Thresholds for Some Basic Properties

Eight Lectures on Random Graphs: Lecture 2

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Graphs and Properties

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- *Monotone* = adding edges cannot violate it.

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- $\mathcal{G}_{n,p}$ = random order-n graph with edge probability p.
- *Whp* = with high probability (approaching 1 as $n \to \infty$).
- *Markov's Inequality*:
  for a random variable $X \geq 0$ and a real $a > 0$

$$\Pr[\,X \geq a\,] \leq \frac{\mathrm{E}[\,X\,]}{a}.$$

Monotone Properties

Theorem For any monotone $\mathcal{A}$ and $p_1 \leq p_2$

$$\Pr\left[\,\mathcal{G}_{n,p_1} \in \mathcal{A}\,\right] \leq \Pr\left[\,\mathcal{G}_{n,p_2} \in \mathcal{A}\,\right].$$

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$$\Pr[\mathcal{G}_{n,p_1} \in \mathcal{A}] \leq \Pr[\mathcal{G}_{n,p_2} \in \mathcal{A}].$$

Proof Define $p_0 \in [0,1]$ by

$$p_1 + (1 - p_1)\, p_0 = p_2.$$

Let $G_1 \in \mathcal{G}_{n,p_1}$ and $G_0 \in \mathcal{G}_{n,p_0}$. Then $G_1 \cup G_0 \sim \mathcal{G}_{n,p_2}$.

$$\Pr[G_1 \in \mathcal{A}] \leq \Pr[G_1 \cup G_0 \in \mathcal{A}]. \blacksquare$$

Thresholds

$p_0 = p_0(n)$ is a *threshold* for a monotone property $\mathcal{A}$ if $\forall p(n)$

$$\Pr\left[\, \mathcal{G}_{n,p} \in \mathcal{A} \,\right] \to \begin{cases} 0, & \text{if } p/p_0 \to 0, \\ 1, & \text{if } p/p_0 \to \infty. \end{cases}$$

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Example $p_0 = \frac{1}{n}$ is a threshold for having a cycle. Indeed,

- if $p = o(1/n)$, then

$$\Pr\left[\,\exists\,\text{cycle}\,\right] \le \mathrm{E}\left[\,\#\text{cycles}\,\right] = \sum_{i\ge 3}^{n} \binom{n}{i} \frac{(i-1)!}{2} p^i \le \sum_{i\ge 3} (np)^i \to 0.$$

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- if $p > \frac{2+\varepsilon}{n}$, then $\mathrm{E}[\,e(G)\,] = p\binom{n}{2} > (2+\varepsilon)\,\frac{n-1}{2}$.
  By Chernoff's bound, whp $e(G) \ge n$.

Monotone $\Rightarrow$ $\exists$ Threshold

Theorem (Bollobás-Thomason'87) Every non-trivial monotone property $\mathcal{A}$ has a threshold.

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Proof Choose $p_0 = p(1/2)$, i.e.

$$\Pr[\,\mathcal{G}_{n,p_0} \in \mathcal{A}\,] = 1/2.$$

p_0 exists as $f(p) = \Pr[\,\mathcal{G}_{n,p} \in \mathcal{A}\,]$ is a polynomial with $f(0) = 0$ and $f(1) = 1$.

$p_0 = p(1/2)$ is a threshold

Given $\varepsilon > 0$, let $(1-\varepsilon)^m < 1/2$. Let $p < p_0/m$. Let $G_1, \dots, G_m \in \mathcal{G}_{n,p}$ and

$$H = G_1 \cup \cdots \cup G_m \sim \mathcal{G}_{n,1-(1-p)^m}.$$

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$$H = G_1 \cup \cdots \cup G_m \sim \mathcal{G}_{n,1-(1-p)^m}.$$

As $1 - (1-p)^m \leq pm \leq p_0$,

$$\frac{1}{2} \leq \Pr[\, H \notin \mathcal{A} \,] \leq \Pr[\, \forall i \; G_i \notin \mathcal{A} \,] = \big(1 - \Pr[\, \mathcal{G}_{n,p} \in \mathcal{A} \,]\big)^m.$$

$\Rightarrow \Pr[\, \mathcal{G}_{n,p} \in \mathcal{A} \,] < \varepsilon$.

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$\Rightarrow \Pr[\,\mathcal{G}_{n,p} \in \mathcal{A}\,] < \varepsilon$.

Other direction: take $\mathcal{G}_{n,p_0} \cup \cdots \cup \mathcal{G}_{n,p_0}$. ∎

Connectivity Property $\mathcal{C}$

Idea 1: Connectivity $=\exists$ spanning tree

$$\mathrm{E}\,[\,\#\,\text{spanning trees}\,] = n^{n-2}\cdot p^{n-1}.$$

The "window" is

$$p = (1+o(1))\,\frac{1}{n}.$$

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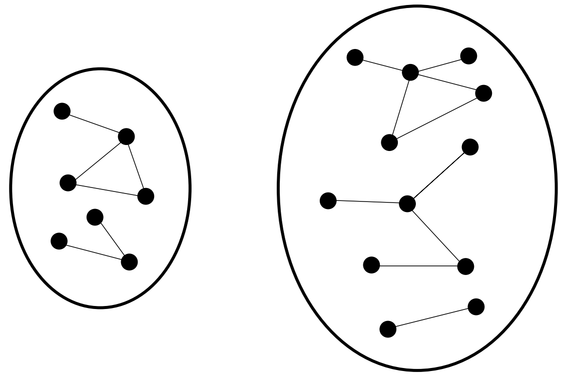
$\mathrm{E}\left[\,\#\,\text{spanning trees}\,\right] \to \infty \;\not\Rightarrow\; G \in \mathcal{C}$.

Idea 2: Look at Cuts

Cut:



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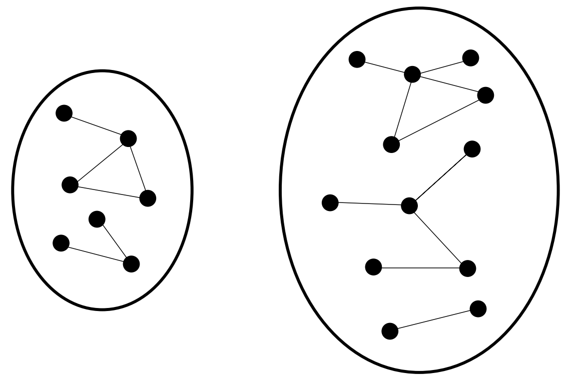

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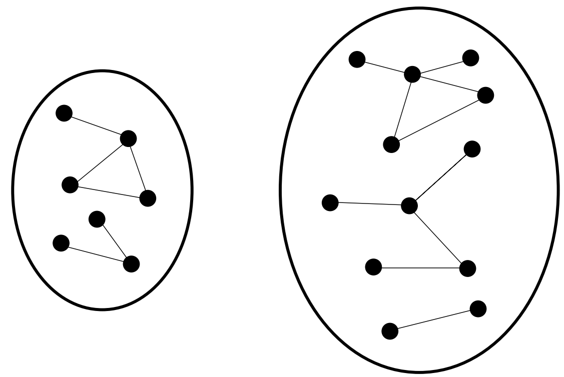

Cuts or Isolated Components?

$C_k = \# \, k$-components

Observation: $G \in \mathcal{C}$ iff $C_k = 0 \; \forall k \in [1, n/2]$.

Connectivity Threshold

Theorem (Erdős-Renyi'60) Let

$$p = \frac{\log n}{n} + \frac{c}{n}.$$

Then $\Pr[\mathcal{G}_{n,p} \in \mathcal{C}] \to \begin{cases} e^{-e^{-c}}, & |c| = O(1), \\ 0, & c \to -\infty, \\ 1, & c \to +\infty. \end{cases}$

In particular, $p_0(n) = \frac{\log n}{n}$ is a threshold for connectivity.

$$p = \frac{\log n}{n} + \frac{O(1)}{n}$$

Let $\sum := \sum_{k=2}^{\lfloor n/2 \rfloor}$.

$$\Pr\left[\sum C_k \geq 1\right] \leq \mathrm{E}\left[\sum C_k\right]$$

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$$\left[\binom{n}{k} \leq (\mathrm{e}n/k)^k \ \& \ (1-x) \leq \mathrm{e}^{-x}\right]$$

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$$\left[\binom{n}{k} \leq (\mathrm{e}n/k)^k \ \& \ (1-x) \leq \mathrm{e}^{-x}\right]$$

$$\leq n \sum \left(O(\log n)\, \mathrm{e}^{-np+kp}\right)^k \to 0.$$

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$$\leq n \sum \left(O(\log n)\, \mathrm{e}^{-np+kp}\right)^k \to 0.$$

Thus whp $C_2 = \cdots = C_{\lfloor n/2 \rfloor} = 0$.

$p = \frac{\log n}{n} + \frac{O(1)}{n}$ (cont.)

Thus, whp $\mathcal{G}_{n,p} \in \mathcal{C}$ iff $C_1 = 0$ (i.e. no isolated vertices).

$p = \frac{\log n}{n} + \frac{O(1)}{n}$ **(cont.)**

Thus, whp $\mathcal{G}_{n,p} \in \mathcal{C}$ iff $C_1 = 0$ (i.e. no isolated vertices).

It is enough to prove $\Pr[\, C_1 = 0 \,] \to \mathrm{e}^{-\mathrm{e}^{-c}}$ because

$$\begin{aligned} 0 &\leq \Pr[\, C_1 = 0 \,] - \Pr[\, C \in \mathcal{C} \,] \\ &\leq \Pr[\, \exists i \in [2, n/2]\ C_i > 0 \,] \;\to\; 0. \end{aligned}$$

Poisson Distribution with Mean μ

n independent trials, $\Pr[\text{success}] = \frac{\mu}{n}$, constant μ.

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$$\Pr[\,i\ \text{successes}\,] = \binom{n}{i} p^i\,(1-p)^{n-i} \to \frac{\mu^i\,\mathrm{e}^{-\mu}}{i!}.$$

Isolated Vertices

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The k-th Factorial Moment:

$$M_k[X] = \mathrm{E}\left[\, (X)_k \,\right] = \mathrm{E}\left[\, X(X-1)\ldots(X-k+1) \,\right].$$

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For fixed k

$$M_k[C_1] = (n)_k\,(1-p)^{k(n-1)-\binom{k}{2}} \to (\mathrm{e}^{-c})^k = M_k[\mathrm{Poisson}(\mathrm{e}^{-c})].$$

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This is known to imply that $C_1 \to \mathrm{Poisson}(\mathrm{e}^{-c})$. In particular,

$$\Pr\,[\,C_1 = 0\,] \to \mathrm{e}^{-\mathrm{e}^{-c}}. \;\blacksquare$$

Sharp Threshold

Connectivity Threshold

p_0 is a *sharp threshold* for a monotone $\mathcal{A}$ if $\forall\,\varepsilon > 0$ whp

$$\mathcal{G}_{n,(1-\varepsilon)p_0} \notin \mathcal{A} \text{ and } \mathcal{G}_{n,(1+\varepsilon)p_0} \in \mathcal{A}.$$

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Examples,

- connectivity: sharp,
- having a triangle: not sharp,
- having a cycle: ‘one-sided sharp’.

Friedgut's Theorem

Note: Sharp threshold $\Rightarrow \frac{\partial}{\partial p}\Pr[\mathcal{G}_{n,p} \in \mathcal{A}] \neq O\left(\frac{1}{p}\right)$.

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Theorem (Friedgut'99) If for a monotone $\mathcal{A}$

$$\frac{\partial}{\partial p}\Pr\left[\,\mathcal{G}_{n,p} \in \mathcal{A}\,\right] = O\left(\frac{1}{p}\right),$$

then $\forall\, \varepsilon > 0$ there is a finite family $\mathcal{F}$ of graphs such that $\forall\, n, p$

$$\Pr\left[\,\mathcal{G}_{n,p} \in \mathcal{A} \,\triangle\, \{\text{an } \mathcal{F}\text{-subgraph}\}\,\right] \leq \varepsilon. \blacksquare$$

Applying Friedgut's Theorem

Difficult to apply: the type of $\mathcal{A} \cup \mathcal{B}$ depends on which one appears 'earlier'.

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Theorem (Achlioptas-Friedgut'99) For fixed $k \geq 3$ k-colorability has a sharp threshold.

Model $\mathcal{G}_{n,M}$

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Theorem For a monotone $\mathcal{A}$,

$$\Pr\left[\mathcal{G}_{n,M} \in \mathcal{A}\right]$$

is a non-decreasing function of M. ∎

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Theorem For any non-trivial monotone $\mathcal{A}$, there is a *threshold* M_0, that is,

$$\Pr\left[\,\mathcal{G}_{n,M} \in \mathcal{A}\,\right] \to \begin{cases} 0, & \text{if } M/M_0 \to 0, \\ 1, & \text{if } M/M_0 \to \infty. \end{cases}$$

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Theorem Let $M = \frac{\log n + c}{n}\binom{n}{2}$. Then

$$\Pr\left[\,\mathcal{G}_{n,M} \in \mathcal{C}\,\right] \to \begin{cases} \mathrm{e}^{-\mathrm{e}^{-c}}, & |c| = O(1), \\ 0, & c \to -\infty, \\ 1, & c \to +\infty. \end{cases}$$

In particular, $M_0 = n\,\log n$ is a threshold for connectivity.

Connectivity of $\mathcal{G}_{n,M}$

Proof Enough to consider $|c| = O(1)$. Take small $\varepsilon > 0$. Let

$$p = \frac{\log n + c - \varepsilon}{n}.$$

Take $G \in \mathcal{G}_{n,p}$. Let $l = M - e(G)$. If $l \geq 0$, let

$$H = G + l \text{ random edges}; \quad H \sim \mathcal{G}_{n,M}.$$

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By Chernoff's bound, $\Pr[\, e(G) > M \,] \to 0$. Hence,

$$\Pr\left[\, \mathcal{G}_{n,M} \in \mathcal{C} \,\right] \geq \Pr[\, G \in \mathcal{C} \,] - \Pr[\, e(G) > M \,] \geq \mathrm{e}^{-\mathrm{e}^{-c+\varepsilon}} - o(1).$$

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$$\Pr\left[\, \mathcal{G}_{n,M} \in \mathcal{C} \,\right] \geq \Pr[\, G \in \mathcal{C} \,] - \Pr[\, e(G) > M \,] \geq \mathrm{e}^{-\mathrm{e}^{-c+\varepsilon}} - o(1).$$

Upper bound: remove edges from $\mathcal{G}_{n,p}$, $p = \frac{\log n + c + \varepsilon}{n}$. ∎

Hitting Time Version

- *Random graph process*:

$$\begin{aligned} G_0 &= n \text{ isolated vertices;} \\ G_{M+1} &= G_M + \text{ a random edge.} \end{aligned}$$

- *Hitting time* $\tau\,[\,\mathcal{A}\,] = \min\{M : G_M \in \mathcal{A}\}$.

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Theorem (Erdős-Renyi'60) Whp $\tau\,[\,\delta \geq 1\,] = \tau\,[\,\mathcal{C}\,]$.

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Theorem (Erdős-Renyi'60) Whp $\tau\,[\,\delta \geq 1\,] = \tau\,[\,\mathcal{C}\,]$.

Proof Let $\mathcal{B} = \{H : \delta \geq 1 \;\&\; H \notin \mathcal{C}\}$.

Idea 1:

$$\Pr\,[\,\exists M : G_M \in \mathcal{B}\,] \leq \sum_M \Pr\left[\,G_{n,M} \in \mathcal{B}\,\right] \not\to 0.$$

Idea 2: Using $G_M \subset G_{M+1}$

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Fix large $c > 0$. Let $m_\pm = \lfloor \frac{\log n \pm c}{n} \rfloor$.

$$\begin{aligned}
\Pr[\,\exists M : G_M \in \mathcal{B}\,] &\leq \Pr[\,\exists M \leq m_- \; \delta(G_M) \geq 1\,] \\
&+ \Pr[\,\exists M \in (m_-, m_+) \; G_M \in \mathcal{B}\,] \\
&+ \Pr[\,\exists M \geq m_+ \; G_M \notin \mathcal{C}\,] \\
&= p_- + p_0 + p_+
\end{aligned}$$

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$$\begin{aligned} \Pr[\,\exists M : G_M \in \mathcal{B}\,] &\le \Pr[\,\exists M \le m_- \; \delta(G_M) \ge 1\,] \\ &+ \Pr[\,\exists M \in (m_-, m_+) \; G_M \in \mathcal{B}\,] \\ &+ \Pr[\,\exists M \ge m_+ \; G_M \notin \mathcal{C}\,] \\ &= p_- + p_0 + p_+ \end{aligned}$$

Now,

$$\begin{aligned} p_+ &= \Pr[\,\mathcal{G}_{n,m_+} \notin \mathcal{C}\,] \;=\; 1 - \mathrm{e}^{-\mathrm{e}^{-c}} + o(1), \\ p_- &= \Pr[\,\delta(\mathcal{G}_{n,m_-}) \ge 1\,] \;=\; \mathrm{e}^{-\mathrm{e}^{c}} + o(1). \end{aligned}$$

The Old Trick

- **Recall:** $\mathcal{B} = \{H : \delta \geq 1 \;\&\; H \notin \mathcal{C}\}$.
- **Aim:** $\Pr\left[\,\exists M \in (m_-, m_+)\; G_M \in \mathcal{B}\,\right] \to 0$.

Let

$$\begin{aligned} p &= \frac{\log n - c - \varepsilon}{n}, \\ G &\in \mathcal{G}_{n,p}. \end{aligned}$$

Counting Components (Again)

Lemma Whp $C_1 \leq \log n$ and $C_2 = \cdots = C_{\lfloor n/2 \rfloor} = 0$, i.e. G consists of at most $\log n$ isolated vertices and one component.

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Lemma Whp $C_1 \leq \log n$ and $C_2 = \cdots = C_{\lfloor n/2 \rfloor} = 0$, i.e. G consists of at most $\log n$ isolated vertices and one component.

Proof

$$\mathrm{E}\left[\, C_1 \,\right] = n(1-p)^{n-1} \leq \mathrm{e}^{c+\varepsilon} + o(1) = O(1).$$

So $\Pr\left[\, C_1 > \log n \,\right] < \frac{\mathrm{E}[\, C_1 \,]}{\log n} \to 0$.

We already proved that whp $C_2 = \cdots = C_{\lfloor n/2 \rfloor} = 0$. ∎

Process between m_- and m_+

$$
\begin{aligned}
\Pr\left[\, \exists M \in (m_-, m_+) : G_M \in \mathcal{B} \,\right] &\leq \Pr\left[\, e(G) > m_- \,\right] \\
&+ \Pr\left[\, C_1 > \log n \,\right] \\
&+ \Pr\left[\, \exists k \in [2, n/2]\ C_k = 0 \,\right] \\
&+ m_+ \frac{\binom{\log n}{2}}{\binom{n}{2} - o(n^2)} \ \rightarrow\ 0.
\end{aligned}
$$

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&+ \Pr\left[\,C_1 > \log n\,\right] \\
&+ \Pr\left[\,\exists k \in [2, n/2]\; C_k = 0\,\right] \\
&+ m_+ \frac{\binom{\log n}{2}}{\binom{n}{2} - o(n^2)} \;\to\; 0.
\end{aligned}$$

Putting all together: whp $\tau\left[\,\mathcal{C}\,\right] = \tau\left[\,\delta \geq 1\,\right]$. $\blacksquare$

Some Spanning Subgraphs

$$\Pr[\mathcal{G}_{n,p} \in \mathcal{A}] \to \begin{cases} 0, & c \to -\infty, \\ 1, & c \to +\infty, \end{cases}$$

- **Erdős & Renyi'66:**
  $\mathcal{A} = \{$perfect matching$\}$, n even, $p = \frac{\log n + c}{n}$.
- **Korshunov'83, Komlós & Szemerédi'83:**
  $\mathcal{A} = \{$Hamiltonian$\}$, $p = \frac{\log n + \log\log n + c}{n}$.
- **Riordan'00:**
  $\mathcal{A} = \{d$-dimensional cube$\}$, $n = 2^d$, $p = \frac{1}{4} + c\,\frac{\log d}{d}$.

Perfect Matchings

$\mathcal{G}_{n,n,p}$: random subgraph of $K_{n,n}$, $\Pr[\,\text{edge}\,] = p$
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Theorem (Erdős & Renyi'64) Let $p = \frac{\log n + c}{n}$ and $G \in \mathcal{G}_{n,n,p}$.
Then

$$\Pr[\,G \text{ has a matching}\,] \to \mathrm{e}^{-2\mathrm{e}^{-c}}.$$

In particular, $p_0 = \frac{\log n}{n}$ is a sharp threshold.

Using Hall's Theorem

Proof No matching $\Leftrightarrow \exists\, S$ s.t.

- $|S| = |\Gamma(S)| + 1$,
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$$\Pr[\,\exists \text{ such } S : |S| \geq 2\,] \;\leq\; \mathrm{E}[\,\#\text{ such } S\,]$$

$$\leq\; 2 \sum_{s=2}^{\lceil n/2 \rceil} \binom{n}{s} \binom{n}{s-1} \binom{s}{2}^{s-1} p^{2s-2} (1-p)^{s(n-s+1)} = o(n\mathrm{e}^{-pn}).$$

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$\mathrm{E}[\,C_1\,] = 2n(1-p)^n \to 2\,\mathrm{e}^{-c}$. As before $C_1 \to \mathrm{Poisson}(2\mathrm{e}^{-c})$. ∎

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$$\begin{aligned}
\Pr\,[\,\exists |S| : |S| = 2\,] &\leq \Pr\,[\,G_{m_-} \text{ has such 2-element } S\,] \\
&+ \Pr\,[\,C_1(G_{m_-}) > \log n\,] \\
&+ \log n \Pr\,[\,\text{such 2-element } S \text{ is created}\,] \\
&\to 0. \; \blacksquare
\end{aligned}$$

General Spanning Subgraphs

Theorem (Alon-Füredi'92) Let $v(H) = n$, $\Delta(H) \leq d$ $D = d^2 + 1$ and $G \in \mathcal{G}_{n,p}$.

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If $\frac{np^d}{D} - \log n \to \infty$, then whp $H \subset G$.

Proof Let $F = H^2$; $\Delta(F) < D$.

Lemma (Hajnal-Szemerédi'70):
$\exists$ F-stable sets $V_1 \cup \cdots \cup V_D = V(F)$, each $|V_i| = \frac{n}{D} \pm 1$.

Take $V(G) = U_1 \cup \cdots \cup U_D$ with $|U_i| = |V_i|$.

Partial H-Embeddings

Build $f_i : V_1 \cup \cdots \cup V_i \longrightarrow U_1 \cup \cdots \cup U_i$ inductively.
Let $m = |V_{i+1}| = |U_{i+1}|$ and

$$F = \{\, (u, v) : u \in U_{i+1},\ v \in V_{i+1}\ \Gamma_H(v) \subset f_i(\Gamma_G(u))\,\}.$$

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Observe $F \sim \mathcal{G}_{m,m,\geq p^d}$.

$$\begin{aligned} \Pr[\,\text{FAIL}\,] &= \Pr[\,\text{no matching}\,] \\ &= O(m\,\mathrm{e}^{-pm}) \;=\; o(1/D). \end{aligned}$$

So, whp f_i exists $\forall\ i \in [D]$, i.e. $H \subset G$. ∎

Random Edge-Weights

- **The model:** Random independent weights w_e, $e \in \binom{[n]}{2}$, each uniformly distributed in $(0,1)$.
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Theorem (Frieze'85)

1. $\mathrm{E}\,[\,w(T)\,] \to \zeta(3)$.
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Proof of 1. Let $G_p = \big([n], \{e : w_e \le p\}\big) \sim \mathcal{G}_{n,p}$.

An Expression for $w(T)$

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where $\kappa(G) = \#$ components of G.

Computing $\mathrm{E}\left[\, w(T) \,\right]$

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$$= \sum_{k\geq 1}^{n^{1/3}} \frac{n^k k^{k-2}}{k!} \times \frac{(k-1)!\,(k(n-k))!}{(k(n-k+1))!} \;\to\; \sum_{k\geq 1} \frac{1}{k^3}.$$

Above the Connectivity Threshold

Lemma Let $p = \frac{3 \log n}{n}$. Then

$$\Pr\left[\, \mathcal{G}_{n,p} \notin \mathcal{C} \,\right] = o(1/n).$$

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Proof As before, we argue that

$$\sum_{k=1}^{\lfloor n/2 \rfloor} \mathrm{E}\left[\,C_k\,\right] = O(n\,\mathrm{e}^{-pn}) = o(1/n). \quad \blacksquare$$

Hence, $\mathrm{E}\left[\,w(T \setminus G_{3\log n/n})\,\right] \leq n \times o(1/n) \to 0$.

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