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Finding Hamilton Cycles in Sparse Random Graphs

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We describe a polynomial time ($O(n^3 \log n)$) algorithm which has a high probability of finding hamilton cycles in two classes of random graph which have constant average degree: the m-out model and the random regular graph model. We also show how the algorithm can be used to find a large cycle in a sparse random graph. © 1988 Academic Press, Inc.

INTRODUCTION

The past few years have seen some important progress with respect to the problem of the existence of hamilton cycles in random graphs. The paper of Komlós and Szemerédi [15] gave the exact threshold for the existence of hamilton cycles in the random graph $G_{n,m}$, tightening the result of Pósa [16]. Bollobás, Fenner and Frieze [7] described a polynomial time algorithm HAM for finding hamilton cycles which gives a constructive proof of the result in [15]. This improved the previous results of Angluin and Valiant [2] and Shamir [18].

In [13] we studied random travelling salesman problems and gave modifications to HAM which enabled us to prove that it has a high probability of success on graphs with considerably fewer edges than needed for the threshold [15] provided the minimum degree is high enough. In this paper we continue this development for three classes of random graph with constant average degree.

We first consider a variation on the class of graphs studied by Fenner and Frieze [9]. Let $v \in V_n = \{1, 2, ..., n\}$ independently make m random choices $c(v, i) \in V_n$, $i = 1, 2, ..., m$. These choices are not necessarily distinct. This is done independently for each $v \in V_n$. Then $D(n, m) =$ the multigraph $(V_n, E(n, m))$, where $E(n, m) = \{(v, c(v, i)): v \in V_n,\ 1 \leqslant i \leqslant m$ and

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$v \neq c(v, i)\}$), i.e., we ignore orientation in $(v, c(v, i))$, but do not coalesce multiple edges or remove loops.

The main result of [9] is $\lim_{n \to \infty} \Pr(D(n, 23)$ is hamiltonian$) = 1$.

The proof is existential and we make the following

Conjecture. $\lim_{n \to \infty} \Pr(D(n, 3)$ is hamiltonian$) = 1$.

Our first result gets a little closer to this conjecture:

THEOREM 1.1. *There is an* $O(n^3 \log n)$ *time algorithm* HAM1 *which satisfies*

$$\lim_{n \to \infty} \Pr(\text{HAM1 } \textit{finds a hamilton cycle in } D(n, 10)) = 1$$

We next consider random regular graphs. Here we let $R(n, r)$ denote a random regular graph chosen uniformly from the set of graphs on V_n which are regular of degree r. Bollobás [5] and Fenner and Frieze [10] independently gave proofs that there is a constant r_0 such that for any constant $r \geqslant r_0$

$$\lim_{n \to \infty} \Pr(R(n, r) \text{ is hamiltonian}) = 1.$$

The smaller value of r_0 was 796 of [10]. This paper improves this, but more importantly gives a polynomial time constructive proof. More specifically we have

THEOREM 1.2. *There is an* $O(n^3 \log n)$ *time algorithm* HAM2 *which satisfies*

$$\lim_{n \to \infty} \Pr(\text{HAM2 } \textit{finds a hamilton cycle in } R(n, r)) = 1$$

for any constant $r \geqslant 85$

It is reasonable to conjecture that $r_0 = 3$, especially as Richmond, Robinson and Wormald [17] have proved the corresponding result for the bipartite case.

Our final result concerns the random graph $G_{n,p}$, $p = c/n$, c constant, which has vertex set V_n and in which each of the $\binom{n}{2}$ above possible edges independently has probability p of being included and $1 - p$ of being excluded. Several papers [1, 11, 4, 6, 12] have been concerned with the length of the longest path or cycle in $G_{n,p}$. The strongest result, for large c, is given in [12]. For a graph G let $\lambda(G) =$ the length of the longest cycle in G. In [12] we show that

$$\lim_{n \to \infty} \Pr(\lambda(G_{n,p}) = (1 - ce^{-c}(1 + \varepsilon(c))n) = 1,$$

where $\lim_{c \to \infty} \varepsilon(c) = 0$.

The proof in [12] was again existential. Our final result is

THEOREM 1.3. *There is an* $O(n^3 \log n)$ *time algorithm* CYCLEFIND *which satisfies*

$$\lim_{n\to\infty} \Pr(\text{CYCLEFIND } \textit{constructs a cycle of length}$$

$$(1 - ce^{-c}(1+\varepsilon(c)))n) = 1.$$

(The result is still valid if $c = c(n) \to \infty$.)

Notation. We give some notation that is used throughout the paper.

A graph G has vertex set $V = V(G)$ and edge set $E = E(G)$. It has minimum degree $\delta(G)$ and $v \in V$ has degree $d_G(v)$.

If $S \subseteq V$ then $G[S] = (S, E_S)$, where $E_S = \{e \in E: e \subseteq S\}$. Also $N(S, G) = \{w \in V - S: \exists v \in S \text{ such that } (v, w) \in E\}$.

An event E_n will be said to occur *almost surely* (a.s.) if $\lim_{n\to\infty} \Pr(E_n) = 1$.

ALGORITHM HAM

The following idea has been used by many authors: given a path $P = (v_1, v_2, ..., v_k)$ plus an edge $e = (v_k, v_i)$, where $1 \leqslant i \leqslant k-2$, we can create another of length $k-1$ by deleting edge (v_i, v_{i+1}) and adding e. Thus let

$$\text{ROTATE}(P, e) = (v_1, v_2, ..., v_i, v_k, v_{k-1}, ..., v_{i+1}) \text{ and } \text{NEW}(P, e) = v_{i+1}.$$

v_1 is called the *fixed endpoint*, v_k is called the *rotated endpoint* and e is called the *rotation edge* of the rotation.

The algorithm we describe proceeds by a number of stages. At the beginning of the kth stage we have a path P_k of length k, with endpoints w_0, w_1. We try to extend P_k from w_1. If we fail but $(w_0, w_1) \in E(H)$ then assuming connectivity we can find a longer path. Failing this, we do a sequence of rotations with w_0 as a fixed endpoint, which creates new paths that we can try to extend or close. We apply the same construction to all these paths and so on until we have succeeded in obtaining a path of length $k+1$ or we have run out of paths to rotate. We then take this set of paths and treat each of them like P_k but using w_0 as the first rotated endpoint.

We construct our sequence of paths in a "depth-first" manner. Suppose the "current" path is Q. One end u will be kept fixed. Suppose its other end v has neighbours $x_1, x_2, ..., x_p$, where $x_1 \in Q$. We replace Q by ROTATE(Q, (u, x_1)) and continue with this "new" Q before considering x_2 and the "old" Q, which will be done after backtracking.

The above procedures are all perfectly natural. We now come to a somewhat unnatural procedure. It is included because without it we cannot make our proofs work. We would like someday to avoid this trick but at present we cannot. Our algorithm HAM assumes the partition of the edges $E(H)$ of the input graph H into 2 sets E_+ and E_-. The edges in E_- can only be used to close cycles. We define $H_+ = (V(H), E_+)$.

We now give a formal description:

HAM's input is a connected graph H with $\delta(H) \geqslant 2$ plus a partition of its edges into E_+ and E_-. We also assume that the input includes specific orderings of the vertex adjacency lists, i.e., for each $v \in V(H)$ HAM is given a total ordering of the set $N(v, H)$ of neighbours of v in H. In this context $\min_v(X)$, for $X \subseteq N(v, H)$, is the first vertex of X appearing in the adjacency lists for v.

```
Algorithm HAM
begin
  let P_0 be the degenerate path consisting of v_1 = min(V(H)) alone
    k := 0;

L0:
  begin [stage k begins here]
    longerpathfound := false;

L1:
  let P_k have endpoints w_0, w_1 where w_0 ∈ P_{k-1} and w_1 ∉ P_{k-1}; [of course
  when k = 1 w_0 = w_1 = v_1.
  storepaths := true; [When this variable is true the new paths generated
  by rotations during SEARCH are stored for later.
  END_k := {w_1} [We keep track of the endpoints of some paths.
  SEARCH(P_k, w_0, P); [Do rotations with w_0 as fixed endpoint
  if longerpathfound then [i.e. if SEARCH has found a path longer than
  P_k
  begin
    k := k + 1; P_k := P; goto LO
  end else
  storepaths := false;  [No need to store paths now
  for w ∈ END_k do      [SEARCH(P_k, w_0, P) constructs a set of paths
                        [{P(w_0, w): w ∈ END_k} where P(w_0, w)
                        [has endpoints w_0 and w.
  begin
    SEARCH(P(w_0, w), w, P);
    if longerpathfound then goto L1
  end;
  terminate unsuccessfully [successful termination with a hamilton cycle
      [occurs in SEARCH
  end;
  end;
```

```
procedure SEARCH(Q, u, P);
begin
   let v be the endpoint of Q other than u;
   DFS(u, *)                        [* is used as a marker here.
end;
procedure DFS(v, y); [vertex y is such that ROTATE(Q, {v, y})
      [reverses the rotation made immediately prior to this call of DFS.
begin
L2:
   let X = {x ∉ Q: {v, x} ∈ E_+};
   if X ≠ ∅ then
   begin
      x := min_v(X); p := Q + {v, x}; longerpathfound := true [extension
   end else
L3:
   if {u, v} ∈ E(H) then
   begin      [cycle extension
      let C be the cycle Q + {u, v};
      if C is a hamilton cycle then terminate HAM successfully
   else
   begin
      starting from u, let a be the first vertex along Q which is adjacent in
      H_+ to some vertex not in C; let B = {x ∈ C: {a, x} ∈ E_+};
      b := min_a(B); let a_1 and a_2 be the neighbours of a on C where a_1 < a_2;
      P := P + {a, b} − {a, a_1}; longerpathfound := true
      end
   end
   else
   begin
      let X = {x_1, x_2, ..., x_p};
      for i = 1 to p do
      if e = {v, x_i} ∈ E_+ and not longerpathfound and e has not been used
         previously as a rotation edge in the current execution of SEARCH
         then
      begin
         Q := ROTATE(Q, {v, x_i}); v' := NEW(Q, {v, x_i});
         if storepaths and v' ∉ END_k then
         begin
            END_k := END_k ∪ {v'}; P(w_0, v') := Q
         end;
         DFS(NEW(Q, {w, x_i}), x_i)
      end;
   end
```

if $y \neq *$ **then** $Q := \text{ROTATE}(Q, \{v, y\})$ [backtrack to the parent path
end;

Running Time of HAM

The running time of HAM is dominated by a factor dependent on the number of rotations. Using the idea of [2] we may do each rotation in $O(\log n)$ time. Each execution of SEARCH requires $\leqslant |E_+|$ rotations, each of the $\leqslant n$ stages requires $\leqslant n$ executions of SEARCH giving $O(n^3 \log n)$ time overall as in our examples $|E_+| = O(n)$ a.s.

For the remainder of this section we consider those executions of HAM that terminate unsuccessfully. In particular suppose that HAM terminates unsuccessfully in stage k. Let

$$\text{END}(H) = \text{END}_k \cup \{w_0\}$$

and

> $\text{END}(H, w) = \{v\colon v$ is an endpoint of a path created during the execution of $\text{SEARCH}(P(w_0, w), w, P)\}$

for $w \in \text{END}(H)$.

The following lemma is clear.

Lemma 2.1. *If* HAM *terminates unsuccessfully then* $w \in \text{END}(H)$, $v \in \text{END}(H, w)$ *implies that* $(v, w) \notin E(H)$.

A set $X \subseteq E_-$ is *deletable* if no $e \in X$ is used to close a cycle at statement L3 during the execution of HAM on H. The following lemma is also clear.

Lemma 2.2. *If* HAM *terminates unsuccessfully,* X *is deletable,* $H_X = (V_n, E(H) - X)$ *and the adjacency lists of* H_X *conform with those of* H *then* HAM *terminates unsuccessfully on* H_X *in stage* k. *Furthermore* $\text{END}(H_X) = \text{END}(H)$ *and* $\text{END}(H_X, w) = \text{END}(H, w)$ *for* $w \in \text{END}(H)$.

We will need to show that $|\text{END}(H)|$ is a.s. large. This will always be shown to follow from

Lemma 2.3. *If* HAM *terminates unsuccessfully then*

$$|N(\text{END}(H), H_+)| < 2|\text{END}(H)|. \tag{2.1a}$$

$$|N(\text{END}(H, w), H_+)| < 2|\text{END}(H, w)|. \tag{2.1b}$$

Proof. We modify the argument of Pósa [16]. We prove (2.1a); an almost identical argument will prove (2.1b). To prove (2.1a) we show that

$$x \in N(\mathrm{END}(H), H_+) \text{ implies } \exists y \in \mathrm{END}(H) \text{ such that } (x, y) \text{ is an edge of } P_k. \tag{2.2}$$

Suppose $x \in N(\mathrm{END}(H), H_+)$, $z \in \mathrm{END}(H)$, $e = (x, z) \in E_+ - P_k$ and neither of the neighbours u_1, u_2 of x on P_k is in $\mathrm{END}(H_X)$. $x \neq w_0$ since stage k terminated unsuccessfully. Eventually HAM creates a path P with z as an endpoint and e will be considered for rotation. It will not have been used before as $x \notin \mathrm{END}(H)$ and P will contain both edges (x, u_1), (x, u_2) because when an edge is deleted by a rotation one of the vertices is placed in $\mathrm{END}(H)$. Thus e will be used to rotate and at least one of u_1, u_2 is in $\mathrm{END}(H)$—contradiction. ∎

M-OUT

This section is devoted to the proof of Theorem 1.1. Let m be a fixed integer; m will be 9 for the main result. We construct an edge coloured $D(n, m+1)$ as the union of $D(n, m)$, with red edges plus an independent $D(n, 1)$ with blue edges.

The input to HAM is $H = D(n, m+1)$ with (1) adjacency lists in random order (note that the adjacency list for v can contain two copies of a vertex w if v chooses w and w chooses v) and (2) $E_+ = E(m, n) = \{\text{red edges of } H\}$ and $E_- = E(n, 1) = \{\text{blue edges of } H\}$.

We first note

LEMMA 3.0 [8]. *$D(n, m)$ is a.s. connected for $m \geq 2$.*

The next lemma shows how we aim to prove Theorem 1.1.

LEMMA 3.1. *Suppose that the following are true a.s. for $\alpha = \alpha(m)$, $\beta = \beta(m)$, $0 < \beta < \alpha < 1$:*

$$S \subseteq V_n,\ |S| \leq \alpha n \text{ implies that } |N(D, S)| \geq 2|S| \tag{3.1a}$$

where $D = D(n, m)$.

HAM *applied to $D(n, m+1)$ makes fewer than βn cycle extensions.* (3.1b)

$$(1-\beta) > (1-\alpha)^{(\alpha-\beta)/(1-\beta)}. \tag{3.1c}$$

Then HAM *a.s. finds a hamilton cycle in $D(n, m+1)$.*

Proof. We use a variation of the colouring argument of [9]. Let $\omega = \lceil \log n \rceil$ and $\varepsilon > 0$ be small. Let Y be a random ω-subset of V_n. For $y \in Y$ let $x(y)$ be the vertex chosen by y in the construction of E_-. Let $X = X(Y) = \{(y, x(y)) \colon y \in Y\}$. We define two events.

$E_1 = \{$(3.1a) holds for H_+, (3.1b) holds and HAM fails on $D(n, m+1)\}$.

$E_2 = E_1 \cap \{X$ is deletable and $|Y \cap \mathrm{END}(H)| \geqslant \gamma\omega\}$

where $\gamma = (1-\varepsilon)(\alpha-\beta)/(1-\beta)$.

The lemma follows from two inequalities plus the fact that ε is arbitrary.

$$\Pr(E_2 \mid E_1) \geqslant (1-o(1))(1-\beta)^{\omega}. \tag{3.2a}$$

$$\Pr(E_2) \leqslant (1-\alpha)^{\gamma\omega}. \tag{3.2b}$$

For then $\Pr(E_1) = o(1)$ and $\Pr(\text{HAM fails}) \leqslant 1 - \Pr((3.1\text{a})$ and $(3.1\text{b})) + \Pr(E_1)$.

Proof of (3.2a). Given $D(n, m+1)$ and an ordering of the adjacency lists such that E_1 occurs, we have $|\mathrm{END}(H)| \geqslant \alpha n$ from Lemma 2.3. The probability of E_2 is then easily seen to be at least $(1-o(1))(1-\beta)^{\omega}$.

Proof of (3.2b). We prove this by showing that

$$\Pr(E_2 \mid H_X) \leqslant (1-\alpha)^{\gamma\omega} \tag{3.3}$$

(by $(\ |H_X)$ we mean that we are given Y, the $m+1$ vertex choices for $v \notin Y$ and the m vertex choices for $v \in Y$). If E_2 occurs then Lemmas 2.1 and 2.2 and (3.1a) imply that

HAM fails on H_X, $|Y \cap \mathrm{END}(H_X)| \geqslant \gamma\omega$ and
$|\mathrm{END}(H_X, w)| \geqslant \alpha n$ for $w \in \mathrm{END}(H_X)$. (3.4a)

$y \in Y \cap \mathrm{END}(H_X)$ implies $x(y) \notin \mathrm{END}(H_X, y)$. (3.4b)

It is important to note that, given H_X, although Y is determined the choices $x(y)$, $y \in Y$ are arbitrary and hence equally likely.

Now $\Pr(E_2 \mid H_X) = 0$ if H_X does not satisfy (3.4a) and so assume that (3.4a) is satisfied. In this case

$$\Pr(E_2 \mid H_X) \leqslant \Pr((3.4\text{b}) \mid H_X) \leqslant (1-\alpha)^{|Y \cap \mathrm{END}(H_X)|}$$

and (3.3) follows. ∎

LEMMA 3.2. *Let*

$$\phi_m(x) = \frac{3^{mx} x^{(m-3)x} (1-x)^{m(1-3x)}}{2^{2x} (1-3x)^{m(1-3x)}}.$$

If $m \geq 4$, $\alpha < (2m-4)/(6m-3)$, *and* $\phi_m(\alpha) < 1$ *then* (3.1a) *holds a.s.*

Proof. Let

$$u_k = \frac{n!}{k!(2k)!(n-3k)!} \left(\frac{3k}{n}\right)^{mk} \left(1 - \frac{k}{n}\right)^{m(n-3k)}.$$

Then $\Pr((3.1a))$ fails$) \leq \sum_{k=1}^{\lfloor \alpha n \rfloor} u_k \leq \sum_{k=1}^{\lfloor \alpha n \rfloor} \phi_m(k/n)^n$ on using Stirling's inequalities.

Now one can see that for some constant A_m, $\phi_m(x) \leq (A_m x)^x$. Thus $\sum_{k=1}^{\lfloor 1/2A_m \rfloor} u_k \leq \sum_{k=1}^{\lfloor 1/2A_m \rfloor} (A_m k/n)^k = O(1/n)$. On the other hand, by differentiating $\log \phi_m(x)$ twice we find

$$\frac{\phi_m''(x)}{\phi_m(x)} - \left(\frac{\phi_m'(x)}{\phi(x)}\right)^2 = \frac{2m}{(1-x)^2} + \frac{3m}{1-x} + \frac{m-3}{x} - \frac{9}{1-3x}$$
$$\geq (6m-3) - \frac{9}{(1-3x)}.$$

Thus the lemma's assumptions imply ϕ_m is convex in $[0, \alpha]$. Since $\phi_m(1/2A_m) < 1$ the result follows. ∎

LEMMA 3.3. *Let* $\varepsilon > 0$ *be arbitrary. Then* HAM *a.s. makes no more than* $(1+\varepsilon)n/(2m-1)$ *cycle extensions.*

Proof. Consider the start of stage k, in particular the first execution of L2. Here $v = w_1$ and

$$\Pr(X = \varnothing \mid \text{previous history}) \leq \left(\frac{k}{n}\right)^{m-1} \left(1 - \frac{1}{n}\right)^{m(n-k)}$$
$$\leq u_k = \left(\frac{k}{n}\right)^{m-1} e^{-m(1-k/n)}. \qquad (3.5)$$

To see this we observe: at any stage of the algorithm P_k contains two types of vertex, **live** and **dead**. A vertex w is dead if at some time previously, at statement L2 we found $X = \varnothing$ for $v = w$. For such a vertex we know that it has no neighbours in $Q_k = V_n - P_k$. If w is a live vertex than all we know is that each time w has appeared as v in L2, $X \neq \varnothing$. Let $L_k = \{\text{live vertices}\}$. Consider now the choices made by vertices in Q_k. The only way they have been conditioned is that they do not choose in $P_k - L_k$, i.e., their chances of choosing in L_k have increased. Furthermore, we have no information about which vertices in L_k are chosen by $x \in Q_k$, for when the algorithm establishes such a choice x immediately becomes an endpoint of the current path. Now w_1 has just been added to P_k. By the above we know that given the previous history the probability that the ($\geqslant$) $m-1$ choices of w_1 that are not yet known to us are all in P_k is $\leqslant (k/n)^{m-1}$ and independently the probability that no vertex in Q_k chooses w_1 is $\leqslant (1-1/n)^{m(n-k)}$, and (3.5) is verified.

It follows that the number of times we **do not** make an **immediate** extension from w_1 at the start of stage k is stochastically dominated by the sum of n 0–1 random variables $Z_1, Z_2, ..., Z_n$, where $\Pr(Z_k = 1) = u_k$. Thus, using Theorem 1 of Hoeffding [14],

$$\Pr\left(\sum_{i=1}^{n} Z_i \geqslant (1+\varepsilon) \sum_{i=1}^{n} u_i\right) \leqslant e^{-\varepsilon^2 \sum_{i=1}^{n} u_i/3} = o(1).$$

Hence for large n HAM a.s. makes fewer than

$$(1+\varepsilon) \sum_{i=1}^{n} u_i \leqslant (1+2\varepsilon) n \int_0^1 x^{m-1} e^{-m(1-x)}\, dx \tag{3.6}$$

$$= (1+2\varepsilon) n \int_0^1 (1-y)^{m-1} e^{-my}\, dy \tag{3.7}$$

$$\leqslant (1+2\varepsilon) n \int_0^1 e^{-(2m-1)y}\, dy$$

and the result follows as ε is arbitrary. ∎

To prove Theorem 1.1 we take $m = 9$. $\alpha = 0.27$ satisfies the conditions of Lemma 3.2. Taking $\beta = 1/17$ from Lemma 3.3 and applying Lemma 3.1 yields the theorem. (Note that using the exact value for the integral in (3.7) does not reduce the value of m. Now if $X = \varnothing$ on the first execution of L2 in stage k then w_1's choices are random in L_k. Using this we can reduce m to 7 and replace 10 by 8 in Theorem 1.1. This requires us to obtain a.s. upper bounds for the number of dead vertices at any stage and use a computer to estimate integrals numerically. We judge that it is not worth reproducing the entire argument here.)

Regular Graphs

This section is devoted to the proof of Theorem 1.2. We must first describe how the edges of $R(n, r)$ are partitioned into E_+ and E_-. We let each $v \in V_n$ independently choose one edge randomly from its r incident edges and place it in E^*. Thus the same edge can be chosen twice.

Next let $W = \{v \in V_n\colon v \text{ is incident with } > r/2 \text{ edges of } E^*\}$ and then $E_- = \{e \in E^*\colon e \cap W = \emptyset\}$. Having made this partition we put the adjacency lists in random order and apply HAM. In order to prove Theorem 1.2 we need a model for studying $R(n, r)$. Let $\mathrm{REG}(n, r)$ be the set of r-regular graphs with vertex set V_n and consider the model defined in Bollobás [3]. Let $D_1, D_2, ..., D_n$ be disjoint sets with $|D_i| = r$ and set $D = \bigcup_{i=1}^n D_i$ and $2m = |D| = rn$. A *configuration* C is a partition of D into m pairs, the edges of C. Let Φ be the set of all $\xi(m) = (2m)!(2^{-m}m!)$ configurations. Turn Φ into a probability space by giving all members of Φ the same probability. For $C \in \Phi$ let $g(C)$ be the multi-graph with vertex set V_n in which i is joined to j whenever C has an edge with one end-vertex in D_i and the other in D_j. Clearly $\mathrm{REG}(n, r) \subseteq g(\Phi)$ and $|g^{-1}(G)| = r!^n$ for every $R(n, r) \in \mathrm{REG}(n, r)$.

Let Q be a property of the graphs in $\mathrm{REG}(n, r)$ and let Q^* be a property of the configurations in Φ. Suppose these properties are such that for $G \in \mathrm{REG}(n, r)$ and $C \in g^{-1}(G)$ the configuration C has Q^* if and only if G has Q. All we shall need from [3] is that if almost every C has Q^* then almost every G has Q.

We shall thus be able to prove the theorem if we can show that HAM applied to a multigraph $g(C)$, C chosen randomly from Φ, almost surely finds a hamilton cycle.

In terms of configurations our partition of the edges of C is done as follows: suppose that $D_i = \{(i-1)r + t\colon t = 1, 2, ..., r\}$ for $i = 1, 2, ..., n$. Let $A_n = \{(i-1)r + 1\colon i = 1, 2, ..., n\}$ and $C^* = \{e \in C\colon e \cap A_n \neq \emptyset\}$.

Let $D_i^* = D_i \cap \bigcup_{e \in C^*} e$ and $C_i = \{e \in C^*\colon e \cap D_i \neq \emptyset\}$ for $i \in V_n$. Let $W = \{i \in V_n\colon |D_i^*| > r/2\}$ and then let $C_- = \bigcup_{i \notin W} C_i$ and $C_+ = C - C_-$ and let $B_n = A_n \cap \bigcup_{e \in C_-} e$.

For C chosen randomly from $g^{-1}(\mathrm{REG}(n, r))$, taking $E_+ = g(C_+)$ and $E_- = g(C_-)$ yields an $R(n, r)$ with the same random edge partition as that given at the start of this section. We now prove the equivalent of Lemmas 3.1–3.3.

LEMMA 4.1. *Suppose that the following are true a.s. for* $\alpha = \alpha(r)$, $\beta = \beta(r)$, $\gamma = \gamma(r)$, $0 < \beta + \gamma < \alpha < 1$.

$$S \subseteq V_n,\ |S| \leqslant \alpha n \text{ implies that } |N(S, g(C_+))| \geqslant 2|S|; \tag{4.1a}$$

$$\text{HAM applied to } g(C) \text{ makes fewer than } \beta n \text{ cycle extensions}; \tag{4.1b}$$

$$|B_n| \geqslant (1-\gamma)n; \tag{4.1c}$$

$$1 - \frac{\beta}{1-\gamma} > \left(1 - \frac{\delta}{2}\right) e^{\delta/2}, \tag{4.1d}$$

where $\delta = (\alpha - \beta - \gamma)/(1 - \beta - \gamma)$. *Then* HAM *a.s. finds a hamilton cycle in* $R(n, r)$.

Proof. Let $\omega = \lceil \log n \rceil^2$ and $\varepsilon > 0$ be small. For $x \in D$ let $p(x, C)$ be the element of D paired with x by C. Also let $h(x)$ be defined by $x \in D_{h(x)}$.

Let Y be a random ω-subset of B_n, $Z = \{h(y): y \in Y\}$ and $X = \{e \in C: e \cap Y \neq \varnothing\}$. We define two events:

$$E_1 = \{(4.1a) \text{ holds for } C_+, (4.1b)\ (4.1c) \text{ hold and HAM fails on } C\}$$

$$E_2 = E_1 \cap \{X \text{ is deletable and (i) } |Z \cap \text{END}(g(C))| \geqslant \xi\omega,$$

$$\text{(ii) } |Z \cap \text{END}(g(C), w)| \geqslant \xi\omega \text{ for } w \in \text{END}(g(C))\},$$

where $\xi = \xi(n)$ is such that $\xi\omega$ is the smallest even integer $\geqslant (1 - \varepsilon)\,\delta\omega$.

The lemma follows from two inequalities as before.

$$\Pr(E_2 \mid E_1) \geqslant (1 - o(1)) \left(1 - \frac{\beta}{1 - \gamma}\right)^{\omega} \tag{4.2a}$$

$$\Pr(E_2) \leqslant (1 + o(1))((1 - \xi/2)\, e^{\xi/2})^{\omega}. \tag{4.2b}$$

The proof of (4.2a) is essentially the same as that for (3.2a).

To prove (4.2b) we show

$$\Pr(E_2 \mid C_X) \leqslant (1 + o(1))((1 - \xi/2)\, e^{\xi/2})^{\omega}, \qquad \text{where} \quad C_X = C - X.$$

If E_2 occurs then Lemmas 2.1–2.3 and (4.1a) imply that

$$\text{HAM fails on } g(C_X),\ |Z \cap \text{END}(g(C_X))| \geqslant \xi\omega \text{ and}$$
$$|Z \cap \text{END}(g(C_X), w)| \geqslant \xi\omega \text{ for } w \in \text{END}(g(C_X)). \tag{4.3a}$$

$$z = h(y) \in Z \cap \text{END}(g(C_X)) \text{ implies } h(p(y, C)) \notin \text{END}(g(C_X), z). \tag{4.3b}$$

It is important now to note that, given C_X, Y is determined but the elements of X are paired up arbitrarily.

Now $\Pr(E_2 \mid C_X) = 0$ if C_X does not satisfy (4.3a) and so assume that (4.3a) is satisfied. In this case

$$\begin{aligned} \Pr(E_2 \mid C_X) &\leqslant \Pr((4.3b \mid C_X) \\ &\leqslant (2\omega - \xi\omega)^{\xi\omega/2} \zeta((2\omega - \xi\omega)/2)/\zeta(\omega) \\ &= (1 + o(1))((1 - \xi/2)\, e^{\xi/2})^{\omega}. \end{aligned}$$

using $|X| \leqslant 2\omega$ and that the "first" $\xi\omega/2$ points of $Z \cap \text{END}(g(C_X))$ have at most $2\omega - \xi\omega$ choices of points to be paired with. The lemma follows. ∎

LEMMA 4.2. *Let*

$$\phi_r(x) = \frac{(3x)^{(r-1)x/2} e^{(1-3x)/6}}{x^x (2x)^{2x} (1-3x)^{1-3x}}.$$

If $r \geqslant 60$, $0 \leqslant \alpha \leqslant (r-7)/(3r-3)$ *and* $\phi_r(\alpha) < 1$ *then* (4.1a) *holds a.s.*

Proof. Now

$$\Pr((4.1\text{a fails}) \leqslant \sum_{k=1}^{\lfloor \alpha n \rfloor} \frac{n!}{k!(2k)!(n-3k)!} \pi_k,$$

where $\pi_k = \Pr(N(V_k, g(C_+)) \subseteq \{k+1, k+2, ..., 3k\})$.

SMALL k. Suppose first that $1 \leqslant k \leqslant \varepsilon_r n$, where $\varepsilon_r = (2^8 12^{-r})^{1/(r-12)}/3$. Since the minimum degree in $g(C_+)$ is at least $r/2$ we have

$$\begin{aligned}\pi_k &\leqslant \binom{rk}{\lceil rk/4 \rceil} \Pr(\{1, 2, ..., \lceil rk/4 \rceil\} \text{ are paired by } C \text{ in } \{1, 2, ..., 3rk\}) \\ &\leqslant \binom{rk}{\lceil rk/4 \rceil} \left(\frac{3k}{n}\right)^{\lceil rk/4 \rceil} \\ &\leqslant \left(\frac{12ek}{n}\right)^{kr/4}.\end{aligned}$$

A routine calculation using

$$\frac{n!}{k!(2k)!(n-3k)!} \leqslant \frac{n^{3k}}{e^{3k} k^k (2k)^{2k}}$$

now yields

$$\sum_{k=1}^{\lfloor \varepsilon_r n \rfloor} \frac{n!}{k!(2k)!(n-3k)!} \pi_k = o(1).$$

LARGE k. We consider the pairings made by all the points in $\bigcup_{i=1}^k D_i - A_n$. This yields

$$\pi_k \leqslant \left(\frac{3(r-1)k+n}{rn}\right)^{(r-1)k/2} \leqslant \left(\frac{3k}{n}\right)^{(r-1)k/2} e^{(n-3k)/6}$$

and hence, on using. Stirling's inequalities,

$$\sum_{k=\lceil \varepsilon_r n \rceil}^{\lfloor \alpha n \rfloor} \frac{n!}{k!(2k)!(n-3k)!} \pi_k \leqslant \sum_{k=\lceil \varepsilon_r n \rceil}^{\lfloor \alpha n \rfloor} \phi_r(k/n)^n. \tag{4.4}$$

We next use

$$\frac{\phi''_m(x)}{\phi_m(x)} - \left(\frac{\phi'_m(x)}{\phi(x)}\right)^2 = \frac{r-7}{2x} - \frac{9}{1-3x}$$

to show that r is convex in $[0, (r-7)/(3r-3)]$. One then checks that $\phi_r(x) \leqslant \psi_r(x) = (3x)^{(r-1)x/2}(3e^{1/6})$ and $\psi_r(1/r) < 1$ if $r \geqslant 41$. The lemma now follows from (4.4) provided $1/r < \varepsilon_r$ and this holds if $r \geqslant 60$. ∎

LEMMA 4.3.

$$|B_n| \geqslant \left(1 - 2r\left(\frac{2e}{r-2}\right)^{r/2-1}\right) n \qquad \text{a.s.}$$

Proof. If $|D_1^*| > r/2$ then at least $r/2 - 1$ out of $\{2, 3, ..., r\}$ are paired by C with elements of A_n. We deduce therefore that

$$\Pr(|D_1^*| > r/2) \leqslant \binom{r}{\lceil r/2 \rceil - 1} r^{-(\lceil r/2 \rceil - 1)} \leqslant \left(\frac{2e}{r-2}\right)^{r/2-1}.$$

Hence

$$E(|W|) \leqslant \left(\frac{2e}{r-2}\right)^{r/2-1} n.$$

One may similarly show that $\operatorname{Var}(|W|) = O(n)$. Thus the Chebycheff inequality shows that $|W| \leqslant 2E(|W|)$ a.s. Now use the fact that $|B_n| \geqslant n - r|W|$. ∎

LEMMA 4.4. *Let $\varepsilon > 0$ be arbitrary. Then* HAM *a.s.* makes not more than $(1+\varepsilon)n/(r-3)$ *cycle extensions.*

Proof. Consider the start of stage k, in particular the first execution of L2. Here $v = w_1$ and

$$\Pr(X = \varnothing \mid \text{previous history}) \leqslant \left(\frac{(r-2)k + (n-k)}{(r-2)k + r(n-k)}\right)^{r-2}. \tag{4.5}$$

To see this we can assume that the previous history gives us all pairings in C that only involve elements of D_i, $i \in P_{k-1}$. Assume that this leaves $d \leqslant (r-2)(k-1)+1$ elements of D_i, $i \in P_{k-1}$ unaccounted for. One point of D_v is known to be paired with a point of D_j, $j \in P_{k-1}$. The remaining points of D are paired arbitrarily. We consider $r-2$ such points of $D_v - A_n$. The probability that each of these is paired with one of the d points previously mentioned or the $n-k$ points of A_n associated with vertices not in P_k is bounded above by the RHS of (4.5).

Arguing as in Lemma 3.3 we then see that HAM a.s. makes fewer than

$$(1+\varepsilon)\int_0^1 \left(\frac{(r-2)x+(1-x)}{(r-2)x+r(1-x)}\right)^{r-2} dx$$

cycle extensions. The result follows on substituting $x=1-y$ in the integral and using

$$\frac{(r-2)-(r-3)\,y}{(r-2)+2y} \leqslant \left(1-\frac{(r-3)\,y}{r-2}\right) \leqslant e^{-(r-3)y/(r-2)}. \quad \blacksquare$$

To obtain the theorem we use $\alpha = 0.309$ in Lemma 4.1. Everything goes through for $r \geqslant 85$.

Sparse Random Graphs

This section is devoted to the proof of Theorem 1.3. Let $G = G_{n,p}$. The idea is to define a "large" set $V^* \subseteq V_n$ such that the graph $H = G[V^*]$ is a.s. hamiltonian. This is what is done in [12].

Construction of V^*

Step 1. The 2-core of G is the largest set $S \subseteq V_n$ such that $\delta(G[S]) \geqslant 2$. It exists because if $\delta(G[S_i]) \geqslant 2$ for $i=1, 2$ then $\delta(G[S_1 \cup S_2]) \geqslant 2$.

The following algorithm constructs the 2-core TWOC:

begin
  TWOC := V_n;
  while $\delta(G[\text{TWOC}]) < 2$ **do** TWOC := TWOC $- \{w \in \text{TWOV}$:
  $d_{G[\text{TWOC}]}(w) < 2\}$
end.

On termination TWOC is the 2-core. This is because one can easily show inductively that each iteration removes vertices not in the 2-core. Note also that no cycle of G contains a vertex of V_n-TWOC.

The remaining steps remove vertices so that $S \subseteq V^*$, $|N(S, H)| < 2|S|$ implies that S is large.

Let $v \in V_n$ be *small* if $d_G(v) \leqslant c/10$ and *large* otherwise.

Step 2.

begin
  SMALL $= \{$small vertices$\}$, $X := \varnothing$;
  repeat
    $S := \{v \in V_n - X\colon |N(v, G) \cap (X \cup \text{SMALL})| \geqslant 2\}$;
    $X := X \cup S$
  until $S = \varnothing$
end

Then let $Y = \{ y \in V_n : d_G(y) = 2 \text{ and } N(y, G) \cap X \neq \emptyset \}$ and

$$W = \bigcup_{t=1}^{4} W_t,$$

where $W_t = \{v \in \text{SMALL}: \exists w \in \text{SMALL} \text{ and a path of length } t \text{ from } v \text{ to } w \text{ in } G\}$. (We allow $v = w$ when $t = 3$ or 4.)

We finally define $V^* = \text{TWOC} - (W \cup X \cup Y)$ and $H = G[V^*]$. The reasons for the exact definition of W, X, Y are made clear by the proofs in [12]. $O(n^3)$ time is ample for the construction of H. The following results are proved in [12].

LEMMA 5.1. *For large enough c we a.s. have*

(a) $|V^*| \geqslant n(1 - (1 + \varepsilon(c))\, ce^{-c})$, *where* $\lim_{c \to 0} \varepsilon(c) = 0$.

(b) $|\text{SMALL}| \leqslant ne^{-2c/3}$

(c) $S \subseteq \text{LARGE} = V_n\text{-SMALL}$, $|S| \leqslant n/12$ *implies* $|N(S, G)| \geqslant 6|S|$.

(d) $S \subseteq V_n$, $n/12 \leqslant |S| \leqslant n/2$ *implies* $|\{(v, w) \in E(G): v \in S, w \notin S\}| \geqslant c|S|/15$.

(e) $S \subseteq V^*$, $|S| \leqslant n/12$ *implies* $|N(S, H)| \geqslant 2|S|$.

(f) $S \subseteq V_n$, $|S| \geqslant ne^{-c}$ *implies* $|\{e \in E(G): e \cap S \neq \emptyset\}| < 4c|S|$.

Note now that by construction

$$d_H(v) \geqslant d_G(v) - 1 \qquad \text{for} \quad v \in V^*. \tag{5.1}$$

To complete the description of CYCLEFIND we must describe the partition of $E(H)$ into E_+ and E_-. To do this we first construct $\hat{E} \subseteq E(G)$ as follows: independently for each $e = (v, w) \in E(G)$ we do a v-experiment and a w-experiment both of which have probability $1/\sqrt{2}$ of success. If both succeed we include e in $\hat{E}$. Thus we can view $\hat{E}$ as $E(\hat{G})$, where $\hat{G} = G_{n,p/2}$. Next let $\hat{H} = \hat{G}[V^*]$, where V^* is defined in terms of G. Let $\widehat{\text{LARGE}} = \{v \in V^*: d_{\hat{H}}(v) > c/20\}$.

We let $E_- = \{e \in E: e \subseteq \widehat{\text{LARGE}}\}$.

LEMMA 5.2. *For large enough c we a.s. have*

(a) $S \subseteq V^*$, $|S| \leqslant n/12$ *implies* $|N(S, H_+)| \geqslant 2|S|$.

(b) H_+ *is connected.*

(c) $|E_-| \geqslant cn/5$.

Proof. (a) Let now $S \subseteq V^*$ with $|S| \leqslant n/12$. Let $S = S_1 \cup S_2$, where $S_2 = S \cap \text{LARGE}$. Now

$$|N(S, H_+)| = |N(S_1, H_+)| + |N(S_2, H_+)| - |N(S, H_+) \cap S_2| - |N(S_2, H_+) \cap S_1| - |N(S_1, H_+) \cap N(S_2, H_+)|. \quad (5.2)$$

But, assuming the conditions of Lemma 4.1 hold,

$$|N(S_1, H_+)| = |N(S_1, H)| \geqslant 2|S_1| \quad \text{by Lemma 5.1(e).} \quad (5.3a)$$

$$|N(S_2, H_+)| \geqslant |N(S_2, H)| \geqslant |N(S_2, G)| - |S_2| \quad \text{as } V^* \cap X = \varnothing$$
$$\geqslant 5|S_2| \quad \text{by Lemma 5.1(c) with } c \text{ replaced by } c/2. \quad (5.3b)$$

$$|N(S_1, H_+) \cap S_2| \leqslant |S_2| \quad (5.3c)$$

$$|N(S_2, H_+) \cap S_1| \leqslant |S_2| \quad \text{as} \quad V^* \cap W = \varnothing \quad (5.3d)$$

$$|N(S_1, H_+) \cap N(S_2, H_+)| \leqslant |S_2) \quad \text{as} \quad V^* \cap W = \varnothing. \quad (5.3e)$$

Equations (5.2) and (5.3) together imply (a).

(b) Suppose that H_+ contains a component A, where $|A| \leqslant |V^*|/2$ and let $B = V^* - A$. By (a) of this lemma we know that $|A| \geqslant n/4$ a.s. Lemma 5.1(a) and (f) imply that for large c

$$|E(G) - E(H)| \leqslant 5ce^{-c}n \text{ a.s.} \quad (5.4)$$

But then Lemma 5.1(d) implies that H a.s. contains at least $c|A|/15 - 5ce^{-c}n \geqslant cn/100$ (large c) edges joining A and B. Conditional on this event the probability that none of these edges is included in E is no more than $2^{-cn/100}$. Thus

$$\Pr(H_+ \text{ is not connected}) \leqslant 2^n 2^{-cn/100} + o(1)$$
$$= o(1) \qquad \text{large } c.$$

(c) An application of the Chebycheff inequality shows that

$$|\{v \in V_n : d_G(v) \leqslant c/2 + 1\}| \leqslant ne^{-c/2} \qquad \text{a.s.} \quad (5.5)$$

Equations (5.1), (5.5) and Lemma 5.1(a) imply that for large c

$$|\{v \in V^* : d_H(v) > c/2\}| \geqslant n(1 - 2e^{-c/2}) \qquad \text{a.s.} \quad (5.6)$$

Since $|E(G)|$ is binomially distributed with mean $cn/2$ it is easy to show

$$|E(G)| \geqslant 7cn/15 \qquad \text{a.s.}$$

and using (5.4) we have, for large c,

$$|E(H)| \geqslant 6cn/13 \qquad \text{a.s.} \tag{5.7}$$

Let $A = \{e = (v, w) \in E(H)\colon d_H(v), d_H(w) > c/2\}$. It follows from (5.6) and (5.7) that for large c

$$|A| \geqslant 5cn/11 \qquad \text{a.s.} \tag{5.8}$$

Now let $\hat{A} = A \cap \hat{E}$. We have

$$|E_-| \geqslant |A| - \sum_{v \in V^*} d_H(v)\, x(v) \tag{5.9}$$

where

$$x(v) = 1 \quad \text{if } d_H(v) \geqslant 3c \text{ or } d_H(v) \leqslant c/2 \text{ or there are at least } d_H(v) - c/20 \text{ successfull } v\text{-experiments}$$
$$= 0 \quad \text{otherwise.}$$

Now (5.8) plus the fact that $e \in A$ is independently placed in $\hat{A}$ with probability 1/2 yields

$$|\hat{A}| \geqslant 5cn/23 \qquad \text{a.s.} \tag{5.10}$$

Now, by construction, the random variables $x(v)$, $v \in V^*$ are independent and

$$\begin{aligned}\Pr(x(v) = 1) &\leqslant \binom{d_H(v)}{\lfloor c/20 \rfloor} \left(\frac{1}{\sqrt{2}}\right)^{d_H(v) - c/20} \qquad \text{for} \quad c/2 < d_H(v) < 3c.\\ &\leqslant \left(\frac{20ed_H(v)}{c - 20}\right)^{c/20} 2^{-(d_H(v) - c/20)/2}\\ &\leqslant 2^{-c/20} \qquad \text{for large } c.\end{aligned}$$

The independence of the $x(v)$'s then implies

$$\sum_{\substack{v \in V^*\\ c/20 < d_H(v) < 3c}} d_H(v)\, x(v) \leqslant 2^{1 - c/20} n \qquad \text{a.s.} \tag{5.11}$$

Now

$$\sum_{\substack{v \in V^*\\ d_H(v) \geqslant 3c}} d_H(v) \leqslant \Delta = \sum_{\substack{v \in V_n\\ d_G(v) \geqslant 3c}} d_G(v). \tag{5.12}$$

But

$$E(\Delta)=\sum_{k=\lceil 3c\rceil n}^{n-1} k\binom{n-1}{k}p^k(1-p)^{n-1-k}$$

$$\leqslant 6cn\binom{n}{\lceil 3c\rceil}p^{\lceil 3c\rceil}(1-p)^{n-1-\lceil 3c\rceil}\leqslant n(e/3)^{3c}$$

Also it is not difficult to show that $\mathrm{Var}(\Delta)=O(n)$ and thus, using the Chebycheff inequality,

$$\Delta\leqslant 2(e/3)^{3c}n \qquad \text{a.s.} \tag{5.13}$$

Equation (5.5) plus (5.9)–(5.13) yield (c) for large c. ∎

Having constructed H we will of course apply HAM. In this case the adjacency lists need not be randomised. Indeed we can assume that they are in increasing order. We finish our proof as before. Having generated $\hat{E}$ we generate $X\subseteq\hat{E}$ by independently including e in X with probability $p_1=\log n/n$. Our two events are

$E_1=\{$the conditions of Lemmas 5.1 and HAM fails on $H\}$

$E_2=E_1\cap\{X$ is deletable and no edge of X is incident with any $v\in V_n-V^*$ or v such that $d_G(v)\leqslant c/2+1\}$

The theorem follows from

$$\Pr(E_2\,|\,E_1)\leqslant(1-p)^{2n} \qquad \text{for } c \text{ large} \tag{5.14}$$

$$\Pr(E_2)\leqslant(1-pp_1/2)^{n^2/300}. \tag{5.15}$$

Proof of (5.14). HAM makes fewer than n cycle extensions and given E_1, Lemma 5.1(a) and (f) and (5.5) imply there are fewer than $5ce^{-c/2}n$ edges incident with a vertex of V_n-V^*.

Proof of (5.15). As usual we show that

$$\Pr(E_2\,|\,G_X)\leqslant(1-pp_1/2)^{n^2/300}, \qquad \text{where} \quad G_X=(V_n, E(G)-X). \tag{5.16}$$

Now if E_2 occurs then applying the method used to construct H from G will produce H_X from G_X. Furthermore HAM will fail on H_X and

$$\begin{aligned}&\text{(i)} \quad |\mathrm{END}(H_X)|\geqslant n/12.\\ &\text{(ii)} \quad |\mathrm{END}(X_X, v)|\geqslant n/12 \qquad \text{for} \quad v\in\mathrm{END}(H_X)\end{aligned} \tag{5.17}$$

and of course

$$e\in Y=\{(v, w): v\in\mathrm{END}(H_X), w\in\mathrm{END}(H_X, v)\} \tag{5.18}$$

implies $e \notin X$. Note that even (5.17) is determined by G_X, not G. Now $\Pr(E_2 \mid G_X) = 0$ if (5.17) does not hold and so assume it does. Note next that if G_X is given then X is a random subset of $E(G_X)$, where $e \in E(G_X)$ is independently included with probability $pp_1/(2(1-p)) \geqslant pp_1/2$. But then

$$\Pr(E_2 \mid G_X) \leqslant \Pr((5.18) \mid G_X) \leqslant (1 - pp_1/2)^{|Y|} \leqslant (1 - pp_1/2)^{n^2/300}$$

and the theorem follows.

Conclusion

We have extended the results of [7, 13] to random graphs with constant average degree. The most important open problems are to (1) reduce the values of 10 and 85 in Theorems 1.1 and 1.2 to 3; (2) modify CYCLEFIND so that it a.s. finds the longest cycle in $G_{n,p}$ and get an asymptotic expression for this length; (3) remove the necessity for partitioning $E(H)$ into E_+ and E_-; and (4) extend all these results to digraphs.

Note added in proof. T. Luczak and the author have now reduced the 10 of Theorem 1 to 5.

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