ yields a properly colored copy of $C_n^{(k)}(\ell)$.*

Note that for loose Hamilton cycles (i.e. $\ell = 1$) the above results are optimal up to the values of c and c'. Theorem 1.2 is trivially optimal, as the largest maximum degree can be at most $r = \binom{n-1}{k-1} \sim \frac{1}{(k-1)!}n^{k-1}$. To see that also Theorem 1.1 is optimal up to the constant for $\ell = 1$ consider any coloring of $K_n^{(k)}$ using each color precisely

$$r = \frac{\binom{n}{k}}{\frac{n}{k-1} - 1} \sim \frac{k-1}{k!}n^{k-1}$$

times, and thus using only $\frac{n}{k-1} - 1$ colors altogether. Such a coloring is r-bounded and, clearly, there is no rainbow copy of $C_n^{(k)}(1)$.

Problem 1.3 For all $k \ge 3$ and $\ell = 1$, determine $\sup c$ and $\sup c'$ over all values of c and, respectively, c' for which Theorems 1.1 and 1.2 hold.

We believe that Theorems 1.1 and 1.2 are optimal up to the constants also for $\ell \ge 2$, that is, we believe that the answer to the following question is positive.

Problem 1.4 For all $k \ge 3$ and $2 \le \ell \le k-1$, does there exist an r-bounded (r-degree bounded) coloring ϕ of $K_n^{(k)}$ such that $r = \Theta(n^{k-l})$ and no copy of $C_n^{(k)}(\ell)$ is rainbow (properly colored)?

As some evidence supporting our belief, consider the bipartite version of both problems for $k = 3$ and $\ell = 2$. Let $K_{n,2n}^{(3)} = (V_1, V_2, E)$, where $|V_1| = n$, $|V_2| = 2n$ and $E = \{e \subset V_1 \cup V_2 \ : \ |e \cap V_i| = i, \quad i = 1,2\}$. To every edge e assign the pair $e \cap V_2$ as its color. Clearly, every color appears exactly n times and hence such a coloring is n-bounded (and thus n-degree bounded). Finally, note that every tight Hamilton cycle in $K_{n,2n}^{(3)}$ induces a cyclic sequence of vertices with a repeated pattern of two vertices from V_2 followed by one vertex from V_1. Hence, there is a pair of consecutive edges with the same color (actually, there are n such pairs), and so no copy of a properly colored (or rainbow) $C_n^{(3)}(2)$ exists.

2 The proofs

We will need a special version of the Lovász Local Lemma. A similar result was already established in [1, 9, 14]. Contrary to the above results, in our formulation of the lemma we avoid conditional probabilities so that we do not need to make a priori assumptions that certain events have positive probability.

Lemma 2.1 *Let $A_1, A_2, \ldots, A_m$ be events in an arbitrary probability space Ω. For each $1 \le i \le m$, let $[m] \setminus \{i\} = X_i \cup Y_i$ be a partition of the index set $[m] \setminus \{i\}$ and let*

$$d = \max\{|Y_i| : 1 \le i \le m\}.$$

If for each $1 \le i \le m$ and all $X \subseteq X_i$

$$\Pr\left(A_i \cap \bigcap_{j \in X} \overline{A_j}\right) \le \frac{1}{4(d+1)} \Pr\left(\bigcap_{j \in X} \overline{A_j}\right) \tag{1}$$

then $\Pr\left(\bigcap_{i=1}^{m} \overline{A_i}\right) > 0$. (We adopt the convention that $\bigcap_{j \in \emptyset} \overline{A_j} = \Omega$.)

Proof. We prove by induction on $t = 1, \ldots, m$, that for every $T \subseteq [m]$, $|T| = t$, and for every $i \in T$, setting $S = T \setminus \{i\}$, we have

$$\Pr\left(\bigcap_{i \in T} \overline{A_i}\right) > 0 \quad \text{and} \quad \Pr\left(A_i \Big| \bigcap_{i \in S} \overline{A_i}\right) \le \frac{1}{2(d+1)}. \tag{2}$$

For $t = 1$ we apply the (1) with $X = \emptyset$, obtaining for each i that $\Pr(A_i) \le \frac{1}{4(d+1)}$, equivalently $\Pr\left(\overline{A_i}\right) \ge 1 - \frac{1}{4(d+1)} > 0$, which confirms (2) for $t = 1$.

Now, assume truth for some t, $1 \le t \le m-1$, and consider a set $T = \{i\} \cup S$, where $i \notin S$ and $|S| = t$. Set $X = S \cap X_i$ and $Y = S \cap Y_i$, and observe that $S = X \cup Y$ and $|Y| \le |Y_i| \le d$. By the induction assumption $\Pr(\bigcap_{j \in S} \overline{A_j}) > 0$. If $Y = \emptyset$ (and thus $X = S$), by our assumption (1),

$$\Pr\left(A_i \Big| \bigcap_{j \in S} \overline{A_j}\right) \le \frac{1}{4(d+1)}.$$

Otherwise, $|X| < |S| = t$ and, again by (1) (in the numerator) and the induction assumption (in the denominator) we argue that

$$\Pr\left(A_i \Big| \bigcap_{j \in S} \overline{A_j}\right) = \frac{\Pr\left(A_i \cap \bigcap_{j \in Y} \overline{A_j} \Big| \bigcap_{j \in X} \overline{A_j}\right)}{\Pr\left(\bigcap_{j \in Y} \overline{A_j} \Big| \bigcap_{j \in X} \overline{A_j}\right)} \le \frac{\frac{1}{4(d+1)}}{1 - |Y|\frac{1}{2(d+1)}} \le \frac{1}{2(d+1)}.$$

Thus,

$$\Pr\left(\bigcap_{j \in T} \overline{A_j}\right) = \Pr\left(\overline{A_i} \Big| \bigcap_{j \in S} \overline{A_j}\right) \Pr(\bigcap_{j \in S} \overline{A_j}) > 0,$$

which completes the proof of Lemma 2.1. □

The proofs of Theorems 1.1 and 1.2 extend some ideas introduced by Albert, Frieze and Reed in [1] and are based on the following technical result.

Proposition 2.2 *For all $1 \leq \ell < k$ there exist constants $\delta = \delta(k,\ell)$, $0 < \delta < 1$, such that for every pair e, f of edges of $K_n^{(k)}$ with $|e \cap f| \leq \ell$ and for every set X of pairs g, h of edges of $K_n^{(k)}$ satisfying $(g \cup h) \cap (e \cup f) = \emptyset$ the following holds. Let*

- *$\mathcal{C}$ be the set of all copies C of $C_n^{(k)}(k-1)$ in $K_n^{(k)}$ such that $\{g, h\} \not\subseteq C$ for all $\{g, h\} \in X$, and*
- *$\mathcal{C}_1 = \{C \in \mathcal{C} \,:\, \{e, f\} \subset C\}$.*

Then, if $\mathcal{C}_1 \neq \emptyset$, one can find a disjoint *family $\{\mathcal{S}_C : C \in \mathcal{C}_1\}$ of sets of copies of $C_n^{(k)}(k-1)$ from $\mathcal{C}$ (indexed by the copies $C \in \mathcal{C}_1$) such that for all $C \in \mathcal{C}_1$ we have $|\mathcal{S}_C| \geq \delta n^{2k-\ell-1}$.*

First we show that Proposition 2.2 together with Lemma 2.1 imply both Theorem 1.1 and Theorem 1.2.

Proof of Theorem 1.1. Let $1 \leq \ell < k$ and set $c = \frac{\delta}{10k}$, where $\delta = \delta(k,\ell)$ is the constant given by Proposition 2.2. Fix a $cn^{k-\ell}$-bounded coloring ϕ of $K_n^{(k)}$ and define

$$M = \{\{e, f\} \,:\, e, f \in K_n^{(k)}, \quad |e \cap f| \leq \ell \text{ and } \phi(e) = \phi(f)\}.$$

Moreover, for every pair $\{e, f\} \in M$ set

$$A_{e,f} = \{C \subset K_n^{(k)} \,:\, C \cong C_n^{(k)}(k-1) \text{ and } \{e, f\} \subset C\}.$$

In order to prove Theorem 1.1 it suffices to show that

$$\bigcap_{\{e,f\} \in M} \overline{A_{e,f}} \neq \emptyset. \tag{3}$$

Indeed, if (5) is true then there is a tight Hamilton cycle $C \cong C_n^{(k)}(k-1)$ such that for every pair of its edges e and f with $|e \cap f| \leq \ell$ we have $\phi(e) \neq \phi(f)$. Since, by assumption, $k - \ell$ divides n, C contains a copy of $C_n^{(k)}(\ell)$ which is rainbow, as required.

To prove (5) we apply the probabilistic method and Lemma 2.1. To this end, for a given pair $\{e, f\} \in M$ let

$$Y_{e,f} = \{\{e', f'\} \in M : \{e', f'\} \neq \{e, f\} \text{ and } (e \cup f) \cap (e' \cup f') \neq \emptyset\}$$

and

$$X_{e,f} = M \setminus (Y_{e,f} \cup \{e, f\}).$$

To estimate d, we bound from above the size of $Y_{e,f}$ as follows. For given edges e and f we can find at most $2kn^{k-1}$ edges e' sharing a vertex from $e \cup f$. For every such e' we have at most $cn^{k-\ell}$ candidates for f', since e' and f' must have the same color. Thus,

$$d = \max_{\{e,f\} \in M} |Y_{e,f}| \leq 2ckn^{2k-\ell-1} \leq \frac{\delta n^{2k-\ell-1}}{4} - 1.$$

Now, let us consider a uniform probability space consisting of all tight Hamilton cycles $C \cong C_n^{(k)}(k-1)$ in $K_n^{(k)}$. In order to prove (5), and thus finish the proof of Theorem 1.1, it suffices to show that

$$\Pr\left(\bigcap_{\{e,f\}\in M} \overline{A_{e,f}}\right) > 0. \tag{4}$$

Thus, it remains to verify assumption (1) of Lemma 2.1. Fix $\{e,f\} \in M$ and $X \subseteq X_{e,f}$. If $\mathcal{C}_1 = \emptyset$, there is nothing to prove. Otherwise, by Proposition 2.2,

$$\frac{\Pr\left(A_{e,f} \cap \bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}\right)}{\Pr\left(\bigcap_{\{e',f'\}\in X} \overline{A_{e',f'}}\right)} = \frac{|\mathcal{C}_1|}{|\mathcal{C}|} \leq \frac{|\mathcal{C}_1|}{\sum_{C\in\mathcal{C}_1} |\mathcal{S}_C|} \leq \frac{1}{\delta n^{2k-\ell-1}} \leq \frac{1}{4(d+1)}.$$

Hence, we are in position to apply Lemma 2.1 with $m = |M|$, $A_i := A_{e,f}$, $X_i := X_{e,f}$, and $Y_i := Y_{e,f}$, and conclude that (4), and consequently (5) holds. This completes the proof of Theorem 1.1. □

Proof of Theorem 1.2. This proof goes along the lines of the proof of Theorem 1.1. Let $c' = \frac{\delta}{10k^2}$, where $\delta = \delta(k,\ell)$ is the constant given by Proposition 2.2. Fix a $dn^{k-\ell}$-degree bounded coloring of $K_n^{(k)}$. Here we slightly modify the definition of M. Let

$$M = \{\{e,f\} \,:\, e,f \in K_n^{(k)}, \quad 1 \leq |e\cap f| \leq \ell \text{ and } \phi(e) = \phi(f)\}.$$

As before,

$$A_{e,f} = \{C \subset K_n^{(k)} \,:\, C \cong C_n^{(k)}(k-1) \text{ and } \{e,f\} \subset C\}.$$

and in order to prove Theorem 1.2 it suffices to show that

$$\bigcap_{\{e,f\}\in M} \overline{A_{e,f}} \neq \emptyset. \tag{5}$$

Indeed, if (5) is true then there is a tight Hamilton cycle $C \cong C_n^{(k)}(k-1)$ such that for every pair of its edges e and f with $1 \leq |e\cap f| \leq \ell$ we have $\phi(e) \neq \phi(f)$. Since, by assumption, $k-\ell$ divides n, C contains a copy of $C_n^{(k)}(\ell)$ which is properly colored, as required.

We define sets $Y_{e,f}$ and $X_{e,f}$ as before and recalculate the upper bound on $|Y_{e,f}|$. For given edges e and f we can find at most $2kn^{k-1}$ edges e' sharing a vertex from $e \cup f$. For every such e' we have at most $c'kn^{k-\ell}$ candidates for f' since e' and f' intersect and have the same color. Thus,

$$|Y_{e,f}| \leq 2c'k^2n^{2k-\ell-1} \leq \frac{\delta n^{2k-\ell-1}}{4} - 1.$$

The rest of the proof is identical to the proof of Theorem 1.1 and therefore is omitted. □

3 Proof of Proposition 2.2

Let e and f be given edges in $K_n^{(k)}$ such that $|e\cap f| \leq \ell$ and let $C \in \mathcal{C}_1$ be a tight Hamilton cycle containing e and f and missing at least one edge from each pair $\{g,h\} \in X$. We describe two constructions depending on the size of $e\cap f$.

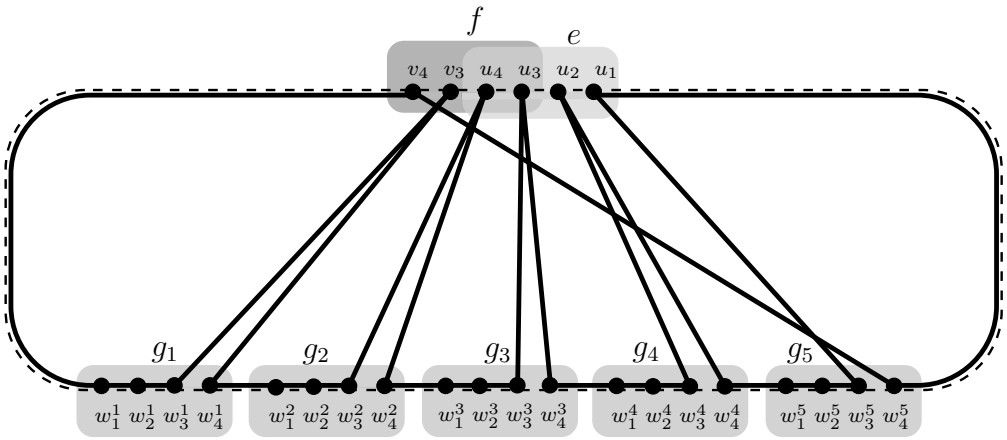

Figure 2: The first construction for $k = 4$ and $a = 2$. The dashed and solid lines denote C and $\tilde{C}$, respectively.

Construction 1: for $2 \leq |e \cap f| \leq \ell$.
Let $|e \cap f| = a$ and let $e = (u_1, \ldots, u_k)$ and $f = (v_1, \ldots, v_k)$ be such that $u_{k-a+1} = v_1, u_{k-a+2} = v_2, \ldots, u_k = v_a$. This way we fix an orientation of C where e precedes f. Let $P = C \setminus \{e \cup f\}$ be the segment of C between f and e of length $n - 2k + a$. We select arbitrarily $2k - a - 1$ vertex disjoint edges $g_1, \ldots, g_{2k-a-1}$ from P, so that C is of the form $e \rightsquigarrow f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g_{2k-a-1} \rightsquigarrow e$, where the symbol $\rightsquigarrow$ indicates a path between the given edges. Clearly, we have $\Omega(n^{2k-a-1}) = \Omega(n^{2k-\ell-1})$ choices for g_i's.

Let $g_i = (w_1^i, \ldots, w_k^i)$ for $1 \leq i \leq 2k - a - 1$, where we list the vertices of g_i in the order of appearance on P. In order to create a cycle $\tilde{C} \in \mathcal{S}_C$, we remove all edges which contain at least one vertex from $(e \cup f) \setminus \{u_1, v_k\}$ and also all edges whose first vertex (in the order induced by C) is w_j^i for $1 \leq i \leq 2k-a-1$ and $j = 1, \ldots, k-1$. After this removal, the vertices in the set $\{u_2, \ldots, u_k, v_{a+1}, \ldots, v_{k-1}\}$ become isolated and the remains of the cycle C form a collection of vertex disjoint paths $v_k \rightsquigarrow w_{k-1}^1, w_k^1 \rightsquigarrow w_{k-1}^2, w_k^2 \rightsquigarrow w_{k-1}^3, \ldots, w_k^{2k-a-1} \rightsquigarrow u_1$.

To create $\tilde{C}$, we connect the above paths by absorbing the isolated vertices. Formally, we define $\tilde{C}$ as the following sequence of vertices (see Fig. 2):

$$
\begin{aligned}
v_k &\rightsquigarrow w_1^1, w_2^1, \ldots, w_{k-1}^1, v_{k-1}, w_k^1 \\
&\rightsquigarrow w_1^2, w_2^2, \ldots, w_{k-1}^2, v_{k-2}, w_k^2 \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w_1^{k-a-1}, w_2^{k-a-1}, \ldots, w_{k-1}^{k-a-1}, v_{a+1}, w_k^{k-a-1} \\
&\rightsquigarrow w_1^{k-a}, w_2^{k-a}, \ldots, w_{k-1}^{k-a}, u_k, w_k^{k-a} \\
&\rightsquigarrow w_1^{k-a+1}, w_2^{k-a+1}, \ldots, w_{k-1}^{k-a+1}, u_{k-1}, w_k^{k-a+1} \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w_1^{2k-a-2}, w_2^{2k-a-2}, \ldots, w_{k-1}^{2k-a-2}, u_2, w_k^{2k-a-2} \\
&\qquad \rightsquigarrow w_1^{2k-a-1}, w_2^{2k-a-1}, \ldots, w_{k-1}^{2k-a-1}, u_1 \rightsquigarrow w_k^{2k-a-1}, v_k.
\end{aligned}
$$

It is easy to check that every new edge intersects a vertex from $e \cup f$. Thus, $\tilde{C} \setminus C$ contains

no edge from any pair of edges belonging to X. Moreover, note that different choices of g_i yield different cycles $\tilde{C}$. Thus, $|\mathcal{S}_C| = \Omega(n^{2k-\ell-1})$.

It remains to show that for any two tight Hamilton cycles $C \neq C' \in \mathcal{C}_1$ we have $\mathcal{S}_C \cap \mathcal{S}_{C'} = \emptyset$. In order to prove it, one can reverse the above procedure and uniquely determine C and the edges $g_1, g_2, \ldots, g_{2k-a-1}$ from $\tilde{C}$. Note that we do not know the order in which the vertices of e and f are traversed by C.

To reconstruct C, we first find in $\tilde{C}$ a unique $e \rightsquigarrow f$ path with no endpoint in $e \cap f$, say Q. From this we deduce that $u_1 = Q \cap e$, $v_k = Q \cap f$ and w_k^{2k-a-1} is the last vertex on Q before v_k. Now we start at v_k and follow $\tilde{C}$ in the direction opposite to w_k^{2k-a-1}. Before we reach u_1 we will intersect $f \cup e$ exactly $2k-a-2$ times. This way we restore the vertices $v_{k-1}, v_{k-2}, \ldots, v_{a+1}, u_k, u_{k-1}, \ldots, u_2$ (in the order of appearance on C). Note that every one of these vertices is adjacent to two vertices w_{k-1}^j and w_k^j for some $1 \leq j \leq 2k-a-2$. Consequently, we can uncover all edges g_i and hence C itself.

Construction 2: for $|e \cap f| \leq 1$.
Here we show a stronger result, namely, we construct a family $\mathcal{S}_C$ of size $\Omega(n^{2(k-1)})$. Let $e = (u_1, \ldots, u_k)$ and $f = (v_1, \ldots, v_k)$. Note that it might happen that $u_k = v_1$ if $|e \cap f| = 1$. Let P be a segment of vertices between e and f of size $\Omega(n)$. For given two vertices x and y denote by $d(x, y)$ the number of vertices on P in the segment between x and y. Now we select $2(k-1)$ vertex disjoint edges $g_1, \ldots, g_{2(k-1)}$ from P so that C is of the form $e \rightsquigarrow f \rightsquigarrow g_1 \rightsquigarrow \cdots \rightsquigarrow g_{2(k-1)} \rightsquigarrow e$ and

$$d(v_k, w_{k-1}^1) < d(w_k^1, w_{k-1}^2) + 1 < d(w_k^2, w_{k-1}^3) + 1 < \cdots < d(w_k^{k-2}, w_{k-1}^{k-1}) + 1. \tag{6}$$

This way we fix an orientation of C. (The above sequence of inequalities will be needed later to establish the orientation of C from $\tilde{C}$.) Clearly, we have $\Omega(n^{2(k-1)})$ choices for g_i's.

Let $g_i = (w_1^i, \ldots, w_k^i)$ for $1 \leq i \leq 2(k-1)$, where we list the vertices of g_i in the order of appearance on P. In order to create a cycle $\tilde{C} \in \mathcal{S}_C$, we remove all edges which contain at least one vertex from $(e \cup f) \setminus \{u_1, v_k\}$ and also all edges whose first vertex (in the order induced by C) is w_j^i for $1 \leq i \leq 2(k-1)$ and $j = 1, \ldots, k-1$. After this removal, the vertices in the set $\{u_2, \ldots, u_{k-1}, v_2, \ldots, v_{k-1}\}$ become isolated and the remains of the cycle C form a collection of vertex disjoint paths $v_k \rightsquigarrow w_{k-1}^1, w_k^1 \rightsquigarrow w_{k-1}^2, w_k^2 \rightsquigarrow w_{k-1}^3, \ldots, w_k^{2(k-1)} \rightsquigarrow u_1$, and $u_k \rightsquigarrow v_1$. (The latter may be degenerated to the set of isolated vertices.)

To create $\tilde{C}$, we connect the above paths by absorbing the isolated vertices. Formally, we define $\tilde{C}$ as the following sequence of vertices (see Fig. 3):

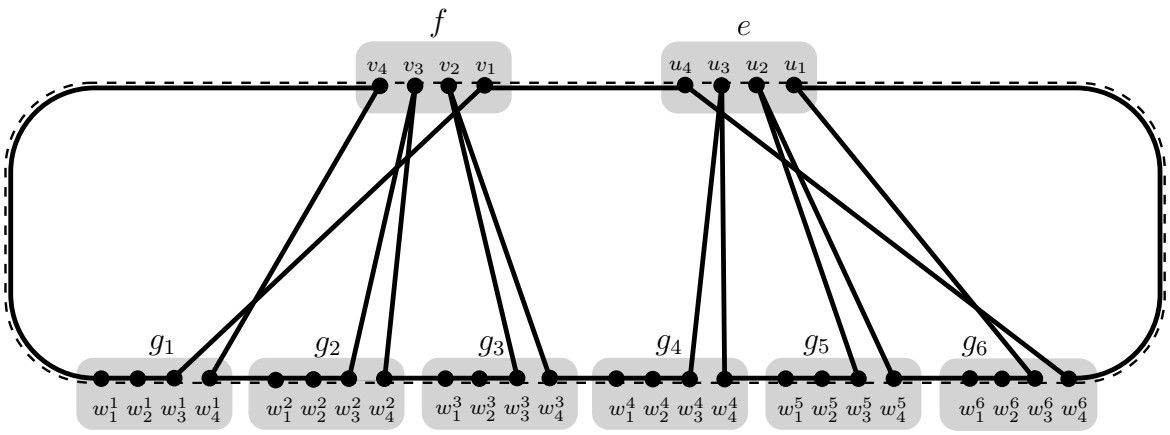

Figure 3: The second construction for $k = 4$. The dashed and solid lines denote C and $\tilde{C}$, respectively.

$$
\begin{aligned}
v_1, w^1_{k-1}, w^1_{k-2}, \ldots, w^1_1 &\rightsquigarrow v_k, w^1_k \\
&\rightsquigarrow w^2_1, w^2_2, \ldots, w^2_{k-1}, v_{k-1}, w^2_k \\
&\rightsquigarrow w^3_1, w^3_2, \ldots, w^3_{k-1}, v_{k-2}, w^3_k \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w^{k-1}_1, w^{k-1}_2, \ldots, w^{k-1}_{k-1}, v_2, w^{k-1}_k \\
&\rightsquigarrow w^k_1, w^k_2, \ldots, w^k_{k-1}, u_{k-1}, w^k_k \\
&\rightsquigarrow w^{k+1}_1, w^{k+1}_2, \ldots, w^{k+1}_{k-1}, u_{k-2}, w^{k+1}_k \\
&\rightsquigarrow \ldots \\
&\rightsquigarrow w^{2k-3}_1, w^{2k-3}_2, \ldots, w^{2k-3}_{k-1}, u_2, w^{2k-3}_k \\
&\qquad \rightsquigarrow w^{2k-2}_1, w^{2k-2}_2, \ldots, w^{2k-2}_{k-1}, u_1 \rightsquigarrow w^{2k-2}_k, u_k \rightsquigarrow v_1.
\end{aligned}
$$

It is easy to check that every new edge intersects a vertex from $e \cup f$. Thus, $\tilde{C} \setminus C$ contains no edge from any pair of edges belonging to X. Moreover, note that different choices of g_i yield different cycles $\tilde{C}$. Thus, $|\mathcal{S}_C| = \Omega(n^{2(k-1)})$.

It remains to show that for any two tight Hamilton cycles $C \neq C' \in \mathcal{C}_1$ we have $\mathcal{S}_C \cap \mathcal{S}_{C'} = \emptyset$. In order to prove it, one can reverse the above procedure and uniquely determine C and the edges $g_1, g_2, \ldots, g_{2(k-1)}$ from $\tilde{C}$. Note that we do not know the order in which the vertices of e and f are traversed by C.

Note that there are exactly two $e \rightsquigarrow f$ paths in $\tilde{C}$, say Q_1 and Q_2. One with u_k and v_1 as its endpoints and the second one with u_{k-1} and v_2 as the endpoints. Our goal is to determine vertex v_1. Once this is known then as in Construction 1 we can uncover all edges g_i and hence C itself.

We assume for a while that $v_1 = Q_1 \cap f$. Then we start at v_1 and follow $\tilde{C}$ in the direction opposite to the second endpoint of Q_1. Before we reach edge e we will intersect

edge f exactly $k-1$ times. This way we pretend that we restore vertices $v_k, v_{k-1}, \ldots, v_2$ (in the order of appearance). Let $\tilde{d}(x,y)$ be the number of vertices on $\tilde{C}$ between vertices x and y. Note that $d(v_k, w_{k-1}^1) = \tilde{d}(v_1, v_k) - 1$ and $d(w_k^j, w_{k-1}^{j+1}) = \tilde{d}(v_{k-j+1}, v_{k-j}) - 2$ for $1 \leq j \leq k-2$. Now we check if (6) holds. If so then Q_1 is really the path with endpoints u_k and v_1; otherwise Q_2 is the one.

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