SCALAR ALGEBRAS AND QUATERNIONS: AN APPROACH BASED ON THE ALGEBRA AND TOPOLOGY OF FINITE-DIMENSIONAL REAL LINEAR SPACES, PART 1

RAY E. ARTZ

Abstract. This expository paper addresses the category of scalar algebras, *i.e.,* real finite-dimensional division algebras. An original proof shows that these algebras possess natural inner products. The inner products are then used as an essential ingredient in a proof of the Frobenius classification according to which every scalar algebra is isomorphic to one of the algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. Several applications of scalar algebras and quaternions are described.

Contents

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Introduction

This paper is Part 1 of a two-part expository article dealing with foundations and applications of scalar algebras with emphasis on quaternionic scalar algebras. The term "scalar algebra" is used here to denote the category of finite-dimensional not-necessarily-commutative field (sometimes called "skew field" or "division algebra") extensions of $\mathbb{R}$. By a well-known classification theorem of Frobenius, all commutative scalar algebras are isomorphic to either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, and all non-commutative scalar algebras are isomorphic to the "division algebra" $\mathbb{H}$ of quaternions. The theorem will be stated and proved in this part of the article.

The approach taken in this article is to treat scalar algebras in general, and quaternions in particular, as real linear spaces endowed with additional structure by prescription of a multiplication operation subject to certain algorithms. Accordingly, we lean heavily on results related to the topology and geometry of linear spaces and spaces of linear mappings.

Roughly, Part 1 of the article addresses scalar algebras and Part 2 addresses quaternions. More specifically, Part 1 develops a unified theory of scalar algebras of which quaternionic algebras happen to be just one class. The most unique feature of this development is a demonstration that scalar algebras have natural structures as inner-product spaces. This inner product is then used to study the basic geometric properties of scalar algebras. For example, we see that scalar algebras decompose as sums of orthogonal subspaces which represent real and imaginary parts, and we see that the scalar-algebra multiplication operation is conformal (that is, it preserves perpendicularity). These properties lead directly to a proof of the Frobenius classification that finishes the Part 1 theoretical development. Part 1 concludes with a description of an important application of scalar algebras to group representation theory.

Part 2 will address features of quaternionic scalar algebras which are specific to that class. The primary topic will be automorphisms of quaternionic scalar algebras and the their relationship to rotations in 3-dimensional Euclidean spaces.

1. Notation and terminology

The notation and terminology of [Nol87] is used in this paper. In particular, when dealing with linear spaces and mappings, we use notation, terminology, and results of Chapters 1 and 2 of [Nol87] extensively and often without explicit reference. We explain immediately below some notation and terminology that we judge may be unfamiliar to a large number of readers, but [Nol87] (which has excellent indexes of symbols and notations) should be consulted when clarification is needed. When dealing with multilinear mappings and functions, we use notation and terminology from [Nol94]. However, we will be more explicit in our usages since that book is not complete at the time this article is being prepared.

We use the symbol ":=" to indicate equality by definition. (See [Nol87], Sect. 00.) We denote by $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ the sets of real, complex, and quaternionic numbers. We denote by $\mathbb{N}$ the set of all natural numbers and by $\mathbb{P}$ the set of all **positive** real numbers (both of which we consider to contain zero). A superscript $^\times$ indicates the removal of zero; in particular $\mathbb{N}^\times$ and $Po^\times$ denote the sets of all **strictly-positive** natural and real numbers, resp. For $n \in \mathbb{N}^\times$, we denote by $n^]$ the set consisting of the first n members of $\mathbb{N}^\times$, *i.e.*, $n^] := \{\, i \in \mathbb{N}^\times \mid 1 \le i \le n \,\}$.

Given a mapping ϕ and subsets $\mathcal{A}$ of its **domain** $\operatorname{Dom}\phi$ and $\mathcal{B}$ of its **codomain** $\operatorname{Cod}\phi$, we denote the **image** of $\mathcal{A}$ under ϕ by

$$\phi_>(\mathcal{A}) := \{\, \phi(x) \mid x \in \mathcal{A} \,\}$$

and the **pre-image** of $\mathcal{B}$ under ϕ by

$$\phi^<(\mathcal{B}) := \{\, x \in \operatorname{Dom}\phi \mid \phi(x) \in \mathcal{B} \,\}.$$

We denote the **range** of ϕ by $\operatorname{Rng}\phi := \phi_>(\operatorname{Dom}\phi)$. When $f_>(\mathcal{A}) \subset \mathcal{B}$, we define the **adjustment** $f|_{\mathcal{A}}^{\mathcal{B}} \colon \mathcal{A} \to \mathcal{B}$ by the rule

$$\phi|_{\mathcal{A}}^{\mathcal{B}}(x) := \phi(x) \quad \text{for all } x \in \mathcal{A}.$$

We use various abbreviated forms of adjustments as follows:

$$\phi|_{\mathcal{A}} := \phi|_{\mathcal{A}}^{\operatorname{Cod} f}, \quad \phi|^{\mathcal{B}} := \phi|_{\operatorname{Dom} f}^{\mathcal{B}}, \quad \phi_{|\mathcal{A}} := \phi|_{\mathcal{A}}^{\mathcal{A}}.$$

The first of these forms is also called a **restriction**. We will use the last form only when $\mathcal{A}$ is **invariant** under ϕ, *i.e.*, when $\phi_>(\mathcal{A}) \subset \mathcal{A} \subset \operatorname{Cod}\phi$. When $\operatorname{Rng}\phi$ satisfies the appropriate conditions, we abbreviate

$$\phi|_{\operatorname{Rng}} := \phi|_{\operatorname{Rng}\phi}, \quad \phi|^{\operatorname{Rng}} := \phi|^{\operatorname{Rng}\phi}, \quad \phi_{|\operatorname{Rng}} := \phi_{|\operatorname{Rng}\phi}.$$

(All these notations can be applied under less restrictive conditions on $\mathcal{A}$, $\mathcal{B}$, and $\operatorname{Rng}\phi$: see [Nol87], Sect. 03.)

If ϕ is bijective (*i.e.,* both injective and surjective), then it is **invertible** and we denote its **inverse** by $\phi^{\leftarrow}\colon \operatorname{Cod}\phi \to \operatorname{Dom}\phi$. If the codomain of a mapping is $\mathbb{R}$, we will usually call the mapping a *function*.

Given mappings ϕ and ψ such that $\operatorname{Dom}\phi = \operatorname{Cod}\psi$, we define the **composite** $\phi\circ\psi\colon \operatorname{Dom}\psi \to \operatorname{Cod}\phi$ of ϕ and ψ by the rule $(\phi\circ\psi)(x) := \phi\left(\psi(x)\right) \quad \text{for all } x \in \operatorname{Dom}\psi$.

Given a set $\mathcal{A}$, we denote by $\mathbf{1}_{\mathcal{A}}$ its **identity** mapping and we denote by $1_{\mathcal{A}}\colon \mathcal{A} \to \mathbb{R}$ the constant function on $\mathcal{A}$ which takes the value $1 \in \mathbb{R}$. Given sets $\mathcal{A}$ and $\mathcal{B}$, we denote the collection of all mappings with domain $\mathcal{A}$ and codomain $\mathcal{B}$ by $\operatorname{Map}(\mathcal{A},\mathcal{B})$. Given a mapping $\phi\colon \mathcal{A}\times\mathcal{B} \to \mathcal{C}$ of two arguments and a member $x \in \mathcal{A}$, we define $\phi(x,\cdot)\colon \mathcal{B} \to \mathcal{C}$ by $\phi(x,\cdot)(y) := \phi(x,y)$. Given $y \in \mathcal{B}$, we define $\phi(\cdot,y)\colon \mathcal{A} \to \mathcal{C}$ in a parallel fashion.

Throughout this paper, we use the term **linear space** to mean "linear space over the field $\mathbb{R}$ of real numbers. Linear spaces are *not* automatically assumed to be finite-dimensional. (Most of the linear spaces discussed in this paper will be finite-dimensional, but we will have occasion to deal with certain infinite-dimensional spaces of continuous functions.) We say a linear space is **non-zero** if it properly includes its subspace $\{\mathbf{0}\}$.

Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{V}$ be linear spaces. We denote the space of linear mappings from $\mathcal{V}$ to $\mathcal{V}$ by $\operatorname{Lin}(\mathcal{V},\mathcal{W})$ and we abbreviate $\operatorname{Lin}\mathcal{V} := \operatorname{Lin}(\mathcal{V},\mathcal{V})$. If a linear mapping $\mathbf{L} \in \operatorname{Lin}(\mathcal{V},\mathcal{V})$ is invertible, we call it a **linear isomorphism**. The set of all linear isomorphisms in $\operatorname{Lin}(\mathcal{V},\mathcal{W})$ is denoted by $\operatorname{Lis}(\mathcal{V},\mathcal{W})$, and we abbreviate $\operatorname{Lis}\mathcal{V} := \operatorname{Lis}(\mathcal{V},\mathcal{V})$. Whenever convenient, we use "multiplicative" notation when dealing with linear mappings. For example, given $\mathbf{v} \in \mathcal{V}$, $\mathbf{L} \in \operatorname{Lin}(\mathcal{V},\mathcal{W})$, $\mathbf{M} \in \operatorname{Lin}(\mathcal{U},\mathcal{V})$, and a subset $\mathcal{S}$ of $\operatorname{Lin}\mathcal{V}$, we often abbreviate $\mathbf{Lv} := \mathbf{L}(\mathbf{v})$, $\mathbf{L}\mathcal{S} := \mathbf{L}_{>}(\mathcal{S})$, $\mathbf{LM} := \mathbf{L}\circ\mathbf{M}$, and, if $\mathbf{L}$ is invertible, $\mathbf{L}^{-1} := \mathbf{L}^{\leftarrow}$. We follow [Nol87] in calling a member of $\operatorname{Lin}\mathcal{V}$ a **lineon**. If a lineon $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ is invertible, we call it a **linear automorphism**. We denote the **trace** of $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ by $\operatorname{tr}\mathbf{L}$, the **determinant** of $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ by $\det\mathbf{L}$, and the **nullspace** of $\mathbf{L} \in \operatorname{Lin}(\mathcal{V},\mathcal{W})$ by $\operatorname{Null}\mathbf{L} := \mathbf{L}_{<}\left(\{\mathbf{0}\}\right)$.

By a **list basis** for a finite-dimensional linear space $\mathcal{V}$, we mean a basis $(\,\mathbf{e}_i \mid i \in n^{]}\,)$, where $n := \dim\mathcal{V}$. Let $\mathcal{S}$ be a subset of $\mathcal{V}$. We denote the **linear span** of $\mathcal{S}$ by $\operatorname{Lsp}\mathcal{S}$.

We consider finite-dimensional linear spaces to be equipped with their **usual topologies**. (See note 1.) Given finite-dimensional linear spaces $\mathcal{V}$ and $\mathcal{V}$, we call a continuous invertible mapping $\phi\colon \mathcal{V} \to \mathcal{W}$ a **homeomorphism** if its inverse is also continuous. The space of continuous real-valued functions on $\mathcal{V}$ is a linear space and is denoted $\operatorname{Co}\mathcal{V}$.

We denote the dual of a finite-dimensional linear space $\mathcal{V}$ by $\mathcal{V}^*$. We use the symbol "$\cong$" to denote the *identification* of two objects. For example, whenever we equip a finite-dimensional linear space $\mathcal{V}$ with an inner-product, thereby making it an **inner-product space**, we make the usual natural identification (see [Nol87], Sect. 41) $\mathcal{V}^* \cong \mathcal{V}$. In such a case, the **transpose** $\mathbf{L}^\top$ of a lineon $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ is itself a lineon in $\operatorname{Lin}\mathcal{V}$. Also in such a case, we denote by $\operatorname{Sym}\mathcal{V}$, [$\operatorname{Skew}\mathcal{V}$, resp.] the spaces of **symmetric** [**skew**] lineons, namely the lineons $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ for which $\mathbf{L}^\top = \mathbf{L}$ [$\mathbf{L}^\top = -\mathbf{L}$]. In this paper, the term **inner-product** will mean *genuine (*i.e., *positive) inner-product* and the term **inner-product space** will mean *genuine inner-product space.* (See [Nol87], Ch. 4.) When $\mathcal{V}$ and $\mathcal{V}$ are inner-product spaces, we denote by $\operatorname{Orth}(\mathcal{V},\mathcal{W})$ the space of **orthogonal** members of $\operatorname{Lin}(\mathcal{V},\mathcal{W})$, namely the linear isomorphisms $\mathbf{U} \in \operatorname{Lis}\mathcal{V}$ for which $\mathbf{L}^\top = \mathbf{L}^{-1}$. We abbreviate $\operatorname{Orth}\mathcal{V} := \operatorname{Orth}(\mathcal{V},\mathcal{V})$. We call members of $\operatorname{Orth}^+\mathcal{V} := \{\, \mathbf{U} \in \operatorname{Orth}(\mathcal{V},\mathcal{V}) \mid \det(\mathbf{U}) = 1 \,\}$ **proper orthogonal lineons.**

Let $\mathcal{S}$, $\mathcal{T}$ be subsets of an inner-product space $\mathcal{V}$ with inner-product "$\cdot$". The **tensor product** $\mathbf{u} \otimes_{\mathcal{V}} \mathbf{v} \in \operatorname{Lin}\mathcal{V}$ of $\mathbf{u}$ and $\mathbf{v}$ is defined by

$$(\mathbf{u} \otimes_{\mathcal{V}} \mathbf{v})\mathbf{w} := (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \qquad \text{for all } \mathbf{w} \in \mathcal{V}.$$

We say that $\mathcal{S}$ is **orthogonal** to $\mathcal{T}$ and write $\mathcal{S} \perp \mathcal{T}$ if $\mathbf{u} \cdot \mathbf{v} = 0$ for all $\mathbf{u} \in \mathcal{S}$, $\mathbf{v} \in \mathcal{V}$. We define

$$\mathcal{S}^\perp := \{\, \mathbf{v} \in \mathcal{V} \mid \mathbf{u} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{u} \in \mathcal{S} \,\}.$$

If $\mathcal{S}$ is a subspace of $\mathcal{V}$, then we call $\mathcal{S}^\perp$ the **orthogonal supplement** of $\mathcal{S}$. Let $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

2. Preliminaries

Our purpose in this section is to collect some specific mathematical content that will be needed in this paper, but that is not covered in the first several chapters of [Nol87]. Nearly all of this material is covered either in later chapters of [Nol87] or in [Nol94]. The reader is cautioned that because [Nol94] is in a draft state at the time this article is being written, specific references such as section numbers in that book may change over time. To mitigate that issue, references to material drawn from [Nol94] will include extra information such as chapter titles and theorem names whenever feasible. We also note that much of the material presented in this section has been tailored to the specific needs of the current article and is hence presented in a more narrow mathematical context here than in the cited books.

2.1. **Determinants.** The following proposition, not proved here, supplies all that we need to know about the **determinant** function

$$\det\colon \operatorname{Lin}\mathcal{V} \to \mathbb{R}.$$

This material is probably very familiar to the reader. If not, see Sect. 14 of [Nol94] for definition and proofs. All but Property (2.1a) are explicitly proved there (specifically in *Basic Rules of the Determinant*), and it follows very directly from the *Theorem on Characterization of the Determinant* that is presented there.

Proposition 2.1. *Let $\mathcal{V}$ be a finite-dimensional linear space of dimension $n \geq 1$. Then the determinant function* $\det\colon \operatorname{Lin}\mathcal{V} \to \mathbb{R}$ *satisfies*

$$\det \in \operatorname{Co}\mathcal{V}; \tag{2.1a}$$

$$\mathbf{L} \in \operatorname{Lis}\mathcal{V} \Longleftrightarrow \det(\mathbf{L}) \in \mathbb{R}^{\times} \quad \text{for all } \mathbf{L} \in \operatorname{Lin}\mathcal{V}; \tag{2.1b}$$

$$\det(\mathbf{L}\mathbf{M}) = \det(\mathbf{L})\det(\mathbf{M}) \quad \text{for all } \mathbf{L}, \mathbf{M} \in \operatorname{Lin}\mathcal{V}; \tag{2.1c}$$

$$\det(\mathbf{L}^{-1}) = (\det(\mathbf{L}))^{-1} \text{ for all } \mathbf{L} \in \operatorname{Lis}\mathcal{V}; \tag{2.1d}$$

$$\det(\xi\mathbf{L}) = \xi^n \det(\mathbf{L}) \quad \text{for all } \xi \in \mathbb{R},\ \mathbf{L} \in \operatorname{Lin}\mathcal{V}; \tag{2.1e}$$

$$\det(\mathbf{1}_{\mathcal{V}}) = 1. \tag{2.1f}$$

Proposition 2.2. *Let $\mathcal{V}$ be a finite-dimensional inner-product space of dimension $n \geq 1$. Then*

$$\det(\mathbf{L}^{\top}) = \det(\mathbf{L}^{\top}) \qquad \text{for all } \mathbf{L} \in \operatorname{Lin}\mathcal{V}; \tag{2.2}$$

$$\det(\mathbf{U}) \in \{1, -1\} \qquad \text{for all } \mathbf{U} \in \operatorname{Orth}\mathcal{V}. \tag{2.3}$$

Proof. Since $\mathcal{V}$ is an inner-product space, we identify $\mathcal{V}^*$ with $\mathcal{V}$ itself and consider $\mathbf{L}^{\top}$ to be member of $\operatorname{Lin}\mathcal{V}$ for all $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$. Then it can be seen that property (2.2) is one of the *Basic Rules of the Determinant* ([Nol94], Sec. 14) cited above. Let $\mathbf{U} \in \operatorname{Orth}\mathcal{V}$, so $\mathbf{U}^\top = \mathbf{U}^{-1}$. Then it follows from (2.2) and (2.1d) that $\det(\mathbf{L}) = (\det(\mathbf{L}))^{-1}$. This establishes (2.3). □

Proposition 2.3. *Let $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$, and let $(\mathcal{U}_i \mid i \in I)$ be an orthogonal decomposition of $\mathcal{V}$, all of whose terms are $\mathbf{L}$-spaces. Then*

$$\det(\mathbf{L}) = \prod_{i \in I} \det(\mathbf{L}_{|\mathcal{U}_i}). \tag{2.4}$$

Proof. See Prop. 5 of Ch. 14, *Determinants*, of [Nol94]. □

2.2. **Integration.** The next lemma supplies all that we will need regarding integration on finite-dimensional spaces. Specifically, it addresses properties of integrals of continuous real-valued functions on bounded convex sets. A full discussion of *integration* in this context, even provision of rigorous definitions, would be out of place and a distraction here. The reader may accept the lemma based on prior knowledge of integration theory or see App. A below for definitions and a proof based on integration theory as presented in [Nol94]. Also, see Note 2.

Lemma 2.4. *Let $\mathcal{V}$ be a finite-dimensional linear space, and let $\mathcal{C}$ be a bounded convex subset of $\mathcal{V}$ with non-empty interior. Then there is a linear functional $\int_{\mathcal{C}}$ on the function space $\operatorname{Co}\mathcal{V}$ such that for all $f \in \operatorname{Co}\mathcal{V}$ and all $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$:*

$$f_{|\mathcal{C}} > 0 \quad \Rightarrow \quad \int_{\mathcal{C}} f \; > 0; \tag{2.5a}$$

$$\mathbf{L}\,\mathcal{C} = \mathcal{C} \quad \Rightarrow \quad \int_{\mathcal{C}} f \circ \mathbf{L} = \int_{\mathcal{C}} f; \tag{2.5b}$$

In (2.5a), the condition "$f|_{\mathcal{C}} > 0$" is interpreted to mean that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathcal{C}$ and that $f(\mathbf{x}) > 0$ for some $\mathbf{x} \in \mathcal{C}$.

3. Algebras and scalar algebras

By an **algebra**, we mean a (real) linear space $\mathcal{A}$ equipped with additional structure by the prescription a bilinear operation

$$(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{ab} \colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

which we call **multiplication**. (See Note 3.) If the multiplication operation is *commutative*, we call $\mathcal{A}$ is a **commutative algebra**. If its multiplication operation is *associative* we call $\mathcal{A}$ an **associative algebra**.

Let $\mathcal{A}$ and $\mathcal{A}'$ be algebras and let $\mathbf{L} \in \operatorname{Lin}(\mathcal{A}, \mathcal{A}')$. If $\mathbf{L}$ is invertible (hence a *linear isomorphism*) and also respects multiplication (*i.e.,* if $(\mathbf{La})(\mathbf{Lb}) = \mathbf{ab}$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}$), then we call $\mathbf{L}$ an **algebra isomorphism** and say that $\mathcal{A}$ and $\mathcal{A}'$ are **isomorphic as algebras**.

Let $\mathcal{A}$ be an algebra and let $(\, \mathbf{e}_i \mid i \in I \,)$ be a basis for $\mathcal{A}$. We call the doubly-indexed family $(\, \mathbf{e}_i \mathbf{e}_j \mid (i, j) \in I \times I \,)$ (whose terms are the products of pairs of basis elements) the **multiplication table** of the multiplication operation on $\mathcal{A}$ with respect to the given basis. Since algebra multiplication is bilinear, the multiplication operation on $\mathcal{A}$ is completely determined by its multiplication table. Also, if $\mathcal{A}$ is a linear space with basis $(\, \mathbf{e}_i \mid i \in I \,)$, then any given table

$$(\, \mathbf{v}_{ij} \in \mathcal{A} \mid (i, j) \in I \times I \,)$$

determines a bilinear multiplication operation on $\mathcal{A}$ which satisfies

$$\mathbf{e}_i \mathbf{e}_j = \mathbf{v}_{ij} \quad \text{for all } i, j \in I.$$

An algebra $\mathcal{A}$ can have at most one element $\mathbf{u}$ with the property that $\mathbf{uv} = \mathbf{vu} = \mathbf{v}$ for all $\mathbf{v} \in \mathcal{A}$. (If there are two such elements, say $\mathbf{u}_1$ and $\mathbf{u}_2$, then $\mathbf{u}_1 = \mathbf{u}_1 \mathbf{u}_2 = \mathbf{u}_2$). We call such an element a **unity**. If $\mathcal{A}$ contains such element, we call it *the* unity of the $\mathcal{A}$ and denote it by $\mathbf{1}^{\mathcal{A}}$ or, more commonly, simply by $\mathbf{1}$. We consider it to be part of the structure of $\mathcal{A}$ and say that $\mathcal{A}$ is an **algebra with a unity**. We observe the typographical distinctions among the unity $\mathbf{1}$, the identity mapping $\mathbf{1}_{\mathcal{A}} \colon \mathcal{A} \to \mathcal{A}$, and the real-valued constant function $1_{\mathcal{A}} \colon \mathcal{A} \to \mathbb{R}$.

Let $\mathcal{A}$ be an algebra with a unity, and let $\mathbf{a} \in \mathcal{A}$. If $\mathbf{b} \in \mathcal{A}$ satisfies $\mathbf{ab} = \mathbf{ba} = 1$, then we say that $\mathbf{b}$ is **a reciprocal** of $\mathbf{b}$. If every member of an algebra $\mathcal{A}$ has a reciprocal, we say that $\mathcal{A}$ is an **algebra with reciprocals**. If $\mathcal{A}$ is an associative algebra, then a member $\mathbf{a} \in \mathcal{A}$ can have at most one reciprocal; if it has one, we denote it by $\mathbf{a}^{-1}$. (Again, see Note 3.)

Definition 3.1. By a **scalar algebra**, we mean a finite-dimensional associative algebra with reciprocals whose unity is not equal to its zero. We call members of a scalar algebra **scalars**. (See Note 4.)

It is clear that the fields $\mathbb{R}$ and $\mathbb{C}$ of real and complex numbers (equipped with their usual multiplication operations) are commutative scalar algebras. An object of primary interest in this paper is the non-commutative scalar algebra called the algebra $\mathbb{H}$ of *quaternions*. According to a well-know theorem by Frobenius (proved herein as Thm. 7.1), every scalar algebra is isomorphic to one of these three. We will describe the three individually in some detail in Sect. 4.

For the remainder of this section, let $\mathcal{A}$ be a scalar algebra.

Definition 3.2. We define the **left-** and **right-multiplication** operators

$$\ell_{\mathcal{A}} \colon \mathcal{A} \to \operatorname{Map}(\mathcal{A}, \mathcal{A}), \qquad \mathfrak{r}_{\mathcal{A}} \colon \mathcal{A} \to \operatorname{Map}(\mathcal{A}, \mathcal{A})$$

by

$$(\ell_{\mathcal{A}}\mathbf{a})\mathbf{b} := (\mathfrak{r}_{\mathcal{A}}\mathbf{b})\mathbf{a} := \mathbf{ab} \qquad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}. \tag{3.1}$$

Proposition 3.3. *The mapping $\ell_{\mathcal{A}}$ has the following properties:*

$$\ell_{\mathcal{A}} \in \operatorname{Lin}(\mathcal{A}, \operatorname{Lin}\mathcal{A}) \tag{3.2a}$$

$$\ell_{\mathcal{A}} \text{ is injective;} \tag{3.2b}$$

$$\mathbf{a} \in \mathcal{A}^{\times} \iff \ell_{\mathcal{A}}\mathbf{a} \in \operatorname{Lis}\mathcal{A} \quad \text{for all } \mathbf{a} \in \mathcal{A}; \tag{3.2c}$$

$$\ell_{\mathcal{A}}(\mathbf{ab}) = (\ell_{\mathcal{A}}\mathbf{a})(\ell_{\mathcal{A}}\mathbf{b}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}; \tag{3.2d}$$

$$\ell_{\mathcal{A}}(\mathbf{a}^{-1}) = (\ell_{\mathcal{A}}\mathbf{a})^{-1} \quad \text{for all } \mathbf{a} \in \mathcal{A}^{\times}; \tag{3.2e}$$

$$\ell_{\mathcal{A}}\mathbf{1} = \mathbf{1}_{\mathcal{A}}. \tag{3.2f}$$

Proof. Property (3.2a) follows directly from the bilinearity of multiplication in $\mathcal{A}$. Let distinct $\mathbf{a}, \mathbf{b}, \in \mathcal{A}$ be given. Then $(\ell_{\mathcal{A}}\mathbf{a})\mathbf{1} = \mathbf{a} \neq \mathbf{b} = (\ell_{\mathcal{A}}\,\mathbf{b})\mathbf{1}$, so $\ell_{\mathcal{A}}\mathbf{a} \neq \ell_{\mathcal{A}}\,\mathbf{b}$. This establishes (3.2b). The rest follows directly from (3.1) and the definition of a scalar algebra. ☐

Proposition 3.4. *The mapping $\mathfrak{r}_{\mathcal{A}}$ has the following properties:*

$$\mathfrak{r}_{\mathcal{A}} \in \operatorname{Lin}(\mathcal{A}, \operatorname{Lin}\mathcal{A}) \tag{3.3a}$$

$$\mathfrak{r}_{\mathcal{A}} \text{ is injective;} \tag{3.3b}$$

$$\mathbf{a} \in \mathcal{A}^{\times} \iff \mathfrak{r}_{\mathcal{A}}\mathbf{a} \in \operatorname{Lis}\mathcal{A} \quad \text{for all } \mathbf{a} \in \mathcal{A}; \tag{3.3c}$$

$$\mathfrak{r}_{\mathcal{A}}(\mathbf{ab}) = (\mathfrak{r}_{\mathcal{A}}\mathbf{b})(\ell_{\mathcal{A}}\mathbf{a}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}; \tag{3.3d}$$

$$\mathfrak{r}_{\mathcal{A}}(\mathbf{a}^{-1}) = (\mathfrak{r}_{\mathcal{A}}\mathbf{a})^{-1} \quad \text{for all } \mathbf{a} \in \mathcal{A}^{\times}; \tag{3.3e}$$

$$\mathfrak{r}_{\mathcal{A}}\mathbf{1} = \mathbf{1}_{\mathcal{A}}. \tag{3.3f}$$

Proof. The proof follows the same logic as the proof of Prop. 3.3. ☐

Proposition 3.5. *Left- and right- multiplications commute;* i.e.,

$$(\ell_{\mathcal{A}}\mathbf{a})(\mathfrak{r}_{\mathcal{A}}\mathbf{b}) = (\mathfrak{r}_{\mathcal{A}}\mathbf{b})(\ell_{\mathcal{A}}\mathbf{a}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}. \tag{3.4}$$

Proof. This follows immediately from Def. 3.2 and the associativity of the multiplication operation on $\mathcal{A}$. $\square$

Remark. In view of the Prop. 3.3, the space $\mathcal{A}$ and its image under $\ell_{\mathcal{A}}$, namely $\ell_{\mathcal{A}}\mathcal{A}$, are naturally isomorphic as algebras. We could use this natural isomorphism to identify $\mathcal{A}$ with $\ell_{\mathcal{A}}\mathcal{A}$. In order to avoid confusion, we won't do so at this time. However, the possibility of the identification motivates much of our development.

Remark. In view of the Prop. 3.4, the space $\mathcal{A}$ and its image under $\mathscr{r}_{\mathcal{A}}$, namely $\mathscr{r}_{\mathcal{A}}\mathcal{A}$, are naturally "antimorphic" as algebras (by this we mean that the order of the operands in the multiplication operation is reversed). This makes right-multiplication slightly less convenient for our purposes. We will make little use of the right-multiplication operation until we take up scalar-algebra automorphisms in Part 2 of this article. However, we will continue to develop the properties that will be needed then.

4. The standard scalar algebras: $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$

Our purpose in this section is to present the three standard examples of scalar algebras, namely the field $\mathbb{R}$ of *real numbers*, the field $\mathbb{C}$ of *complex numbers*, and the algebra $\mathbb{H}$ of *quaternions*. Each of these standard algebras is conventionally (albeit sometimes tacitly) considered to have additional structure beyond its structure as a scalar algebra. Indeed, each is normally considered to be equipped with a list-basis which conforms to a specific multiplication table. In the case of $\mathbb{R}$, the list basis is simply (1). (We denote it by $(\mathbf{1})$ when we wish to consider $\mathbb{R}$ as an example of a scalar algebra.) In the cases of $\mathbb{C}$ and $\mathbb{H}$, the list bases are conventionally denoted by $(\mathbf{1}, \mathbf{i})$ and $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$, resp., or by slight variants thereof. The standard list basis for $\mathbb{R}$ is natural in the sense that it is completely determined by the algebraic structure of $\mathbb{R}$. Such is not the case for the standard bases of $\mathbb{C}$ and $\mathbb{H}$, however: they require arbitrary choices. Another way to say this is to say that $\mathbb{C}$ and $\mathbb{H}$, considered as scalar algebras, admit automorphisms other than their identity mappings. For that reason, we only speak of *the* field $\mathbb{C}$ of complex numbers or *the* algebra $\mathbb{H}$ of quaternions after we have equipped the algebras with list bases which conform to their conventional multiplication tables. (See Note 5.)

We will demonstrate an example of each of the three classes of algebras equipped with its standard list basis and its standard multiplication table.

Our strategy in each case is as follows.

(i) We specify $n \in \mathbb{N}^{\times}$, we put $I := n^{]}$, and we assign convenient names to the terms of the standard basis $(\, \delta_I^I \mid i \in I \,)$ for $\mathbb{R}^I$. We define the algebra $\mathcal{A}$ as the linear space $\mathbb{R}^I$ given additional structure by the prescription of a multiplication operation determined by a prescribed multiplication table $(\, \delta_I^I \delta_I^I \mid (i,j) \in I \times I \,)$. In each case it will be clear that δ_1^I is a unity, so $\mathcal{A}$ is an algebra with a unity. Also, this will justify our consistent assigning of the name $\mathbf{1} := \delta_1^I$.

(ii) We use the prescribed multiplication table to show that every non-zero member of $\mathcal{A}$ has a reciprocal.

(iii) For each $i \in I$, we use the prescribed multiplication table to explicitly determine the matrix $\ell_{\mathcal{A}} \delta_I^I \in \operatorname{Lin} \mathcal{A} = \operatorname{Lin} \mathbb{R}^I \cong \mathbb{R}^{I \times I}$ for each $i \in I$. We then show by direct computation that the list $(\, \ell_{\mathcal{A}} \delta_I^I \mid i \in I \,)$ of matrices is linearly independent, so $\ell_{\mathcal{A}}|^{\mathrm{Rng}} \colon \mathcal{A} \to \ell_{\mathcal{A}} \mathcal{A}$ is a linear isomorphism and that $(\, \ell_{\mathcal{A}} \delta_I^I \mid i \in I \,)$ is a list-basis for $\ell_{\mathcal{A}} \mathcal{A}$.

(iv) We show by direct computation that $\ell_{\mathcal{A}}$ respects the prescribed multiplication table in the sense that $(\ell_{\mathcal{A}}\delta_I^I)(\ell_{\mathcal{A}}\delta_j^I) = \ell_{\mathcal{A}}(\delta_I^I\delta_j^I)$ for all $i, j \in I$. Here, the multiplication on the left side of the equality is composition in $\operatorname{Lin}\mathcal{A}$, which is equivalent here to matrix multiplication in $\mathbb{R}^{I\times I}$. Since $\ell_{\mathcal{A}}$ is linear and both matrix multiplication and the prescribed multiplication in $\mathcal{A}$ are bilinear, it follows that

$$(\ell_{\mathcal{A}}\mathbf{a})(\ell_{\mathcal{A}}\mathbf{b}) = \ell_{\mathcal{A}}(\mathbf{ab}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}. \tag{4.1}$$

This shows that $\ell_{\mathcal{A}}\mathcal{A}$ is stable under composition and is hence (because linear composition of lineons is both bilinear and associative) an algebra.

(v) From (iii) and (4.1), we conclude that $\ell_{\mathcal{A}}\big|^{\mathrm{Rng}} \colon \mathcal{A} \to \ell_{\mathcal{A}}\mathcal{A}$ is an algebra isomorphism. Since, by (ii) and (iv), resp., $\mathcal{A}$ is an algebra-with-reciprocals and $\ell_{\mathcal{A}}\mathcal{A}$ is an associative algebra, it follows that the both are (naturally isomorphic as) scalar algebras.

Remark. It is clear that it is not necessary to make use of the left-multiplication operator $\ell_{\mathcal{A}}$ in these examples, since we could verity multiplicative associativity directly from the prescribed multiplication tables. However, that would involve (at least in the case of the third example, namely the algebra of quaternions) some tedious calculation involving triples of vectors. Also, the strategy we have chosen serves to illuminate our use of the left-multiplication operator in the general development of scalar algebras which follows.

Example 4.1 (The algebra $\mathbb{R}$ of real numbers)*.* For the sake of consistency, we follow the prescribed strategy even though, in this case, many of the steps are obvious to the point of being potentially confusing. Let $n = 1$, so $I = \{1\}$, and $\mathbb{R}^I = \mathbb{R}^1 = \mathbb{R}$.

(i) We assign convenient alternative names for the terms (in this case, just one term, namely δ_1^I) of the standard list basis on $\mathbb{R}^I = \mathbb{R}^1$ as follows:

$$(\mathbf{1}) := (\delta_1^I) = (\, \delta_i^I \mid i \in I\,).$$

We make $\mathcal{A} := \mathbb{R}^I$ an algebra by prescribing the following multiplication table:

$$\begin{array}{c|c} & \mathbf{1} \\ \hline \mathbf{1} & \mathbf{1} \end{array}\,; \tag{4.2}$$

we note that $\mathbf{1}$ is a unity as promised.

(ii) Let $\mathbf{a} \in \mathbb{R}^I$ such that $\mathbf{a} \neq \mathbf{0}$. Since $(\mathbf{1}) = (\delta_1^I)$ is a basis for $\mathbb{R}^I$, we may choose $a \in \mathbb{R}^\times$ such that $\mathbf{a} = a\mathbf{1}$. It is clear from (4.2) that $\mathbf{a}^2 = a^2 > 0$. Hence, putting $\mathbf{b} := (1/a^2)\mathbf{a}$, we have $\mathbf{ab} = \mathbf{ba} = \mathbf{1}$, so $\mathbf{b}$ is a reciprocal of $\mathbf{a}$. Since $\mathbf{a} \in \mathbb{R}^I$ was arbitrary, $\mathcal{A}$ is an algebra with reciprocals.

(iii) We determine the left-multiplication matrix corresponding to $\mathbf{1}$ to be

$$\ell_{\mathcal{A}}\mathbf{1} = \begin{bmatrix} 1 \end{bmatrix} ,$$

and note that $(\ell_{\mathcal{A}}\mathbf{1})$ is indeed a list basis for the space $\mathbb{R}^{I\times I}$ of 1-by-1 matrices.

(iv) Clearly, $\ell_{\mathcal{A}}\mathbf{1}$ is a (hence *the*) unity in the matrix algebra $\ell_{\mathcal{A}}\mathcal{A} = \mathbb{R}^{I\times I}$ of 1-by-1 matrices. This shows that $\ell_{\mathcal{A}}$ respects the prescribed multiplication table (4.2) and hence respects multiplication in the sense of (4.1). Hence $\ell_{\mathcal{A}}\mathcal{A}$ is a sub-algebra of $\mathbb{R}^{I\times I}$; indeed, in this case, the two are identical.

(v) We conclude that $\mathbb{R}^1$, when equipped with the multiplication table (4.2), is a scalar algebra. We of course call it the *algebra of real numbers.*

Example 4.2 (The algebra $\mathbb{C}$ of complex numbers). We again follow the prescribed strategy. Let $n = 2$, so $I = \{1, 2\}$, and $\mathbb{R}^I = \mathbb{R}^2$.

(i) We assign convenient alternative names for the terms of the standard list basis on $\mathbb{R}^I = \mathbb{R}^2$ as follows:

$$(\mathbf{1}, \mathbf{i}) := (\delta_1^I, \delta_2^I) = (\, \delta_i^I \mid i \in I \,).$$

We make $\mathcal{A} := \mathbb{R}^I$ an algebra by prescribing the following multiplication table:

$$\begin{array}{c|cc} & \mathbf{1} & \mathbf{i} \\ \hline \mathbf{1} & \mathbf{1} & \mathbf{i} \\ \mathbf{i} & \mathbf{i} & -\mathbf{1} \end{array} \tag{4.3}$$

we note that $\mathbf{1}$ is a unity as promised.

(ii) Let $\mathbf{a} \in \mathbb{R}^I$ such that $\mathbf{a} \neq \mathbf{0}$. Since $(\mathbf{1}, \mathbf{i})$ is a basis for $\mathbb{R}^I$, we may choose $a_{\mathbf{1}}, a_{\mathbf{i}} \in \mathbb{R}$ such that $\mathbf{a} = a_{\mathbf{1}}\mathbf{1} + a_{\mathbf{i}}\mathbf{i}$. Put

$$\operatorname{Re}\mathbf{a} := a_{\mathbf{1}}\mathbf{1}; \quad \operatorname{Im}\mathbf{a} := a_{\mathbf{i}}\mathbf{i}; \quad \mathbf{a}^\dagger := \operatorname{Re}\mathbf{a} - \operatorname{Im}\mathbf{a}.$$

(These notations will be introduced more generally later; for now, they may be regarded simply as convenient abbreviations.) It is clear that $(\operatorname{Im}\mathbf{a})^2 = -(\operatorname{Im}\mathbf{a}) \cdot (\operatorname{Im}\mathbf{a})\mathbf{1}$ and that $(\operatorname{Re}\mathbf{a})^2 = (\operatorname{Re}\mathbf{a}) \cdot (\operatorname{Re}\mathbf{a})\mathbf{1}$, where "$\cdot$" represents the usual inner product on $\mathbb{R}^4$. Hence

$$\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} = \operatorname{Re}\mathbf{a}^2 - \operatorname{Im}\mathbf{a}^2 = a_{\mathbf{1}}^2\mathbf{1} + (\operatorname{Im}\mathbf{a} \cdot \operatorname{Im}\mathbf{a})\mathbf{1} = (\mathbf{a} \cdot \mathbf{a})\mathbf{1}.$$

Since $\mathbf{a} \neq \mathbf{0}$, we have $\mathbf{a} \cdot \mathbf{a} \neq 0$. Putting $\mathbf{b} := (\mathbf{a} \cdot \mathbf{a})^{-1}\mathbf{a}^\dagger$, we have $\mathbf{a}\mathbf{b} = \mathbf{b}\mathbf{a} = \mathbf{1}$, so $\mathbf{b}$ is a reciprocal of $\mathbf{a}$. Since $\mathbf{a} \in \mathbb{R}^I$ was arbitrary, $\mathcal{A}$ is an algebra with reciprocals.

(iii) We determine the left-multiplication matrix corresponding to $\mathbf{1}$ to be

$$\ell_{\mathcal{A}}\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \ell_{\mathcal{A}}\mathbf{i} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{4.4}$$

Let $a_{\mathbf{1}}, a_{\mathbf{i}} \in \mathbb{R}$ be given, and suppose that

$$a_{\mathbf{1}}\,\ell_{\mathcal{A}}\mathbf{1} + a_{\mathbf{i}}\,\ell_{\mathcal{A}}\mathbf{i} = \begin{bmatrix} a_{\mathbf{1}} & -a_{\mathbf{i}} \\ a_{\mathbf{i}} & a_{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then $(a_{\mathbf{1}}, a_{\mathbf{i}}) = (0,0)$. Since $a_{\mathbf{1}}, a_{\mathbf{i}} \in \mathbb{R}$ were arbitrary, it follow that the list $(\mathbf{1}, \mathbf{i})$ is linearly independent and hence a basis for $\ell_{\mathcal{A}}\mathcal{A}$. Also, $\ell_{\mathcal{A}}\big|^{\mathrm{Rng}} \colon \mathcal{A} \to \ell_{\mathcal{A}}\mathcal{A}$ is a linear isomorphism.

(iv) Clearly, $\ell_{\mathcal{A}}\mathbf{1}$ is a (hence *the*) unity in the matrix algebra $\ell_{\mathcal{A}}\mathcal{A} \subset \mathbb{R}^{I\times I}$. This verifies that $\ell_{\mathcal{A}}$ respects the first row and first column of the multiplication table (4.3). We easily compute via matrix multiplication that $(\ell_{\mathcal{A}}\mathbf{i})(\ell_{\mathcal{A}}\mathbf{i}) = -\ell_{\mathcal{A}}\mathbf{1}$., This completes the verification that $\ell_{\mathcal{A}}$ respects the prescribed multiplication table (4.3) and hence respects multiplication in the sense of (4.1). Hence $\ell_{\mathcal{A}}\mathcal{A}$ is a sub-algebra of $\mathbb{R}^{I\times I}$ and so is itself an associative algebra.

(v) We conclude that $\mathbb{R}^2$, when equipped with the multiplication table (4.3), is a scalar algebra. We of course call it the *algebra of complex numbers.*

Example 4.3 (The algebra $\mathbb{H}$ of quaternions). We continue to follow the prescribed strategy. Let $n = 4$, so $I = \{1,2,3,4\}$, and $\mathbb{R}^I = \mathbb{R}^4$.

(i) We assign convenient alternative names for the terms of the standard list basis on $\mathbb{R}^I = \mathbb{R}^4$ as follows:

$$(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}) := (\delta_1^I, \delta_2^I, \delta_3^I, \delta_4^I) = (\,\delta_i^I \mid i \in I\,).$$

We make $\mathcal{A} := \mathbb{R}^I$ an algebra by prescribing the following multiplication table:

(4.5)

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

we note that $\mathbf{1}$ is a unity as promised.

(ii) Let $\mathbf{a} \in \mathbb{R}^I$ such that $\mathbf{a} \neq \mathbf{0}$. Since $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is a basis for $\mathbb{R}^I$, we may choose $a_{\mathbf{1}}, a_{\mathbf{i}}, a_{\mathbf{j}}, a_{\mathbf{k}} \in \mathbb{R}$ such that $\mathbf{a} = a_{\mathbf{1}}\mathbf{1} + a_{\mathbf{i}}\mathbf{i} + a_{\mathbf{j}}\mathbf{j} + a_{\mathbf{k}}\mathbf{k}$. Put

$$\operatorname{Re}\mathbf{a} := a_{\mathbf{1}}\mathbf{1}; \quad \operatorname{Im}\mathbf{a} := a_{\mathbf{i}}\mathbf{i} + a_{\mathbf{j}}\mathbf{j} + a_{\mathbf{k}}\mathbf{k}; \quad \mathbf{a}^\dagger := \operatorname{Re}\mathbf{a} - \operatorname{Im}\mathbf{a}.$$

(These notations will be introduced more generally later; for now, they may be regarded simply as convenient abbreviations.) It is easy to see from (4.5) that

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}; \quad \mathbf{ij} + \mathbf{ji} = \mathbf{jk} + \mathbf{kj} = \mathbf{ki} + \mathbf{ik} = \mathbf{0}.$$

It follows directly that $(\operatorname{Im}\mathbf{a})^2 = -(\operatorname{Im}\mathbf{a}\cdot\operatorname{Im}\mathbf{a})\mathbf{1}$, and it is clear that $(\operatorname{Re}\mathbf{a})^2 = (\operatorname{Re}\mathbf{a}\cdot\operatorname{Re}\mathbf{a})\mathbf{1} = {a_{\mathbf{1}}}^2\mathbf{1}$, where "$\cdot$" represents the usual inner product on $\mathbb{R}^4$. Hence $\mathbf{a}\mathbf{a}^\dagger = \mathbf{a}^\dagger\mathbf{a} = (\operatorname{Re}\mathbf{a})^2 - (\operatorname{Im}\mathbf{a})^2 = {a_{\mathbf{1}}}^2\mathbf{1} + (\operatorname{Im}\mathbf{a}\cdot\operatorname{Im}\mathbf{a})\mathbf{1} = (\mathbf{a}\cdot\mathbf{a})\mathbf{1}$. Since $\mathbf{a} \neq \mathbf{0}$, we have $\mathbf{a}\cdot\mathbf{a} \neq 0$. Hence, putting $\mathbf{b} := (\mathbf{a}\cdot\mathbf{a})^{-1}\mathbf{a}^\dagger$, we have $\mathbf{ab} = \mathbf{ba} = \mathbf{1}$, so $\mathbf{b}$ is a reciprocal of $\mathbf{b}$. Since $\mathbf{a} \in \mathbb{R}^I$ was arbitrary, $\mathcal{A}$ is an algebra with reciprocals.

(iii) We determine the left-multiplication matrices corresponding to $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ to be

$$\ell_{\mathcal{A}}\mathbf{1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad \ell_{\mathcal{A}}\mathbf{i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

$$\ell_{\mathcal{A}}\mathbf{j} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}; \quad \ell_{\mathcal{A}}\mathbf{k} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Let $a_{\mathbf{1}}, a_{\mathbf{i}}, a_{\mathbf{j}}, a_{\mathbf{k}} \in \mathbb{R}$ be given, and suppose that

$$a_{\mathbf{1}}\,\ell_{\mathcal{A}}\mathbf{1} + a_{\mathbf{i}}\,\ell_{\mathcal{A}}\mathbf{i} + a_{\mathbf{j}}\,\ell_{\mathcal{A}}\mathbf{j} + a_{\mathbf{k}}\,\ell_{\mathcal{A}}\mathbf{k} = \begin{bmatrix} a_{\mathbf{1}} & -a_{\mathbf{i}} & -a_{\mathbf{j}} & -a_{\mathbf{k}} \\ a_{\mathbf{i}} & a_{\mathbf{1}} & -a_{\mathbf{k}} & a_{\mathbf{j}} \\ a_{\mathbf{j}} & a_{\mathbf{k}} & a_{\mathbf{1}} & -a_{\mathbf{i}} \\ a_{\mathbf{k}} & -a_{\mathbf{j}} & a_{\mathbf{i}} & a_{\mathbf{1}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $(a_{\mathbf{1}}, a_{\mathbf{i}}, a_{\mathbf{j}}, a_{\mathbf{k}}) = (0, 0, 0, 0)$. Since $a_{\mathbf{1}}, a_{\mathbf{i}}, a_{\mathbf{j}}, a_{\mathbf{k}} \in \mathbb{R}$ were arbitrary, it follow that the list $(\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$ is linearly independent and hence a basis for $\ell_{\mathcal{A}}\mathcal{A}$. Also, $\ell_{\mathcal{A}}|^{\mathrm{Rng}} : \mathcal{A} \to \ell_{\mathcal{A}}\mathcal{A}$ is a linear isomorphism.

(iv) Clearly, $\ell_{\mathcal{A}}\mathbf{1}$ is a (hence *the*) unity in the matrix algebra $\ell_{\mathcal{A}}\mathcal{A} \subset \mathbb{R}^{I\times I}$. This verifies that $\ell_{\mathcal{A}}$ respects the first row and first column of the multiplication table (4.5). We easily compute via matrix multiplication that

$$(\ell_{\mathcal{A}}\mathbf{i})(\ell_{\mathcal{A}}\mathbf{i}) = (\ell_{\mathcal{A}}\mathbf{j})(\ell_{\mathcal{A}}\mathbf{j}) = (\ell_{\mathcal{A}}\mathbf{k})(\ell_{\mathcal{A}}\mathbf{k}) = -\ell_{\mathcal{A}}\mathbf{1}.$$

With just a bit more effort, we compute (we can appeal to cyclic symmetry for assistance)

$$(\ell_{\mathcal{A}}\mathbf{i})(\ell_{\mathcal{A}}\mathbf{j}) = \ell_{\mathcal{A}}\mathbf{k};\ (\ell_{\mathcal{A}}\mathbf{j})(\ell_{\mathcal{A}}\mathbf{k}) = \ell_{\mathcal{A}}\mathbf{i};\ (\ell_{\mathcal{A}}\mathbf{k})(\ell_{\mathcal{A}}\mathbf{i}) = \ell_{\mathcal{A}}\mathbf{j}.$$

We observe that $\ell_{\mathcal{A}}\mathbf{i}$, $\ell_{\mathcal{A}}\mathbf{j}$, and $\ell_{\mathcal{A}}\mathbf{k}$ are skew-symmetric, so we can take transposes in the three equalities above to obtain

$$(\ell_{\mathcal{A}}\mathbf{j})(\ell_{\mathcal{A}}\mathbf{i}) = -\ell_{\mathcal{A}}\mathbf{k};\ (\ell_{\mathcal{A}}\mathbf{k})(\ell_{\mathcal{A}}\mathbf{j}) = -\ell_{\mathcal{A}}\mathbf{i};\ (\ell_{\mathcal{A}}\mathbf{i})(\ell_{\mathcal{A}}\mathbf{k}) = -\ell_{\mathcal{A}}\mathbf{j}.$$

This completes the verification that $\ell_{\mathcal{A}}$ respects the entire prescribed multiplication table (4.5) and hence respects multiplication in the sense of (iv). Hence $\ell_{\mathcal{A}}\mathcal{A}$ is a sub-algebra of $\mathbb{R}^{I\times I}$ and so is itself an associative algebra.

(v) We conclude that $\mathbb{R}^4$, when equipped with the multiplication table (4.5), is a scalar algebra. We of course call it the *algebra of quaternions.*

5. Inner product

Throughout this section, let $\mathcal{A}$ be a scalar algebra.

Definition 5.1. We define $\mu_{\mathcal{A}} \colon \mathcal{A} \to \mathbb{R}$ by

$$\mu_{\mathcal{A}}(\mathbf{a}) := |\det(\ell_{\mathcal{A}}\mathbf{a})|^{(1/\dim \mathcal{A})} \qquad \text{for all } \mathbf{a} \in \mathcal{A}, \tag{5.1}$$

and, for all $\rho \in \mathbb{P}^{\times}$, we define sets

$$\mathrm{S}_{\mathcal{A},\rho} := \{\, \mathbf{a} \in \mathcal{A} \mid \mu_{\mathcal{A}}(\mathbf{a}) = \rho \,\}; \tag{5.2a}$$

$$\mathrm{B}_{\mathcal{A},\rho} := \{\, \mathbf{a} \in \mathcal{A} \mid \mu_{\mathcal{A}}(\mathbf{a}) < \rho \,\}; \tag{5.2b}$$

$$\mathrm{C}_{\mathcal{A},\rho} := \mathrm{Cxh}(\mathrm{B}_{\mathcal{A},\rho}). \tag{5.2c}$$

We abbreviate $\mathrm{S}_{\mathcal{A}} := \mathrm{S}_{\mathcal{A},1}$, $\mathrm{B}_{\mathcal{A}} := \mathrm{B}_{\mathcal{A},1}$, $\mathrm{C}_{\mathcal{A}} := \mathrm{C}_{\mathcal{A},1}$.

Remark. The symbols μ, S, and B are intended to suggest *magnitude*, *sphere*, and *ball*. However, it would be premature to use these terms since we have not yet made the case that $\mu_{\mathcal{A}}$ is the magnitude function induced by an inner product. That will be established in Thm. 5.5.

Proposition 5.2. *The mapping $\mu_{\mathcal{A}}$ is continuous and the following properties:*

$$\mathbf{a} = \mathbf{0} \iff \mu_{\mathcal{A}}(\mathbf{a}) = 0 \quad \text{for all } \mathbf{a} \in \mathcal{A}; \tag{5.3a}$$

$$\mu_{\mathcal{A}}(\mathbf{1}) = 1 \tag{5.3b}$$

$$\mu_{\mathcal{A}}(\mathbf{ab}) = \mu_{\mathcal{A}}(\mathbf{a})\mu_{\mathcal{A}}(\mathbf{b}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}; \tag{5.3c}$$

$$\mu_{\mathcal{A}}(\xi\mathbf{a}) = |\,\xi|\, \mu_{\mathcal{A}}(\mathbf{a}) \quad \text{for all } \xi \in \mathbb{R},\ \mathbf{a} \in \mathcal{A}. \tag{5.3d}$$

Proof. These properties follow immediately from Def. 5.1, Prop. 2.1, and Prop. 3.3. □

Proposition 5.3. *For all $\rho \in \mathbb{P}^{\times}$, the sets $\mathrm{S}_{\mathcal{A},\rho}$, $\mathrm{B}_{\mathcal{A},\rho}$, and $\mathrm{C}_{\mathcal{A},\rho}$ are fixed under left- and right-multiplication by members of $\mathrm{S}_{\mathcal{A}}$. That is,*

$$\mathbf{u}\,\mathrm{S}_{\mathcal{A},\rho} = \mathrm{S}_{\mathcal{A},\rho}\,\mathbf{u} = \mathrm{S}_{\mathcal{A},\rho} \qquad \text{for all } \rho \in \mathbb{P}^{\times},\ \mathbf{u} \in \mathrm{S}_{\mathcal{A}}; \tag{5.4a}$$

$$\mathbf{u}\,\mathrm{B}_{\mathcal{A},\rho} = \mathrm{B}_{\mathcal{A},\rho}\,\mathbf{u} = \mathrm{B}_{\mathcal{A},\rho} \qquad \text{for all } \rho \in \mathbb{P}^{\times},\ \mathbf{u} \in \mathrm{S}_{\mathcal{A}}; \tag{5.4b}$$

$$\mathbf{u}\,\mathrm{C}_{\mathcal{A},\rho} = \mathrm{C}_{\mathcal{A},\rho}\,\mathbf{u} = \mathrm{C}_{\mathcal{A},\rho} \qquad \text{for all } \rho \in \mathbb{P}^{\times},\ \mathbf{u} \in \mathrm{S}_{\mathcal{A}}; \tag{5.4c}$$

Proof. Properties (5.4a) and (5.4b) follow immediately from (5.3c). Left- and right-multiplication operations in $\mathcal{A}$ are linear, so they respect the taking of convex hulls (see [Nol87], Sect. 37, Prop. 5). Hence (5.4c) follows directly from (5.4b). □

Proposition 5.4. *For all $\rho \in \mathbb{P}^{\times}$, the convex set $\mathrm{C}_{\mathcal{A},\rho}$ is bounded and has a non-empty interior.*

Proof. Let $\rho \in \mathbb{P}^{\times}$. The set $\mathrm{B}_{\mathcal{A},\rho}$ is clearly open and contains $\mathbf{0}$. The set $\mathrm{C}_{\mathcal{A},\rho}$ includes $\mathrm{B}_{\mathcal{A},\rho}$ and hence has a non-empty interior.

Choose a *norm* (see [Nol87], Sect. 51) ν on $\mathcal{A}$. Then $\mathcal{N} := \nu^{<}(\{1\})$ is closed and bounded, hence compact. Since $\mathbf{1} \neq \mathbf{0}$, we have $\dim \mathcal{A} > 0$, so $\mathcal{N}$ is non-empty. Hence, since $\mu_{\mathcal{A}}$ is continuous by Prop. 5.2, we may choose $\mathbf{b} \in \mathcal{N}$ such that $\xi := \mu_{\mathcal{A}}(\mathbf{b}) = \min(\mu_{\mathcal{A}>}(\mathcal{N}))$. Then

$$\nu(\mathbf{a}) = 1 \quad \Longrightarrow \quad \mu_{\mathcal{A}}(\mathbf{a}) \geq \xi \qquad \text{for all } \mathbf{a} \in \mathcal{A}.$$

It follows by (5.3d) that

$$\nu(\mathbf{a}) \geq 1 \quad \Longrightarrow \quad \mu_{\mathcal{A}}(\mathbf{a}) \geq \xi\nu(\mathbf{a}) \geq \zeta \qquad \text{for all } \mathbf{a} \in \mathcal{A}.$$

Hence

$$\mu_{\mathcal{A}}(\mathbf{a}) < \xi \quad \Longrightarrow \quad \nu(\mathbf{a}) < 1 \qquad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{5.5}$$

Since $\nu(\mathbf{b}) > 0$, we have $\mathbf{b} \neq \mathbf{0}$; it follows by (5.3a) that $\xi > 0$. Put $\zeta := \rho/\xi$. By scaling (5.5), we have $\mathrm{B}_{\mathcal{A},\rho} \subset \{\, \mathbf{a} \in \mathcal{A} \mid \nu(\mathbf{a}) < \zeta \,\}$. Hence $\mathrm{B}_{\mathcal{A},\rho}$ is bounded. Since $\{\, \mathbf{a} \in \mathcal{A} \mid \nu(\mathbf{a}) < \zeta \,\}$ is convex, it also includes the convex hull $\mathrm{C}_{\mathcal{A},\rho}$ of $\mathrm{B}_{\mathcal{A},\rho}$, so $\mathrm{C}_{\mathcal{A},\rho}$ is bounded as well. □

Theorem 5.5. *There is exactly one inner product* $\mathrm{ip}_{\mathcal{A}}$ *on* $\mathcal{A}$ *that satisfies*

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{a},\mathbf{a}) = (\mu_{\mathcal{A}}(\mathbf{a}))^2 \qquad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{5.6}$$

This inner product is invariant under $\mathrm{S}_{\mathcal{A}}$ *in the sense that*

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{au},\mathbf{bu}) = \mathrm{ip}_{\mathcal{A}}(\mathbf{a},\mathbf{b}) = \mathrm{ip}_{\mathcal{A}}(\mathbf{ua},\mathbf{ub}) \qquad \text{for all } \mathbf{u} \in \mathrm{S}_{\mathcal{A}} \text{ and all } \mathbf{a},\mathbf{b} \in \mathcal{A}. \tag{5.7}$$

Proof. A symmetric bilinear mapping $\mathbf{B} \colon \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ on a linear space is determined by its associated quadratic form $\mathbf{v} \mapsto \mathbf{B}(\mathbf{v},\mathbf{v}) \colon \mathcal{V} \to \mathbb{R}$ (see [Nol87], Sect. 41). Hence, there can be at most one bilinear mapping $\mathrm{ip}_{\mathcal{A}}$ on $\mathcal{A}$ that satisfies (5.6). Further, it is clear that a symmetric bilinear mapping that satisfies (5.6) is non-degenerate and positive, hence an inner product.

Choose an inner product on $\mathcal{A}$. For each pair $(\mathbf{a},\mathbf{b}) \in \mathcal{A}\times\mathcal{A}$, define $\mathbf{B}_{(\mathbf{a},\mathbf{b})} \in \mathrm{Co}\,\mathcal{A}$ by

$$\mathbf{B}_{(\mathbf{a},\mathbf{b})}(\mathbf{v}) := \mathbf{B}(\mathbf{av},\mathbf{bv}) \qquad \text{for all } \mathbf{v} \in \mathcal{A}. \tag{5.8}$$

Also, for each $\mathbf{b} \in \mathcal{A}$, define $\mathbf{B}_{(\cdot\,,\mathbf{b})} \in \mathrm{Lin}(\mathcal{A},\mathrm{Co}\,\mathcal{A})$ by

$$\mathbf{B}_{(\cdot\,,\mathbf{b})}(\mathbf{a}) := \mathbf{B}_{(\mathbf{a},\mathbf{b})} \qquad \text{for all } \mathbf{a} \in \mathcal{A}.$$

By Prop. 5.4, the convex set $\mathrm{C}_{\mathcal{A}}$ is bounded with non-empty interior. Hence, by Lemma 2.4, we may choose a linear functional $\int_{\mathrm{C}_{\mathcal{A}}}$ on $\mathrm{Co}\,\mathcal{A}$

which satisfies

$$f|_{\mathrm{C}_{\mathcal{A}}} > 0 \quad \Rightarrow \quad \int_{\mathrm{C}_{\mathcal{A}}} f \; > 0; \tag{5.9a}$$

$$\mathbf{L}\,\mathrm{C}_{\mathcal{A}} = \mathrm{C}_{\mathcal{A}} \quad \Rightarrow \quad \int_{\mathrm{C}_{\mathcal{A}}} f \circ \mathbf{L} = \int_{\mathrm{C}_{\mathcal{A}}} f; \tag{5.9b}$$

Choose $\xi \in \mathbb{R}$ such that $0 < \xi < 1$, so $\xi\mathbf{1} \in \mathrm{C}_{\mathcal{A}}$. Since $\mathbf{B}$ is an inner product and $\xi\mathbf{1} \neq \mathbf{0}$, we have $\mathbf{B}_{(\mathbf{1},\mathbf{1})}(\xi\mathbf{1}) > 0$. Hence $\mathbf{B}_{(\mathbf{1},\mathbf{1})}\big|_{\mathrm{C}_{\mathcal{A}}} > 0$. By (5.9a), we have

$$\gamma := \int_{\mathrm{C}_{\mathcal{A}}} \mathbf{B}_{(\mathbf{1},\mathbf{1})} > 0. \tag{5.10}$$

Define $\mathrm{ip}_{\mathcal{A}} \colon \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ by

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{a},\mathbf{b}) := (1/\gamma) \int_{\mathrm{C}_{\mathcal{A}}} \mathbf{B}_{(\mathbf{a},\mathbf{b})} \qquad \text{for all } \mathbf{b},\, \mathbf{b} \in \mathcal{A}. \tag{5.11}$$

Since a composite of linear mappings is linear,

$$\mathrm{ip}_{\mathcal{A}}(\,\cdot\,,\mathbf{b}) = \left(\int_{\mathrm{C}_{\mathcal{A}}}\right) \circ \mathbf{B}_{(\,\cdot\,,\mathbf{b})} \in \mathrm{Lin}(\mathcal{A},\mathbb{R}) \qquad \text{for all } \mathbf{b} \in \mathcal{A}.$$

By the same argument, $\mathrm{ip}_{\mathcal{A}}(\mathbf{a},\,\cdot\,) \in \mathrm{Lin}(\mathcal{A},\mathbb{R})$ for all $\mathbf{a} \in \mathcal{A}$, so $\mathrm{ip}_{\mathcal{A}}$ is bilinear. Since $\mathbf{B}$ is symmetric, so is $\mathrm{ip}_{\mathcal{A}}$.

Let $\mathbf{u} \in \mathrm{S}_{\mathcal{A}}$. In view of Prop. 5.3, the automorphism $\ell_{\mathcal{A}}\mathbf{u} \in \mathrm{Lin}\,\mathcal{A}$ satisfies $(\ell_{\mathcal{A}}\mathbf{u})\,\mathrm{C}_{\mathcal{A}} = \mathrm{C}_{\mathcal{A}}$. It follows by (5.9b) that

$$\int_{\mathrm{C}_{\mathcal{A}}} \mathbf{B}_{(\mathbf{a},\mathbf{b})} = \int_{\mathrm{C}_{\mathcal{A}}} \big(\mathbf{B}_{(\mathbf{a},\mathbf{b})} \circ (\ell_{\mathcal{A}}\mathbf{u})\big) = \int_{\mathrm{C}_{\mathcal{A}}} \mathbf{B}_{(\mathbf{au},\mathbf{bu})} \quad \text{for all } \mathbf{b},\, \mathbf{b} \in \mathcal{A}.$$

This establishes the first equality of (5.7); the second equality follows by the same reasoning applied to right-multiplication by $\mathbf{u}$.

In view of (5.11) and (5.10), we have

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{1},\mathbf{1}) := (1/\gamma) \int_{\mathrm{C}_{\mathcal{A}}} \mathbf{B}_{(\mathbf{1},\mathbf{1})} = 1. \tag{5.12}$$

By (5.7) and (5.12), we have

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{u},\mathbf{u}) = \mathrm{ip}_{\mathcal{A}}(\mathbf{1u},\mathbf{1u}) = \mathrm{ip}_{\mathcal{A}}(\mathbf{1},\mathbf{1}) = 1 \quad \text{for all } \mathbf{u} \in \mathrm{S}_{\mathcal{A}}. \tag{5.13}$$

Let $\mathbf{a} \in \mathcal{A}^{\times}$, put $\xi := \mu_{\mathcal{A}}(\mathbf{a}) > 0$ and put $\mathbf{u} := (1/\xi)$. Then $\mathbf{u} \in \mathrm{S}_{\mathcal{A}}$, so , in view of (5.13), we have

$$\mathrm{ip}_{\mathcal{A}}(\mathbf{a},\mathbf{a}) = \xi^2\,\mathrm{ip}_{\mathcal{A}}(\mathbf{u},\mathbf{u}) = \xi^2 = (\mu_{\mathcal{A}}(\mathbf{a}))^2\,. \tag{5.14}$$

This establishes (5.6). $\square$

Convention. We shall henceforth consider $\mathcal{A}$ (as well as any other scalar algebra under consideration) to be equipped with additional structure by the prescription of its natural inner product (as determined by Thm. 5.5), and thereby consider it to be an *inner-product space*. Further, we use the inner product $\mathrm{ip}_{\mathcal{A}}$ to identify the dual space $\mathcal{A}^*$ of $\mathcal{A}$ with $\mathcal{A}$ itself in the usual manner (see [Nol87], Sect. 41).

We denote the inner product on $\mathcal{A}$ by $\mathrm{ip}_{\mathcal{A}}$ when necessary, but shall henceforth usually use the conventional abbreviated notation

$$\mathbf{a}\cdot\mathbf{b} := \mathrm{ip}_{\mathcal{A}}(\mathbf{a},\mathbf{b}) \qquad \text{for all } \mathbf{b},\ \mathbf{b}\in\mathcal{A}. \tag{5.15}$$

It is clear from Thm. 5.5 that $\mu_{\mathcal{A}}$ is indeed the **magnitude** function associated with the inner product $\mathrm{ip}_{\mathcal{A}}$. We shall henceforth use the conventional notation

$$|\,\mathbf{a}| := \mu_{\mathcal{A}}(\mathbf{a}) = \sqrt{\mathbf{a}\cdot\mathbf{a}} \qquad \text{for all } \mathbf{a}\in\mathcal{A}.$$

In accordance with convention for the standard scalar algebras $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, we call $|\,\mathbf{a}|$ the **absolute value** of $\mathbf{a}\in\mathcal{A}$. For convenience, we re-state Prop. 5.2 using this notation.

Proposition 5.6. *The mapping* $\mathbf{a}\mapsto|\,\mathbf{a}|\colon\mathcal{A}\to\mathbb{R}$ *is continuous and has the following properties:*

$$\mathbf{a}=\mathbf{0} \iff |\,\mathbf{a}|=0 \quad \text{for all } \mathbf{a}\in\mathcal{A}; \tag{5.16a}$$

$$|\,\mathbf{1}|=1 \tag{5.16b}$$

$$|\,\mathbf{ab}|=|\,\mathbf{a}|\,|\,\mathbf{b}| \quad \text{for all } \mathbf{a},\mathbf{b}\in\mathcal{A}; \tag{5.16c}$$

$$|\,\xi\mathbf{a}|=|\,\xi|\,|\,\mathbf{a}| \quad \text{for all } \xi\in\mathbb{R},\ \mathbf{a}\in\mathcal{A}. \tag{5.16d}$$

It is also clear that, for every $\rho\in\mathbb{P}$, the sets $\mathrm{B}_{\mathcal{A},\rho}$ and $\mathrm{S}_{\mathcal{A},\rho}$ are, resp., the **ball** and **sphere** of radius ρ in $\mathcal{A}$. Further, since $\mathrm{B}_{\mathcal{A},\rho}$ is convex, we have $\mathrm{C}_{\mathcal{A},\rho}=\mathrm{B}_{\mathcal{A},\rho}$. We are now justified in using the various notations for balls and spheres in inner-product spaces (and the closely-related Euclidean spaces) that are introduced in [Nol87], Sects. 42 and 46. Of these, we will have occasion to use only one in what follows, namely the notation $\mathrm{Usph}\,\mathcal{V}$ for the **unit sphere** of an inner-product space $\mathcal{V}$. In particular, having shown that

$$\mathrm{S}_{\mathcal{A}}=\mathrm{Usph}\,\mathcal{A},$$

we shall henceforth use the latter notation instead of the former. We shall call members of $\mathrm{Usph}\,\mathcal{A}$ **unit scalars** or simply **units** (see Note 6).

By Thm. 5.5 that left (and right) multiplications by units respect the inner product $\mathrm{ip}_{\mathcal{A}}$ and hence are orthogonal lineons; other left and right multiplications differ from orthogonal lineons by a positive scale factor other than 1 and are thus not orthogonal. Hence

$$\ell_{\mathcal{A}}\mathrm{Usph}\,\mathcal{A} = \ell_{\mathcal{A}}\mathcal{A} \cap \mathrm{Orth}\,\mathcal{A}; \tag{5.17a}$$

$$\mathscr{r}_{\mathcal{A}}\mathrm{Usph}\,\mathcal{A} = \mathscr{r}_{\mathcal{A}}\mathcal{A} \cap \mathrm{Orth}\,\mathcal{A}. \tag{5.17b}$$

Since we have made the identification $\mathcal{A}^* \cong \mathcal{A}$, we also have the identification $\mathrm{Lin}\,\mathcal{A}^* \cong \mathrm{Lin}\,\mathcal{A}$. Hence, for $\mathbf{A} \in \mathrm{Lin}\,\mathcal{A}$, we have $\mathbf{A}^\top \in \mathrm{Lin}\,\mathcal{A}$. We note that by (5.17) we have

$$(\ell_{\mathcal{A}}\mathbf{u})^\top = (\ell_{\mathcal{A}}\mathbf{u})^{-1} = \ell_{\mathcal{A}}\mathbf{u}^{-1} \in \ell_{\mathcal{A}}\mathcal{A} \qquad \text{for all } \mathbf{u} \in \mathrm{Usph}\,\mathcal{A}; \tag{5.18a}$$

$$(\mathscr{r}_{\mathcal{A}}\mathbf{u})^\top = (\mathscr{r}_{\mathcal{A}}\mathbf{u})^{-1} = \mathscr{r}_{\mathcal{A}}\mathbf{u}^{-1} \in \mathscr{r}_{\mathcal{A}}\mathcal{A} \qquad \text{for all } \mathbf{u} \in \mathrm{Usph}\,\mathcal{A}. \tag{5.18b}$$

It follows immediately that

$$(\ell_{\mathcal{A}}\mathbf{a})^\top \in \ell_{\mathcal{A}}\mathcal{A}, \quad (\mathscr{r}_{\mathcal{A}}\mathbf{a})^\top \in \mathscr{r}_{\mathcal{A}}\mathcal{A} \qquad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{5.19}$$

It is customary to define an induced inner product on $\mathrm{Lin}\,\mathcal{A}$ via the following (or an equivalent) definition (see [Nol87], Sect. 44):

$$\mathbf{A} \cdot \mathbf{B} := \mathrm{tr}(\mathbf{A}\mathbf{B}^\top) \qquad \text{for all } \mathbf{A}, \mathbf{B} \in \mathrm{Lin}\,\mathcal{A}. \tag{5.20}$$

It would appear that each of the algebras $\ell_{\mathcal{A}}\mathcal{A}$ and $\ell_{\mathcal{A}}\mathcal{A}$ now has two natural inner products: one acquired as the image of the inner-product space $\mathcal{A}$ under the linear isomorphism $\ell_{\mathcal{A}}|^{\mathrm{Rng}}$ or $\mathscr{r}_{\mathcal{A}}|^{\mathrm{Rng}}$ (as appropriate), the other acquired as a subspace of $\mathrm{Lin}\,\mathcal{A}$. The next proposition shows that, in each case, the two are identical except for a scale factor.

Proposition 5.7. *Put* $\gamma := \dim \mathcal{A}$. *Then the linear isomorphisms*

$$\ell_{\mathcal{A}}|^{\mathrm{Rng}} : \mathcal{A} \to \ell_{\mathcal{A}}\mathcal{A} \quad \text{and} \quad \mathscr{r}_{\mathcal{A}}|^{\mathrm{Rng}} : \mathcal{A} \to \mathscr{r}_{\mathcal{A}}\mathcal{A}$$

satisfy

$$(\ell_{\mathcal{A}}\mathbf{a}) \cdot (\ell_{\mathcal{A}}\mathbf{b}) = \gamma\, \mathbf{a} \cdot \mathbf{b} = (\mathscr{r}_{\mathcal{A}}\mathbf{a}) \cdot (\mathscr{r}_{\mathcal{A}}\mathbf{b}) \quad \text{for all } \mathbf{a}, \mathbf{b} \in \mathcal{A}. \tag{5.21}$$

Proof. Let $\mathbf{u} \in \mathrm{Usph}\,\mathcal{A}$. In view of (5.18), we have

$$(\ell_{\mathcal{A}}\mathbf{u}) \cdot (\ell_{\mathcal{A}}\mathbf{u}) = \mathrm{tr}\left((\ell_{\mathcal{A}}\mathbf{u})(\ell_{\mathcal{A}}\mathbf{u})^{-1}\right) = \mathrm{tr}\,\mathbf{1}_{\mathcal{A}} = \gamma.$$

Let $\mathbf{a} \in \mathcal{A}$, and choose $\alpha \in \mathbb{R}$, $\mathbf{u} \in \mathrm{Usph}\,\mathcal{A}$ such that $\mathbf{a} = \alpha\mathbf{u}$. Then

$$(\ell_{\mathcal{A}}\mathbf{a}) \cdot (\ell_{\mathcal{A}}\mathbf{a}) = \alpha^2(\ell_{\mathcal{A}}\mathbf{u}) \cdot (\ell_{\mathcal{A}}\mathbf{u}) = \alpha^2\gamma = \gamma\, \mathbf{a} \cdot \mathbf{a}.$$

Since symmetric bilinear functions are determined by their associated quadratic forms (see [Nol87], Sect. 27), this establishes the first equality of (5.21). The second is established by similar reasoning. □

6. Geometric and algebraic structure

Throughout this section, let $\mathcal{A}$ be a scalar algebra.

In view of (5.19) and the fact that the mapping $\ell_{\mathcal{A}|\mathrm{Rng}}$ is an isomorphism, we can define the **conjugation operator** $\dagger_{\mathcal{A}} \in \operatorname{Lin}\mathcal{A}$ by

$$\dagger_{\mathcal{A}}\mathbf{a} := \left(\ell_{\mathcal{A}|\mathrm{Rng}}\right)^{-1}(\ell_{\mathcal{A}}\mathbf{a})^{\top} \qquad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{6.1}$$

Whenever convenient, we abbreviate $\mathbf{a}^{\dagger} := \dagger_{\mathcal{A}}\mathbf{a}$. We call $\mathbf{a}^{\dagger}$ the **conjugate** of $\mathbf{a} \in \mathcal{A}$. It is sometimes convenient to have (6.1) restated as follows:

$$\ell_{\mathcal{A}}\,\mathbf{a}^{\dagger} = (\ell_{\mathcal{A}}\mathbf{a})^{\top} \qquad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{6.2}$$

Proposition 6.1. $\operatorname{Usph}\mathcal{A} = \{\,\mathbf{u} \in \mathcal{A} \mid \mathbf{u}^{\dagger} = \mathbf{u}^{-1}\,\}$.

Proof. This follows immediately from (5.17a), (5.18), and (6.2). □

Reversing the reasoning used to prove Prop. 6.1, and appealing to (5.17b), and (5.18), we have

$$\mathfrak{r}_{\mathcal{A}}\,\mathbf{a}^{\dagger} = (\mathfrak{r}_{\mathcal{A}}\mathbf{a})^{\top} \qquad \text{for all } \mathbf{a} \in \mathcal{A}; \tag{6.3}$$

in addition to (6.2).

Proposition 6.2. *For all* $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{A}$*, we have*

$$(\mathbf{ab}) \cdot \mathbf{c} = \mathbf{b} \cdot (\mathbf{a}^{\dagger}\mathbf{c}); \tag{6.4a}$$

$$(\mathbf{a}^{\dagger})^{\dagger} = \mathbf{a}; \tag{6.4b}$$

$$\mathbf{1}^{\dagger} = \mathbf{1}; \tag{6.4c}$$

$$(\mathbf{ab})^{\dagger} = \mathbf{b}^{\dagger}\mathbf{a}^{\dagger}; \tag{6.4d}$$

$$\mathbf{a} \neq \mathbf{0} \quad \Rightarrow \quad (\mathbf{a}^{\dagger})^{-1} = (\mathbf{a}^{-1})^{\dagger} \tag{6.4e}$$

$$\mathbf{a}^{\dagger} \cdot \mathbf{b}^{\dagger} = \mathbf{a} \cdot \mathbf{b} \tag{6.4f}$$

$$\mathbf{aa}^{\dagger} = \mathbf{a}^{\dagger}\mathbf{a} = |\,\mathbf{a}|^{2}\mathbf{1}; \tag{6.4g}$$

$$\mathbf{a} \neq \mathbf{0} \quad \Rightarrow \quad \mathbf{a}^{\dagger} = |\,\mathbf{a}|^{2}\mathbf{a}^{-1}. \tag{6.4h}$$

Proof. In this proof, we make repeated use of (6.2) and Prop. 5.7 without further reference.

In view of (21.3) of [Nol87] (which is essentially the definition of transposition for the case at hand), we have

$$((\ell_{\mathcal{A}}\mathbf{a})\mathbf{b}) \cdot \mathbf{c} = \mathbf{b} \cdot \big((\ell_{\mathcal{A}}\mathbf{a})^{\top}\mathbf{c}\big) = \mathbf{b} \cdot \big((\ell_{\mathcal{A}}\mathbf{a}^{\dagger})\mathbf{c}\big)\,.$$

Property (6.4a) follows immediately.

Property (6.4b) follows from the easily-proved fact that $(\mathbf{A}^{\top})^{\top} = \mathbf{A}$ for all $\mathbf{A} \in \operatorname{Lin}\mathcal{V}$ whenever $\mathcal{V}$ is a finite-dimensional inner-product space. Properties (6.4c) and (6.4d) follow from Prop. 1 of Sect. 21 of [Nol87]. Property (6.4e) follows from the same proposition together with the fact that every non-zero member of $\mathcal{A}$ has a reciprocal. Property (6.4f) follows from Prop. 1 of Sect. 44 of [Nol87].

Property (6.4g) follows from Prop. 6.1 since for every $\mathbf{a} \in \mathcal{A}$ we may choose $\mathbf{u} \in \operatorname{Usph}\mathcal{A}$ such that $\mathbf{a} = |\,\mathbf{a}|\mathbf{u}$; then (6.4h) follows immediately from (6.4g). $\square$

Definition 6.3. We define lineons $\operatorname{Re}, \operatorname{Im} \in \operatorname{Lin}\mathcal{A}$ by

(6.5a) $$\operatorname{Re}\mathbf{a} := (1/2)(\mathbf{a} + \mathbf{a}^\dagger) \qquad \text{for all } \mathbf{a} \in \mathcal{A},$$

(6.5b) $$\operatorname{Im}\mathbf{a} := (1/2)(\mathbf{a} - \mathbf{a}^\dagger) \qquad \text{for all } \mathbf{a} \in \mathcal{A}.$$

We call members of their ranges

(6.6a) $$\operatorname{Re}\mathcal{A} := \{\, \operatorname{Re}\mathbf{a} \mid \mathbf{a} \in \mathcal{A} \,\},$$

(6.6b) $$\operatorname{Im}\mathcal{A} := \{\, \operatorname{Im}\mathbf{a} \mid \mathbf{a} \in \mathcal{A} \,\},$$

real and imaginary scalars, resp. For $\mathbf{a} \in \mathcal{A}$, we call $\operatorname{Re}\mathbf{a}$ and $\operatorname{Im}\mathbf{a}$ the **real part** and **imaginary part**, resp., of $\mathbf{b}$.

Proposition 6.4. *Let* $\mathbf{a} \in \mathcal{A}$. *Then*

(6.7a) $$\mathbf{a} \in \operatorname{Re}\mathcal{A} \quad \Leftrightarrow \quad \ell_{\mathcal{A}}\mathbf{a} \in \operatorname{Sym}\mathcal{A} \quad \Leftrightarrow \quad \mathfrak{r}_{\mathcal{A}}\mathbf{a} \in \operatorname{Sym}\mathcal{A};$$

(6.7b) $$\mathbf{a} \in \operatorname{Im}\mathcal{A} \quad \Leftrightarrow \quad \ell_{\mathcal{A}}\mathbf{a} \in \operatorname{Skew}\mathcal{A} \quad \Leftrightarrow \quad \mathfrak{r}_{\mathcal{A}}\mathbf{a} \in \operatorname{Skew}\mathcal{A};$$

Proof. This follows immediately from (6.2) and (6.3). $\square$

It is clear from (6.5) that for all $\mathbf{a} \in \mathcal{A}$,

(6.8a) $$\mathbf{a} = \operatorname{Re}\mathbf{a} + \operatorname{Im}\mathbf{a};$$

(6.8b) $$\mathbf{a}^\dagger = \operatorname{Re}\mathbf{a} - \operatorname{Im}\mathbf{a}.$$

Proposition 6.5. *The subspaces* $\operatorname{Re}\mathcal{A}$ *and* $\operatorname{Im}\mathcal{A}$ *are orthogonal supplements in* $\mathcal{A}$. *That is,* $\operatorname{Re}\mathcal{A} \perp \operatorname{Im}\mathcal{A}$ *and* $\operatorname{Re}\mathcal{A} + \operatorname{Im}\mathcal{A} = \mathcal{A}$.

Proof. Orthogonality can be shown via direct computation using (6.10) and (6.4f). This implies that $\operatorname{Re}\mathcal{A} \cap \operatorname{Im}\mathcal{A} = \{\mathbf{0}\}$. Then supplementarity follows by (6.8a). $\square$

It follows immediately from Prop. 6.5 that

(6.9a) $$\operatorname{Re}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \operatorname{Re}\mathbf{a} = \mathbf{a} \,\} = \{\, \mathbf{a} \in \mathcal{A} \mid \operatorname{Im}\mathbf{a} = \mathbf{0} \,\},$$

(6.9b) $$\operatorname{Im}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \operatorname{Im}\mathbf{a} = \mathbf{a} \,\} = \{\, \mathbf{a} \in \mathcal{A} \mid \operatorname{Re}\mathbf{a} = \mathbf{0} \,\}.$$

Proposition 6.6. *We have*

(6.10a) $$\operatorname{Re}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \mathbf{a}^\dagger = \mathbf{a} \,\},$$

(6.10b) $$\operatorname{Im}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \mathbf{a}^\dagger = -\mathbf{a} \,\}.$$

Proof. This follows from (6.9) and (6.6). □

Proposition 6.7. *We have*

$$\operatorname{Re}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \mathbf{a}^2 \in \mathbb{P}\mathbf{1} \,\}; \tag{6.11a}$$

$$\operatorname{Im}\mathcal{A} = \{\, \mathbf{a} \in \mathcal{A} \mid \mathbf{a}^2 \in -\mathbb{P}\mathbf{1} \,\}. \tag{6.11b}$$

Further,

$$\mathbf{a}^2 = |\,\mathbf{a}|^2\mathbf{1} \qquad \text{for all } \mathbf{a} \in \operatorname{Re}\mathcal{A}; \tag{6.12a}$$

$$\mathbf{a}^2 = -|\,\mathbf{a}|^2\mathbf{1} \qquad \text{for all } \mathbf{a} \in \operatorname{Im}\mathcal{A}. \tag{6.12b}$$

Proof. Let $\mathbf{u} \in \operatorname{Usph}\mathcal{A}$. By Prop. 6.1 and Prop. 6.6, we have

$$\mathbf{u} \in \operatorname{Re}\mathcal{A} \quad \Leftrightarrow \quad \mathbf{u} = \mathbf{u}^\dagger = \mathbf{u}^{-1} \quad \Leftrightarrow \quad \mathbf{u}^2 = \mathbf{1} \tag{6.13a}$$

$$\mathbf{u} \in \operatorname{Im}\mathcal{A} \quad \Leftrightarrow \quad \mathbf{u} = -\mathbf{u}^\dagger = -\mathbf{u}^{-1} \quad \Leftrightarrow \quad \mathbf{u}^2 = -\mathbf{1}. \tag{6.13b}$$

Since every scalar is a multiple of a unit scalar, the results follow by appropriate scaling. □

Proposition 6.8. *Let* $\mathbf{a}, \mathbf{b} \in \operatorname{Im}\mathcal{A}$. *Then*

$$\mathbf{ab} \in \operatorname{Re}\mathcal{A} \quad \Leftrightarrow \quad \mathbf{ba} = \mathbf{ab}; \tag{6.14a}$$

$$\mathbf{ab} \in \operatorname{Im}\mathcal{A} \quad \Leftrightarrow \quad \mathbf{ba} = -\mathbf{ab}. \tag{6.14b}$$

Proof. In view of (6.4d), and (6.10b) we have $\mathbf{ba} = \mathbf{b}^\dagger\mathbf{a}^\dagger = (\mathbf{ab})^\dagger$. The results follow by (6.10a) and (6.10b). □

Remark. The conclusions of Prop. 6.8 also hold under the hypothesis that $\mathbf{a}, \mathbf{b} \in \operatorname{Re}\mathcal{A}$. However, there is not much point in asserting that fact because, in view of the next proposition, every real scalar is a multiple of $\mathbf{1}$.

The next theorem is key to the structure of scalar algebras.

Theorem 6.9. $\operatorname{Re}\mathcal{A}$ *is 1-dimensional. Further,*

$$\operatorname{Re}\mathcal{A} = \mathbb{R}\mathbf{1}; \tag{6.15a}$$

$$\operatorname{Re} = \mathbf{1} \otimes \mathbf{1}; \tag{6.15b}$$

$$\operatorname{Re}\mathbf{a} = (\mathbf{1}\cdot\mathbf{a})\mathbf{1} \quad \text{for all } \mathbf{a} \in \mathcal{A}. \tag{6.15c}$$

Proof. Let $\mathbf{a} \in \mathcal{A}$. On the one hand, suppose that $\mathbf{a} \in \mathbb{R}\mathbf{1}$. Then $\ell_{\mathcal{A}}\mathbf{a} \in \mathbb{R}\ell_{\mathcal{A}}\mathbf{1} = \mathbb{R}\mathbf{1}_{\mathcal{A}}$ and hence is symmetric; it follows that $\mathbf{b}$ is real. On the other hand, suppose that $\mathbf{b}$ is real. Then $\ell_{\mathcal{A}}\mathbf{a}$ is symmetric, so we may choose an eigenvalue α for it. Then $\ell_{\mathcal{A}}\mathbf{a} - \alpha\mathbf{1}_{\mathcal{A}}$ is singular, so $\mathbf{a} - \alpha\mathbf{1}$ has no reciprocal, so $\mathbf{a} = \alpha\mathbf{1}$. Thus $\mathbf{a} \in \mathbb{R}\mathbf{1}$. This establishes (6.15a). It follows that $(\mathbf{1})$ is an orthonormal basis for the range (namely $\mathbb{R}\mathbf{1}$) of Re. In view of Prop. 6.5, we have (6.15b), which immediately implies (6.15c). □

Corollary 6.10. *Let* $\mathbf{a}, \mathbf{b} \in \operatorname{Im}\mathcal{A}$ *such that* $\mathbf{a} \cdot \mathbf{b} = 0$. *Then*

$$\mathbf{ab} \in \operatorname{Im}\mathcal{A} \tag{6.16a}$$

$$\mathbf{ab} = -\mathbf{ba}; \tag{6.16b}$$

$$(\mathbf{ab}) \cdot \mathbf{a} = (\mathbf{ab}) \cdot \mathbf{b} = 0. \tag{6.16c}$$

Proof. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be as stated. By (6.4a) and (6.10b), we have $\mathbf{1} \cdot (\mathbf{ab}) = \mathbf{a}^\dagger \cdot \mathbf{b} = -\mathbf{a} \cdot \mathbf{b} = 0$. It follows by (6.15c) that $\operatorname{Re}(\mathbf{ab}) = \mathbf{0}$, so, by (6.9b), $\mathbf{ab}$ is imaginary. This establishes (6.16a), and (6.16b) follows by (6.14b).

By (6.4a) and (6.4g), we have $(\mathbf{ab}) \cdot \mathbf{a} = \mathbf{b} \cdot (\mathbf{a}^\dagger \mathbf{a}) = |\,\mathbf{a}|^2 \mathbf{b} \cdot \mathbf{1} = 0$. Since $\mathbf{ba} = -\mathbf{ab}$ by Prop. 6.8, it follows (reversing the roles of $\mathbf{b}$ and $\mathbf{b}$) that $(\mathbf{ab}) \cdot \mathbf{b} = -(\mathbf{ba}) \cdot \mathbf{b} = 0$. □

The next corollary is the heart of the upcoming Frobenius classification theorem.

Corollary 6.11. *Let* $(\mathbf{a}, \mathbf{b})$ *be an orthogonal pair in* $\operatorname{Im}\mathcal{A}$*, such that* $\mathbf{a} \neq \mathbf{0}$ *and* $\mathbf{b} \neq \mathbf{0}$. *Then the triple* $(\mathbf{a}, \mathbf{b}, \mathbf{ab})$ *is an orthogonal basis for* $\operatorname{Im}\mathcal{A}$.

Proof. Let $(\mathbf{a}, \mathbf{b})$ be as stated. In view of Cor. 6.10, we have $(\mathbf{ab}) \in \operatorname{Im}\mathcal{A}$ and the triple $(\mathbf{a}, \mathbf{b}, \mathbf{ab})$ is an orthogonal list. In view of (5.16c), we have $|\,\mathbf{ab}| = |\,\mathbf{a}||\,\mathbf{b}| \neq 0$, so $\mathbf{ab} \neq \mathbf{0}$. It follows that the list $(\mathbf{a}, \mathbf{b}, \mathbf{ab}) \in (\operatorname{Im}\mathcal{A})^3$ is linearly independent.

Let $\mathbf{c} \in \operatorname{Im}\mathcal{A} \cap \{\mathbf{a}, \mathbf{b}, \mathbf{ab}\}^\perp$. Applying (6.14b) to the imaginary pair $(\mathbf{ab}, \mathbf{c})$, we obtain

$$\mathbf{abc} = (\mathbf{ab})\mathbf{c} = -\mathbf{c}(\mathbf{ab}) = -\mathbf{cab}.$$

However, applying (6.14b) twice, first to the pair $(\mathbf{b}, \mathbf{c})$ and then to the pair $(\mathbf{a}, \mathbf{c})$, we obtain

$$\mathbf{abc} = -\mathbf{acb} = \mathbf{cab}.$$

It follows that $\mathbf{abc} = \mathbf{0}$; left multiplication by $\mathbf{b}^{-1}\mathbf{a}^{-1}$ yields $\mathbf{c} = \mathbf{0}$. Hence $\operatorname{Im}\mathcal{A} \cap \{\mathbf{a}, \mathbf{b}, \mathbf{ab}\}^\perp = \{\mathbf{0}\}$, so $\{\mathbf{a}, \mathbf{b}, \mathbf{ab}\}$ spans $\operatorname{Im}\mathcal{A}$ and $(\mathbf{a}, \mathbf{b}, \mathbf{ab})$ is a basis for $\operatorname{Im}\mathcal{A}$. □

7. Frobenius classification

When interpreting the following theorem, recall that we make the identifications $\mathbb{R} \cong \mathbb{R}^1$, $\mathbb{C} \cong \mathbb{R}^2$ and $\mathbb{H} \cong \mathbb{R}^4$ when $\mathbb{R}^1$, $\mathbb{R}^2$ and $\mathbb{R}^4$ are considered to be equipped with product operations determined by the multiplication tables presented in Sect. 4. Also, it should be recalled that in those tables, whenever the symbols $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are used, they are simply alternative names for the terms of the standard basis $(\,\delta^I_I \mid i \in I\,)$ of $\mathbb{R}^I$ for appropriate I. In the following theorem, the same symbols are used for certain members of a scalar algebra $\mathcal{A}$ in accordance with the hypotheses of the theorem. There is an obvious intentional correspondence between the two collections of symbols, but there is also a potential for confusion. To avoid confusion, we will use the standard names $\{\,\delta^I_I \mid i \in I\,\}$, and not use the alternative names, when referring to the basis vectors of $\mathbb{R}^I$ in the statement and proof of this theorem.

Theorem 7.1. ***(Frobenius classification)*** *Let $\mathcal{A}$ be a scalar algebra. Then $\dim\mathcal{A} \in \{1,2,4\}$ and $\mathcal{A}$ conforms to one of three standard algebraic structures as follows.*

- *If $\dim\mathcal{A} = 1$, then the one-term list $(\mathbf{1})$ is (in a trivial sense) an orthonormal basis for $\mathcal{A}$. Further, putting $I := \{1\}$, the linear mapping $\mathbf{L}\colon \mathcal{A} \to \mathbb{R}^I$ determined by*

$$\mathbf{L1} = \delta^I_1 \tag{7.1}$$

  is a scalar-algebra isomorphism.
- *If $\dim\mathcal{A} = 2$, then for each $\mathbf{i} \in \operatorname{Im}\mathcal{A} \cap \operatorname{Usph}\mathcal{A}$ (of which there are exactly two), the pair $(\mathbf{1},\mathbf{i})$ is an orthonormal basis for $\mathcal{A}$. Further, putting $I := \{1,2\}$, the linear mapping $\mathbf{L}\colon \mathcal{A} \to \mathbb{R}^I \cong \mathbb{C}$ determined by*

$$\mathbf{L1} = \delta^I_1, \quad \mathbf{Li} = \delta^I_2 \tag{7.2}$$

  is a scalar-algebra isomorphism.
- *If $\dim\mathcal{A} = 4$, then for every orthonormal pair $(\mathbf{i},\mathbf{j}) \in (\operatorname{Im}\mathcal{A} \cap \operatorname{Usph}\mathcal{A})^2$ (of which there are infinitely many), the quadruple $(\mathbf{1},\mathbf{i},\mathbf{j},\mathbf{k})$, where $\mathbf{k} := \mathbf{ij}$, is an orthonormal basis for $\mathcal{A}$. Further, putting $I := \{1,2,3,4\}$, the linear mapping $\mathbf{L}\colon \mathcal{A} \to \mathbb{R}^I \cong \mathbb{H}$ determined by*

$$\mathbf{L1} = \delta^I_1, \quad \mathbf{Li} = \delta^I_2, \quad \mathbf{Lj} = \delta^I_3, \quad \mathbf{Lk} = \delta^I_4 \tag{7.3}$$

  is a scalar-algebra isomorphism.

Proof. Let $\mathcal{A}$ be as stated. Our definition of scalar algebras insures that $\dim\mathcal{A} \in \mathbb{N}^\times$. Put $n := \dim\mathcal{A}$ and put $I := n^]$. Note that by(5.16b), we have $|\mathbf{1}| = 1$. It follows by Thm. 6.9 that $(\mathbf{1})$ is an orthonormal basis for $\operatorname{Re}\mathcal{A}$.

If $\dim\mathcal{A} = 1$, put $(\,\mathbf{e}_i \mid i \in I\,) := (\mathbf{1})$. Since $\dim\mathcal{A} = 1$ we have $\mathcal{A} = \operatorname{Re}\mathcal{A}$, so the list $(\,\mathbf{e}_i \mid i \in I\,)$ is an orthonormal basis for $\mathcal{A}$.

If $\dim\mathcal{A} = 2$, let $\mathbf{i} \in \operatorname{Im}\mathcal{A} \cap \operatorname{Usph}\mathcal{A}$ and put $(\,\mathbf{e}_i \mid i \in I\,) := (\mathbf{1}, \mathbf{i})$. Since $\dim\mathcal{A} = 2$, we have $\dim\operatorname{Im}\mathcal{A} = 1$ by Prop. 6.5. We have $|\mathbf{i}| = 1$ because $\mathbf{i} \in \operatorname{Usph}\mathcal{A}$, so $(\mathbf{i})$ is an orthonormal basis for $\operatorname{Im}\mathcal{A}$. It follows by Prop. 6.5 that the list $(\,\mathbf{e}_i \mid i \in I\,)$ is an orthonormal basis for $\mathcal{A}$.

If $\dim\mathcal{A} > 2$, let $(\mathbf{i}, \mathbf{j}) \in (\operatorname{Im}\mathcal{A} \cap \operatorname{Usph}\mathcal{A})^2$ such that $\mathbf{i} \cdot \mathbf{j} = 0$, put $\mathbf{k} := \mathbf{ij}$, and put $(\,\mathbf{e}_i \mid i \in I\,) := (\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k})$. Then $|\mathbf{i}| = |\mathbf{j}| = 1$ because both are in $\operatorname{Usph}\mathcal{A}$. It follows by Cor. 6.11 that $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ is an orthonormal basis for $\operatorname{Im}\mathcal{A}$ and then by Prop. 6.5 that $(\,\mathbf{e}_i \mid i \in I\,)$ is an orthonormal basis for $\mathcal{A}$.

We have shown that $\dim\mathcal{A} \in \{1, 2, 4\}$, and that, in each case, $(\,\mathbf{e}_i \mid i \in I\,)$, as we have defined it, is an orthonormal basis for $\mathcal{A}$. It follows immediately that in each case, the linear mapping $\mathbf{L} \in \operatorname{Lin}(\mathcal{A}, \mathbb{R}^I)$ determined by

$$\mathbf{L}_i\mathbf{e}_i = \delta_I^I \qquad \text{for all } i \in I \tag{7.4}$$

is a linear isomorphism.

To show that $\mathbf{L}$ is an algebra isomorphism, it remains to shown that $\mathbf{L}$ respects the product operation, *i.e.,* that for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}$, we have

$$\mathbf{L}(\mathbf{ab}) = (\mathbf{La})(\mathbf{Lb}).$$

For this, it suffices to show that, for all $i, j \in I$, we have

$$(\mathbf{Le}_i)(\mathbf{Le}_j) = \mathbf{L}(\mathbf{e}_i\mathbf{e}_j). \tag{7.5}$$

To begin, we note that for any value of $n := \dim\mathcal{A}$, since $\mathbf{e}_1 = \mathbf{1}$, we have $\mathbf{e}_1\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_1 = \mathbf{e}_i$ for all $i \in I$. The tables in Sect. 4 show the corresponding property $\delta_1^I\delta_I^I = \delta_I^I\delta_1^I = \delta_I^I$ for all $i \in I$. Hence (7.5) holds whenever $i = 1$ or $j = 1$. This immediately shows that (7.5) holds for all $i, j \in I$ when $n = 1$.

Next, we note that if $i > 1$ then $\mathbf{e}_i$ is an imaginary unit and hence,in view of (6.12b), that $(\mathbf{e}_i)^2 = -\mathbf{1} = -\mathbf{e}_1$. Inspection of the tables in Sect. 4 shows that $(\delta_I^I)^2 = -\delta_1^I$ for all $i \in I$, so (7.5) holds whenever $i = j$. This, together with we have already shown, shows that (7.5) holds for all $i, j \in I$ when $n = 2$.

Finally, suppose that $n = 4$. We still need to show that (7.5) holds when $i, j \in \{2, 3, 4\}$ and $i \neq j$. Equivalently, we need to show

$$\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j}, \tag{7.6a}$$

$$\mathbf{ji} = -\mathbf{k}, \quad \mathbf{ki} = -\mathbf{i}, \quad \mathbf{ik} = -\mathbf{j}. \tag{7.6b}$$

We have $\mathbf{k} = \mathbf{ij}$ by definition. Left multiplication by $\mathbf{i}$ and right multiplication by $\mathbf{j}$ yield, resp., $\mathbf{ik} = -\mathbf{j}$ and $\mathbf{kj} = -\mathbf{i}$. This establishes three of the equalities. The remaining three follow by (6.16b). $\square$

Corollary 7.2. *The inner products on* $\mathbb{R}$*,* $\mathbb{C}$*, and* $\mathbb{H}$ *considered as scalar algebras are the same as the standard inner products on their underlying linear spaces* $\mathbb{R}^1$*,* $\mathbb{R}^2$*, and* $\mathbb{R}^4$*.*

Proof. In each of the three cases $n \in \{1, 2, 4\}$, the standard basis $(\, \delta_I^I \mid i \in n^\rfloor \,)$ of $\mathbb{R}^n$ was shown to be the image under a scalar-algebra isomorphism (namely $\mathbf{L} \in \mathrm{Lin}(\mathcal{A}, \mathbb{R}^n)$) of a basis $(\, \mathbf{e}_I \mid i \in n^\rfloor \,)$ that was orthonormal as determined by the scalar algebra structure on $\mathcal{A}$. $\square$

8. Scalar algebras and group representations

Our purpose in this section is to show that the category we call "scalar algebras" is not artificial: the category arises naturally in the context of group representation theory. Some of the material in this section (in particular, remarks regarding topological groups) is presented informally and goes beyond the scope of [Nol87], but nothing in the rest of this document depends on the material presented here.

A **representation** ρ of a group $\mathcal{G}$ on a finite-dimensional linear space $\mathcal{V}$ is a group homomorphism

$$\rho\colon \mathcal{G} \to \operatorname{Lis}\mathcal{V}.$$

For the remainder of this section, let $\mathcal{G}$ be a group, let $\mathcal{V}$ be a finite-dimensional linear space, and let ρ be a representation of $\mathcal{G}$ on $\mathcal{V}$.

A linear subspace $\mathcal{U}$ of $\mathcal{V}$ with the property that $\mathbf{G}\mathcal{U} \subset \mathcal{U}$ for all $\mathbf{G} \in \operatorname{Rng}\rho$ is said to be ρ-invariant. (See Note 9.) If $\mathcal{U}$ is a ρ-invariant subspace of $\mathcal{V}$, then we can define a representation $\rho_{\mathcal{U}}$ of $\mathcal{G}$ on $\mathcal{U}$ by appropriate *adjustment*

$$\rho_{\mathcal{U}}(\mathbf{g}) := \rho(\mathbf{g})_{|\mathcal{U}} \qquad \text{for all } \mathbf{g} \in \mathcal{G}.$$

We say, in this case, that $\rho_{\mathcal{U}}$ is a **subrepresentation** of ρ.

If $\mathcal{V}$ has no ρ-invariant subspaces other than $\{\mathbf{0}\}$ and $\mathcal{V}$ itself (and hence admits no proper non-trivial subrepresentations), then $\mathcal{V}$ is said to be **irreducible**.

A lineon $\mathbf{L} \in \mathcal{V}$ is said to be **ρ-linear** or a **ρ-lineon** if

$$\mathbf{GL} = \mathbf{LG} \qquad \text{for all } \mathbf{g} \in \operatorname{Rng}\rho.$$

The set of all ρ-lineons is denoted by $\operatorname{Lin}_\rho \mathcal{V}$. (Again, see Note 9.) It is very easy to show that

(1) $\mathbf{1}_{\mathcal{V}} \in \operatorname{Lin}_\rho \mathcal{V}$;
(2) $\operatorname{Lin}_\rho \mathcal{V}$ is a subalgebra of $\operatorname{Lin}\mathcal{V}$ and hence an associative algebra.

The following result (well-known, though not usually stated in terms of scalar algebras) is essentially a corollary of a very useful result in group representation theory known as *Schur's Lemma*. (See [BtD85], Thm. (1.10) of II-1 and Thm.(6.7) of II-6.)

Proposition 8.1. *If ρ is irreducible, then $\operatorname{Lin}_\rho \mathcal{V}$ is a scalar algebra.*

Proof. Suppose that $\operatorname{Lin}_\rho \mathcal{V}$ is not a scalar algebra. Then we can choose a non-zero non-invertible $\mathbf{L} \in \operatorname{Lin}_\rho \mathcal{V}$. Denote by $\mathcal{U}$ the nullspace of $\mathbf{L}$, that is, $\mathcal{U} := L^{<}\{\mathbf{0}\}$. Let $\mathbf{u} \in \mathcal{U}$ and let $\mathbf{G} \in \operatorname{Rng}\rho$. Then

$$\mathbf{L}(\mathbf{Gu}) = \mathbf{G}(\mathbf{Lu}) = \mathbf{0},$$

so $\mathbf{Gu} \in \mathcal{U}$. Hence the subspace $\mathcal{U}$ is ρ-invariant. Since $\{\mathbf{0}\} \subsetneqq \mathcal{U} \subsetneqq \mathcal{V}$, the representation ρ is not irreducible. The result follows. $\square$

Remark. Classifying all possible representations of a given group is one of the standard problems of group representation theory. The importance of the preceding result is that every irreducible representation ρ on $\mathcal{V}$ can immediately be put into one of three classes, depending upon whether its algebra $\mathrm{Lin}_\rho \mathcal{V}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. The irreducible representation is then said to be of *real*, *complex*, or *quaternionic type*. This is an important step towards a complete classification. The study of group representations of quaternionic type is one of, and perhaps *the*, most important application of the study of quaternions in modern mathematics.

Remark. The importance of the preceding result is more pronounced in certain important special cases such as when the group $\mathcal{G}$ has additional structure as a compact topological group and the representation ρ is required to be a continuous mapping as well as a group homomorphism. In that case, $\mathcal{V}$ can be endowed with a natural ρ-invariant inner-product and every finite-dimensional representation of $\mathcal{G}$ can be decomposed as a sum (in an appropriate sense) of irreducible representations on orthogonal subspaces of $\mathcal{V}$. Hence the overall classification of representations of such topological groups is reduced to the problem of classifying irreducible groups. It may be of interest that the demonstration of the existence of the natural inner-product just mentioned inspired the proof of the existence of a natural inner product for a scalar algebra which was provided in Sect. 5 (also, see Note 8). See [BtD85] or other standard references on representation of compact topological groups or compact Lie groups for details.

Remark. Everything just said regarding representations of compact topological groups applies to finite groups because finite groups can be regarded as topological groups with the discrete topology. In this case every mapping $\rho \colon \mathcal{G} \to \mathcal{V}$ is automatically continuous, so topological issues can be ignored. Also, the demonstration of a ρ-invariant inner product on $\mathcal{V}$ is technically simpler (though similar in spirit) because it requires only ordinary summation over the elements of a finite group instead of the more sophisticated theory of integration on a topological group or a manifold. (Compact topological groups are naturally Lie groups, hence manifolds.)

Appendix A. Integration

Throughout this section, let $\mathcal{V}$ be a finite-dimensional linear space.

Our purpose in this section is to provide a proof of Lemma 2.4, which, for convenience, we re-state as Lemma A.9 below. This material is adapted from [Nol94], Sect. 41.

Intuitively, we call a subset $\mathcal{S}$ of $\mathcal{V}$ **negligible** if it can be covered by a finite collection of *cells* whose total volume can be made arbitrarily small. When $\mathcal{V}$ is a topological space, we call a mapping $f\colon \mathcal{V} \to \mathcal{W}$ **almost continuous** if every bounded subset of the set of all discontinuities of f is negligible. (See [Nol94], Sect. 41. *Negligible sets; almost continuous functions* for precise definitions.)

We use the **notation** $\mathrm{Bnb}\,\mathcal{V}$ to denote the collection of all bounded subsets of $\mathcal{V}$ which have negligible boundary. We use the **notation** $\mathrm{Co}\,\mathcal{V}$ to denote the collection of continuous real-valued functions on $\mathcal{V}$. We use the **notation** $\mathrm{Ac}\,\mathcal{V}$ to denote the collection of almost continuous real-valued functions on $\mathcal{V}$ and the **notation** $\mathrm{Bbac}\,\mathcal{V}$ to denote the collection of almost continuous real-valued functions on $\mathcal{V}$ which have bounded range and bounded support. (The **support** of $f\colon \mathcal{V} \to \mathbb{R}$ is the set $\{\, \mathbf{x} \in \mathcal{V} \mid f(\mathbf{x}) \neq 0 \,\}$.) We use the **notation** $\mathrm{ch}_{\mathcal{S}}\colon \mathcal{V} \to \mathbb{R}$ for the **characteristic function** of the subset $\mathcal{S}$ of $\mathcal{V}$, *i.e.,* the real-valued function given by

$$\mathrm{ch}_{\mathcal{S}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in \mathcal{S}; \\ 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

We record for later reference several propositions which are either stated explicitly as results in Sect. 41 of [Nol94], or are easy consequences of results stated there.

Proposition A.1. *Every bounded convex subset of $\mathcal{V}$ has a negligible boundary and is hence a member of $\mathrm{Bnb}\,\mathcal{V}$.*

Proposition A.2. *A given subset $\mathcal{S}$ of $\mathcal{V}$ belongs to $\mathrm{Bnb}\,\mathcal{V}$ if and only if its characteristic function $\mathrm{ch}_{\mathcal{S}}$ belongs to $\mathrm{Bbac}\,\mathcal{V}$.*

Proposition A.3. *The set $\mathrm{Bbac}\,\mathcal{V}$ is a subspace of the linear space of real-valued functions on $\mathcal{V}$. The set $\mathrm{Co}\,\mathcal{V}$ is a subspace of $\mathrm{Bbac}\,\mathcal{V}$.*

Proposition A.4. *Let $\mathcal{S} \in \mathrm{Bnb}\,\mathcal{V}$ and let $f \in \mathrm{Ac}\,\mathcal{V}$ be given such that $f_{>}(\mathcal{S})$ is bounded. Then $f\,\mathrm{ch}_{\mathcal{S}} \in \mathrm{Bbac}\,\mathcal{V}$. (Here, $f\,\mathrm{ch}_{\mathcal{S}}$ is just the value-wise product of the real-valued functions f and $\mathrm{ch}_{\mathcal{S}}$.)*

The following material regarding integrals is adapted from Sect. 42, *Integrals and Volumes*, of [Nol94]. We note the use of the "dummy-variable" integration **notation** where convenient: $\int f(x)\,dx := \int f$.

(Prop. A.6 is not explicitly stated in [Nol94], but is a direct consequence of integrals being positive and non-zero.)

Definition A.5. A non-zero linear functional $\int$ on the function space $\mathrm{Bbac}\,\mathcal{V}$ is called an **integral** on $\mathcal{V}$ if it is positive in the sense that

$$f \geq 0 \quad \Rightarrow \quad \int f \geq 0 \qquad \text{for all } f \in \mathrm{Bbac}\,\mathcal{V} \tag{A.2}$$

and translation-invariant in the sense that

$$\int f(\mathbf{x}+\mathbf{v})\,d\mathbf{x} = \int f \qquad \text{for all } f \in \mathrm{Bbac}\,\mathcal{V},\ x \in \mathcal{V}. \tag{A.3}$$

Proposition A.6. *Let $\int$ be an integral on $\mathcal{V}$. Then $\int$ is strictly positive in the following sense: if $f \in \mathrm{Bbac}\,\mathcal{V}$ satisfies $f \geq 0$ and if $f(\mathbf{x}) > 0$ at some continuity point $\mathbf{x} \in \mathcal{V}$ of f, then $\int f > 0$.*

Theorem A.7. *There is an integral $\int$ on $\mathcal{V}$. It is unique up to a strictly positive scale factor in the sense that $\{\, c\int \mid c \in \mathbb{P}^{\times} \,\}$ is the collection of all integrals on $\mathcal{V}$.*

Proposition A.8. *Let $\int$ be an integral on $\mathcal{V}$ and let $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$ be a linear isomorphism. Then there is $c \in \mathbb{P}^{\times}$ such that*

$$\int f \circ \mathbf{L} = c \int f \qquad \text{for all } f \in \mathrm{Bbac}\,\mathcal{V}. \tag{A.4}$$

Lemma A.9. *Let $\mathcal{C}$ be a bounded convex subset of $\mathcal{V}$ with non-empty interior. Then there is a linear functional $\int_{\mathcal{C}}$ on the function space $\mathrm{Co}\,\mathcal{V}$ such that for all $f \in \mathrm{Co}\,\mathcal{V}$ and all $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$:*

$$f|_{\mathcal{C}} > 0 \quad \Rightarrow \quad \int_{\mathcal{C}} f > 0; \tag{A.5a}$$

$$\mathbf{L}\,\mathcal{C} = \mathcal{C} \quad \Rightarrow \quad \int_{\mathcal{C}} f \circ \mathbf{L} = \int_{\mathcal{C}} f; \tag{A.5b}$$

In (A.5a), the condition "$f|_{\mathcal{C}} > 0$" is interpreted to mean that $f(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathcal{C}$ and that $f(\mathbf{x}) > 0$ for some $\mathbf{x} \in \mathcal{C}$.

Proof. By Thm. A.7, we may choose an integral $\int$ on $\mathcal{V}$. The convex set $\mathcal{C}$ is compact, hence bounded, hence, by A.1, a member of $\mathrm{Bnb}\,\mathcal{V}$. It follows by Prop. A.2 and Prop. A.4 that $f\ \mathrm{ch}_{\mathcal{C}} \in \mathrm{Bbac}\,\mathcal{V}$ for every $f \in \mathrm{Co}\,\mathcal{V}$. It is thus meaningful to define a mapping $\int_{\mathcal{C}} \colon \mathrm{Co}\,\mathcal{C} \to \mathbb{R}$ by

$$\int_{\mathcal{C}} f := \int (f\ \mathrm{ch}_{\mathcal{C}}) \qquad \text{for all } f \in \mathrm{Co}\,\mathcal{V}.$$

The linearity of $\int_{\mathcal{C}}$ follows immediately from Def. A.5 Suppose that $f \geq> 0$ and that $f(\mathbf{x}) > 0$ for some $\mathbf{x} \in \mathcal{C}$. Since $\mathcal{C}$ is convex with non-empty interior, it is easy to show that $f(\mathbf{y}) > 0$ for some continuity point of $f\ \mathrm{ch}_{\mathcal{C}}$. Then (A.5a) follows from Prop. A.6.

Now let $\mathbf{L} \in \mathrm{Lin}\,\mathcal{V}$ be given such that $\mathbf{L}\mathcal{C} = \mathcal{C}$. Since $\mathcal{C}$ has a non-empty interior, $\mathbf{L}$ is invertible. Hence $\mathbf{L}^{<}(\mathcal{C}) = \mathcal{C}$ and $\mathrm{ch}_{\mathcal{C}} \circ \mathbf{L} = \mathrm{ch}_{\mathbf{L}^{<}(\mathcal{C})} = \mathrm{ch}_{\mathcal{C}}$. It follows by A.8 that we may choose $c \in \mathbb{P}^{\times}$ such that

$$\int (f \circ \mathbf{L})\,\mathrm{ch}_{\mathcal{C}} = \int (f\ \mathrm{ch}_{\mathcal{C}}) \circ \mathbf{L} = c \int (f\ \mathrm{ch}_{\mathcal{C}}) \qquad \text{for all } f \in \mathrm{Co}\,\mathcal{V}.$$

Applying this to $f := \mathrm{ch}_{\mathcal{C}}$, we see that $c = 1$. This establishes (A.5b). $\square$

Notes

[1] By the term **usual topology**, we mean the topology whose properties are described in [Nol87], Ch. 5. For the sake of the reader more comfortable with other developments of topological properties, we note that every norm-induced topology on a finite-dimensional linear space is identical to its *usual topology.* We do not need to deal with any other topologies or more general topological spaces in this paper.

[2] As the reader may notice, the conditions imposed in Lemma 2.4 (restated in App. A as Lemma A.9) are not sufficient to guarantee that the linear functional denoted there by "$\int_{\mathcal{C}}$" is actually "integration over $\mathcal{C}$" in the usual sense (*i.e.,* that it is induced by "restricting to $\mathcal{C}$" a translation-invariant "volumetric" integral "$\int$"). This does not affect our use of the lemma to prove Thm. 5.5, since we need only the imposed conditions. It is possible to add additional conditions to insure that "$\int_{\mathcal{C}}$" is indeed "integration over $\mathcal{C}$" in the usual sense. In fact, such integration is used in the proof of Lemma 2.4 in App. A).

[3] We use the term "algebra" here for what is sometimes called "algebra over the field $\mathbb{R}$". It is easy to show (using the term "algebra" in either sense) that an associative algebra is an algebra with reciprocals and a non-zero unity if and only it is a *division algebra* (as usually defined) and that an associative commutative algebra is an algebra with reciprocals and a non-zero unity if and only it is a *field* (see Sect. 06 of [Nol87] for the definition of a field).

[4] There is unfortunately no concise conventional name for the category of objects which exactly encompasses the rings which are ring-isomorphic to one of the "standard" rings $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ of real, complex, and quaternionic numbers, resp. Roughly, these are objects which are finite-dimensional ring-extensions (equivalently, skew-field extensions) of $\mathbb{R}$ which would qualify to be fields (see [Nol87], Sect. 06) except that their multiplicative operations are not required to be commutative. We use the term *scalar algebra* because members of objects of this category are often "scalars" in the context of the theory of linear spaces. It is common to speak of, for example, a "linear space over" $\mathbb{C}$ or $\mathbb{H}$; then members of $\mathbb{C}$ or $\mathbb{H}$ are called "scalars". Such linear spaces are of special importance in the theory of group representations (see, for example, [BtD85] and Sect. 8 of this document.)

[5] We could make this distinction more formal by defining a category of *algebras-with-bases*, and then being able to speak of isomorphisms of algebras-with-bases. It did not seem to be especially useful here, but could be when making distinctions between, for example, linear spaces over the algebra of quaternions versus linear spaces over a scalar algebra which is simply isomorphic (as an algebra) to the algebra of quaternions.

[6] We are aware that the term *units* has other meanings, but it is very convenient here.

[7] The conjugate of $\mathbf{b}$ is often denoted by $\overline{\mathbf{a}}$. In fact, conjugates of complex numbers are so denoted in [Nol87]. That notation seemed awkward in this paper.

[8] The reader familiar with the theory of representations of compact topological groups or Lie groups will recognize the argument we use to develop the natural inner product $\mathrm{ip}_{\mathcal{A}}$ on $\mathcal{A}$. The only fundamental difference between the argument used in that theory and the argument used here is that the former relies on more advanced theories of integration, namely theories of integration on manifolds or locally compact groups (see [BtD85] or [Nac65], resp.) Those theories would permit us to integrate directly on the group $\mathrm{S}_{\mathcal{A}}$ since it is easy to show that $\mathrm{S}_{\mathcal{A}}$ is a compact subgroup of the Lie group $\mathcal{A}^{\times}$. Here, because of special circumstances, we can modify the conventional arguments so as to appeal only to the more elementary theory of ordinary (*i.e.,* "volumetric") integration on finite-dimensional flat spaces.

[9] Actually, the terminology "$\mathcal{G}$-invariant" is commonly used instead of our "ρ-invariant", but we avoid the common terminology because it is ambiguous. For the same reason, we use "ρ-lineon" instead of "$\mathbf{G}$-linear operator", "$\mathrm{Lin}_\rho \mathcal{V}$" instead of "$\mathrm{Lin}_{\mathcal{G}} \mathcal{V}$", etc.

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13140 Doyles Court, Apple Valley, MN 55124, USA
E-mail address: ray.e.artz@gmail.com