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SHORT COMMUNICATION

A BILINEAR PROGRAMMING FORMULATION OF THE 3-DIMENSIONAL ASSIGNMENT PROBLEM

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1. Introduction

The three-dimensional assignment problem described here is a generalisation of the classical (two-dimensional) assignment problem. It arises when three sets of entities, e.g. students, teachers and projects, have to be matched together to maximise some objective.

As an integer programming problem it becomes:

$$\text{maximise} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ijk}\, x_{ijk}\,,$$

$$\text{subject to} \quad \sum_{j=1}^{n} \sum_{k=1}^{p} x_{ijk} = 1 \qquad i = 1, ..., m\,,$$

$$\sum_{i=1}^{m} \sum_{k=1}^{p} x_{ijk} \leqslant b_j \qquad j = 1, ..., n\,,$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ijk} \leqslant c_k \qquad k = 1, ..., p\,,$$

$$x_{ijk} = 0 \text{ or } 1.$$

We can assume without loss of generality that $b_1, ..., b_n, c_1, ..., c_p$ are positive integers. The above problem would arise if each student i

had to be assigned a project j and a supervisor k and if a_{ijk} was the 'suitability' of this assignment.

2. A bilinear program

A solution to the above problem consists of choosing for each index i indices j_i, k_i such that

$$x_{ij_ik_i} = 1 \qquad i = 1, ..., m,$$

$$x_{ijk} = 0 \qquad \text{otherwise.}$$

We can therefore formulate the problem in terms of vectors (y_{ij}), (z_{ik}) such that $y_{ij_i} = 1$, $z_{ik_i} = 1$ and the remaining values are zero.

We can show without difficulty that

Theorem 1. *There is a one-to-one correspondence between solutions to the three-dimensional assignment problem and problem* QI *below.*

This correspondence preserves the value of the objective function and so the two problems are equivalent.

$$\text{QI:} \quad \text{maximise} \quad \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} a_{ijk}\, y_{ij}\, z_{ik},$$

$$\text{subject to} \quad \sum_{j=1}^{n} y_{ij} = 1 \qquad i = 1, ..., m, \tag{2.1}$$

$$\sum_{i=1}^{m} y_{ij} \leqslant b_j \qquad j = 1, ..., n, \tag{2.2}$$

$$\sum_{k=1}^{p} z_{ik} = 1 \qquad i = 1, ..., m, \tag{2.3}$$

$$\sum_{i=1}^{m} z_{ik} \leqslant c_k \qquad k = 1, ..., p, \tag{2.4}$$

$$y_{ij}, z_{ik} = 0 \text{ or } 1.$$

Now let V (resp., V_I) be the set of non-negative (resp., (0, 1)) vectors satisfying (2.1) and (2.2) and define W (resp., W_I) similarly with respect to (2.3) and (2.4). Then it is known that V_I (resp., W_I) is the set of vertices of V (resp., W). We note also that for fixed $Y \in V_I$ the objective function becomes linear in $Z \in W_I$. It is therefore possible to drop the 0–1 constraints on Y, Z and consider the problem Q:

$$\text{Q:}\quad \begin{aligned} &\text{maximise} \quad \varphi(Y, Z) = \sum_i \sum_j \sum_k a_{ijk}\, y_{ij}\, z_{ik}\,, \\ &\text{subject to} \quad Y \in V, \qquad Z \in W\,. \end{aligned}$$

Q is therefore a bilinear programming problem with 0–1 solutions. We can state an obvious necessary condition for (Y^0, Z^0) to be optimal for Q.

Theorem 2. *A necessary condition for a pair* (Y^0, Z^0) *to be an optimal solution to problem* Q *is*

$$\begin{aligned} \varphi(Y^0, Z^0) &= \max\,(\varphi(Y, Z^0)\colon\ Y \in V) \\ &= \max\,(\varphi(Y^0, Z)\colon Z \in W)\,. \end{aligned} \tag{2.5}$$

We note that the maximisations in (2.5) are both two-dimensional assignment problems that can be solved efficiently by well-known methods.

We constructed an algorithm in FORTRAN IV which alternately maximised on Y for the current Z, and maximised on Z for the current Y. The algorithm was tested on some problems involving students, teachers and projects, where $a_{ijk} = 1$ (resp., 0) if the assignment (i, j, k) was suitable (resp., unsuitable). The problem was to find the maximum number of suitable triples for each problem. The results for 3 problems are given in Table 1.

Table 1

m	n	p	s	t	q
43	16	36	43	23	5
69	25	40	63	28	5
114	61	135	110	292	7

s = number of suitable assignments in best solution found.
t = time in seconds on an ICL 1904A.
q = number of two-dimensional assignment problems solved.

3. Conclusion

The above approach leads to a fast method of finding what appear to be good solutions to large problems that could not be solved by existing integer programming, branch and bound methods. The method can be extended to assignment problems of any dimension in which a feasible solution contains one non-zero variable for each distinct value of the first index.

We note finally that the objective function for Q can be replaced by

$$\tfrac{1}{2} \sum_i \sum_j \sum_k a_{ijk} \, (y_{ij} + z_{ik})(y_{ij} + z_{ik} - 1)$$

without changing its value at any integer point. If $a_{ijk} \geqslant 0$, as may be assumed without loss of generality, then this function is convex and one could try the algorithm of Hoang Tuy [3].

For work by others on branch and bound, etc., see [1, 2, 4].

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