Finding maximum matchings in random regular graphs in linear expected time

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Abstract

In a seminal paper on finding large matchings in sparse random graphs, Karp and Sipser [12] proposed two algorithms for this task. The second algorithm has been intensely studied, but due to technical difficulties, the first algorithm has received less attention. Empirical results in [12] suggest that the first algorithm is superior. In this paper we show that this is indeed the case, at least for random k-regular graphs. We show that w.h.p. the first algorithm will find a matching of size $n/2 - O(\log n)$ in a random k-regular graph, $k = O(1)$. We also show that the algorithm can be adapted to find a maximum matching w.h.p. in $O(n)$ time, as opposed to $O(n^{3/2})$ time for the worst-case.

1 Introduction

Given a graph $G = (V, E)$, a matching M of G is a subset of edges such that no vertex is incident to two edges in M. Finding a maximum cardinality matching is a central problem in algorithmic graph theory. The most efficient algorithm for general graphs is that given by Micali and Vazirani [13] and runs in $O(|E||V|^{1/2})$ time.

In a seminal paper, Karp and Sipser [12] introduced two simple greedy algorithms for finding a large matching in the random graph $G_{n,m}, m = cn/2$ for some positive constant $c > 0$. Let us call them Algorithms 1 and 2 as they are in [12]. Algorithm 2 is simpler than Algorithm 1 and has been intensely studied: see for example Aronson, Frieze and Pittel [1], Bohman and Frieze [3], Balister and Gerke [2] or Bordenave and Lelarge [6]. In particular, [1] together with Frieze and Pittel [9] shows that w.h.p. Algorithm 2 finds a matching that is within

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$\tilde{\Theta}(n^{1/5})$ of the optimum, when applied to $G_{n,m}$. Subsequently, Chebolu, Frieze and Melsted [5] showed how to use Algorithm 2 as a basis for a linear expected time algorithm, when c is sufficiently large.

Algorithm 2 proceeds as follows (a formal definition of Algorithm 1 is given in the next section). While there are isolated vertices it deletes them. After which, while there are vertices of degree one in the graph, it chooses one at random and adds the edge incident with it to the matching and deletes the endpoints of the edge. Otherwise, if the current graph has minimum degree at least two, then it adds a random edge to the matching and deletes the endpoints of the edge.

In the same paper Karp and Sipser proposed another algorithm for finding a matching that also runs in linear time. This was Algorithm 1, described below. The algorithm is more careful with vertices of degree two and sequentially reduces the graph until it reaches the empty graph. Then it unwinds some of the actions that it has taken and grows a matching which is then output. Even though it was shown empirically to outperform Algorithm 2, it has not been rigorously analyzed.

In this paper we analyze Algorithm 1 in the case where the underlying graph G is a random k-regular graph. We will prove

Theorem 1. *For $3 \leq k = O(1)$ let $G_{n,k}$ denote a simple random k-regular graph with vertex set $[n]$. Then,*

(a) Algorithm 1 finds a matching of size $n/2 - O(\log n)$, w.h.p.

(b) Algorithm 1 can be modified to find a maximum matching in $O(n)$ time w.h.p. and in expectation.

We will in fact prove something stronger by analysing Algorithm 1 on random graphs with a fixed degree sequence that satisfies a certain property. For a degree sequence $\mathbf{d}$, let $n_i(\mathbf{d})$ be the number of occurrences of i in $\mathbf{d}$. We define the set of $(3,k)$-*dominant* degree sequences $\mathcal{C}_{3,k}$ to be those with minimum degree at least three and maximum degree k that satisfy

$$\mathcal{C}_{3,k} := \{\mathbf{d} : \mathcal{D}_{k,j}(\mathbf{d}) \text{ holds for all } 3 < j \leq k\} \tag{1}$$

where

$$\mathcal{D}_{k,j}(\mathbf{d}) := \{n_j(\mathbf{d}) \geq 1.3 n_{j-1}(\mathbf{d}) - (\log^2 n - k) n^{0.8}/2^j\}. \tag{2}$$

(The factor 1.3 is obviously chosen to make the proof below correct. In what follows we often replace 1.3 by α. We delineate the essential properties of this class that are of use to us at the end of Section 2.)

We will slightly abuse the above notation by writing $G \in \mathcal{C}_{3,k}$ to mean that G has a degree sequence $\mathbf{d} \in \mathcal{C}_{3,k}$. Observe that the degree sequence of a k-regular graph is $(3,k)$-dominant. We prove Theorem 1 by proving

Theorem 2. *Let G be a random graph with degree sequence $\mathbf{d} \in \mathcal{C}_{3,k}$ for some $k = O(1)$. Then,* *(a) Algorithm 1 finds a matching of size $n/2 - O(\log n)$, w.h.p.*

(b) Algorithm 1 can be modified to find a maximum matching in $O(n)$ time w.h.p. and in expectation.

We analyze the REDUCE-CONSTRUCT algorithm. Its description is given in the next section. Under certain assumptions, see below, Algorithm 1 and REDUCE-CONSTRUCT are equivalent.

We divide the rest of the paper as follows: We give a description of REDUCE-CONSTRUCT in Section 2. In addition we define the main quantity of interest which we call *excess* and also the typical behaviour that REDUCE-CONSTRUCT exhibits which we call good hyperactions. We state our main lemmas and use them to prove Theorem 2. In Section 3 we prove lemmas pertaining to the excess and the hyperactions. The proof of Theorem 2 uses an inductive argument. The base case is proven in Section 5.1. Its inductive step is proven by two lemmas, in Sections 4 and 5 respectively.

2 The REDUCE-CONSTRUCT Algorithm: overview

Algorithm 1, given in [12], can be split into two parts. The first part repeatedly contracts edges/vertices until the graph is empty. Then the second part unwinds part of this contraction and constructs a matching which is then output.

To reduce the graph, Algorithm 1 proceeds as follows:

(1) First, while there are vertices of degree 0 or degree 1 the algorithm removes them along with any edge incident to them. The edges removed at this stage will be part of the output matching.

(2) Second, while there are vertices of degree 2 the algorithm contracts them along with their two neighbors. That is the induced path (x, y, z) is replaced by a single contracted vertex y_c whose neighbors are those of x, z other than y. The description in [12] does not explicitly say what to do with loops or multiple edges created by this process. In any case, such creations are very rare. We say a little more on this in Section 2.1.

(3) Finally if the graph has minimum degree 3 then a random vertex is chosen among those of maximum degree and then a random edge incident to that vertex is deleted. These edges will not be used in the unwinding. The aim is to delete a random non-essential edge. Here we see a different philosophy to that of Algorithm 2, where when there is no forced move we add a random edge to our matching.

(4) In the unwinding, if we have so far constructed a matching containing an edge $\{y_c, \xi\}$ incident with y_c and ξ is a neighbor of x then in our matching we replace this edge by $\{x, \xi\}$ and $\{y, z\}$. If there is no matching edge so far chosen incident with y_c then we add an arbitrary one of $\{x, y\}$ or $\{y, z\}$ to our matching.

The idea of the algorithm is that it is possible to make optimal decisions about vertices of degree at most two. In other words, a maximum matching in the graph G' yielded by operations (1), (2) on a graph G, can be used to construct a maximum matching in the graph G itself. Operation (3) can lead to errors, but as we will see, errors come from the existence of small cycles and these are rare.

2.1 The Reduce-Construct algorithm: details

The precise algorithm that we analyze is called Reduce-Construct. The description of Algorithm 1 given in [12] is not explicit in how to deal with loops and multiple edges, as they arise. We remove loops immediately, but keep the multiple edges until removed by other operations.

We assume that our input (multi-)graph $G = G([n], E)$ has degree sequence $\mathbf{d}$ and is generated by the configuration model of Bollobás [4]. Let $W = [2\nu]$, $2\nu = \sum_{i=1}^{n} d(i)$, be our set of *configuration points* and let Φ be the set of *configurations* i.e. functions $\phi : W \mapsto [n]$ such that $|\phi^{-1}(i)| = d(i)$ for every $i \in [n]$. Given $\phi \in \Phi$ we define the graph $G_\phi = ([n], E_\phi)$ where $E_\phi = \{\{\phi(2j-1), \phi(2j)\} : j \in [\nu]\}$. Choosing a function $\phi \in \Phi$ uniformly at random yields a random (multi-)graph G_ϕ with degree sequence $\mathbf{d}$.

It is known that conditional on G_ϕ being simple, i.e. having no loops or multiple edges, it is equally likely to be any simple graph that has degree sequence $\mathbf{d}$. Also, if the maximum degree is bounded then the probability that G_ϕ is simple is bounded below by a positive quantity that is independent of n. Thus results on this model can be translated immediately to random simple graphs.

Reduced-Construct, displayed shortly, has as an input (i) $G_0 = G_\phi$ where we condition on there being no loops, (ii) a logical condition Ξ that dictates when the reduction phase of the algorithm ends, and (iii) a pre-specified matching algorithm "Match($\cdot$)" that takes as input a graph and outputs a matching. Ξ can be as simple as "G_i is not the empty graph" or "G_i has at least so many vertices/edges left".

Reduce-Construct can be naturally split into two parts: the Reduce and Construct algorithms corresponding to the lines 3-12 and 14-22 respectively.

Lines 1–6 should be as expected by the reader. Lines 7–9 describe what is called an "auto-correction contraction". Suppose we have vertices u, v, w where u is joined to vertex w by two parallel edges and by a sigle edge to vertex v. Now u has degree three, but from the point of view of finding a matching, u should be treated as having degree two. An auto-correction does this as the need arises.

Algorithm 1 Reduce-Construct

1: Input: G_0, $\Xi = true$, Match.
2: Reduce
3: $i = 0$.
4: **while** Ξ **do**
5: **if** $\delta(G_i) = 0$ **then** Perform a **vertex-0 removal**: choose a vertex of degree 0 and remove it from V_i.
6: **else if** $\delta(G_i) = 1$ **then** perform a **vertex-1 removal**: choose a random vertex v of degree 1 and remove it along with its neighbor w and any edge incident to w.
7: **else if** $\delta(G_i) = 2$ **then** perform a **contraction**: choose a random vertex v of degree 2. Then replace v and its neighbors $N(v)$ by a single vertex v_c. For $u \in V \setminus (\{v\} \cup N(v))$, u is joined to v_c by as many edges as there are in G_i from u to $\{v\} \cup N(v)$. Remove any loops created.
8: **else** perform a **max-edge removal**: choose a random vertex of maximum degree and remove a random edge incident with it.
9: **end if**
10: **if** the last action was a max-edge removal, say the removal of edge $\{u, v\}$ and in the current graph (after the edge removal) we have the degree $d(v) = 2$ and v is joined to a single vertex w by a pair of parallel edges **then** perform an **auto correction contraction**: add back to the graph the edge $\{u, v\}$ and then contract u, v, w into a single vertex. Remove any loops created.
11: **end if**
12: $i = i + 1$ and let G_i be the current graph.
13: **end while**
14: Construct
15: Now unwind and construct a matching.
16: Set $\tau_{end} = i$, $M_0 =$Match$(G_{\tau_{end}})$.
17: **for** $j = 1$ to τ_{end} **do**
18: **if** $\delta(G_{\tau_{end}-j}) = 1$ **then** let v be the vertex of degree 1 that was chosen at the $(\tau_{end} - j)th$ step of the while loop at line 3 and let e be the edge that is incident to v in $G_{\tau_{end}-j}$. Then, set $M_{\tau_{end}-j} = M_{\tau_{end}-j+1} \cup \{e\}$.
19: **else if** at step $(\tau_{end} - j)th$ of the While loop at line 3, a contraction or an auto-contraction was performed **then** let v be the vertex of degree 2 that was selected in $G_{\tau_{end}-j}$. Let $N(v) = \{u, w\}$ and v_c be the vertex resulting from the contraction of u, v, w.
20: **if** v_c is not covered by $M_{\tau_{end}-j+1}$ **then** $M_{\tau_{end}-j} = M_{\tau_{end}-j+1} \cup \{u, v\}$
21: **else** Assume that $\{v_c, z\} \in M_{\tau-j+1}$ for some $z \in V(G_{\tau-j+i})$. Without loss of generality assume that in $G_{\tau_{end}-j}$, z is connected to u. Set $M_{\tau_{end}-j} = M_{\tau_{end}-j+1} \cup \{\{v, w\}, \{u, z\}\}) \setminus \{v_c, z\}$.
22: **end if**
23: **else** Set $M_{\tau_{end}-j} = M_{\tau_{end}-j+1}$.
24: **end if**
25: **end for**
26: Output $M = M_{\tau_{end}}$.

For a diagram illustrating an auto-correction contraction, see the first diagram in the appendix.

During the execution of REDUCE we only reveal edges (pairs of the form $(\phi(2j-1), \phi(2j))$: $j \in [\nu]$) of G_ϕ as the need arises in the algorithm. Moreover the algorithm removes any edges that are revealed. Thus if we let $\mathbf{d}_i$ be the degree sequence of G_i then, given $\mathbf{d}_i$ we have that G_i is uniformly distributed among all configurations with degree sequence $\mathbf{d}_i$ and no loops.

Call a contraction that is performed by REDUCE in Line 6 and involves only 2 vertices *bad* i.e. one where $u = N(v)$. Otherwise call it *good*.

For an initial graph $G = G_0$ and $j \in \{0, 1, ..., \tau_{end}\}$ denote by $R_0(G, j)$ and $R_{2b}(G, j)$ the number of times that REDUCE has performed a vertex-0 removal and a bad contraction respectively until it generates G_j. For a graph G and a matching M denote by $\kappa(G, M)$ the number of vertices that are not covered by M. The following Lemma determines the quality of the output of the REDUCE-CONSTRUCT algorithm.

Lemma 3. *Let G be a graph and M be the output of the Reduce-Backtrack algorithm applied to G. Then, for $j \geq 0$,*

$$\kappa(G, M) = R_0(G, j) + R_{2b}(G, j) + \kappa(G_j, M_j). \tag{3}$$

Proof. Let $G = G_0, G_1, ..., G_{\tau_{end}}$ be the sequence of graphs produced by REDUCE and let $M_j, M_{j-1}, ..., M_0 = M$ be the sequence of matchings produced by CONSTRUCT. For $i \leq j$ let $R_0(G, j, i)$ and $R_{2b}(G, j, i)$ be the number of vertex-0 removals and bad contractions performed by REDUCE going from G_{j-i} to G_j. We will prove that for every $0 \leq i \leq j$,

$$\kappa(G_{j-i}, M_{j-i}) = R_0(G, j, i) + R_{2b}(G, j, i) + \kappa(G_j, M_j). \tag{4}$$

Taking $i = j$ yields the desired result.

For $i = 0$, equation (3) holds as $R_0(G, j, 0) = R_{2b}(G, j, 0) = 0$. Assume inductively that (3) holds for $i = k - 1$ where k satisfies $0 < k \leq j$. For $i = k$, if a max-edge deletion was performed on G_{j-k} then $|V_{j-k}| = |V_{j-k+1}|$. Furthermore, $R_0(G, j, k) = R_0(G, j, k-1)$ and $R_{2b}(G, j, k) = R_{2b}(G, j, k-1)$ and hence (3) continues to hold. If a vertex-0 deletion or a bad contraction was performed on G_{j-k} then $|V_{j-k}| = |V_{j-k+1}| + 1$ and $M_{j-k} = M_{j-k+1}$. In the case of a vertex-0 deletion we have $R_0(G, j, k) = R_0(G, j, k-1) + 1$ and $R_{2b}(G, j, k) = R_{2b}(G, j, k-1)$ and both sides of (4) increase by one. In the case of a bad contraction we have $R_0(G, j, k) = R_0(G, j, k-1)$ and $R_{2b}(G, j, k) = R_{2b}(G, j, k-1) + 1$ and again both sides of (4) increase by one. Finally if a good contraction or a vertex-1 removal was performed on G_{j-k} then $R_0(G, j, k) = R_0(G, j, k-1)$ and $R_{2b}(G, j, k) = R_{2b}(G, j, k-1)$. At the same time we have that $\kappa(G_{j-i}, M_{j-i}) = \kappa(G_{j-i+1}, M_{j-i+1})$, completing the induction. □

By introducing the auto correction contraction we replace a number of bad contractions (those that would had followed a max-edge removal) with good ones. As a result we reduce $R_{2b}(G, \tau_{end})$, consequently $\kappa(G, M)$. Note that we do not claim that all bad contractions can be dealt with in this way. We only show later that other instance of bad contractions are very unlikely.

2.2 Organizing the actions taken by REDUCE

We do not analyze the effects of each action taken by REDUCE individually. Instead we group together sequences of actions, into what we call *hyperactions*, and we analyze the effects of the individual hyperactions. Hyperactions take a graph G of minimum degree at least 3 to another smaller graph G' with minimum degree at least 3. We divide them into *good* and *bad* hyperactions. Only bad hyperactions increase R_0 or R_{2b}. Thus if REDUCE only executes good hyperactions and ends up with a graph with a (near-)perfect matching then Lemma 3 implies that the initial graph has a (near-)perfect matching. Here a (near-)perfect matching of a graph G is a matching of size $\lfloor |V(G)|/2 \rfloor$. Furthermore, CONSTRUCT will produce such a matching.

We construct a sub-sequence $\Gamma_0 = G, \Gamma_1, \ldots, \Gamma_\tau$ of $G_0, G_1, ..., G_{\tau_{end}}$. Every hyperaction, starts with a max-edge removal and it consists of all the actions taken until the next max-edge removal or until $G_{\tau_{end}}$ is reached. We let Γ_i be the graph that results from performing the first i hyperactions. Thus Γ_i is the ith graph in the sequence $G_0, G_1, ..., G_{\tau_{end}}$ that has minimum degree at least 3 and going from Γ_i to Γ_{i+1} REDUCE performs a max-edge removal followed by a sequence of vertex-0, vertex-1 removals and contractions. Thus $\Gamma_0, \Gamma_1, ..., \Gamma_{\tau-1}$ consists of all the graphs in the sequence $G_0, G_1, ..., G_{\tau_{end}-1}$ with minimum degree at least 3.

2.3 Excess and hyperactions of interest

The central quantity of this paper is the *excess* which we denote by $ex_\ell(\cdot)$. For a graph G, with degrees $d(v), v \in V(G)$, and a positive integer ℓ we let

$$ex_\ell(G) := \sum_{v \in V(G)} (d(v) - \ell)\mathbb{I}(d(v) > \ell).$$

$ex_\ell(G)$ can be thought as a distance measure of the degree sequence of G from the set of degree sequences of size $|V(G)|$ with maximum degree ℓ.

Hyperactions of interest
For the analysis of REDUCE we consider 7 distinct hyperactions (sequences of actions) which we call hyperactions of Type 1,2,3,4,5,33 and 34 respectively. We have put some diagrams of these hyperactions at the end of the paper. In the case that the maximum degree is larger than 3 we consider the following hyperactions:

Type 1: A single max-edge removal,

Type 2: A max edge-removal followed by an auto correction contraction.

Type 3: A single max-edge removal followed by a good contraction.

Type 4: A single max-edge removal followed by 2 good contractions. In this case we add the restriction that there are exactly 6 distinct vertices v, u, x_1, x_2, w_1, w_2 involved in this hyperaction and they satisfy the following: (i) v is a vertex of maximum degree, it is adjacent to u and $\{u, v\}$ is removed during the max-edge removal,

(ii) $d(u) = d(x_1) = d(x_2) = 3$, (iii) $N(u) = \{v, x_1, x_2\}$, $N(x_1) = \{u, x_2, w_1\}$ and $N(x_2) = \{u, x_1, w_2\}$. (Thus $\{u, x_1, x_2\}$ form a triangle.) The two contractions have the same effect as contracting $\{u, x_1, x_2, w_1, w_2\}$ into a single vertex.

In the case that the maximum degree equals 3 we consider the following hyperactions:

Type 5: A max-edge removal followed by 2 good contractions that interact. In this case the 5 vertices u, v, x_1, x_2, z involved in the hyperaction satisfy the following: (i) $\{u, v\}$ is the edge removed by the max-edge removal, (ii) $N(v) = \{u, x_1, x_2\}$, $N(u) = \{v, x_1, z\}$, (so $\{u, v, x_1\}$ form a triangle), (iii) $|(N(x_1) \cup N(x_2) \cup N(z)) \setminus \{u, v, x_1, x_2, z\}| \geq 3$. This hyperaction has the same effect as contracting all of $\{u, v, x_1, x_2, z\}$ into a single vertex.

Type 33: A max-edge removal followed by 2 good contractions that do not interact. There are 6 distinct vertices involved v, v_1, v_2, u, u_1, u_2 where $N(u) = \{v, u_1, u_2\}$ and $N(v) = \{u, v_1, v_2\}$. During the max-edge removal $\{u, v\}$ is removed. Thereafter each of the 2 sets of vertices $\{v, v_1, v_2\}$ and $\{u, u_1, u_2\}$ is contracted to a single vertex.

Type 34: A max-edge removal followed by 3 good contractions. There are 8 distinct vertices involved $v, v_1, v_2, v, u, u_1, u_2, w_1, w_2$. During the max-edge removal $\{u, v\}$ is removed. The conditions satisfied by v, u, u_1, u_2, w_1, w_2 and the actions that are performed on them are the same as the ones in a hyperaction of Type 4 except that now v has degree 3 before the hyperaction. In addition $\{v, v_1, v_2\}$ is contracted into a single vertex.

We divide hyperactions of Type 3 into three classes based on the number of loops created. Assume that during a hyperaction of Type 3 the set $\{v, a, b\}$ is contracted, v is the contracted vertex and v_c is the new vertex. Note that in general, $d(v_c) = d(a) + d(b) - 2 - 2\eta_{a,b}$, where $\eta_{a,b}$ is the number of parallel edges incident with a, b. Once the contraction takes place those edges are turned into loops and then removed by the algorithm. We say that such a hyperaction is of **Type 3a** if $d(v_c) = d(a) + d(b) - 2$ (0 loops are created), is of **Type 3b** if $d(v_c) = d(a) + d(b) - 4$ (1 loop is created) and is of **Type 3c** if $d(v_c) < d(a) + d(b) - 4$ (at least 2 loops are created).

With the exception of a hyperaction of Type 3c, where $\eta_{a,b} \geq 2$, we refer to the hyperactions of interest as good hyperactions. We call any hyperaction that is not good, including a hyperaction of Type 3c, bad.

We next state two lemmas. The proof of Lemma 4 is deferred to Section 3. It states that as long as the excess stays "small" the algorithm only performs good hyperactions, w.h.p. Later we will close the cycle by showing, for the class of graphs that we consider, that as long as good hyperactions are performed the excess stays small.

Lemma 4. *Let $0 \leq i < \tau$ and assume that Γ_i satisfies $ex_\ell(\Gamma_i) \leq \log^2 |V(\Gamma_i)|$ for some $3 \leq \ell = O(1)$. Then with probability $1 - o(|V(\Gamma_i)|^{-1.9})$ the hyperaction that* Reduce *applies to Γ_i is good. In addition, a hyperaction of Type 1,3a or 33 is applied with probability $1 - o(|V(\Gamma_i)|^{-0.9})$.*

Given Lemma 4 we can now prove the following:

Lemma 5. *Let $\omega = \omega(n)$ be a function of n that tends to infinity as n tends to infinity. For $\ell \in \mathbb{N}$ let $Q_{\ell,\omega}(G)$ be the event that* Reduce *only applies good hyperactions to every graph Γ' of the sequence $\Gamma_0, \Gamma_1, ..., \Gamma_\tau - 1$ that satisfies $ex_\ell(\Gamma') \le \log^2 |V(\Gamma')|$ and $|E(\Gamma')| \ge \omega$. Then*

$$\mathbf{Pr}(Q_{\ell,\omega}(G)) = 1 - o(\omega^{-0.9}).$$

Furthermore if Γ' is such a graph then Reduce *applies a bad hyperaction to Γ' with probability $o(\omega^{-1.9})$, while it applies a hyperaction of Type 1, 3a, 33 to Γ' with probability $1-o(\omega^{-0.9})$.*

Proof. For $0 \le i \le \tau - 1$, $ex_\ell(\Gamma_i) \le \log^2 |V(\Gamma_i)|$ implies,

$$2|E(\Gamma_i)| \le \ell|V(\Gamma_i)| + \log^2 |V(\Gamma_i)| \le 2\ell|V(\Gamma_i)|. \tag{5}$$

In addition $|E(\Gamma_i)|$ is decreasing with respect to i. Therefore the probability the event $Q_{\ell,\omega}(G)$ does not occur is bounded by

$$\sum_{i:|E(\Gamma_i)|\ge\omega} o(|V(\Gamma_i)|^{-1.9}) \le \sum_{i:|E(\Gamma_i)|\ge\omega} o(|E(\Gamma_i)|^{-1.9}) \le \sum_{j\ge\omega} j^{-1.9} = o(\omega^{-0.9}).$$

The second part of Lemma 5 follows directly from (5) and the second part of Lemma 4. □

2.4 Proof of Theorem 2

For $\ell \in \mathbb{N}$ define the stopping times

$$\tau_\ell := \min\{i : \Gamma_i \text{ has maximum degree } \ell \text{ or } |E(\Gamma_i)| \le n^{0.9}\}. \tag{6}$$

The first step in the proof of Theorem 2 is the following lemma:

Lemma 6. *Let $8 \le k = O(1)$ and $\Gamma_0 = G$ be a random (multi)-graph with degree sequence $\mathbf{d} \in \mathcal{C}_{3,k}$, n vertices, maximum degree k, minimum degree at least 3 and no loops. Then with probability $1 - o(n^{-0.5})$,*

i) the first $\tau_{k-1} - 1$ hyperactions applied by Reduce *to Γ_0 are good,*

ii) $\Gamma_i \in \mathcal{C}_{3,k-1}$ for $i \le \tau_{k-1}$,

iii) $|E(\Gamma_{\tau_{k-1}})| \ge (1 - 4/k)|E(\Gamma_0)| = \Omega(n)$,

It follows from this lemma that w.h.p. Reduce contracts G to a graph of maximum degree at most 7. And that along the way, the existence of a perfect matching has been preserved. Furthermore, the graph that remains has $\Omega(n)$ vertices. After this, the following lemma shows that w.h.p. Reduce lowers the maximum degree to 4 via good hyperactions.

Lemma 7. *Let $k \in \{5, 6, 7\}$ and $\Gamma_0 = G$ be a random (multi)-graph with degree sequence $\mathbf{d} \in \{3, 4, ..., k\}^n$. Then with probability $1 - o(n^{-0.5})$,*

i) the first $\tau_{k-1} - 1$ hyperactions applied by REDUCE *to Γ_0 are good,*
ii) $|E(\Gamma_{\tau_{k-1}})| \geq |E(\Gamma_0)|/10^{25} = \Omega(n)$,

It follows from Lemmas 6 and 7 that w.h.p. a sequence of good hyperactions will reduce the maximum degree to at most 4. And then we have need to deal with the case where the degrees of G are 3 or 4. For this we show that once again w.h.p. only good hyperactions are executed, this time until we have a graph with $\omega = o(n), \omega \to \infty$ vertices. We show in addition that as long as the remaining graph has $\omega(n)$ vertices, it has a (near-)perfect matching w.h.p. This is summarised in the following lemma.

Lemma 8. *Let $\Gamma_0 = G$ be a random (multi)-graph with vertex degrees 3 or 4 only. Suppose that $\omega = \Omega(\log n)$ and that the stopping condition Ξ in* REDUCE *is $|V(G_i)| \leq \omega$. Then with probability $1 - O(\omega^{-3/4})$*

(i) REDUCE *only executes good hyperactions.*

(ii) On termination, $\Gamma_{\tau_{end}}$ has a (near) perfect matching.

Proof of Theorem 2a: We prove Theorem 2a where we assume that Algorithm 1 treats loops and parallel edge the same way that REDUCE-CONSTRUCT does i.e. it is equivalent to the REDUCE-CONSTRUCT algorithm applied to G with the stopping condition Ξ being $E(G_i) \neq \emptyset$. Let G_j be the first graph in the sequence satysfying $|V(G_j)| \leq \log n$ and $\Delta(G_j) \leq 4$. Then, by taking $\omega = \log n$ in Lemma 8, Lemmas 6, 7 and 8 imply that w.h.p. $R(G, \tau_{end}) = R_{2b}(G, \tau_{end}) = 0$. Substituting in (3) gives that w.h.p.

$$\kappa(G, M) = \kappa(G_{\tau_{end}}, M_{\tau_{end}}) \leq |V(G_{\tau_{end}})| \leq \log n.$$

Hence w.h.p. REDUCE-BACKTRACK matches at least $n - \log n$ vertices. □

Proof of Theorem 1b: We prove Theorem 1b where the modified version of Algorithm 1 corresponds to a 2-phase algorithm $\mathcal{A}$. In its first phase, $\mathcal{A}$ applies the REDUCE-CONSTRUCT algorithm with the stopping condition Ξ being $|V(G_i)| \leq n^{2/3}$ and $\Delta(G_i) \leq 4$ and the matching algorithm Match being the Micali-Vazirani algorithm. Lemmas 6, 7 and 8 imply that with probability $1-o(n^{-1/2})$, Γ_τ has a (near) perfect matching. Such a matching is then found in $O((n^{2/3})^{3/2}) = O(n)$ time by Match and expanded to a (near)-perfect matching of the original graph G.

In the event that the output matching is not (near) perfect then $\mathcal{A}$ proceeds to its second phase where it applies the Micali-Vazirani algorithm to the original graph and finds a maximum matching.

The complexity of the first and the second phase of $\mathcal{A}$ are $O(n + (n^{2/3})^{3/2})$ and $O(n^{3/2})$ respectively. Hence $\mathcal{A}$ outputs a maximum matching in $O(n + n^{-1/2} \cdot n^{3/2}) = O(n)$ expected time. □

We briefly discuss the important properties of $\mathcal{C}_{3,k}$. These are (a) p_3 says small (hence the excess stays small and only good hyperactions occur. Thus we can control most things. (b) The graph belongs to $C_{3,k-1}$ up to the point where its maximum degree becomes $k-1$. Hence we can invoke (a) (i.e p_3 is small) throughout the analysis and we can apply an induction argument.

3 Notation - Preliminaries Results

We start this section by displaying various pieces of notation that we will use in later sections for ease of reference. Later, in Section 3.2, we state and prove results about the excess and hyperactions. These results are used in multiple later Sections.

3.1 Notation

$\Gamma_0, \Gamma_1, ..., \Gamma_\tau$ is a sequence of graphs that is generated by REDUCED. $\Gamma_0 = G$ is the input and Γ_τ is the empty graph. Every graph in the sequence has minimum degree 3 except the last one. To go from Γ_i to Γ_{i+1} REDUCE performs a hyperaction which may be one of the hyperactions of Interest (a.k.a. good hyperactions), listed in Subsection 2.2. Furthermore given the degree sequence $\mathbf{d}_i$ of Γ_i we have that Γ_i is a random (multi)-graph with degree sequence Γ_i and no loops.

Observe that at every hyperaction a max-edge removal is performed, therefore $e_i \leq e_0 - i$ for $i \leq \tau$. Thus, if our initial graph has n vertices and maximum degree k then $2e_i \leq kn - 2i$. $e_\tau \geq 0$ implies that every $i \leq \tau$ satisfies

$$i \leq \tau \leq kn/2. \tag{7}$$

For a graph G, $j, \ell \in \mathbb{N}$:

- $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees of G respectively,
- $n_j(G)$ is the number of vertices of G of degree j,
- $n_j(\mathbf{d})$ is the number of times element j appears in $\mathbf{d}$, for a degree sequence $\mathbf{d}$,
- $p_j(G) := \frac{jn_j(G)}{2|E(G)|}$,
- $p_{>j}(G) := \sum_{h>j} p_h(G)$,
- $ex_\ell(G) := \sum_{v \in V(G)} (d(v) - \ell)\mathbb{I}(d(v) > \ell)$,
- $\mathbf{d}(G)$ is the degree sequence of G.

We denote by $\delta_i, \Delta_i, n_{j,i}, p_{j,i}, p_{>j,i}$ and $ex_{\ell,i}$ the corresponding quantities of Γ_i. Furthermore we let $e_i := |E(\Gamma_i)|$ and $n_i := |V(\Gamma_i)|$.

Observe that from the definitions (1), (2), it follows that $\mathcal{C}_{3,\ell} \subseteq \mathcal{C}_{3,\ell-1}$. We denote $\mathcal{D}_{\ell,j}(\mathbf{d}_i)$ by $\mathcal{D}_{\ell,j,i}$.

Given the sequence $\Gamma_0, \Gamma_1, ..., \Gamma_\tau$, for $3 \le j = O(1)$ we define the following stopping times, in addition to τ_j in (6).

- $t_j := \min\{i : \Gamma_i \notin \mathcal{C}_{3,j} \text{ or } e_i \le n^{0.9}\}$ and
- $t_j^* := \min\{\tau_j, t_j\}$.

For a function $\omega = \omega(n)$ that tends to infinity as n tends to infinity we define

$$\sigma_\omega := \min\{i : e_i \le \omega\}.$$

We later show that if $\Gamma_0 \in \mathcal{C}_{3,k} \subseteq \mathcal{C}_{3,k-1}$ then w.h.p. $t_{k-1}^* = \tau_{k-1} < t_{k-1}$ for $k \ge 8$.

For $\ell \in \mathbb{N}$ and a stopping time σ we let $F_{k,\ell,\sigma}(G)$ be the event that

i) $ex_{\ell,i} \le \log^2 n_i$ for $0 \le i < \sigma$,

ii) Reduce applies a good hyperaction to Γ_i for $0 \le i < \sigma$,

iii) for every $i \le \sigma$ there exists z_i, $i - \log^2 n_i/(k-2) \le z_i < i$ such that $ex_{\ell,z_i} = 0$.

We let $Q_{\ell,\omega}(G)$ be the event that Reduce applies a good hyperaction to every graph Γ' of the sequence $\Gamma_0, \Gamma_1, ..., \Gamma_\tau$ that satisfies $ex_\ell(\Gamma') \le \log^2 n_i$ and $e(\Gamma') \ge \omega$.

We are going to use the following Azuma-Hoeffding inequality ([11]), in multiple places:

Lemma 9. *Let $b \in \mathbb{N}$. For $i_1 \le i < i_2$ let X_i be a random variable that is bounded by b and let $Y_i = \mathbf{E}[X_i|\Gamma_i]$. Then for any $t > 0$,*

$$\mathbf{Pr}\left(\left|\sum_{j=i_1}^{i_2-1}(Y_j - X_j)\right| > t\right) \le 2\exp\left\{-\frac{t^2}{2b^2(i_2-i_1)}\right\}.$$

Notation 10. *We sometimes write $A \le_O B$ in place of $A = O(B)$ for aesthetic purposes.*

3.2 Preliminary Results

We start by proving Lemma 4.

Notation 11. *Let K be an arbitrary positive integer and $b \in \{0,1\}$. For a random graph G and $v \in V(G)$, let $\mathcal{B}_K(G, v, b)$ be the event that G spans a subgraph that contains v, spans $a \le K$ vertices and $a + b$ edges.*

We show that REDUCE either performs one of the good hyperactions given in Section 2.3 or $\mathcal{B}_K(G, v, 1)$ occurs where v is the vertex of maximum degree chosen by REDUCE.

Lemma 12. *Let K be an arbitrary fixed positive integer. Let $\mathbf{d}$ be a degree sequence of length n and G be a random graph with degree sequence $\mathbf{d}$ and no loops. Assume that $ex_\ell(G) \leq \log^2 n$ for some $3 \leq \ell = O(1)$ and let $b \in \{0, 1\}$. Then,*

$$\mathbf{Pr}(\mathcal{B}_K(G, v, b)) = o(n^{-0.9-b}).$$

Proof. $ex_\ell(G) \leq \log^2 n$ implies that G has no loops with probability bounded below by a positive constant (see for example [7]). Hence the condition of having no loops can be ignored in the proof that events have probability $o(1)$. The condition also implies that $\Delta = \Delta(G) \leq \ell + ex_\ell(G) \leq \ell + \log^2 n$.

Let $2m = \sum_{i=1}^n d(i) \leq \ell n + ex_\ell(G) = \Theta(n)$. Then for vertex v and for $b = 0, 1$ the probability that G spans a subgraph that covers v, spans $a \leq K$ vertices and $a + b$ edges can be bounded above by

$$\begin{aligned}
&\leq_O \sum_{a=2}^{K} \binom{n}{a-1} (\Delta a)^{2(a+b)} \frac{(2m - 2(a+b))!}{2^{m-(a+b)}(m - (a+b))!} \times \frac{2^m m!}{(2m)!} \\
&\leq_O \sum_{a=2}^{K} n^{a-1} \Delta^{2(a+b)} \frac{m(m-1)...(m-(a+b)+1)}{2m(2m-1)...(2m-2(a+b)+1)} \\
&\leq_O \sum_{a=2}^{K} n^{a-1} \Delta^{2(a+b)} m^{-(a+b)} \\
&= o(n^{-0.9-b}).
\end{aligned}$$

□

We will now drop the subscript K from $\mathcal{B}$. Taking $K = 20$ will easily suffice for the rest of the proof. Thus $\mathcal{B}(G, v, b) = \mathcal{B}_{20}(G, v, b)$ from now on.

Proof of Lemma 4. As might be expected, this involves a lengthy case analysis that depends on the local structure of Γ_i at a vertex of maximum degree. Let v be the vertex of maximum degree chosen by REDUCE and let u be the vertex adjacent to v such that $\{u, v\}$ is chosen for removal. We will show that if $\mathcal{B}(\Gamma_i, v, 1)$ does not occur then REDUCE performs one of the hyperactions given in Section 2.3. Also observe that if a good hyperaction occurs but not one of Type 1, 3a or 33 then $\mathcal{B}(\Gamma_i, v, 0)$ occurs. Also note that if a hyperaction of Type 3c occurs, corresponding to a bad hyperaction, then $\mathcal{B}(\Gamma_i, v, 1)$ occurs. The probability estimates are deferred to Lemma 12 where it is shown that $\mathbf{Pr}(\mathcal{B}(\Gamma_i, v, 0)) = o(|V(\Gamma_i)|^{-0.9})$ and $\mathbf{Pr}(\mathcal{B}(\Gamma_i, v, 1)) = o(|V(\Gamma_i)|^{-1.9})$.

Case A: $d(v) \geq 4$.
If $d(u) \geq 4$ then a hyperaction of Type 1 is performed. Thus assume $d(u) = 3$ and consider the cases where $|N(u)| = 1, 2, 3$, (recall that we allow parallel edges but not self-loops).

Case A1: $|N(u)| = 1$.
u is connected to v by 3 parallel edges and so $\mathcal{B}(\Gamma_i, v, 1)$ occurs.

Case A2: $|N(u)| = 2$.
Let $N(u) = \{v, u'\}$ and $S = \{u, u', v\}$ and note that $d(S) \geq 10$. Let $T = (N(u') \cup N(v)) \setminus S$. If $|T| \leq 2$ then either S spans more than 3 edges or $S \cup T$ spans at least 7 edges. In both cases $\mathcal{B}(\Gamma_i, v, 1)$ occurs. Assume then that $|T| \geq 3$. Now exactly one of $\{u, u'\}$, $\{u, v\}$ is repeated, else $\mathcal{B}(\Gamma_i, v, 1)$ occurs. If $\{u, u'\}$ is repeated then we perform an auto correction contraction resulting to a hyperaction of Type 2 . If $\{u, v\}$ is repeated then we contract the remaining path (u', u, v). Hence we have performed a hyperaction of Type 3b.

Case A3: $|N(u)| = 3$.
Let $N(u) = \{v, x_1, x_2\}$ and $T = (N(x_1) \cup N(x_2)) \setminus \{u, x_1, x_2\}$.

Sub-case A3a: $|T| \leq 1$.
$\{u, x_1, x_2\} \cup T$ spans at least $(2 + d(x_1) + d(x_2) + |T|)/2 \geq 4 + |T|/2$ edges and the event $\mathcal{B}(\Gamma_i, v, 1)$ occurs.

Sub-case A3b: $T = \{w_1, w_2\}$.
Let Γ_i' denote the graph obtained by deleting the edge $\{u, v\}$ from Γ_i. If v is at distance less than 6 from $\{u\} \cup N(u)$ in Γ_i' then $\mathcal{B}(\Gamma_i, v, 1)$ occurs. To see this consider the subgraph H spanned by $\{u, v, x_1, x_2, w_1, w_2, y\}$ and the vertices on the shortest path P from v to u in Γ_i'. Here y is the neighbor of v on P. It must contain at least two distinct cycles. One spanned by each of $\{u, x_1, x_2, w_1, w_2\}$ and $\{u, v, \} \cup V(P)$.

Thus we may assume that v is at distance at least 6 from $\{u\} \cup N(u)$ in Γ'. Now, if there is no edge from x_1 to x_2 then $\{v, u, x_1, x_2, w_1, w_2\}$ spans at least 7 edges and so $\mathcal{B}(\Gamma_i, v, 1)$ occurs. Thus we may additionally assume that $N(x_1) = \{u, x_2, w_1\}$, $N(x_2) = \{u, x_1, w_2\}$ and $v \notin \{w_1, w_2\} \cup N(w_1) \cup N(w_2)$. We may also assume that $\{w_1, w_2\}$ is not an edge of Γ, for otherwise $\{u, x_1, x_2, w_1, w_2\}$ contains two distinct cycles and $\mathcal{B}(G_i, v, 1)$ occurs. The algorithm Reduce proceeds by contracting u, x_1, x_2 into a single vertex x'. x' has degree 2 and then Reduce proceeds by performing a contraction of x', w_1, w_2 into a new vertex w'. Let $S = N\{w_1, w_2\} \setminus \{x_1, x_2\}$. If $|S| \leq 3$ then $\mathcal{B}(\Gamma_i, u, 1)$ occurs. To see this observe that w_1, w_2 must then have a common neighbor w_3 say. Consider the subgraph H spanned by $\{u, x_1, x_2, w_1, w_2, w_3\}$. H contains at least 7 edges and 6 vertices. If $|S| \geq 4$ then the new vertex has degree 4 and the sequence of actions taken by Reduce corresponds to a hyperaction of Type 4.

Sub-case A3c: $|T| \geq 3$. After the removal of $\{v, u\}$ we contract $\{u, x_1, x_2\}$ into a single vertex of degree at least 3, hence a hyperaction of Type 3 is performed.

Case B: $d(v) = d(u) = 3$.
Case B1: In Γ_i', u and v are at distance at least 4.
Let N' refer to neighborhoods in Γ_i'. If $|N'(N'(u))|$ and $|N'(N'(v))| \leq 3$ then $\mathcal{B}(\Gamma_i, v, 1)$ occurs. Thus we can assume that either $|N'(N'(u))| = 4$ and/or $|N'(N'(v))| = 4$. If both $|N'(N'(u))|, |N'(N'(v))| = 4$ then *Reduce* will perform 2 good contractions and this amounts to a hyperaction of Type 33. Assume then that $|N'(N'(u))| = 4$ and that $|N'(N'(v))| \leq 3$.

If $|N'(N'(v))| = 2$ or 3 then $\mathcal{B}(\Gamma_i, v, 0)$ occurs and *Reduce* will perform 2 good contractions amounting to a hyperaction of Type 33 or 34. Finally, if $|N'(N'(v))| = 1$ then $\mathcal{B}(\Gamma_i, v, 1)$ occurs.

Case B2: In Γ_i', u and v are at distance 3.
In Γ_i there is a cycle C of length 4 containing u, v. If $|N'(N'(u))| \leq 3$ or $|N(N(v))| \leq 3$ or $|N'(u) \cap N'(N'(v))| > 1$ or $|N'(v) \cap N'(N'(u))| > 1$ then $\mathcal{B}(\Gamma_i, v, 1)$ occurs. This is because the graph spanned by $\{u, v\} \cup N(u) \cup N(v) \cup N(N(u) \cup N(N(v))$ in Γ_i will contain a cycle distinct from C. Assume this is not the case. W.l.o.g we may assume that after the max-edge removal of $\{u, v\}$ we have a contraction of $\{u\} \cup N(u)$ followed by a contraction of $\{v\} \cup N(v)$. Observe that neither contraction Reduces the size of $N(N(u))$ or $N(N(v))$. Thus Reduce performs a hyperaction of Type 33.

Case B3: In Γ_i', u and v are at distance 2.
In the case that u, v have 2 common neighbors in Γ_i' we see that $\mathcal{B}(\Gamma_i, v, 1)$ occurs. Assume then that they have a single common neighbor x_1. Let z, x_2 be the other neighbors of u, v respectively. Then either $\mathcal{B}(\Gamma_i, v, 1)$ occurs or Reduce performs a hyperaction of Type 5.

Case B4: In Γ_i', u and v are at distance 1.
So here we have that $\{u, v\}$ is a double edge in Γ_i. Let x, y be the other neighbors of u, v repectively in Γ. Assuming that $\mathcal{B}(\Gamma_i, v, 1)$ does not occur, Reduce performs a max-edge removal followed by a single good contraction and this will be equivalent to a hyperaction of Type 3, involving the contraction of one of x, u, v or u, v, y. □

Now that we have proved Lemma 4 we must verify its key assumption viz. that $ex_\ell(\Gamma)$ remains small for $\ell = O(1)$. This is the aim of Lemmas 13 and 16. In these two lemmas we study how the good hyperactions effect the expected changes of $n_{r,i}$ and $ex_{\ell,i}$ respectively. As discussed earlier given the degree sequence of G_i, $\mathbf{d}_i$, we have that G_i is uniformly distributed among all configurations with degree sequence $\mathbf{d}_i$ and no loops. The mild conditioning resulting from imposing the condition that Γ_i has no loops is insignificant and results in constant factors. These will be insignificant as they will only affect tems of value $o(1)$. For the clarity of the presentation we omit such factors.

Lemma 13. *Let $4 \leq k = O(1)$. Let $\Gamma_0 = G$ be a random (multi)-graph with degree sequence $\mathbf{d}$, maximum degree k, minimum degree 3 and no loops. Suppose that Γ_i has maximum degree at least k and satisfies $ex_{k,i} \leq \log^2 n_i$. Conditioned on the event that a good hyperaction is applied to Γ_i, we have,*

$$n_{r,i} - n_{r,i+1} \leq 5, \qquad \textit{for } 3 \leq r \leq k-1 \ . \tag{8}$$

Furthermore for $3 \leq r \leq k-2$,

$$\mathbf{E}\left[n_{r,i+1} - n_{r,i} | \Gamma_i\right] = p_{r+1,i} - p_{r,i} + p_{3,i}\left(\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - 2p_{r,i} - p_{r,i}^2\right) \\ - p_{r,i}^3 \mathbb{I}(r=3) + o(n_i^{-0.9}), \tag{9}$$

and for $r = k-1$,

$$\mathbf{E}\left[n_{k-1,i+1} - n_{k-1,i} | \Gamma_i\right] = p_{k,i} - p_{k-1,i} + p_{3,i}\left(\sum_{j_1+j_2-2=k-1} p_{j_1,i}p_{j_2,i} - 2p_{k-1,i}\right) \qquad (10)$$
$$+ \mathbb{I}(\Delta_i = k) + o(n_i^{-0.9}).$$

In addition

$$e_i - e_{i+1} \leq 6, \qquad (11)$$

and

$$\mathbf{E}\left[e_{i+1} - e_i | \Gamma_i\right] = -1 - 2p_{3,i} + o(n_i^{-0.9}) \qquad (12)$$

Proof. Fix r, $3 \leq r \leq k-2$. Throughout this lemma we condition on the event that the ith hyperaction is good. We have that $\Delta_i \geq k > 3$ and so REDUCE performs a hyperaction of Type 1, 2, 3a, 3b or 4 with probability $1-p_{3,i}, o(n_i^{-0.9}), p_{3,i}, o(n_i^{-0.9})$ and $o(n_i^{-0.9})$ respectively. All of the hyperactions start with a max-edge removal. That is a random vertex of maximum degree v is chosen along with a random neighbor u and the edge $\{v, u\}$ is removed. v is a vertex of maximum degree and thus $d(v) \geq k$. We summarize the case analysis that follows in Tables 1 and 2 given below.

If $d(u) > 3$ then a hyperaction of Type 1 occurs. As a result, a vertex of degree $d(v)$ and $d(u)$ respectively becomes of degree $d(v)-1$ and $d(u)-1$ resp. Given the above for $4 \leq r \leq k-2$ we have the following two cases:

• **Case a:** $d(u) = r+1$. Then, $n_{r,i+1} - n_{r,i} = 1$. Case (a) occurs with probability $p_{r+1,i}$. Actually it occurs with probability $p_{r+1,j} + O(n_i^{-1})$. The error term $O(n_i^{-1})$ is absorbed into the $o(n_i^{-0.9})$ term that arises and so we omit it here, hopefully without confusion.

• **Case b:** $d(u) = r$. Then, $n_{r,i+1} - n_{r,i} = -1$. Case (b) occurs with probability $p_{r,i}$.

If $d(u) = 3$ then a hyperaction of Type 2, 3a, 3b or 4 occurs. Assume that a hyperaction of Type 3a occurs, that is u has 3 neighbors in Γ_i, let them be $\{v, x_1, x_2\}$. There is no edge from x_1 to x_2. In this case REDUCE contracts $\{v, x_1, x_2\}$. The new vertex, say v_c, has degree $d(x_1)+d(x_2)-2$. For $r \geq 3$, $n_{j,i}$ is decreased by 1 for every vertex of degree r in $\{v, u, x_1, x_2\}$. And $n_{r,i}$ is increased by 1 for every element of $\{d(v)-1, d(x_1)+d(x_2)-2\}$ that is equal to r.

First we let $4 \leq r \leq k-2$ and consider the following 3 cases:

• **Case c:** $d(x_1) = d(x_2) = r$. Then $n_{r,i+1} - n_{r,i} = -2$. Case (c) occurs with probability $p_{3,i}p_{r,i}^2$.

• **Case d:** $d(x_1) = r, d(x_2) \neq r$ or $d(x_1) \neq r, d(x_2) = r$. Then $n_{r,i+1} - n_{r,i} = -1$. Case (d) occurs with probability $p_{3,i}p_{r,i}(2-p_{r,i})$.

• **Case e:** $d(x_1) + d(x_2) - 2 = r$. Then the new vertex has degree r and $n_{r,i+1} - n_{r,i} = 1$. Case (e) occurs with probability $p_{3,i} \cdot p_{d(x_1),i} \cdot p_{d(x_2),i}$.

If $r = 3$ then the above cases are modified as follows (recall that $d(u) = 3$):

- **Case c′:** $d(x_1) = d(x_2) = 3$. Then $n_{3,i+1} - n_{3,i} = -3$. Case (c′) occurs with probability $p_{3,i}^3$.

- **Case d′:** $d(x_1) = 3, d(x_2) \neq 3$ or $d(x_1) \neq 3, d(x_2) = 3$. Then $n_{3,i+1} - n_{3,i} = -2$. Case (d′) occurs with probability $p_{3,i}^2(2 - p_{3,i})$.

- **Case e′:** $d(x_1), d(x_2) > 3$. Then $n_{3,i+1} - n_{3,i} = -1$. Case (e′) occurs with probability $p_{3,i}(1 - p_{3,i})^2$.

This completes the analysis of Case $r = 3$. From the case analysis above and the definition of the hyperactions it follows that (8) holds for $3 \leq r \leq k - 2$.

For $4 \leq r \leq k - 2$ we summarize the case analysis in Table 1.

Case	$d(u)$	**hyperaction that takes place**	$n_{r,i+1} - n_{r,i}$	**probability of occurring**
Case a	$r + 1$	Type 1	1	$p_{r+1,i}$
Case b	r	Type 1	-1	$p_{r,i}$
Case c	3	Type 3a	-2	$p_{3,i}p_{r,i}^2$
Case d	3	Type 3a	-1	$p_{3,i}p_{r,i}(2 - p_{r,i})$
Case e	3	Type 3a	1	$p_{3,i}\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i}$

Table 1: Case analysis for $4 \leq r \leq k - 2$

A hyperaction of Type 2, 3b or 4 occurs with probability $o(n_i^{-0.9})$. The upper bound in (8) is achieved when a hyperaction of Type 4 takes place and all 5 vertices involved in the contractions have degree 3. Thus,

$$\begin{aligned}
\mathbf{E}\left[n_{r,i+1} - n_{r,i} | \Gamma_i\right] &= p_{r+1,i} - p_{r,i} - 2p_{3,i}p_{r,i}^2 - p_{3,i}p_{r,i}(2 - p_{r,i}) \\
&\quad + p_{3,i} \sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} + o(n_i^{-0.9}) \\
&= p_{r+1,i} - p_{r,i} + p_{3,i}\left(\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - 2p_{r,i} - p_{r,i}^2\right) + o(n_i^{-0.9}).
\end{aligned}$$

If $r = 3$ then Case (b), where we assume $d(u) = r > 3$, does not apply. In place of Table 1 we have Table 2 given below.

Case	$d(u)$	**hyperaction that takes place**	$n_{3,i+1} - n_{3,i}$	**probability of occurring**
Case a	4	Type 1	1	$p_{4,i}$
Case c′	3	Type 3a	-3	$p_{3,i}^3$
Case d′	3	Type 3a	-2	$p_{3,i}^2(2 - p_{3,i})$
Case e′	3	Type 3a	-1	$p_{3,i}(1 - p_{3,i})^2$

Table 2: Case analysis for $r = 3$

A hyperaction of Type 2, 3b or 4 occurs with probability $o(n_i^{-0.9})$. Thus, using the identity $p_{3,i}\sum_{j_1+j_2-2=3} p_{j_1,i}p_{j_2,i} = 0$ ($p_{j,i} = 0$ for $j < 3$) we have,

$$\begin{aligned}\mathbf{E}\left[n_{3,i+1} - n_{3,i}|\Gamma_i\right] &= p_{4,i} - 3p_{3,i}^3 - 2p_{3,i}^2(2 - p_{r,i}) - p_{3,i}(1 - p_{3,i})^2 \\ &\quad + p_{3,i}\sum_{j_1+j_2-2=3} p_{j_1,i}p_{j_2,i} + o(n^{-0.75}) \\ &= p_{4,i} - p_{3,i} + p_{3,i}\left(\sum_{j_1+j_2-2=3} p_{j_1,i}p_{j_2,i} - 2p_{3,i} - p_{3,i}^2\right) - p_{3,i}^3 + o(n_i^{-0.9}).\end{aligned}$$

This completes the verification of (9). The derivation of (8) for $r = k-1$ and (10) follows the same analysis except for the fact that if $d(v) = \Delta_i = k$, then because of the initial max-edge removal, $n_{k-1,i}$ is initially increased by one resulting in the additional $\mathbb{I}(\Delta_i = k)$ term found in (10).

Equation (11) is easy to verify from the definition of the Hyperactions. Finally for (12) we have the following table:

Hyperaction	$e_{i+1} - e_i$	**probability occurring**
Type 1	-1	$1 - p_{3,i}$
Type 3a	-3	$p_{3,i}$
Type 2/3b/4	O(1)	$o(n_i^{-0.9})$

Therefore,

$$\mathbf{E}\left[e_{i+1} - e_i\right] = -(1 - p_{3,i}) - 3p_{3,i} - o(n_i^{-0.9}) = -1 - 2p_{3,i} + o(n_i^{0.9}).$$

□

Corollary 14. *Suppose that $i_1 < i_2$ and assume that the first $i_2 - 1$ hyperactions are good. If $i_2 - i_1 \leq n^{0.8}$ and $e_{i_2-1} \geq n^{0.9}$ then $|p_{j,i_2} - p_{j,i_1}| \leq o(n^{-0.05})$ for all $j \in \mathbb{N}$.*

Proof. This follows directly from (11). □

In the proof of Lemma 6 we will use (10) to control $n_{k-1,i_1} - n_{k-1,i_2}$. We use the following lemma to control the change in the most problematic term appearing in (10), namely of the term $\mathbb{I}(\Delta_i = k)$.

Lemma 15. *Suppose that $8 \leq \Delta_i \leq k = O(1)$ and that σ is a stopping time. Assume that the event $F_{k,k,\sigma}$ occurs. In addition assume that $e_{\sigma-1} \geq n^{0.9}$. Let $i_1 < i_2 \leq \sigma$ be such that $n^{0.7} \leq i_2 - i_1 \leq n^{0.8}$. Then w.h.p.*

$$\sum_{i=i_1}^{i_2-1} \mathbb{I}(\Delta_i = k) \geq (i_2 - i_1)(0.999 - (k-2)p_{3,i_1}). \tag{13}$$

Proof. The occurrence event $F_{k,k,\sigma}$ implies that $ex_{k,i} < \log^2 n_i$ and the ith hyperaction is good for $i < i_2 \leq \tau_\sigma$.

Let $Z_i = d(u_c) - k$ if the ith hyperaction is of Type 2, 3a, 3b or 4, and the new vertex created u_c has degree $d(u_c) > k$. Here the edge $\{u, v\}$ was deleted and v was the selected vertex of maximum degree. Otherwise let $Z_i = 0$.

The inequality $ex_{k,i} < \log^2 n_i$ implies that $p_{d,i} \leq (\log^2 n_i + k)/2e_i = o(n_i^{-0.9} \log^2 n)$ for $k < d \leq \log^2 n_i + k$ and $p_{d,i} = 0$ for $d > \log^2 n_i + k$. Hence for $2 \log^2 n_i + k \geq \ell > k$, $Z_i = \ell - k$ if a Hyperaction of Type 3a took place and the vertices involved had degrees $3, d_1, d_2$ (this occurs with probability $p_{3,i} p_{d_1,i} p_{d_2,i}$) or a hyperaction of Type 2,3b or 4 took place (this occurs with probability $o(n_i^{-0.9})$. Therefore,

$$\mathbf{Pr}(Z_i = \ell - k) = p_{3,i} \sum_{d_1+d_2-2=\ell} p_{d_1,i} p_{d_2,i} + o(n_i^{-0.9}) = p_{3,i} \sum_{\substack{d_1+d_2-2=\ell \\ 3 \leq d_1, d_2 \leq k}} p_{d_1,i} p_{d_2,i} + o(n_i^{-0.9} \log^2 n).$$

The inequality $ex_{k,i} < \log^2 n_i$ implies $\mathbf{Pr}(Z_i = \ell - k) = 0$ for $\ell \geq 2 \log^2 n_i$. Thus,

$$\mathbf{E}\left[Z_i | \Gamma_i\right] = p_{3,i} \sum_{3 \leq j_1, j_2 \leq k} p_{j_1,i} p_{j_2,i} \cdot (j_1 + j_2 - 2 - k) \mathbb{I}(j_1 + j_2 - 2 - k \geq 0) + o(1) \tag{14}$$

$$\leq (k-2) p_{3,i} + o(1) \leq (k-2) p_{3,i_1} + o(1). \tag{15}$$

Equation (15) uses Corollary 14. Now observe that

$$0 \leq ex_{k,i_2} = ex_{k,i_1} + \sum_{i=i_1}^{i_2-1} (ex_{k,i+1} - ex_{k,i}) \leq ex_{k,i_1} + \sum_{i=i_1}^{i_2-1} \{Z_i - (1 - \mathbb{I}(\Delta_i = k))\}. \tag{16}$$

The $-(1 - \mathbb{I}(\Delta_i > k))$ term accounts for the fact that when $\Delta_i > k$, $ex_{k,i}$ is decreased by 1, due to the max-edge removal.

We observe next that if $\sum_{i=i_1}^{i_2-1} \mathbb{I}(\Delta(\Gamma_i) = k) < (i_2 - i_1)(0.999 - (k-2) p_{3,i_1})$, then (16) implies that

$$(i_2 - i_1)(0.999 - (k-2) p_{3,i_1}) > \sum_{i=i_1}^{i_2-1} \mathbb{I}(\Delta_i = k) \geq (i_2 - i_1) - ex_{k,i_1} - \sum_{i=i_1}^{i_2-1} Z_i$$
$$\geq (i_2 - i_1) - \log^2 n_{i_1} - \sum_{i=i_1}^{i_2-1} Z_i.$$

This implies that

$$\sum_{i=i_1}^{i_2-1} Z_i \geq (i_2 - i_1)(0.001 + (k-2) p_{3,i_1}) - \log^2 n_{i_1}. \tag{17}$$

Note that $Z_i \leq 2 \log^2 n$. Then, (15) and (17) together with the Azuma-Hoeffding inequality imply,

$$\mathbf{Pr}\left(\sum_{i=i_1}^{i_2-1} \mathbb{I}(\Delta(\Gamma_i) = k) < (i_2 - i_1)(0.999 - (k-2) p_{3,i_1} \right)$$

$$\le \mathbf{Pr}\left(\sum_{i=i_1}^{i_2-1} Z_i \ge (i_2-i_1)(0.001+(k-2)p_{3,i_1}) - \log^2 n_{i_1}\right)$$
$$\le 2\exp\left\{\frac{((0.001-o(1))(i_2-i_1))^2}{8(i_2-i_1)\log^4 n}\right\} = o(n^{-0.5}).$$

In the last equation we used that $n^{0.7} \le i_2 - i_1$. □

We now proceed to estimate the expected change in $ex_{\ell,i}$ in terms of $p_{3,i}$ and $p_{\ell+1}$. Later in Lemma 17 we will argue that as long as it is negative, w.h.p. only good hyperactions occur.

Lemma 16. *Let $4 \le k = O(1)$. Let $\Gamma_0 = G$ be a random (multi)-graph with degree sequence* $\mathbf{d}$*, maximum degree k, minimum degree 3 and no loops. Suppose that Γ_i has maximum degree at least k and satisfies satisfies $ex_{k,i} \le \log^2 n_i$. Conditioned on the event that a good hyperaction is applied to Γ_i, we have for $4 \le \ell \le k$,*

$$|ex_{\ell,i+1} - ex_{\ell,i}| \le \ell - 3 + \mathbb{I}(ex_{\ell,i} = 0). \tag{18}$$

Moreover, either $ex_{\ell,i} = 0$ or

$$\mathbf{E}\left[ex_{\ell,i+1} - ex_{\ell,i} | \Gamma_i\right] \le -(1-p_{3,i}) - p_{\ell+1,i} - p_{3,i}^3 + (\ell-3)p_{3,i}(1-p_{3,i})^2 + n_i^{-0.9}\log^2 n. \tag{19}$$

Proof. As in Lemma 13 we condition on the event that the ith hyperaction is good. The following case analysis is summarized in Table 3 given below.

Fix $4 \le \ell \le k$. Initially a vertex v of maximum degree is chosen along with a neighbor u and the edge $\{v, u\}$ is removed.

If $d(u) > 3$ then a hyperaction of Type 1 occurs and $ex_{\ell,i}$ is decreased by 1 for each vertex of degree greater than $\ell \ge 3$ in $\{v, u\}$. $d(v) = \Delta_i$ and therefore $d(v) = \Delta_i > \ell$ if and only if $ex_{\ell,i} > 0$. We consider the following cases, depending on the value of $d(u)$:

- **Case 1:** $d(u) > \ell$. Then $d(v) \ge d(u) > \ell$ and $ex_{\ell,i+1} - ex_{\ell,i} = -2$. Case 1 occurs with probability $p_{\ge \ell+1,i}$.

- **Case 2:** $3 < d(u) \le \ell$. If Case 2 occurs then $ex_{\ell,i+1} - ex_{\ell,i} = -\mathbb{I}(ex_{\ell,i} > 0)$. Case 2 occurs with probability $1 - p_{3,i} - p_{\ge \ell+1,i}$.

If $d(u) = 3$ then a hyperaction of Type 2, 3a, 3b or 4 occurs. Assume that a hyperaction of Type 3a occurs, that is u has 3 neighbors in Γ_i. Let them be $\{v, x_1, x_2\}$ and observe that there is no edge from x_1 to x_2. In this case REDUCE contracts $\{v, x_1, x_2\}$. The new vertex v_c has degree $d(x_1) + d(x_2) - 2$. For the change in ex_ℓ, $\ell \ge 3$ we consider 3 cases. In all 3 cases the max-edge removal results in the decrease of $ex_{\ell,i}$ by the amount of $\mathbb{I}(d(v) > \ell)$.

- **Case 3:** $d(x_1) = d(x_2) = 3$. The new vertex v_c has degree 4. Thus it does not contribute to $ex_{\ell,i}$ and $ex_{\ell,i+1} - ex_{\ell,i} = -\mathbb{I}(ex_{\ell,i} > 0)$. Case 3 occurs with probability $p_{3,i}^3$

- **Case 4:** $d(x_1) = 3, d(x_2) \ne 3$. The new vertex v_c contributes to $ex_{\ell,i+1}$ by the amount of $d(v_c) - \ell = (d(x_2) + 3 - 2) - \ell = (d(x_2) + 1) - \ell$ only if $d(v_c) = d(x_2) + 1 > \ell$ i.e. only if

$d(x_2) \geq \ell$. If $d(x_2) > \ell$ then x_2 contributes to $ex_{\ell,i}$ by the amount of $d(x_2) - \ell$. Thus,

$$\begin{aligned} ex_{\ell,i+1} - ex_{\ell,i} &= -\mathbb{I}(d(v) > \ell) + (d(x_2) + 1 - \ell)\mathbb{I}(d(x_2) + 1 > \ell) - (d(x_2) - \ell)\mathbb{I}(d(x_2) > \ell) \\ &= -1 \text{ or } 0. \end{aligned}$$

Case 4 occurs with probability at most $p_{3,i}^2(2 - p_{3,i})$.

• **Case 5:** $d(x_1) = j_1 > 3, d(x_2) = j_2 > 3$. x_h contributes to $ex_{\ell,i}$ by the amount of $(j_h - \ell)\mathbb{I}(j_h > \ell)$ for $h = 1, 2$, while the new vertex has degree $j_1 + j_2 - 2$ and contributes by the amount of

$$((j_1 + j_2 - 2) - \ell)\mathbb{I}(j_1 + j_2 - 2 > \ell).$$

Therefore,

$$\begin{aligned} ex_{\ell,i+1} - ex_{\ell,i} &= -\mathbb{I}(d(v) > \ell) + ((j_1 + j_2 - 2) - \ell)\mathbb{I}(j_1 + j_2 - 2 > \ell) \\ &\quad - (j_1 - \ell)\mathbb{I}(j_1 > \ell) - (j_2 - \ell)\mathbb{I}(j_2 > \ell)) \\ &\leq \ell - 2 - \mathbb{I}(d(v) > \ell) = \ell - 3 + \mathbb{I}(ex_{\ell,i} = 0). \end{aligned} \tag{20}$$

In the last line we used $d(v) = \Delta_i$ implies that $\Delta_i > \ell$ iff $ex_{\ell,i} > 0$. Therefore $\mathbb{I}(d(v) > \ell) = \mathbb{I}(ex_{\ell,i} > \ell) = 1 - \mathbb{I}(ex_{\ell,i} = 0)$. From (20) we can conclude that if $ex_{\ell,i} > 0$ then

$$ex_{\ell,i+1} - ex_{\ell,i} \leq \ell - 3. \tag{21}$$

Case 5 occurs with probability $p_{3,i} \cdot p_{j_1,i} \cdot p_{j_2,i}$ (recall $j_1 = d(x_1) > 3, j_2 = d(x_2) > 3$).

• **Case 6:** A hyperaction of Type 2, 3b or 4 occurs. The analysis is similar to Cases 3, 4 and 5. It follows that

$$ex_{\ell,i+1} - ex_{\ell,i} \leq \ell - 3 + \mathbb{I}(ex_{\ell,i} = 0).$$

Case 6 occurs with probability $o(n_i^{-0.9})$.

We summarize the above case analysis in Table 3. Either $ex_{\ell,i} = 0$ or

Case	$d(u)$	**Hyperaction that takes place**	$ex_{\ell,i+1} - ex_{\ell,i}$	**probability occurring**
Case 1	$\ell + 1 \leq d(v)$	Type 1	-2	$p_{\geq \ell+1,i}$
Case 2	$3 < d(u) \leq \ell$	Type 1	$-\mathbb{I}(ex_{\ell,i} > 0)$	$1 - p_{3,i} - p_{\geq \ell+1}$
Case 3	3	Type 3a	-1	$p_{3,i}^3$
Case 4	3	Type 3a	-1 or 0	$p_{3,i}^2(2 - p_{3,i})$
Case 5	3	Type 3a	$\in [-1, u_\ell]$	$p_{3,i} \sum_{j_1,j_2>3} p_{j_1,i} p_{j_2,i}$
Case 6		Type 3b/4/2	$\in [-1, u_\ell]$	$o(n_i^{-0.9})$

Table 3: Case analysis for $4 \leq \ell \leq k$, $u_\ell = \ell - 3 + \mathbb{I}(ex_{\ell,i} = 0)$

Therefore (18) is satisfied. In addition if $ex_{\ell,i} > 0$ then $-\mathbb{I}(ex_{\ell,i} > 0) = -1$ and $u_\ell = \ell - 3$. Thus either $ex_{\ell,i} = 0$ or

$$\mathbf{E}(ex_{\ell,i+1} - ex_{\ell,i} | \Gamma_i) \leq -2p_{\ell+1,i} - (1 - p_{3,i} - p_{\ell+1,i}) - p_{3,i}^3$$

$$
\begin{aligned}
&+ (\ell-3)\sum_{j_1,j_2>3} p_{j_1,i}p_{j_2,i} + (\ell-3)o(n_i^{-0.9}) \\
&\le -1 - p_{\ell+1,i} - p_{3,i}^3 + (\ell-3)p_{3,i}(1-p_{3,i})^2 + n_i^{-0.9}\log^2 n.
\end{aligned}
$$

□

In Lemma 17 and Corollary 18, using (18) and (19), we show that $F_{k,k,\sigma}$ occurs w.h.p. for various stopping times σ. Hence for the corresponding σ we have that w.h.p. the ith hyperaction is good for $i < \sigma$.

Lemma 17. *Let $k \ge 4$ and $\omega = \omega(n) \to \infty$. Let $\Gamma_0 = G$ be a random (multi)-graph with degree sequence $\mathbf{d}$, n vertices, minimum degree at least 3 and no loops that satisfies $ex_{k,0} = 0$. Let σ be a stopping time such that the inequalities $i < \sigma$ and $0 < ex_{k,i} \le \log^2 n_i$ imply*

$$
\mathbf{E}(ex_{k,i+1} - ex_{k,i} | \Gamma_i, Q_{k,\omega}(G)) < C \qquad \text{and} \qquad e_i \ge \omega. \tag{22}
$$

for some constant $C < 0$. Then with probability $1 - o(\omega^{-0.9})$ the event $F_{k,k,\sigma}(G)$ occurs. (See Section 3.1 for the definitions of the events $F_{k,k,\sigma}, Q_{k,\omega}$.)

Proof. We have $ex_{k,0} = 0$, and so conditioned on $Q_{k,\omega}(G)$ occurring, (18) implies that if $F_{k,k,\sigma}(G)$ does not occur then there exists $i \le \sigma$ such that:

i) $0 \le ex_{k,i-\log^2 n_i/(k-2)} \le k-2$,
ii) $0 < ex_{k,j} < \log^2 n_j$ for $i - \log^2 n_i/(k-2) \le j < i$ and
iii) $ex_{k,i} > 0$.

Indeed, conditioned on $Q_{k,\omega}(G)$ occurring, (18) implies that for $i - \log^2 n_i/(k-2) \le j < i$

$$
ex_{k,j} \le ex_{k,i-\log^2 n_i/(k-2)} + (j - (i - \log^2 n_i/(k-2)))(k-2) \le \log^2 n_i \le \log^2 n_j.
$$

Thus the inequality

$$
\mathbf{E}(ex_{k,i+1} - ex_{k,i} | \Gamma_i, Q_k(G)) < C \text{ holds for } i - \log^2 n_i/(k-2) \le j \le i.
$$

The Azuma-Hoeffding inequality (see Lemma 9) implies that

$$
\begin{aligned}
&\mathbf{Pr}(F_{k,k,\sigma}(G) \text{ does not occur }) \le \\
&\qquad \sum_{\Gamma_i : e_i \ge \omega} \exp\left\{ -\frac{((k-2) - C\cdot(\frac{\log^2 n_i}{k-2} - 1))^2}{2\cdot \frac{\log^2 n_i}{k-2}\cdot (k-2)^2} \right\} + \mathbf{Pr}(\neg Q_{k,\omega}(G)) = o(\omega^{-0.9}).
\end{aligned}
$$

For the second inequality, we used Lemma 5 and the observation that $ex_{k,i} \le \log^2 n_i$ implies $kn_i + \log n_i \ge 2e_i$ and the fact that e_i is decreasing. □

Corollary 18. *(a) Let $\Gamma_0 = G$ be a random graph with degree sequence* $\mathbf{d}$*, minimum degree 3, maximum degree 4 and no loops. Let $\omega = \omega(n)$ be a function of n that tends to infinity as n tends to infinity and $\sigma = \min\{i : |E(\Gamma_i)| \leq \omega\}$. Then, with probability $1 - o(\omega^{-0.9})$ the event $F_{4,4,\sigma}$ occurs.*

(b) For $k \in \{5, 6, 7\}$ let $\Gamma_0 = G$ be a random graph with degree sequence $\mathbf{d}$*, minimum degree 3, maximum degree k and no loops. Then, with probability $1-o(n^{-0.5})$ the event $F_{k,k,\tau_{k-1}}$ occurs.*

(c) For $8 \leq k$ let $\Gamma_0 = G \in \mathcal{C}_{3,k}$ be a random graph with degree sequence $\mathbf{d}$*, minimum degree 3, maximum degree k and no loops. Then, with probability $1-o(n^{-0.5})$ the event $F_{k,k,t_{k-1}}$ occurs.*

Proof. We apply Lemma 17, with $\omega = n^{0.9}$ for parts (b),(c). Recall that for $i < \tau_{k-1}$ we have $e_i \geq n^{0.9}$. Thus it remains to verify that the first condition in (22) is also satisfied in each of the settings of (a), (b) and (c). By setting $\ell = k$, (19) implies

$$\mathbf{E}\left[ex_{k,i+1} - ex_{k,i}|\Gamma_i, Q_k(G)\right] \leq -(1 - p_{3,i}) - p_{3,i}^3 + (k-3)p_{3,i}(1 - p_{3,i})^2 + n_i^{-0.9}. \tag{23}$$

a) Maximizing (23) with $k = 4$ and $p_{3,i} \in [0, 1]$ yields a maximum of $-0.5 + o(1)$, attained at $p_{3,i} = 0.5$.

b) Maximizing (23) with $k \in \{5, 6, 7\}$ and $p_{3,i} \in [0, 1]$ yields a maximum of -0.08791, attained at $k = 7, p_{3,i} = 0.40457$.

c) Lemma 19 (below) is applicable for $i < t_{k-1}$. Thus

$$p_{3,i} \leq \frac{3}{\sum_{j=3}^{k-1} \alpha^{j-3} j} + o(1) \leq \frac{1}{k-2} + o(1). \tag{24}$$

Maximizing (23) over $8 \leq k$ and $p_{3,i} \leq 1/(k-2)$ yields a maximum of $-((k-1)(k-3)^2 + 1 - (k-3)^3)/(k-2)^3 + o(1)$ attained at $p_{3,i} = 1/(k-2)$. □

4 Proof of Lemma 6

We split the proof of Lemma 6 into a series of three Lemmas. The first one, Lemma 20, implies that w.h.p. t_{k-1} is determined by the event $\mathcal{D}_{k-1,k-1,i}$. Observe that $i < t_{k-1}$ implies that $n_{r,i} - \alpha n_{r-i,i} > -(\log^2 n - (k-1))n^{0.8}/2^r$ for $4 \leq r \leq k-2$. The proof of Lemma 20 is based on the fact that if $n_{r,i} - \alpha n_{r-1,i}$ is close to $(\log^2 n - (k-1))n^{0.8}/2^r$ then after the ith hyperaction it will increase in expectation for $i < t_{k-1}$.

In Lemma 21, using similar arguments to those in Lemma 20, we show that $\mathcal{D}_{k-1,k-1,i}$ occurs for $i \leq t^*_{k-1}$. Hence $t^*_{k-1} = \tau_{k-1} < t_{k-1}$. Lemmas 20 and 21 together with part (c) of Corollary 18 imply parts (i) and (ii) of Lemma 6. Part (iv) follows from the definition of the graph sequence $\Gamma_0, \Gamma_1, ..., \Gamma_\tau$.

To prove part (iii) of Lemma 6, as we do in Lemma 22 we first argue that $t^*_{k-1} < 1.5n_{k,0}+n^{0.6}$. Then we use (12) to bound $e_{i+1} - e_i$ in terms of $p_{3,i}$. An upper bound on $p_{3,i}$ is provided by Lemma 19 stated below.

Lemma 19. *Let $p_{3,i} = 3n_3/2e_i$ and assume that $e_i \geq n^{0.9}$. Then, $\Gamma_i \in \mathcal{C}_{3,k-1}$ implies that*

$$p_{3,i} \leq \frac{3}{\sum_{j=3}^{k-1} \alpha^{j-3} j} + o(1).$$

Hence $p_{3,i} \leq 0.051$ for $k \geq 8$.

Proof. $\Gamma_i \in \mathcal{C}_{3,k-1}$ implies that $n_{j,i} \geq \alpha^{j-3} n_{3,i} - o(n^{0.85})$ for $3 \leq j \leq k-1$. Therefore

$$p_{3,i} = \frac{3n_{3,i}}{2e_i} = \frac{3n_{3,i}}{\sum_{j=3}^{k-1} jn_{j,i}} \leq \frac{3}{\sum_{j=3}^{k-1} \alpha^{3-j} j} + o(1). \tag{25}$$

Finally for $k \geq 8$, equation (25) implies $p_{3,i} \leq 0.051$. □

Lemma 20. *Let $8 \leq k = O(1)$. Let $\mathbf{d} \in \mathcal{C}_{3,k} \subseteq \mathcal{C}_{3,k-1}$ be a degree sequence with maximum degree k and minimum degree at least 3. Let $G = G_0$ be a random (multi)-graph with degree sequence $\mathbf{d}$, and no loops. Then the event $\mathcal{D}_{k-1,r,t_{k-1}}$ holds w.h.p. for $4 \leq r \leq k-2$.*

Proof. Fix r, $4 \leq r \leq k-2$. We condition on the event $F_{k,k,t_{k-1}}$ occurring. Corollary 18 states that it occurs w.h.p. Hence for every $0 \leq i < t_{k-1}$ the ith hyperaction is good. Also recall $\Gamma_i \in \mathcal{C}_{3,k-1} \supseteq \mathcal{C}_{3,k}$ and $e_i \geq n^{0.9}$ for $i < t_{k-1}$.

For $4 \leq r \leq k-2$, if the event $\neg\mathcal{D}_{k-1,r,t_{k-1}}$ occurs then

$$n_{r,t_{k-1}} - \alpha n_{r-1,t_{k-1}} < -\frac{(\log^2 n - (k-1))n^{0.8}}{2^r}$$

and

$$n_{r+1,i} - \alpha n_{r,i} \geq -\frac{(\log^2 n - (k-1))n^{0.8}}{2^{r+1}} \text{ for } i < t_{k-1}. \tag{26}$$

For $i < t_{k-1}$ let $X_{r,i} = n_{r,i} - \alpha n_{r-1,i}$. Equation (8) implies,

$$|X_{r,i+1} - X_{r,i}| \leq 12 \tag{27}$$

Equation (27), $\Gamma_0 \in \mathcal{C}_{3,k}$ and $\Gamma_i \in \mathcal{C}_{3,k-1}$ for $i < t_{k-1}$ imply that for

$$t_{k-1} - \frac{n^{0.8}}{12 \cdot 2^r} \leq i \leq t_{k-1} - 1$$

we have

$$n_{r,i} - \alpha n_{r-1,i} \leq n_{r,t_{k-1}} - \alpha n_{r-1,t_{k-1}} + 12(t_{k-1} - i)$$

$$\leq -\frac{(\log^2 n - (k-1))n^{0.8}}{2^r} + 12 \cdot \frac{n^{0.8}}{12 \cdot 2^r} \leq -\frac{(\log^2 n - k)n^{0.8}}{2^r}.$$

Thus

$$n_{r-1,i} \geq \alpha^{-1}\left(n_{r,i} + \frac{(\log^2 n - k)n^{0.8}}{2^r}\right). \tag{28}$$

We now embark on a long chain of calculations in order to show that $\mathbf{E}(X_{r,i+1} - X_{r,i}|\Gamma_i) \geq n^{-0.2}$, see (32).

Using (9), the following holds:

$$\begin{aligned}
\mathbf{E}&(X_{r,i+1} - X_{r,i}|\Gamma_i) = \mathbf{E}(n_{r,i+1} - n_{r,i}|\Gamma_i) - \alpha\mathbf{E}(n_{r-1,i+1} - n_{r-1,i}|\Gamma_i) \\
&= p_{r+1,i} - p_{r,i} + p_{3,i}\left(\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - 2p_{r,i} - p_{r,i}^2\right) \\
&- \alpha p_{r,i} + \alpha p_{r-1,i} - \alpha p_{3,i}\left(\sum_{j_1+j_2-2=r-1} p_{j_1,i}p_{j_2,i} - 2p_{r-1,i} - p_{r-1,i}^2\right) - o(n_i^{-0.9}) \\
&= p_{r+1,i} - (1 + \alpha + 2p_{3,i})p_{r,i} + (\alpha + 2\alpha p_{3,1})p_{r-1,i} \\
&+ p_{3,i}\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - p_{3,i}p_{r,i}^2 - \alpha p_{3,i}\sum_{j_1+j_2-2=r-1} p_{j_1,i}p_{j_2,i} + \alpha p_{3,i}p_{r-1,i}^2 \\
&\geq p_{r+1,i} - (1 + \alpha + 2p_{3,i})p_{r,i} + (\alpha + 2\alpha p_{3,1})p_{r-1,i} - 0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}) - o(n^{-0.9}) \qquad (29) \\
&= \frac{(r+1)n_{r+1,i}}{2e_i} - \frac{(1 + \alpha + 2p_{3,i})rn_{r,i}}{2e_i} + \frac{(\alpha + 2\alpha p_{3,1})(r-1)n_{r-1,i}}{2e_i} \\
&- 0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}) - o(n^{-0.9}) \qquad (30)
\end{aligned}$$

To derive (29) we used

$$\begin{aligned}
p_{3,i}&\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - p_{3,i}p_{r,i}^2 - \alpha p_{3,i}\sum_{j_1+j_2-2=r-1} p_{j_1,i}p_{j_2,i} + \alpha p_{3,i}p_{r-1,i}^2 \\
&\geq p_{3,i}\sum_{j_1+j_2-2=r} p_{j_1,i}p_{j_2,i} - \alpha p_{3,i}\sum_{j_1+j_2-2=r-1} p_{j_1,i}(p_{j_2+1,i} + n^{-0.09})/\alpha \\
&- p_{3,i}p_{r,i}^2 + \alpha p_{3,i}((p_{r,i} + n^{-0.09})/\alpha)^2 \\
&\geq (1/\alpha - 1)p_{3,i}p_{r,i}^2 + o(p_{3,i}) - p_{3,i}\sum_{3 \leq j_1 \leq r} p_{j_1,i}n^{-0.09} \geq -0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}).
\end{aligned}$$

In the second line of the above calculations we used that for $3 \leq b_2 \leq b_1 \leq k-1$, $\Gamma_i \in \mathcal{C}_{3,k-1}$ implies

$$\begin{aligned}
p_{b_1,i} &= \frac{b_1 n_{b_1,i}}{2e_i} \geq \frac{\alpha^{b_1-b_2}b_1 n_{b_2,i} - O(n^{0.8}\log^2 n)}{2e_i} \\
&\geq \frac{\alpha^{b_1-b_2}b_1 b_2 n_{b_2,i}}{2b_2 e_i} - O\left(\frac{n^{0.8}\log^2 n}{2n^{0.9}}\right) \geq \alpha^{b_1-b_2}p_{b_2,i} - n^{-0.09}.
\end{aligned}$$

Using (26) and (28) in order to upper bound $n_{r+1,i}$ and $n_{r-1,i}$ respectively by terms involving only $n_{r,i}$, (30) implies

$$
\begin{aligned}
\mathbf{E}(X_{r,i+1} - X_{r,i}|\Gamma_i) &\geq \frac{(r+1)(\alpha n_{r,i} - (\log^2 n - (k-1))n^{0.8}/2^{r+1})}{2e_i} - \frac{(1+\alpha+2p_{3,i})rn_{r,i}}{2e_i} \\
&+ \frac{(\alpha + 2\alpha p_{3,i})(r-1)(n_{r,i} + (\log^2 n - k)n^{0.8}/2^r)}{\alpha \cdot 2e_i} - 0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}) - o(n_i^{-0.9}) \\
&= \frac{(\alpha - 1 - 2p_{3,i})n_{r,i}}{2e_i} - 0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}) - o(n_i^{-0.9}) \\
&+ \frac{(2(r-1)(1+2p_{3,i})(\log^2 n - k) - (r+1)(\log^2 n - (k-1)))n^{0.8}}{2^{r+2}e_i} \\
&\geq \frac{0.198n_{r,i}}{2e_i} - 0.231p_{3,i}p_{r,i}^2 - o(p_{r,i}) - o(n_i^{-0.9}) + \frac{n^{0.8}\log^2 n}{e_i} \qquad (31) \\
&\geq \frac{0.197p_{r,i}}{r} - \frac{0.231 \cdot 3}{1.3^{k-4}(k-1) + 1.3^{k-5}(k-2)}p_{r,i}^2 + n^{-0.2} - o(n_i^{-0.9}) \geq n^{-0.2}. \qquad (32)
\end{aligned}
$$

In (31) and (32) we used Lemma 19 to bound $p_{3,i}$. In addition to (32), if the event $\neg\mathcal{D}_{k-1,r,t_{k-1}}$ occurs then

$$
X_{r,t_{k-1}} < -\frac{(\log^2 n - (k-1))n^{0.8}}{2^r} \leq X_{r,t_{k-1}-n^{0.8}/12\cdot 2^r}
$$

and hence

$$
X_{r,t_{k-1}} - X_{r,t_{k-1}-n^{0.8}/12\cdot 2^r} = \sum_{j=t_{k-1}-n^{0.8}/12\cdot 2^r}^{t_{k-1}-1} (X_{r,j+1} - X_{r,j}) \leq 0. \qquad (33)
$$

Using (27), (32) and (33), the Azuma-Hoeffding inequality gives us,

$$
\begin{aligned}
&\mathbf{Pr}(\neg\mathcal{D}_{k-1,j,t_{k-1}} \text{ for some } 3 \leq j \leq r-2) \\
&\qquad \leq \sum_{r=4}^{k-2} \mathbf{Pr}\Bigg(\sum_{j=t_{k-1}-n^{0.8}/12\cdot 2^r}^{t_{k-1}-1} X_{r,j+1} - X_{r,j} \leq 0 \Bigg| F_{k,k,t_{k-1}} \Bigg) + \mathbf{Pr}(\neg F_{k,k,t_{k-1}}) \\
&\qquad \leq 2\exp\left\{ -\frac{(n^{-0.2} \cdot n^{0.8}/12 \cdot 2^r)^2}{2 \cdot 12^2 \cdot n^{0.8}/12 \cdot 2^r} \right\} + o(n^{-0.5}) = o(n^{-0.5}).
\end{aligned}
$$

□

We use similar techniques, to those used in the proof of Lemma 20, to prove the following Lemma.

Lemma 21. *Under the same assumptions as in Lemma 20, we have that w.h.p.*

$$
t_{k-1}^* = \tau_{k-1} < t_{k-1}.
$$

Proof. Given Lemma 20 it suffices to show that w.h.p. the inequality

$$
n_{k-1,\tau_{k-1}} - \alpha n_{k-2,\tau_{k-1}} \geq -\frac{(\log^2 n - (k-1))n^{0.8}}{2^{k-1}}. \qquad (34)
$$

holds. Assume otherwise. Then,

$$n_{k-1,\tau_{k-1}} - \alpha n_{k-2,\tau_{k-1}} < -\frac{(\log^2 n - (k-1))n^{0.8}}{2^{k-1}}. \tag{35}$$

Equations (35) and (8) imply that the following inequality holds for $\tau_{k-1} - n^{0.8}/(12 \cdot 2^{k-1}) \leq i \leq \tau_{k-1} - 1$:

$$n_{k-1,i} - \alpha n_{k-2,i} < -\frac{(\log^2 n - (k-1))n^{0.8}}{2^{k-1}} + 12 \cdot i \leq -\frac{(\log^2 n - k)n^{0.8}}{2^{k-1}}. \tag{36}$$

Let $X_{k-1,i} = n_{k-1,i} - \alpha n_{k-2,i}$. In a similar manner to the derivation of (30) from (9), we see that equation (10) implies that we have:

$$\begin{aligned}
\mathbf{E}(X_{k-1,i+1} - X_{k-1,i}|\Gamma_i) \geq & -\frac{(1+\alpha+2p_{3,i})(k-1)n_{k-1,i}}{2e_i} \\
& + \frac{(\alpha+2p_{3,i})(k-2)n_{k-2,i}}{2e_i} + \mathbb{I}(\Delta_i = k) - 0.231 p_{3,i} p_{k-1,i}^2 - o(1) \\
& \geq -\frac{(1+\alpha+2p_{3,i})(k-1)\cdot \alpha n_{k-2,i}}{2e_i} \\
& + \frac{(\alpha+2p_{3,i})(k-2)n_{k-2,i}}{2e_i} + \mathbb{I}(\Delta_i = k) - \frac{0.231\alpha^2 p_{3,i} p_{k-2,i}^2 (k-1)^2}{(k-2)^2} - o(1) \qquad (37) \\
& = -\frac{(\alpha + \alpha^2(k-1) + 2\alpha p_{3,i} + 2(k-2)(\alpha-1)p_{3,i})n_{k-2,i}}{2e_i} \\
& + \mathbb{I}(\Delta_i = k) - \frac{0.231\alpha^2 p_{3,i} p_{k-2,i}^2 (k-1)^2}{(k-2)^2} - o(1) \\
& = -\left(\frac{\alpha^2+\alpha+2\alpha p_{3,i}}{k-2} + \alpha^2 + 2(\alpha-1)p_{3,i}\right) p_{k-2,i} \\
& - \frac{0.231\alpha^2 p_{3,i} p_{k-2,i}^2 (k-1)^2}{(k-2)^2} + \mathbb{I}(\Delta_i = k) - o(1) \\
& \geq -2.25 p_{k-2,i} + \mathbb{I}(\Delta_i = k) - o(1).
\end{aligned}$$

To derive (37) we used the LHS of (36), which implies that $n_{k-1,i} \leq \alpha n_{k-2,i}$. In the last line we used $p_{3,i} \leq 0.051$ (Lemma 19) and the inequality $k \geq 8$.

Let $t_\ell = \tau_{k-1} - n^{0.8}/(12 \cdot 2^k), t_u = \tau_{k-1} - 1$. Corollary 14 and Lemma 15 imply that,

$$\begin{aligned}
\sum_{i=t_\ell}^{t_u} \mathbf{E}(X_{k-1,i+1} - X_{k-1,i}|\Gamma_i) \geq & \sum_{i=t_\ell}^{t_u} (-2.25 p_{k-2,i} + \mathbb{I}(\Delta_i = k) - o(1)) \\
& \geq (t_u - t_\ell)(-2.25 p_{k-2,t_\ell} + 0.999 - (k-2)p_{3,t_\ell} - o(1)) \\
& \geq (t_u - t_\ell)\left(-2.25 p_{k-2,t_\ell} + 0.999 - \frac{3(k-2)}{\sum_{j=3}^{k-1} \alpha^{j-3} j} - o(1)\right) \qquad (38) \\
& \geq 0.01(t_u - t_\ell). \qquad (39)
\end{aligned}$$

For (38) we have used (24) to bound p_{3,t_ℓ} and a bound of $1/2.3$ on p_{3,t_ℓ}. To derive this bound on p_{k-2,t_ℓ} observe that $\alpha n_{k-2,t_\ell} \leq n_{k-1,t_\ell} + o(1)$ implies $\alpha p_{k-2,t_\ell} \leq p_{k-1,t_\ell} + o(1)$. Therefore,

$$p_{k-2,t_\ell} \leq \frac{p_{k-2,t_\ell} + p_{k-1,t_\ell}}{1+\alpha} + o(1) \leq (1+\alpha)^{-1}\left(1 - \sum_{j=3}^{k-3} p_{j,t_\ell}\right) + o(1) \leq \frac{1}{2.3}.$$

On the other hand (34), (36) imply

$$\sum_{j=t_\ell}^{t_u} X_{k-1,j+1} - X_{k-1,j} = X_{k-1,t_u} - X_{k-1,t_\ell} \leq 0.$$

Also (8) implies $|X_{k-1,i+1} - X_{k-1,i}| \leq 12$. Thus the Azuma-Hoeffding inequality gives us,

$$\begin{aligned}\mathbf{Pr}(t_{k-1} \leq \tau_{k-1}) &\leq \mathbf{Pr}\left(\sum_{j=t_l}^{t_u} X_{k-1,j+1} - X_{k-1,j} \leq 0 \middle| F_{k,t_{k-1}}\right) + o(n^{-0.5}) \\ &\leq 2\exp\left\{-\frac{(0.01n^{0.8}/12\cdot 2^{k-1})^2}{2\cdot 12^2\cdot(n^{0.8}/12\cdot 2^{k-1})}\right\} + o(n^{-0.5}) = o(n^{-0.5}).\end{aligned}$$

□

For the final part of Lemma 6 we have

Lemma 22. *Under the same assumptions as in Lemma 20, we have that with probability* $i - o(n^{-0.5})$,

$$\tau_{k-1} \leq 1.5n_{k,0} + n^{0.6} \text{ and } e_{\tau_{k-1}} \geq \left(1 - \frac{4}{k}\right)e_0 = \Omega(n).$$

Proof. We condition on the event $F_{k,k,\tau_{k-1}}$ occurring and we note that Lemmas 20 and 21 imply that the event $F_{k,k,\tau_{k-1}}$ occurs with probability $1 - o(n^{-0.5})$. Using the bound provided by (19) (with $\ell = k-1$) we get:

$$\begin{aligned}\mathbf{E}\left[ex_{k-1,i+1} - ex_{k-1,i}|\Gamma_i\right] &\leq -(1-p_{3,i}) - p_{k,i} - p_{3,i}^3 + (k-4)p_{3,i} + n^{-0.75} \\ &\leq -1 + (k-3)p_{3,i} + n^{-0.75} \leq -0.9.\end{aligned}$$

For the last inequality we use $(k-3)p_{3,i} \leq (k-3)\cdot 3/(\sum_{i=3}^{k-1} i\cdot\alpha^{i-3}) + o(1) \leq 0.33$ for $k \geq 8$ (see Lemma 19). Since $ex_{k-1,0} = n_{k,0}$ and $ex_{k-1,\tau_{k-1}} = n_{k,\tau_{k-1}} = 0$, using (18) and the Azuma-Heoffding inequality we get

$$\begin{aligned}\mathbf{Pr}(\tau_{k-1} \geq 1.5n_{k,0} + n^{0.6}) &\leq \mathbf{Pr}(ex_{k-1,i} > 0 \text{ for } i \leq 1.5n_{k,0} + n^{0.6}) \\ &\leq 2\exp\left\{-\frac{(n_{k,0} - 0.9\cdot(1.5n_{k,0}+n^{0.6}))^2}{2(k-2)^2(1.5n_{k,0}+n^{0.6})}\right\} = o(n^{-0.5}).\end{aligned}$$

For $i \leq \tau_{k-1} < t_{k-1}$, using $p_{3,i} \leq 0.051$ from Lemma 19) and (12) we see that $\mathbf{E}(e_{i+1} - e_i|\Gamma_i) \geq -1.2$. Conditioned on the event $\tau_{k-1} \leq 1.5n_{k,0} + n^{0.6}$ and using (11), the Azuma-Hoeffding inequality gives,

$$\mathbf{Pr}(e_0 - e_{\tau_{k-1}} \geq 1.9n_{k,0} + n^{0.7}) \leq 2\exp\left\{-\frac{(1.9n_{k,0} + n^{0.7} - 1.2(1.5n_{k,0}+n^{0.6}))^2}{2\cdot 6^2\cdot(1.5n_{k,0}+n^{0.6})}\right\}$$

$$= o(n^{-0.5}).$$

Finally, $k \geq 8$ implies,

$$e_{\tau_{k-1}} \geq e_0 - 1.9n_{k,0} - n^{0.7} \geq \frac{1}{2}\sum_{j=3}^{k-1} jn_{j,0} + \frac{1}{2}kn_{k,0} - 1.9n_{k,0} - n^{0.7}$$
$$\geq \left(1 - \frac{4}{k}\right)\frac{1}{2}\sum_{j=3}^{k-1} jn_{j,0} + \frac{1}{2}\left(1 - \frac{4}{k}\right)kn_{k,0} = \left(1 - \frac{4}{k}\right)e_0.$$

□

5 Proof of Lemma 7

In this section, we fix $k \in \{5, 6, 7\}$. Let $\mathbf{d}$ be a degree sequence with minimum degree at least 3 and maximum degree k. We also let $G = \Gamma_0$ be a random graph with degree sequence $\mathbf{d}$ and no loops. For the rest of this section we condition on $F_{k,k,\tau_{k-1}}$ occurring. Corollary 18 states that it occurs with probability $1 - o(n^{-0,5})$.

The proof of Lemma 7 is split into two parts. In the first part, Lemma 24, we let

$$t^* = \min\{i : ex_{k-1,i} \leq 10^{-2}e_i\}$$

and we show that $e_{t^*} \geq e_0/10^{25} = \Omega(n)$. In the second part, Lemma 25, we show that $\tau_{k-1} \leq t^* + 6e_{t^*}/10^2$.

Let

$$X_i = \left((ex_{k-1,i+1} - ex_{k-1,i}) - 2.4\frac{ex_{k-1,i}}{e_i}(e_{i+1} - e_i)\right)$$

Roughly speaking X_i compares the rates of decrease of $ex_{k-1,i}$ and e_i after the ith hyperaction. In Lemma 23 we show that X_i decreases in expectation. Using this fact, we show that after a number of hyperactions the ratio $ex_{k-1,i}/e_i$ decreases. As a consequence we prove that there exists t^* such that $e_{t^*} \geq e_0/10^{25} = \Omega(n)$ and $ex_{k-1,t^*} \leq 10^{-2}e_{t^*}$.

To prove part (ii) of Lemma 7 it suffices to argue that $ex_{k,i}$ is decreased by at least 0.2 in expectation after the ith hyperaction for $i \leq \tau_{k-1}$ (done in Lemma 23). From Lemma 24 we have that $ex_{k-1,t^*} \leq e_{t^*}/100$ and therefore, in expectation, ex_{k-1,t^*} reaches 0 in $e_{t^*}/20$ hyperactions. At the same time the number of edges is decreased by at most 6 per iteration (see (11)) and hence after $e_{t^*}/20$ hyperactions it remains linear in n.

We start with a technical Lemma. Equations (40) and (41) are used in the proofs of Lemmas 25 and 24 respectively.

Lemma 23. *For $i < \tau_{k-1}$*

$$\mathbf{E}\left[ex_{k-1,i+1} - ex_{k-1,i} | \Gamma_i\right] \leq -0.2. \tag{40}$$

Furthermore,

$$\mathbf{E}\left[X_i|\Gamma_i\right] < 0 \text{ and } |X_i| \leq k+11. \tag{41}$$

Proof. $i < \tau_{k-1}$ implies that $ex_{k-1,i} > 0$. By setting $\ell = k-1$ in (19) we get

$$\mathbf{E}\left[ex_{k-1,i+1} - ex_{k-1,i}|\Gamma_i\right] \leq -(1-p_{3,i}) - p_{3,i}^3 + (k-4)p_{3,i}(1-p_{3,i})^2 + n_i^{-0.9}\log^2 n \leq -0.2.$$

The last inequality can be easily verified numerically. Its maximum over $k \in \{5,6,7\}, p_{3,i} \in [0,1]$ is -0.23020 and it is attained at $k=7, p_{3,i} = 0.42265$.

In the high probability event $F_{k,k,\tau_{k-1}}$, we have for $i < \tau_{k-1}$,

$$ex_{k-1,i} = n_{k,i} + ex_{k,i} = n_{k,i} + O(\log^2 n).$$

Also $i < \tau_{k-1}$ implies that $e_i > n^{0.9}$. Thus,

$$\frac{kex_{k-1,i}}{2e_i} = \frac{kn_{k,i} + O(\log^2 n)}{2e_i} = p_{k,i} + o(1). \tag{42}$$

Equations (12) and (42) imply that

$$\begin{aligned}\mathbf{E}\left[\frac{ex_{k-1,i}}{e_i}(e_{i+1}-e_i)\middle|\Gamma_i\right] &= \left(\frac{2p_{k,i}}{k} + o(1)\right)\cdot(-1-2p_{3,i}+o(1)) \\ &= -\frac{2p_{k,i}(1+2p_{3,i})}{k} + o(1).\end{aligned} \tag{43}$$

Equation (19) (with $\ell = k-1$) and (43) imply

$$k\mathbf{E}\left[X_i|\Gamma_i\right] \leq k(-(1-p_{3,i}) - p_{k,i} - p_{3,i}^3 + (k-4)p_{3,i}(1-p_{3,i})^2) + 4.8p_{k,i}(1+2p_{3,i}) + o(1)$$

The maximum of the above expression over $k \in \{5,6,7\}$, $p_{3,i}, p_{k,i}, p_{3,i}+p_{k,i} \in [0,1]$ is $-391/1960$ attained at $k=5$, $p_{3,i} = 99/196$ and $p_{5,i} = 97/196$. equations (11) and (18) imply that $|e_{i+1} - e_i| \leq 6$ and $|ex_{k-1,i+1} - ex_{k-1,i}| \leq k-4$ respectively. Therefore $|X_i| \leq (k-4) + 2.4 \cdot 1 \cdot 6 \leq k+11$. □

Lemma 24. *Let*

$$t^* = \min\{i : ex_{k-1,i} \leq 10^{-2}e_i\}.$$

Then w.h.p. $e_{t^*} \geq e_0/10^{24} = \Omega(n)$.

Proof. We start by proving Claim 1. We later use Claim 1 to show that there exists $t^* < \tau_{k-1}$ such that $ex_{k-1,t^*} \leq e_{t^*}/10^2$ and $e_{t^*} \geq e_0/10^{24} = \Omega(n)$.

Claim 1: W.h.p. for every $j \in \mathbb{N}$ such that $e_j = \Omega(n)$ and $j < \tau_{k-1}$ at least one of the following hold:

i) there exists $s_j \geq j$ such that $e_{s_j} \geq e_j/10^3 = \Omega(n)$ and $ex_{k-1,s_j} \leq e_{s_j}/10^2$

ii) there exists $s_j^* \geq j$ such that $e_{s_j} \geq e_j/10^3 = \Omega(n)$ and $ex_{k-1,s_j}/e_{s_j} \leq 0.5ex_{k-1,j}/e_j$.

Proof of Claim 1: Let $j \in \mathbb{N}$ be such that $e_j = \Omega(n)$. Let

$$s_j := \min\{i \geq j : ex_{k-1,i} \leq 0.11ex_{k-1,j}\}.$$

Then for $j \leq i < s_j$,

$$e_i \geq ex_{k-1,i} \geq 0.11ex_{k-1,j} \geq \frac{0.11e_j}{10^2} = \Omega(n).$$

Thus for $j \leq i < s_j$ the inequalities $e_i = \Omega(n)$, $ex_{k-1,i} > 0$ hold. Therefore $s_j \leq \tau_{k-1}$. Lemma 23 implies that $\mathbf{E}(X_i|\Gamma_i) \leq 0$ and $|X_i| \leq k+11$ for every $i < s_j$. Now $s_j \leq kn/2 \leq 3.5n$. Hence, from the Azuma-Hoeffding Inequality we have,

$$\begin{aligned}\mathbf{Pr}\bigg(\sum_{r=j}^{s_j-2} X_r > n^{0.6}\bigg) &\leq s_j \cdot 2\exp\bigg\{-\frac{(n^{0.6})^2}{2(k+11)^2 \cdot 3.5n}\bigg\} + \mathbf{Pr}(\neg F_{k,\tau_{k-1}}) \\ &\leq 7ne^{-n^{0.19}} + o(n^{-0.5}) = o(n^{-0.5}). \qquad (44)\end{aligned}$$

Now for $j \leq i < s_j$ let

$$Y_i = \Big((ex_{k-1,i+1} - ex_{k-1,i}) - 1.2\frac{ex_{k-1,j}}{e_j}(e_{i+1} - e_i)\Big).$$

Assume that (ii) does not hold and that for $j \leq i < s_j$, $ex_{k-1,i}/e_i > 0.5ex_{k-1,j}/e_j$. In this case (44), the definitions of X_i, Y_i and the fact that e_i is decreasing with respect to i imply that w.h.p.

$$n^{0.6} \geq \sum_{r=j}^{s_j-2} X_i \geq \sum_{r=j}^{s_j-2} Y_i.$$

Hence,

$$\begin{aligned}0.11ex_{k-1,j} \leq ex_{k-1,s_j-1} &= ex_{k-1,j} - \sum_{i=j}^{s_j-2}(ex_{k-1,i} - ex_{k-1,i+1}) \\ &\leq ex_{k-1,j} - \sum_{i=j}^{s_j-2}((ex_{k-1,i} - ex_{k-1,i+1}) + Y_i) + n^{0.6} \\ &\leq ex_{k-1,j} + \sum_{i=j}^{s_j-2} 1.2\frac{ex_{k-1,j}}{e_j}(e_{i+1} - e_i) + n^{0.6} \\ &= ex_{k-1,j} + 1.2\frac{ex_{k-1,j}}{e_j}(e_{s_j-1} - e_j) + n^{0.6} \\ &= -0.2ex_{k-1,j} + 1.2\frac{ex_{k-1,j}}{e_j}e_{s_j-1} + n^{0.6}.\end{aligned}$$

The last equality implies that $0.31ex_{k-1,j} \le 1.2\frac{ex_{k-1,j}}{e_j}e_{s_j-1} + n^{0.6}$ and hence

$$0.25e_j \le e_{s_j-1} + \frac{e_j}{1.2e_{k-1,j}} \cdot n^{0.6}. \tag{45}$$

Now assume that in addition to condition (ii), condition (i) does not hold. Then, $e_j < 10^2 ex_{k-1,j}$. Substituting into (45) gives,

$$0.25e_j \le e_{s_j-1} + 10^2 n^{0.6}. \tag{46}$$

Conditioned on $F_{k,k,\tau_{k-1}}$ the (s_j-1)th hyperaction is good and therefore (11) gives,

$$e_{s_j} \ge e_{s_j-1} - 6. \tag{47}$$

In addition,

$$ex_{k-1,j} = n_{k,i} + O(\log^2 n_i) \le e_j. \tag{48}$$

The definition of s_j, (46), (47), (48) and $e_j = \Omega(n)$ imply,

$$\frac{ex_{k-1,s_j}}{e_{s_j}} \le \frac{0.11ex_{k-1,j}}{0.25e_j - 10^2n^{0.6} - 6} \le 0.5\frac{ex_{k-1,j}}{e_j}, \tag{49}$$

contradicting the assumption that (ii) does not hold. □

By iteratively applying Claim 1 we get that w.h.p. there exists a sequence $0 = \sigma_0, \sigma_1, \sigma_2, ..., \sigma_8$ such that

i) $ex_{k-1,\sigma_i}/e_{\sigma_i} \le 0.5ex_{k-1,\sigma_{i-1}}/e_{\sigma_{i-1}}$ or $ex_{k-1,\sigma_i} \le e_{\sigma_i}/100$ for $i \le 8$ and
ii) $e_{\sigma_i} \ge e_{\sigma_{i-1}}/10^3$ for $i \le 8$.

Supose first that $ex_{k-1,\sigma_i}/e_{\sigma_i} \le 0.5ex_{k-1,\sigma_{i-1}}/e_{\sigma_{hi-1}}$ for $i \le 8$. Then,

$$\frac{ex_{k-1,\sigma_8}}{e_{\sigma_8}} \le 0.5\frac{ex_{k-1,\sigma_7}}{e_{\sigma_7}} \le 0.5^2\frac{ex_{k-1,\sigma_6}}{e_{\sigma_6}} \le \cdots \le 0.5^8\frac{ex_{k-1,\sigma_0}}{e_{\sigma_0}} = 0.5^7\frac{ex_{k-1,\sigma_0}}{2e_{\sigma_0}} \le 0.5^7 \le 0.01.$$

So there exists $t^* = \min\{\sigma_i : ex_{k-1,\sigma_i} \le e_{\sigma_i}/100\}$ and $t^* \le \sigma_8$ and $e_{t^*} \ge e_{\sigma_8} \ge e_0/(10^3)^8 = e_0/10^{24}$. □

Lemma 7 now follows from Lemma 25 and Corollary 18. Corollary 18 states that w.h.p. $F_{k,k,\tau_{k-1}}$ occurs and hence the first $\tau_{k-1} - 1$ hyperactions are good.

Lemma 25. *W.h.p.* $\tau_{k-1} \le t^* + 6 \cdot 10^{-2}e_{t^*}$ *and* $e_{\tau_{k-1}} \ge e_0/2^{25}$.

Proof. Let t^* be as in Lemma 24 and let $Z(\Gamma_{t^*})$ be the event that $ex_{k-1,j} > 0$ for $j \le t_1^* = t^* + 6e_{t^*}/10^2$. If $Z(\Gamma_{t^*})$ occurs, conditioned on $F_{k,k,\tau_{k-1}}(G)$, the first $t^* + 6e_{t^*}/10^2$ hyperactions are good. Thus for $t^* \le j \le t_1^*$ the inequality $|e_{j+1} - e_j| \le 6$ holds (see (18)) which implies

$$e_{t^*+i} \ge e_{t^*} - 6i \ge 0.5e_{t^*} \ge e_0/10^{25} = \Omega(n) \text{ for } i \le 6e_{t^*}/10^2. \tag{50}$$

Moreover (40) and (18) state

$$\mathbf{E}(ex_{k-1,j+1} - ex_{k-1,j}|\Gamma_j) = -0.2 \quad \text{and} \quad |ex_{k-1,j+1} - ex_{k-1,j}| \leq k - 4.$$

So,

$$\mathbf{E}(ex_{k-1,t_1^*} \mid \Gamma_j) = \sum_{j=t^*}^{t_1^*} \mathbf{E}(ex_{k-1,j+1} - ex_{k-1,j}|\Gamma_j) + ex_{k-1,t^*} \leq$$
$$- 0.2 \cdot 6e_{t^*}/10^2 + e_{t^*}/10^2 < -0.2e_{t^*}/10^2.$$

Therefore, since $e_{t^*} = \Omega(n)$, the Azuma-Hoeffding inequality (see Lemma 9) implies

$$\mathbf{Pr}(Z(\Gamma_{t^*})) \leq 2\exp\left\{ -\frac{(0.2e_{t^*}/10^2)^2}{2(6e_{t^*}/10^2)\cdot(k-4)^2} \right\} + o(n^{-0.5}) = o(n^{-0.5}).$$

Hence $\tau_{k-1} \leq t^* + 6 \cdot 10^{-2} e_{t^*}$. Equation (50) implies that $e_{\tau_{k-1}} \leq e_0/10^{25}$. □

5.1 Proof of Lemma 8

Proof. Let $\tau = \tau_{end}$ and Γ_τ be the graph that is generated by Reduce-Construct with the stopping condition Ξ being $|V(G_i)| \leq \omega$ and $\Delta(G_i) \leq 4$. Let $\mathcal{E}$ be the event that there exist $i \leq \tau$ such the first i hyperactions performed by Reduce are good and Γ_i has minimum degree 3 and maximum degree 4 and assume that $\mathbf{Pr}(\mathcal{E}) \geq 1-p$. Let $\mathcal{E}'$ be the event that $n_\tau \geq \omega(n)/2$ and all the hyperactions are good.

Claim 26. $\mathbf{Pr}(\mathcal{E}') \geq 1 - p - o(\omega^{-0.9})$.

Given Claim 26, conditioned on $\mathcal{E}'$, we have that $R_0(G,\tau) = R_{2b}(G,\tau) = 0$. In addition

Theorem 27. *Let* $\mathbf{d}$ *be a degree sequence where* $3 \leq d(i) \leq 4$ *for all* i*. Let* G *be a random graph with degree sequence* $\mathbf{d}$ *and no loops. Then* G *has a (near)-perfect matching with probability* $1 - O(n^{-3/4})$.

The proof of this is given in Section 6. Theorem 27 implies that with probability $1-O(\omega^{-3/4})$, Γ_τ has a (near) perfect matching completing the proof of Lemma 8. □

Proof of Claim 26: Assume that $n_\tau < \omega(n)/2$. Corollary 18 implies that there exists $\tau - \log^2 n_\tau/(k-2) \leq j < \tau$ such that $ex_{4,j} = 0$, implying that $\Delta(G_j) \leq 4$. Equation (11) implies that $n_j - n_{j+1} \leq 6$ for $j \leq \tau$. Therefore

$$n_j \leq n_\tau + 6\log^2 n_\tau \leq \frac{\omega}{2} + 6\log^2\left(\frac{\omega}{2}\right) < \omega,$$

contradicting the definition of τ. Finally Corollary 18 implies that the hyperactions performed on $\Gamma_i, \Gamma_{i+1}, ..., \Gamma_{\tau-1}$ are good with probability $1 - o(\omega^{-0.9})$.

6 Existence of a Perfect Matching

We devote this section to the proof of Lemma 27. As discussed in the previous section it is enough to prove that G is given a degree sequence $\mathbf{d} = (d(1), ..., d(n))$ that satisfies $3 \leq d(i) \leq 4$ for all i, then w.h.p. G has a (near)-perfect matching. We will first assume that n is even and verify Tutte's condition. That is for every $W \subset V$ the number of odd components $q(V/W)$ induced by $V \setminus W$, is at most $|W|$. We split the verification of Tutte's condition into

Lemma 28. *Let $W \subset V$ be a set of minimum size that satisfies $q(V/W) > |W|$. Then with probability $1 - O(n^{-3/4})$, $|W| > 10^{-5}n$.*

Lemma 29. *Let $W \subset V$ be a set of maximum size that satisfies $q(V/W) > |W|$. Then with probability $1 - O(n^{-3/4})$, $|W| < 10^{-5}n$.*

Lemmas 28 and 29 together imply that Tutte's condition is satisfied w.h.p. In the proof of these lemmas we use the following estimates.

Lemma 30. *The number of distinct partitions of a set of size $2r$ into 2-element subsets, denoted by $\phi(2r)$, satisfies $\phi(2r) = \Theta\left(\left(\frac{2r}{e}\right)^r\right)$. Also for $\ell < r$ we have $\phi(2r) \leq 2r^{r-\ell}\left(\frac{2\ell}{e}\right)^\ell$.*

Proof. To generate a matching we first choose a permutation of the $2r$ items and then we pair the $(2i-1)$th item with the $2i$th item. Therefore, using Stirling's approximation we have

$$\phi(2r) = \frac{(2r)!}{2^r r!} = \frac{\Theta(\sqrt{2r}\left(\frac{2r}{e}\right)^{2r})}{\Theta(2^r\sqrt{r}\left(\frac{r}{e}\right)^r)} = \Theta\left(\left(\frac{2r}{e}\right)^r\right).$$

Also

$$\phi(2r) = \frac{(2r)!}{2^r r!} \leq \frac{(2r)^{r-\ell}(2\ell)!}{2^{r-\ell}2^\ell \ell!} \leq 2r^{r-\ell}\left(\frac{2\ell}{e}\right)^\ell,$$

where we have used $(2r)! \leq (2r)^{2(r-\ell)}(2\ell)!$. □

6.1 Proof of Lemma 28:

Let W be a set satisfying $q(V \backslash W) > |W|$ of minimum size and assume $2 \leq w = |W| \leq 10^{-5}n$. We can rule out the case $w = 1$ from the fact that with probability $1 - O(1/n)$, G will be 3-connected, see e.g. the proof of Theorem 10.8 in [7] . Let C_z be a component spanned by $V \setminus W$ of maximum size and let $r = |C_z|$.

Case 1: $r = |C_z| \leq 0.997n$**.** In this case we can partition $V \setminus W$ into two parts V_1, V_2 such that (i) each $V_l, l = 1, 2$ is the union of components of $V \setminus W$, (ii) $|V_1| \geq |V_2|$, and (iii) $|V_2| \geq (n - (r + w))/2 \geq 10^{-3}n$.

Let $d_2 = d(V_2)$ and $d_W = d(W)$. Out of the d_W endpoints in W (i.e. configuration points that correspond to vertices in W), suppose that $\ell \leq d_W$ are matched with endpoints in V_2 and the rest with endpoints in V_1.

For fixed i, w, d_2, d_W the probability that there are sets V_1, V_2, W with $w = |W|, d(W) = d_W$ and $|V_2| = i, d(V_2) = d_2$ satisfying $1 \le w \le 10^{-5}n, 10^{-3}n \le i \le 0.5n$ and $d_W \le 4w$, such that $V_1 \times V_2$ spans no edges is bounded by

$$\begin{aligned}
p_1 &\le \sum_{\ell=0}^{d_W} \binom{n}{i}\binom{n-i}{w}\binom{d_W}{\ell} \frac{\phi(d_2+\ell)\cdot\phi(2m-d_2-\ell)}{\phi(2m)} \\
&\le_O \sum_{\ell=0}^{d_W} \left(\frac{en}{i}\right)^i \left(\frac{en}{w}\right)^w 2^{d_W} \frac{\left(\frac{d_2+\ell}{e}\right)^{(d_2+\ell)/2}\left(\frac{2m-d_2-\ell}{e}\right)^{(2m-d_2-\ell)/2}}{\left(\frac{2m}{e}\right)^m} \\
&\le \sum_{\ell=0}^{d_W} \left(\frac{en}{i}\right)^i \left(\frac{100en}{i}\right)^{i/100} 2^{i/25} \left(\frac{d_2+\ell}{2m}\right)^{(d_2+\ell)/2} \left(1-\frac{d_2+\ell}{2m}\right)^{(2m-d_2-\ell)/2} \\
&\le_O \sum_{\ell=0}^{d_W} \left(\frac{1600(en)^{101}}{i^{101}}\right)^{i/100} \left(\frac{d_2+\ell}{2m}\right)^{(d_2+\ell)/2} \exp\left\{-\frac{d_2+\ell}{2}\left(1-\frac{d_2+\ell}{2m}\right)\right\}
\end{aligned}$$

For the third line we used the fact that $w \le i/100$ and $d_W \le 4w \le i/25$.

Let $f(x) = x^x e^{-x(1-x)}$ and $L(x) = \log f(x)$. Then $L''(x) = x^{-1} + 2$ and so L and hence f is convex for $x > 0$. Now $3i \le d_2 \le 4i$ and $\ell \le d_W \le i/25$ and so $d_2 + \ell \in J = [3i, 4.04i]$. Since

$$\left(\frac{d_2+\ell}{2m}\right)^{(d_2+\ell)/2} \exp\left\{-\frac{d_2+\ell}{2}\left(1-\frac{d_2+\ell}{2m}\right)\right\} = f\left(\frac{d_2+\ell}{2m}\right)^m$$

we see that its maxima are at the endpoints of J. In general $3i \le 3n/2 \le m$. However when $d_2 + \ell = 4.04i$ we have that

$$2m \ge 4.04i + 3(n-i-w) \ge 4.04i + 3(n-1.01i) = 3n + 1.01i. \tag{51}$$

Case 1a: $d_2 + \ell = 3i$**.**
We have $\frac{d_2+\ell}{2m} \le \frac{3i}{3n} \le \frac{1}{2}$ and $(d_2+\ell)(1-\frac{d_2+\ell}{2m}) \ge 3i/2$. Therefore,

$$\begin{aligned}
p_1 &\le_O w \left(\frac{1600(en)^{101}}{i^{101}}\right)^{i/100} \left(\frac{i}{n}\right)^{3i/2} e^{-3i/4} \\
&= w\left(\frac{1600e^{26}}{2^{49}}\left(\frac{2i}{n}\right)^{49}\right)^{i/100} \\
&\le w\left(e^{-1/2}\left(\frac{2i}{n}\right)^{49}\right)^{i/100} \\
&\le we^{-i/200}.
\end{aligned}$$

Case 1b: $d_2 + \ell = 4.04i$**.**
It follows from (51) that $\frac{d_2+\ell}{2m} \le \frac{4.04i}{3n+1.01i} \le 0.577$ where the second inequality uses $i \le n/2$. It follows from this that $(d_2+\ell)(1-\frac{d_2+\ell}{2m})/2 \ge 0.85i$. Hence,

$$p_1 \le_O w\left(\frac{1600(en)^{101}}{i^{101}}\right)^{i/100} \left(\frac{4.04i}{3n+1.01i}\right)^{2.02i} e^{-0.85i}$$

$$
\begin{aligned}
&\leq_O w\left(1600e^{16}\left(\frac{n}{i}\right)^{101}\left(\frac{4.04i}{3n}\right)^{101}\left(\frac{4.04i}{3n+1.01i}\right)^{101}\right)^{i/100}\\
&\leq_O w\left(1600e^{16}\left(\frac{4.04}{3}\cdot 0.577\right)^{101}\right)^{i/100}\\
&\leq_O we^{-i/100}.
\end{aligned}
$$

Therefore the probability that Case 1 is satisfied is bounded by a constant times

$$
\sum_{w=1}^{10^{-5}n}\sum_{i=10^{-3}n}^{0.5n} we^{-i/200} = O(n^{-3/4}).
$$

Case 2: $r = |C_z| \geq 0.997n$. Let $V_1 = V(C_z)$, $V_2 = V \setminus (V_1 \cup W)$. First note that V_2 spans at least w components. Therefore $|V_2| \geq w$. We use the following claim to lower bound $e(V_2 : W)$.

Claim 1 Every vertex in W is adjacent to at least 3 distinct components in $V \setminus W$, and hence to at least 2 vertices in V_2.

Proof of Claim 1: Let $v \in W$ be such that it is adjacent to $t \in \{0,1,2\}$ components in $V \setminus W$. Consider $W' = W \setminus \{v\}$. Thus $|W'| = |W| - 1$. If $t = 0$ then $q(V \setminus W') = q(V \setminus W) + 1$. If $t = 1$ then $q(V \setminus W') \geq q(V \setminus W) - 1$. If $t = 2$ then if the both of the corresponding components have odd size then the new component will also have odd side, while if only one of them has odd size then the new one has even size. Finally if both have even size the new one has odd size. In all three cases the inequality $q(V \setminus W') \geq q(V \setminus W) - 1$ is satisfied. Therefore $q(V \setminus W') \geq q(V \setminus W) - 1 > |W| - 1 = |W'|$ contradicting the minimality of W. □

From Claim 1 it follows that $W : V_2$ spans at least $2w$ edges. We also have that $|V_2| \leq n - r - w \leq 0.003n$ and $i \geq q(V \setminus W) - 1 \geq w$. For fixed $2 \leq w \leq 10^{-5}n$, $3w \leq d_W \leq 4w$ and $w \leq i$ the probability that there exist such sets V_1, V_2, W, $|V_2| = i$, $w = |W|$, $d(W) = d_W$ and $2w \leq \ell = e(V_2 : W) \leq 4w$ is bounded by

$$
\begin{aligned}
&\sum_{\ell=2w}^{d_W}\binom{n}{i}\binom{n-i}{w}\binom{d_W}{\ell}\frac{\phi(d_2+\ell)\cdot\phi(2m-d_2-\ell)}{\phi(2m)}\\
&\leq_O \sum_{\ell=2w}^{d_W}\left(\frac{en}{i}\right)^i\left(\frac{en}{w}\right)^w 2^{4w}\frac{\left(\frac{d_2+\ell}{2}\right)^{(d_2+\ell-2w)/2}\left(\frac{2w}{e}\right)^w\left(\frac{2m-d_2-\ell}{e}\right)^{(2m-d_2-\ell)/2}}{\left(\frac{2m}{e}\right)^m}\\
&= \sum_{\ell=2w}^{d_W}\left(\frac{en}{i}\right)^i\left(\frac{en}{w}\right)^w 2^{4w}\left(\frac{2w}{2m}\right)^w\left(\frac{d_2+\ell}{2m}\cdot\frac{e}{2}\right)^{(d_2+\ell-2w)/2}\left(1-\frac{d_2+\ell}{2m}\right)^{(2m-d_2-\ell)/2}\\
&\leq_O w\left(\frac{en}{i}\right)^i\left(\frac{16e}{3}\right)^w\left(\frac{5i}{3n}\cdot\frac{e}{2}\right)^{3i/2}\\
&= w\left(\frac{e^5}{8}\left(\frac{16e}{3}\right)^{2w/i}\frac{5^3 i}{3^3 n}\right)^{i/2}.
\end{aligned}
$$

For the second line we used the second inequality of Lemma 30. For the fourth line we used that $2w \leq \ell$, $d_2 + \ell \leq 4|V_2| + 4w \leq 0.01204n$ and so $\left(\frac{d_2+\ell}{2m} \cdot \frac{e}{2}\right)^{(d_2+\ell-2w)/2}$ is maximized when d_2, ℓ are as small as possible, that is $d_2 = 3i, \ell = 2w \leq 2i$. Furthermore note that $d_2 + \ell - 2w \geq d_2 \geq 3i$. Therefore the probability that Case 2 is satisfied is bounded by a constant times

$$\sum_{w=2}^{10^{-5}n} \sum_{i=w}^{0.003n} \left(\frac{e^5}{8} \left(\frac{16e}{3}\right)^{2w/i} \frac{5^3 i}{3^3 n} \right)^{i/2}$$
$$\leq_O \sum_{w=2}^{10^{-5}n} \sum_{i=w}^{2w} \left(\frac{C_1 i}{n}\right)^{i/2} + \sum_{w=2}^{10^{-5}n} \sum_{i=2w}^{0.003n} \left(\frac{C_2 i}{n}\right)^{i/2}$$

where $C_1 = 16^2 5^3 e^7/(8 \cdot 3^5), C_2 = 16 \cdot 5^3 e^5/(8 \cdot 3^4)$,

$$\leq \sum_{i=2}^{n^{1/4}} i \left(\left(\frac{C_1}{n^{3/4}}\right)^{i/2} + \left(\frac{C_2}{n^{3/4}}\right)^{i/2} \right) + \sum_{i=n^{1/4}}^{2\cdot 10^{-5}n} i \left(\frac{2C_1}{10^5}\right)^{i/2} + \sum_{i=n^{1/4}}^{0.003n} i \left(\frac{3C_2}{10^3}\right)^{i/2}$$
$$= O(n^{-3/4}).$$

Finally, since G has an even number of vertices, for $W = \emptyset$ we have $|W| = q(V \setminus W) = 0$. □

6.2 Proof of Lemma 29:

Let W be a set satisfying $q(V \setminus W) > |W|$ of maximum size and assume $w = |W| \geq 10^{-5}n$.

Claim 2 No component induced by $V \setminus W$ is a tree with more than one vertex.

Proof of Claim 2: Indeed assume that C_i is such a component. If $|C_i|$ is even then let v be a leaf of C_i and define $W' = W \cup \{v\}$. Then $C_i \setminus \{v\}$ is an odd component in $V \setminus W'$ and $q(V \setminus W') = q(V \setminus W) + 1 > |W| + 1 = |W'|$ contradicting the maximality of W.

Thus assume that $|C_i|$ is odd. Let L_1 be the set of leaves of C_i and L_2 be the neighbors of L_1. Set $W' = W \cup L_2$. Then $|L_1| \geq |L_2|$. Furthermore every vertex in L_1 is an odd component in $V \setminus W'$ and in the case $|L_1| = |L_2|$ then $C_i \setminus (L_1 \cup L_2)$ is also an odd component in $V \setminus W'$. Therefore,

$$\begin{aligned} q(V/W') &= q(V/W) - 1 + |L_1| + \mathbb{I}(|L_1| = |L_2|) \\ &\geq q(V/W) + |L_2| + |L_1| - |L_2| + \mathbb{I}(|L_1| = |L_2|) - 1 \\ &> |W| + |L_2| = |W'|, \end{aligned}$$

contradicting the maximality of W. □

We partition $V \setminus W$ into three sets, W_1, W_2 and W_3, as follows. With the underlying graph being the one spanned by $V \setminus W$, W_1 consists of the isolated vertices in $V \setminus W$, W_2 consists of the vertices spanned by components that contain a cycle and have size $s \leq \frac{1}{10} \log n$ and W_3 consists of the vertices that are spanned by a component of size at least $\frac{1}{10} \log n$. Finally let $W_4 = W_2 \cup W_3$. To lower bound W_1 we use the following claim.

Claim 3: W.h.p. W_4 spans at most $\frac{11w}{\log n}$ components in $V \setminus W$.

Proof of Claim 3: First observe that the number of components spanned by W_2 is smaller than the number of cycles of size at most $\frac{1}{10}\log n$ in G, which we denote by r.

$$\begin{aligned}\mathbf{Pr}(r \geq n^{0.3}) &\leq n^{-0.3} \sum_{q=1}^{0.1\log n} \binom{n}{q} 4^q q! \frac{\phi(2q)\phi(2m-2q)}{\phi(2m)} \\ &\leq_O n^{-0.3} \sum_{q=1}^{0.1\log n} \left(\frac{en}{q}\right)^q 4^q \left(\frac{2q}{e}\right)^q \left(\frac{e}{2m}\right)^q \\ &\leq_O n^{-0.3} \sum_{q=1}^{0.1\log n} \left(\frac{8e}{3}\right)^q \leq_O n^{-0.3} (\log n) 8^{0.1\log n} = o(1).\end{aligned}$$

Hence w.h.p. W_2 spans at most $n^{0.3}$ components. Moreover every component spanned by W_3 has size at least $\frac{1}{10}\log n$. Therefore W_4 spans at most $n^{0.3} + \frac{10w}{\log n} = \frac{(1+o(1))10w}{\log n}$ components in $V \setminus W$. □

Since W_4 spans at most $u = \frac{11w}{\log n}$ components in $V \setminus W$ and no component is a tree it follows that the rest of the components consist of at least $q(V \setminus W) - u > w - u$ isolated vertices that lie in W_1.

For convenience, we move $|W_1| - (w-u)$ vertices from W_1 to W_4. Therefore $|W_1| = w - u$. Let k_1 be the number of vertices of degree 4 in W_1 and $d = d(W) - d(W_1)$. Then $0 \leq d \leq 4w - (3(w-u) + k_1) = w + 3u - k_1$. For fixed $10^{-5}n \leq w \leq 0.5n$ the probability that there exist such sets W, W_1, W_4 is bounded by

$$p_2 \leq \sum_{k_1=0}^{w-u} \sum_{d=0}^{w+3u-k_1} \binom{n}{2w}\binom{2w}{w}\binom{w}{u}\binom{4w}{d} \mathbf{Pr}(d(W) - d(W_1) = d) \tag{52}$$

$$\times (3(w-u)+k_1)! \times \frac{(2m - (6(w-u)+2k_1))!}{2^{m-(3(w-u)+k_1}(m - (3(w-u)+k_1))!} \times \frac{2^m m!}{(2m)!}. \tag{53}$$

Explanation: We first choose the sets W, W_1 and W_4 of size $w, w-u$ and $n - 2w + u$ respectively. This can be done in $\binom{n}{w}\binom{n-w}{w-u} = \binom{n}{2w}\binom{2w}{w}\binom{w}{u}\binom{n-2w+u}{u}^{-1}$ ways, but we ignore the final factor. From the at most $4w$ copies of vertices in W we choose a set $W'' \subset W$ be of size d. We let $W' = W/W''$. These are the $3(w-u)+k_1$ copies of vertices that will be matched with those in W_1, explaining the $(3(w-u)+k_1)!$ factor). We must finally explain the meaning of $p_3 = \mathbf{Pr}(d(W) - d(W_1) = d)$ in this context. Looking at the first three binomial coefficients, their product is the number of ways of first choosing a set X of size $2w$, then choosing a set $W \subseteq X$ of size w and then choosing a subset W_1 of size $w-u$ from $X \setminus W$. Having chosen W, W_1 we will randomly choose a subset H of h vertices in $[n]$ to be of degree 4, the remaining vertices being of degree 3. The parameter h will be a variable in the calculation and the probability p_3 is computed with respect to this random choice of H.

In the calculations that follow we let $a = w/n \geq 10^{-5}$. We also let k_4 be the number of vertices of degree 4 that lie in W_4. We first bound the binomial coefficients, found in the first line.

$$\begin{aligned}\binom{n}{2w}\binom{2w}{w}\binom{w}{u}\binom{4w}{d} &= \binom{n}{2an}\binom{2an}{an}\binom{an}{u}\binom{4an}{d} \\ &\leq 2^{o(n)}\left(\frac{1}{2a}\right)^{2an}\left(\frac{1}{1-2a}\right)^{(1-2a)n} 2^{2an}\left(\frac{4ean}{d}\right)^{d} \\ &= 2^{o(n)}\left(\frac{1}{a}\right)^{2an}\left(\frac{1}{1-2a}\right)^{(1-2a)n}\left(\frac{4ean}{d}\right)^{d}. \end{aligned} \tag{54}$$

For the second line we used the fact that $u \leq u_0$ which implies that $\binom{an}{u} = 2^{o(n)}$. Observe that

$$2m = 6(w-u) + 2k_1 + d + 3(n - 2w + u) + k_4 = 3n + d + 2k_1 + k_4 - 3u. \tag{55}$$

Let $m_0 = d + 2k_1 + k_4 - 3u$. For the terms in line (53) we have

$$\begin{aligned}\frac{(2m)!}{2^m m!} &= \frac{(3n)!}{2^{1.5n}(1.5n)!}\frac{\prod_{i=1}^{m_0}(3n+i)}{2^{m_0/2}\prod_{i=1}^{m_0/2}(1.5n+i)} \geq_O \left(\frac{3n}{e}\right)^{1.5n}\prod_{i=1}^{m_0/2}(3n+(2i-1)). \\ &\geq \left(\frac{3n}{e}\right)^{1.5n} e^{-o(n)}(3n)^{-3u/2}\prod_{i=1}^{d/2+k_1+k_4/2}(3n+(2i-1))\end{aligned}$$

Equation (55) implies that

$$2m - (6(w-u) + 2k_1) = 3(1-2a)n + 3u + k_4 + d.$$

Thus,

$$\begin{aligned}&\frac{(2m - (6(w-u)+2k_1))!}{2^{m-(3(w-u)+2k_1)}(m - (3(w-u)+k_1))!} = \frac{(3(1-2a)n)!}{2^{1.5(1-2a)n}(1.5(1-2a)n)!}\cdot\frac{\prod_{i=1}^{d} 3(1-2a)n+i}{2^{d/2}\prod_{j=1}^{\frac{d}{2}} 1.5(1-2a)n+j} \\ &\quad\times \frac{\prod_{i=1}^{k_4} 3(1-2a)n+d+i}{2^{k_4/2}\prod_{j=1}^{k_4/2} 1.5(1-2a)n+d/2+j}\cdot\frac{\prod_{i=1}^{3u} 3(1-2a)n+d+k_4+i}{2^{3u/2}\prod_{j=1}^{\frac{3u}{2}} 1.5(1-2a)n+d/2+k_4/2+j} \\ &\quad\leq_O \left(\frac{3(1-2a)n}{e}\right)^{1.5(1-2a)n}\prod_{i=1}^{d/2}(3(1-2a)n+(2i-1))\prod_{j=1}^{k_4/2}(3(1-2a)n+d+(2j-1))\cdot(2m)^{3u/2} \\ &\quad\leq_O \left(\frac{3(1-2a)n}{e}\right)^{1.5(1-2a)n}(3(1-2a)n+an/2+o(n))^{d/2}(2m)^{3u/2}\prod_{j=1}^{k_4/2}(3(1-2a)n+d+(2j-1))\end{aligned}$$

For the last inequality we used the Arithmetic Mean-Geometric Mean inequality and the fact that $d/2 \leq an/2 + o(n)$, which follows from $d \leq w + 3u - k_1$.

For the first term of (53) we have

$$(3(w-u)+k_1)! \le \frac{3w!}{(3(w-u))^{3u}} \prod_{i=1}^{k_1}(3(w-u)+i) \le \left(\frac{3an}{e}\right)^{3an} \frac{2^{o(n)}}{n^{3u}} \prod_{i=1}^{k_1}(3(w-u)+i).$$

Thus the expression in (53) is bounded by

$$\begin{aligned}
&2^{o(n)} \left(\frac{3an}{e}\right)^{3an} \frac{1}{n^{3u}} \prod_{i=1}^{k_1}(3(w-u)+i) \\
&\times \left(\frac{3(1-2a)n}{e}\right)^{1.5(1-2a)n} (3(1-2a)n+an/2)^{d/2}(2m)^{3u/2} \prod_{j=1}^{k_4/2}(3(1-2a)n+d+2j-1) \\
&\times \left(\left(\frac{3n}{e}\right)^{1.5n} (3n)^{-3u/2} \prod_{i=1}^{d/2+k_1+k_4/2} (3n+(2i-1))\right)^{-1} \\
&= 2^{o(n)} a^{3an}((1-2a)n)^{1.5(1-2a)n} \left(\frac{6m}{n}\right)^{3u/2} \prod_{i=1}^{d/2} \frac{3(1-2a)n+an/2}{3n+(2i-1)} \\
&\times \prod_{i=1}^{k_1} \frac{3(w-u)+i}{3n+d+(2i-1)} \prod_{i=1}^{k_4/2} \frac{3(1-2a)n+d+2i-1}{3n+d+2k_1+2i-1} \\
&\le_O 2^{o(n)} a^{3an}((1-2a)n)^{1.5(1-2a)n} \prod_{i=1}^{d/2} \frac{3(1-2a)n+an/2}{3n} \prod_{i=1}^{k_1} \frac{3(w-u)+i}{3n+d+(2i-1)} \prod_{i=1}^{k_4/2} 1 \\
&\le_O 2^{o(n)} a^{3an}((1-2a)n)^{1.5(1-2a)n}((1-2a)+a/6)^{d/2} 2^{-k_1} \qquad (56)
\end{aligned}$$

Finally we consider the term $\mathbf{Pr}(d(W)-d(W_1)=d)$ and assume that h vertices of degree 4 were chosen to be included in $W \cup W_1$, so that $d = h + 3u - 2k_1$. Then, because there are $\binom{h}{k_1}\binom{2w-u-h}{(w-u)-k_1}$ out of $\binom{2w}{w-u}$ ways to so distribute the k_1 vertices of degree 4,

$$\begin{aligned}
p_3 = \mathbf{Pr}(d(W)-d(W_1)=d) &= \binom{h}{k_1}\binom{2w-u-h}{(w-u)-k_1} \Big/ \binom{2w-u}{w-u} \\
&\le \binom{h}{k_1}\binom{2w-u-h}{w-u} \Big/ \binom{2w-u}{w-u} = \binom{h}{k_1} \prod_{i=0}^{h-1} \frac{w-i}{2w-u-i} \\
&\le 2^{hH(k_1/h)-h+o(n)} = 2^{k_1} 2^{-k_1+h\cdot H(k_1/h)-h+o(n)}.
\end{aligned}$$

Here $H(x) = -x\log_2(x) - (1-x)\log_2(1-x)$ is the entropy function. For fixed d we have $h = d + 2k_1 + o(n)$. Thus,

$$p_3 \le 2^{o(n)+k_1+df(k_1/d)}, \text{ where } f(x) = -x + (1+2x)H\left(\frac{x}{1+2x}\right) - (1+2x).$$

$f(x)$ has a unique maximum at x^*, the solution to $8x(1+x) = (1+2x)^2$ and $f(x^*) \le -0.771$. Hence

$$p_3 \le 2^{-0.771d+k_1+o(n)}. \qquad (57)$$

Multiplying the bounds in (54), (56), (57) together we have a bound

$$\begin{aligned}
p_2 &\le 2^{o(n)-0.771d+k_1}\left(\frac{1}{a}\right)^{2an}\left(\frac{1}{1-2a}\right)^{(1-2a)n}\left(\frac{4ean}{d}\right)^{d} \\
&\times a^{3an}(1-2a)^{1.5(1-2a)n}\left(1-2a+\frac{a}{6}\right)^{d/2}2^{-k_1} \\
&= 2^{o(n)}\left(\frac{2^{1.229}ean}{d}\right)^{d}a^{an}(1-2a)^{0.5(1-2a)n}\left(1-\frac{11a}{6}\right)^{d/2}
\end{aligned}$$

Thus $p_2 = o(1)$ when $d = o(n)$. Let $d = ban$ for some $0 < b \le 1$. Then,

$$p_2 \le \left\{2^{o(1)}\left(\frac{2^{1.229}e}{b}\left(1-\frac{11a}{6}\right)^{0.5}\right)^{b}a(1-2a)^{0.5(1-2a)/a}\right\}^{an}$$

Let $g(a) = 2^{1.229}e\left(1-\frac{11a}{6}\right)^{0.5}$. When $g(a) < e$ then $\left(\frac{g(a)}{b}\right)^{b}$ is maximized when $b = \frac{g(a)}{e}$ which yields

$$p_2 \le \left\{2^{o(1)}\ e^{2^{1.229}(1-\frac{11a}{6})^{0.5}}a(1-2a)^{0.5(1-2a)/a}\right\}^{an} \le \left(\frac{99}{100}\right)^{an}.$$

The last inequality is most easily verified numerically.

When $g(a) > e$ then $\left(\frac{g(a)}{b}\right)^{b}$ is maximized at $b = 1$. Hence

$$p_2 \le \left\{2^{o(1)}\ 2^{1.229}e\left(1-\frac{11a}{6}\right)^{0.5}a(1-2a)^{0.5(1-2a)/a}\right\}^{an} \le \left(\frac{19}{20}\right)^{an}.$$

The last inequality is most easily verified numerically. Thus the probability that there exists a set W satisfying $q(V \setminus W) > |W|$ of size $w = |W| \le 10^{-5}n$ is bounded by

$$\sum_{w=10^{-5}n}^{0.5n}\left(\frac{99}{100}\right)^{w} = o(1).$$

This only leaves the case of n odd. The reader will notice that in none of the calculations above, did we use the fact that n was even. The Tutte-Berge formula for the maximum size of a matching $\nu(G)$ is

$$\nu(G) = \min_{W\subseteq V}\frac{1}{2}(|V|+|W|-q(V\setminus W)).$$

We have shown that the above expression is at least $|V|/2$ for $W \ne \emptyset$ and so the case of n odd is handled by putting $W = \emptyset$ and $q(W) = 1$.

7 Conclusion

In this paper we have analyzed a variant of a Karp-Sipser algorithm and we have shown that w.h.p. it finds a maximum matching in random $k = O(1)$-regular graphs. A key ingredient was the study $(3, k)$-*dominant* graphs and to show that this notion of *dominant* is self preserving with respect to the algorithm. It is natural to try to extend this approach to prove the correctness of the algorithm applied to $G_{n,p}$, as originally intended [12].

From our analysis the following theorem follows from Lemmas 7 and 8. Starting from a random graph with minimum degree 3 and maximum 7 and after applying Reduce-Construct until a graph is reached with minimum degree 3 and maximum 4. This has a perfect matching w.h.p. which can be lifted to a perfect matching of the original graph.

Theorem 31. *Let d be a degree sequence with minimum degree 3 and maximum degree 7. Let G be a random graph with degree sequence d. Then, w.h.p. G has a (near) perfect matching.*

A natural question therefore is to find the maximum integer N such that you can substitude 7 by N in the above theorem. In a random graph with $0.25n$ vertices of degree 21 and $0.75n$ vertices of degree 3, more than $0.25n$ vertices of degree 3 are expected not to have a neighbor of degree 3. Hence a concentration argument easily implies that $N \leq 20$.

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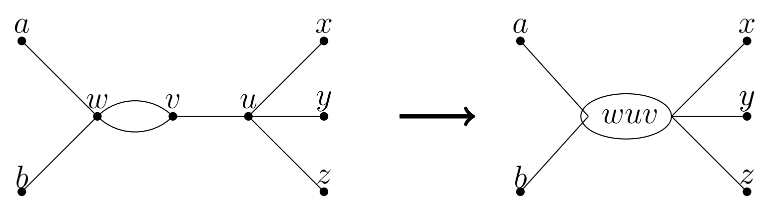
A Diagrams of hyperactions of interest

Type 2.



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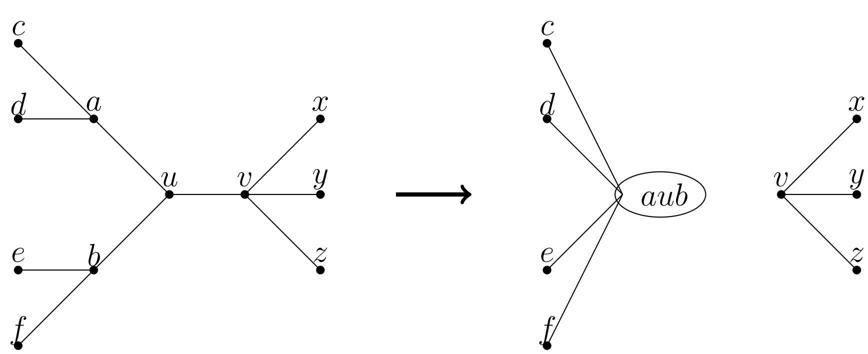
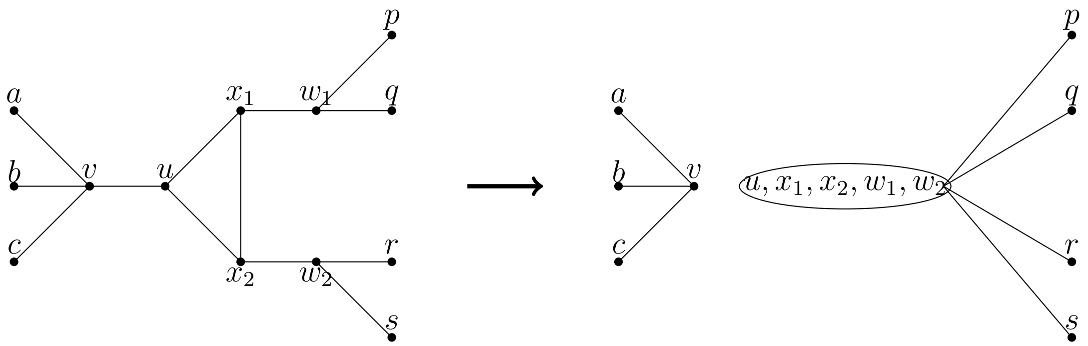


We allow the edge $\{a, b\}$ to be a single edge in this construction. This gives us a Type 3b hyperaction.

Type 4.



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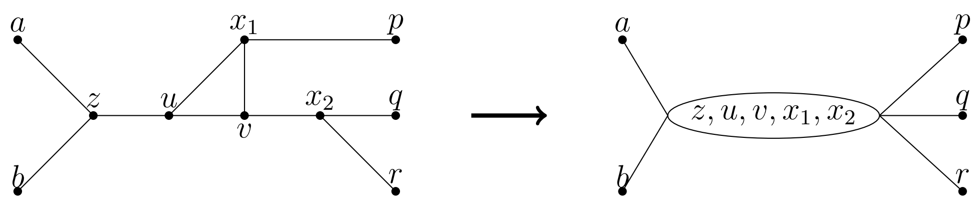

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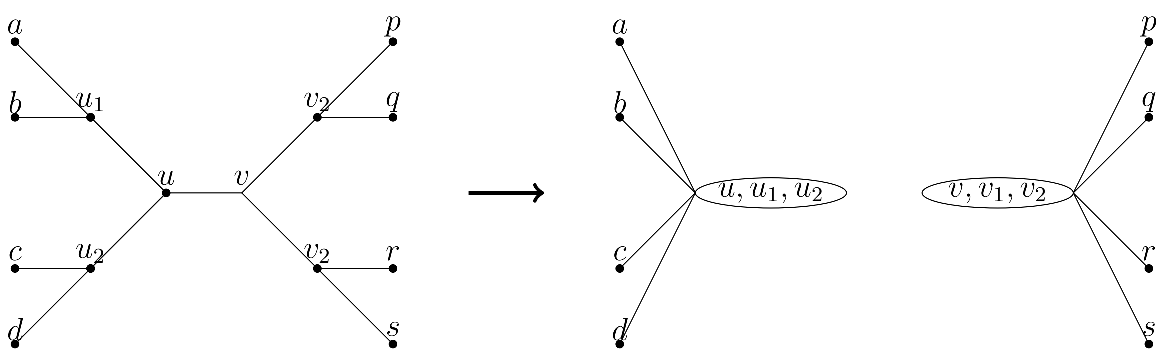

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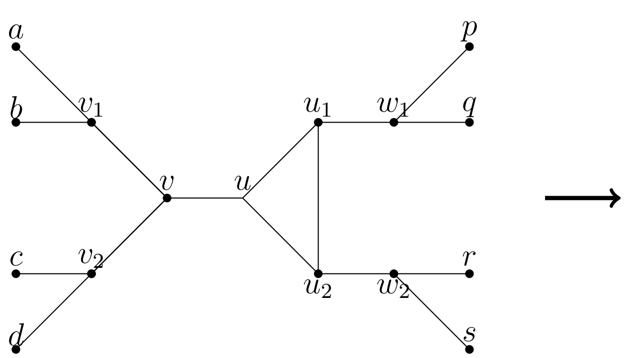

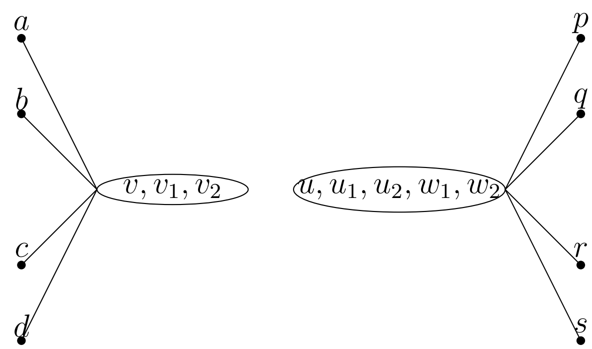
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