On the Conjugacy of Orthogonal Groups.

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0. Introduction

In much of the mathematical literature, there is talk about the (note the definite article) orthogonal group $\mathrm{O}(n, \mathbb{F})$ of degree $n \in \mathbb{N}$ over a field $\mathbb{F}$. This is extremely misleading, because, given a linear space $\mathcal{T}$ of dimension n, one can consider the orthogonal group of any non-degenerate quadratic form $\mathbf{Q}$ on $\mathcal{T}$. In Chapter 6 of the book *Basic Algebra I* [J], Jacobson denotes this orthogonal group by $\mathrm{O}(\mathbf{Q})$. Since there are many such quadratic forms, there are many orthogonal groups. The main purpose of this paper is to study the relations between them and, in particular, determine under what conditions they are conjugate, I was able to find a complete answer in the case when $\mathbb{F}$ is an ordered field. (See Theorem 7 below.[1])

The groups $\mathrm{O}(n, \mathbb{F})$ are usually understood to consist of matrices. My approach to mathematics and mathematical science is uncompromisingly coordinate-free and matrix-free when dealing with concepts. Coordinates and matrices should be used only when dealing with special problems and numerical methods.[2] This is why I believe that $\mathrm{O}(n, \mathbb{F})$ should be put into the trash.

An orthogonal group is determined by a non-degenerate quadratic form or by the corresponding non-degenerate symmetric bilinear form. Since this last 5-word term is much too clumsy for the purposes of this paper, I replaced it by the one-word term *format*.

One could ask under what conditions the symplectic groups, determined by non-degenerate alternating bilinear forms on a given linear space, are conjugate. I believe that, in this case, there is a simple answer: They all are.

The impetus for this paper came from the fact that an infinity of formats (then called *configurations*) must be used to understand deformations in continuum physics and that the corresponding infinity of orthogonal groups are needed to understand what is meant by *material symmetry* and *isotropy*. This insight, to the best of my knowledge, was first employed by me in 1972 in [N1]. For details see Sect.8 below.

The mathematical infrastructure used here is taken from my book *Finite-Dimensional Spaces: Algebra, Geometry, and Analysis, Vol.I* [FDSI]. The first section here is a synopsis of some of the material presented in Chapters 1 and 2 of [FDSI]. In later sections, material from Chapters 3 and 8 is also used.

I use multi-letter symbols such as Sub, Lin, Sym, Lis, Frm, Orth, etc. when they can be interpreted as *isofunctors* and dim, deg, tr, det, qu, ind, sgn^{+}, etc. when they can be interpreted as *natural assignments* as explained in my paper *Isocategories and Tensor Functors* [N2].

Notation and terminology: We denote the set of all natural numbers (including zero) by $\mathbb{N}$, the set of all real numbers by $\mathbb{R}$, the set of all positive reals (including zero) by $\mathbb{P}$, and the set of all integers by $\mathbb{Z}$. The superscript $^{\times}$ is used for the process of removing the zero element from any set that contains a zero. For example, the set of all strictly positive reals (excluding zero) is denoted by $\mathbb{P}^{\times}$. The symbol $\mathbb{F}$ denotes any field in which $1 \neq -1$, i.e, not of characteristic 2. Given $i, k \in \mathbb{Z}$, we denote by $\{i..k\}$ the set of all integers

[1] The easy part of this theorem is the "if" part and may have appeared in the past literature. I believe that the "only if" part is new.

[2] For a more detailed explanation, see part F of the Introduction in my book [FDSI].

between i and k, i.e.,

$$\{i..k\} := \{j \in \mathbb{Z} \mid i \leq j \leq k\}$$

Note that $\{i..k\}$ is empty if $i > k$.

In order to specify a **mapping** $f : \mathcal{A} \longrightarrow \mathcal{B}$, one first has to prescribe two sets, $\mathcal{A}$ and $\mathcal{B}$, and then a definite procedure, called the **evaluation rule** of f, which assigns to each element $a \in \mathcal{A}$ exactly one element $f(a) \in \mathcal{B}$. The set $\mathcal{A}$ is called the **domain** of f and the set $\mathcal{B}$ is called the **codomain** of f. For every set $\mathcal{A}$ we have the **identity mapping** $1_{\mathcal{A}} : \mathcal{A} \longrightarrow \mathcal{A}$ of $\mathcal{A}$, defined by $1_{\mathcal{A}}(a) := a$ for all $a \in \mathcal{A}$.

The mapping $f : \mathcal{A} \longrightarrow \mathcal{B}$ induces a mapping $f_{>} : \text{Sub } \mathcal{A} \longrightarrow \text{Sub}\,\mathcal{B}$, from the set Sub $\mathcal{A}$ of all subsets of $\mathcal{A}$ to the set Sub $\mathcal{B}$ of all subsets of $\mathcal{B}$. It is defined by

$$f_{>}(\mathcal{U}) := \{f(u) \in \mathcal{B} \mid u \in \mathcal{U}\} \quad \text{for all} \;\; \mathcal{U} \in \text{Sub } \mathcal{A}$$

and called the **image mapping** of f. Given $\mathcal{U} \in \text{Sub } \mathcal{A}$, the *restriction* $f|_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathcal{B}$ of f to $\mathcal{U}$ is given by $f|_{\mathcal{U}}(u) := f(u) \quad \text{for all} \quad u \in \mathcal{U}$. Given, in addition, $\mathcal{V} \in \text{Sub } \mathcal{B}$ such that $f_{>}(\mathcal{U}) \subset \mathcal{V}$, the *adjustment* $f|_{\mathcal{U}}^{\mathcal{V}}$ is given by $f|_{\mathcal{U}}^{\mathcal{V}}(u) := f(u) \quad \text{for all} \quad u \in \mathcal{U}$. (This notation is needed, for example, to give a tight and precise formulation of Witt's Extension Theorem in Sect.4 below.)

When we use the term *linear space* we mean a finite-dimensional linear space over a fixed field $\mathbb{F}$.

1. Linear Algebra

Let linear spaces $\mathcal{T}_1$ and $\mathcal{T}_2$ be given. We use the notation $\text{Lin}\,(\mathcal{T}_1, \mathcal{T}_2)$ for the set of all linear mappings from $\mathcal{T}_1$ to $\mathcal{T}_2$. This set also has the structure of a linear space and $\dim \text{Lin}\,(\mathcal{T}_1, \mathcal{T}_2) = \dim(\mathcal{T}_1) \times \dim(\mathcal{T}_2)$. Given $\mathbf{L} \in \text{Lin}\,(\mathcal{T}_1, \mathcal{T}_2)$ and $\mathbf{v} \in \mathcal{T}_1$ we denote by $\mathbf{Lv}$ the element of $\mathcal{T}_2$ that $\mathbf{L}$ assigns to $\mathbf{v}$.

If $\mathbf{L}_1$ and $\mathbf{L}_2$ are both linear mappings such that the composite $\mathbf{L}_1 \circ \mathbf{L}_2$ is meaningful, we will denote this composite simply by $\mathbf{L}_1\mathbf{L}_2$. If a linear mapping $\mathbf{L}$ is invertible, we denote its inverse by $\mathbf{L}^{-1}$. We denote by $\text{Lis}\,(\mathcal{T}_1, \mathcal{T}_2)$ the set of all invertible linear mappings, i.e. **linear isomorphisms**, from $\mathcal{T}_1$ to $\mathcal{T}_2$. This set is non-empty if and only if $\dim \mathcal{T}_1 = \dim \mathcal{T}_2$.

Now let a linear space $\mathcal{T}$ be given. We use the abbreviations

$$\text{Lin } \mathcal{T} := \text{Lin}\,(\mathcal{T}, \mathcal{T}) \;\; \text{and} \;\; \text{Lis } \mathcal{T} := \text{Lis}\,(\mathcal{T}, \mathcal{T}). \tag{1.1}$$

The elements of Lin $\mathcal{T}$ will be called **lineons**[3]. The set Lis $\mathcal{T}$ of linear automorphisms of $\mathcal{T}$ forms a group with respect to composition, called the **linear group** of $\mathcal{T}$.

The **dual** of a linear space $\mathcal{T}$ is defined by

$$\mathcal{T}^* := \text{Lin}(\mathcal{T}, \mathbb{F}\,). \tag{1.2}$$

In accordance with the general rule of denoting the evaluation of linear mappings, the value of $\boldsymbol{\lambda} \in \mathcal{T}^*$ at $\mathbf{v} \in \mathcal{T}$ will be be denoted simply by $\boldsymbol{\lambda}\mathbf{v}$. The dual $\mathcal{T}^{**}$ of the dual space $\mathcal{T}^*$ will be identified with $\mathcal{T}$ in such a way that the value at $\boldsymbol{\lambda} \in \mathcal{T}^*$ of the element of $\mathcal{T}^{**}$ identified with $\mathbf{v} \in \mathcal{T}$ is $\mathbf{v}\boldsymbol{\lambda} := \boldsymbol{\lambda}\mathbf{v}$. We have $\dim \mathcal{T}^* = \dim \mathcal{T}$.

Given $\mathbf{v} \in \mathcal{T}_2$ and $\boldsymbol{\lambda} \in \mathcal{T}_1^*$ we define the **tensor product** $\mathbf{v} \otimes \boldsymbol{\lambda} \in \text{Lin}\,(\mathcal{T}_1, \mathcal{T}_2)$ of $\mathbf{v}$ and $\boldsymbol{\lambda}$ by

$$(\mathbf{v} \otimes \boldsymbol{\lambda})\mathbf{u} := (\boldsymbol{\lambda}\mathbf{u})\mathbf{v} \quad \text{for all} \;\; \mathbf{u} \in \mathcal{T}_1. \tag{1.3}$$

The dual of Lin $\mathcal{T}$ contains a special element $\text{tr} \in (\text{Lin } \mathcal{T})^*$ called the **trace** which is characterized by the property

$$\text{tr}(\mathbf{v} \otimes \boldsymbol{\lambda}) = \boldsymbol{\lambda}\mathbf{v} \quad \text{for all} \;\; \mathbf{v} \in \mathcal{T}, \boldsymbol{\lambda} \in \mathcal{T}^*. \tag{1.4}$$

[3]short for "linear transformations"

Another mapping of interest is the **determinant** $\det : \text{Lin}\ \mathcal{T} \longrightarrow \mathbf{R}$.[4] We use the notation

$$\text{Unim}\,\mathcal{T} := \{\mathbf{L} \in \text{Lis}\ \mathcal{V} \mid \det \mathbf{L} = \pm 1\} \tag{1.5}$$

for the **unimodular group**, which is a subgroup of Lis $\mathcal{T}$, and the notation

$$\text{Unim}^{+}\ \mathcal{T} := \{\mathbf{L} \in \text{Lis}\ \mathcal{V} \mid \det \mathbf{L} = 1\} \tag{1.6}$$

for the **proper unimodular group**, which is a subgroup of Unim $\mathcal{T}$.

To every $\mathbf{L} \in \text{Lin}\,(\mathcal{T}_1, \mathcal{T}_2)$ one can associate exactly one element $\mathbf{L}^\top \in \text{Lin}\,(\mathcal{T}_2^*, \mathcal{T}_1^*)$, called the **transpose** of $\mathbf{L}$, characterized by the condition that

$$\boldsymbol{\lambda}(\mathbf{L}\mathbf{v}) = (\mathbf{L}^\top \boldsymbol{\lambda})\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathcal{T}_1,\ \boldsymbol{\lambda} \in \mathcal{T}_2^*. \tag{1.7}$$

The space $\text{Lin}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the space of all bilinear forms on $\mathcal{T}$. The identification is expressed by

$$\mathbf{G}(\mathbf{u}, \mathbf{v}) = (\mathbf{G}\mathbf{u})\mathbf{v} \qquad \text{for all} \qquad \mathbf{u}, \mathbf{v} \in \mathcal{T}, \quad \mathbf{G} \in \text{Lin}\,(\mathcal{T}, \mathcal{T}^*)\ . \tag{1.8}$$

The subspace

$$\text{Sym}\,(\mathcal{T}, \mathcal{T}^*) := \{\mathbf{G} \in \text{Lin}\,(\mathcal{T}, \mathcal{T}^*) \mid \mathbf{G}^\top = \mathbf{G}\} \tag{1.9}$$

of $\text{Lin}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the space of all symmetric bilinear forms on $\mathcal{T}$. The subspace

$$\text{Skew}\,(\mathcal{T}, \mathcal{T}^*) := \{\mathbf{W} \in \text{Lin}\,(\mathcal{T}, \mathcal{T}^*) \mid \mathbf{W}^\top = \mathbf{W}\} \tag{1.10}$$

of $\text{Lin}\,(\mathcal{T}, \mathcal{T}^*)$ will be identified with the space of all alternating bilinear forms on $\mathcal{T}$.

Given $\mathbf{G} \in \text{Sym}\,(\mathcal{T}, \mathcal{T}^*)$ we consider the mapping qu$\mathbf{G}$ from $\mathcal{T}$ to $\mathbb{F}$ defined by

$$\text{qu}\mathbf{G}(\mathbf{u}) := (\mathbf{G}\mathbf{u})\mathbf{u} \qquad \text{for all} \quad \mathbf{u} \in \mathcal{T} \tag{1.11}$$

and call it the **quadratic form** corresponding to $\mathbf{G}$. By Prop.2 of Sect.27 in [FDSI], this mapping is linear and injective. We denote its range, i.e., the set of all quadratic forms on $\mathcal{T}$, by Qu $\mathcal{T}$, so that

$$\text{qu} : \text{Sym}\,(\mathcal{T}, \mathcal{T}^*) \longrightarrow \text{Qu}\ \mathcal{T} \tag{1.12}$$

is invertible.

2. Formats and Forms

Again, we assume that a linear space $\mathcal{T}$ is given. We use the notation

$$\text{Fmt}\ \mathcal{T} := \text{Sym}\,(\mathcal{T}, \mathcal{T}^*) \cap \text{Lis}\,(\mathcal{T}, \mathcal{T}^*)\ . \tag{2.1}$$

for the set of all symmetric linear isomorphisms from $\mathcal{T}$ to $\mathcal{T}^*$ and call its members **formats** of $\mathcal{T}$. This set is identified with the set of all non-degenerate symmetric bilinear forms on $\mathcal{T}$. For every format $\mathbf{G} \in \text{Fmt}\ \mathcal{T}$, we say that the corresponding quadratic form $\text{qu}\mathbf{G} : \mathcal{T} \longrightarrow \mathbb{F}$ is **non-degenrate** and call it simply the **form** of $\mathbf{G}$.

We have

$$c\mathbf{G} \in \text{Fmt}\ \mathcal{T} \qquad \text{for all} \qquad \mathbf{G} \in \text{Fmt}\ \mathcal{T}\ ,\ c \in \mathbb{F}^{\times}\ . \tag{2.2}$$

and

$$\gamma_{\mathbf{A}}(\mathbf{G}) := \mathbf{A}^\top \mathbf{G} \mathbf{A} \in \text{Fmt}\ \mathcal{T} \qquad \text{for all} \quad \mathbf{G} \in \text{Fmt}\ \mathcal{T}\ ,\ \mathbf{A} \in \text{Lis}\,\mathcal{T}\ . \tag{2.3}$$

The mapping $\gamma : \text{Lis}\ \mathcal{T} \longrightarrow \text{Perm}\,\text{Fmt}\ \mathcal{T}$ defined by (2.3), is an action, as defined by Def.1 in Sect.31 of [FDSI], of the group Lis $\mathcal{T}$ on the the set of all formats. We have

$$\text{qu}(\gamma_{\mathbf{A}}(\mathbf{G})) = \text{qu}\mathbf{G} \circ \mathbf{A} \qquad \text{for all} \quad \mathbf{G} \in \text{Fmt}\ \mathcal{T}\ ,\ \mathbf{A} \in \text{Lis}\,\mathcal{T}\ . \tag{2.4}$$

[4] For a matrix-free definition see Sect.14 of [FDSII].

We say that a subspace $\mathcal{U}$ is **totally singular** relative to a given $\mathbf{G}$ if

$$\mathrm{qu}\mathbf{G}|_{\mathcal{U}} = 0 \quad \text{, i.e.,} \quad \mathrm{qu}\mathbf{G}(\mathbf{u}) = 0 \qquad \text{for all} \qquad \mathbf{u} \in \mathcal{U} \ . \tag{2.5}$$

Let $\mathbf{G} \in \mathrm{Fmt}\,\mathcal{T}$ and $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ be given. It follows from (2.4) and (2.5) that a subspace $\mathcal{U}$ is totally singular relative to a given $\mathbf{G}$ if and only if $\mathbf{A}_{>}(\mathcal{U})$ is totally singular relative to $\gamma_{\mathcal{A}}(\mathbf{G})$.

The **index**[5] of the format $\mathbf{G}$ is defined to be the maximum dimension of all totally singular subspaces of $\mathcal{T}$, and it is denoted by

$$\mathrm{ind}(\mathbf{G}) := \max\{\dim \mathcal{U} \mid \mathcal{U} \text{ is a subspace of } \mathcal{T} \text{ such that } \mathrm{qu}\mathbf{G}|_{\mathcal{U}} = 0\} \ . \tag{2.6}$$

It is easily seen that

$$\mathrm{ind}(\gamma_{\mathbf{A}}(\mathbf{G})) = \mathrm{ind}(\mathbf{G}) \quad \text{for all} \quad \mathbf{G} \in \mathrm{Fmt}\,\mathcal{T} \ , \ \mathbf{A} \in \mathrm{Lis}\,\mathcal{T}. \tag{2.7}$$

We now assume that a format $\mathbf{G} \in \mathrm{Fmt}\,\mathcal{T}$ is given. We use the notations

$$\mathrm{Sym}_{\mathbf{G}}\mathcal{T} := \{\mathbf{G}^{-1}\mathbf{S} \mid \mathbf{S} \in \mathrm{Sym}\,(\mathcal{T}, \mathcal{T}^*)\} \ = \{\mathbf{T} \in \mathrm{Lin}\,\mathcal{T} \mid \mathbf{T}^\top\mathbf{G} = \mathbf{G}\mathbf{T} \ \} \tag{2.8}$$

and

$$\mathrm{Skew}_{\mathbf{G}}\mathcal{T} := \{\mathbf{G}^{-1}\mathbf{S} \mid \mathbf{S} \in \mathrm{Skew}\,(\mathcal{T}, \mathcal{T}^*)\} \ = \{\mathbf{T} \in \mathrm{Lin}\,\mathcal{T} \mid \mathbf{T}^\top\mathbf{G} = -\mathbf{G}\mathbf{T} \ \} \tag{2.9}$$

for the set of all lineons that are **symmetric relative to G** or **skew relative to G**, respectively. Both are subspaces of Lin $\mathcal{T}$ and the Additive Decomposition Theorem (see Sect.89 of [FDSI]) remains valid in this context.

We say that two subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{T}$ are **orthogonal relative to G** if

$$(\mathbf{G}\mathbf{u})\mathbf{v} = \mathbf{0} \qquad \text{for all} \qquad \mathbf{u} \in \mathcal{A}, \ \mathbf{v} \in \mathcal{B} \ . \tag{2.10}$$

We say that a family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{T}$ is **orthogonal relative to G** if its terms are pairwise orthogonal relative to $\mathbf{G}$.

Put $n := \dim \mathcal{T}$. Given $k \in \{0..n\}$ we say that a basis $(\mathbf{b}_i \mid i \in \{1..n\})$ of $\mathcal{T}$ is **G-orthonormal of minus-signature** k if

$$(\mathbf{G}\mathbf{b}_i)\mathbf{b}_j = \begin{cases} 0 & \text{if} \quad i \neq j \ , \\ -1 & \text{if} \quad i = j \in \{1..k\} \ , \\ 1 & \text{if} \quad i = j \in \{(k+1)..n\} \ . \end{cases} \tag{2.11}$$

We say an orthonormal basis in **genuine** if it is of minus-signature 0.

We say that the format $\mathbf{G}$ is **regular** if it admits orthonormal bases. We say k is a **minus-signature** of $\mathbf{G}$ if there is a $\mathbf{G}$-orthonormal basis of minus-signature k. We say that the format $\mathbf{G}$ is **genuine** if 0 is a minus-signsture of $\mathbf{G}$, i.e., if there is genuine $\mathbf{G}$-orthonormal basis.

Remark:. In the case when $\mathbb{F}$ is an ordered field, every format is regular and has exactly one minus-signature, as we will see in the Section 3 below. In the case when $\mathbb{F}$ contains and element whose square is -1, for example when it is the field of complex numbers, then every number in $\{0..n\}$ is a minus-signature of every regular format. If $\mathbb{F}$ is arbitrary, I do not know whether there can be non-regular formats or what the set of minus-signatares for regular formats can be. ∎

Proposition 1: *Given any basis $(\mathbf{b}_i \mid i \in \{1..n\})$ of $\mathcal{T}$ and and $k \in \{0,,n\}$, there is at least one format $\mathbf{G} \in \mathrm{Fmt}\,\mathcal{T}$ such that this basis is $\mathbf{G}$-orthonormal of minus-signature k. It is given by*

$$\mathbf{G} := -\sum_{i \in \{1,,k\}} \mathbf{b}_i^* \otimes \mathbf{b}_i^* + \sum_{i \in \{k+1,,n\}} \mathbf{b}_i^* \otimes \mathbf{b}_i^* \ , \tag{2.12}$$

[5] a.k.a *Witt index*

where the basis $(\mathbf{b}_i^* \mid i \in \{1..n\})$ *of* $\mathcal{T}^*$ *is the dual of the given basis.*

Proof: Using the definition (1.3) and the fact that the dual basis is characterized by $\mathbf{b}_i^* \mathbf{b}_j = \begin{cases} 0 & \text{if} \quad i \neq j \ , \\ 1 & \text{if} \quad i = j, \end{cases}$ (see Sect.23 in [FDSI]), it is clear that $\mathbf{G}$, as defined by (2.12), has the desired property. ▮

Proposition 2: *Let* $\mathbf{G} \in \text{Fmt}\,\mathcal{T}$ *be a format of minus-signature* $k \in \{0..n\}$. *Then another format* $\mathbf{G}' \in \text{Fmt}\,\mathcal{T}$ *is also a format of minus-signature* k *if and only if there is a* $\mathbf{A} \in \text{Lis}\,\mathcal{T}$ *such that* $\mathbf{G}' = \gamma_{\mathbf{A}}(\mathbf{G})$.

Proof: Chose a basis $(\mathbf{b}_i \mid i \in \{1..n\})$ of $\mathcal{T}$ that is $\mathbf{G}$-orthonormal of minus-signature k.

Let $\mathbf{A} \in \text{Lis}\,\mathcal{T}$ be given and put $\mathbf{G}' = \gamma_{\mathcal{A}}(\mathbf{G})$. an easy calculation shows that $(\mathbf{A}\mathbf{b}_i \mid i \in \{1..n\})$ is $\mathbf{G}'$-orthonormal of minus-signature k.

Now assume that $\mathbf{G}' \in \text{Fmt}\,\mathcal{T}$ is a format of minus-signature k and chose a basis $(\mathbf{c}_i \mid i \in \{1..n\})$ of $\mathcal{T}$ that is $\mathbf{G}'$-orthonormal of minus-signature k. There is exactly one $\mathbf{A} \in \text{Lis}\,\mathcal{T}$ such that $\mathbf{c}_i = \mathbf{A}\mathbf{b}_i$ for all $i \in \{1..n\}$ and hence $\mathbf{G}' = \gamma_{\mathcal{A}}(\mathbf{G})$. ▮

3. Formats of linear spaces over ordered fields.

Now we consider the case when $\mathcal{T}$ is a linear space over an ordered field $\mathbb{F}$, for example the field of rational numbers or real numbers.

Let $\mathbf{G} \in \text{Fmt}\,\mathcal{T}$ be given. We say that a subspace $\mathcal{U}$ of $\mathcal{T}$ is

positive regular relative to G if $\text{qu}\mathbf{G}|_{\mathcal{U}}$ is strictly positive, i.e., if

$$\text{qu}\mathbf{G}(\mathbf{v}) > 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathcal{U}^{\times} \ , \tag{3.1}$$

negative reguar relative to G if $\text{qu}\mathbf{G}|_{\mathcal{U}}$ is strictly negative, i.e., if

$$\text{qu}\mathbf{G}(\mathbf{v}) < 0 \qquad \text{for all} \qquad \mathbf{v} \in \mathcal{U}^{\times} \ . \tag{3.2}$$

The greatest among the dimensions of all positive [negative] regular spaces relative to $\mathbf{G}$ is denoted by $\text{sig}^{+}\mathbf{G}$ [$\text{sig}^{-}\mathbf{G}$], respectively. The pair $(\text{sig}^{+}\mathbf{G},\ \text{sig}^{-}\mathbf{G})$ is called the **signature** of the format $\mathbf{G}$. Recall (see (2.6)) that the greatest among the dimensions of all subspaces that are totally singular relative to $\mathbf{G}$ is denoted by $\text{ind}\,\mathbf{G}$ and is called the index of $\mathbf{G}$. Also recall the definition of minus-signatures given in the previous section.

Theorem 1: *Every format is regular and has exactly one minus-signature. Given* $\mathbf{G} \in \text{Fmt}\,\mathcal{T}$*, the minus-signature of* $\mathbf{G}$ *is* $\text{sig}^{-}\mathbf{G}$. *We have*

$$\text{sig}^{+}\mathbf{G} + \text{sig}^{-}\mathbf{G} = \dim\,\mathcal{T} \ , \tag{3.3}$$

and

$$\text{ind}\,\mathbf{G} = \min\{\text{sig}^{+}\mathbf{G},\ \text{sig}^{-}\mathbf{G}\} \ . \tag{3.4}$$

This Theorem is a reformulation and condensation of the results of Sect.47 of [FDSI] for the present context. There it is assumed that the field $\mathbb{F}$ is the real field $\mathbb{R}$, but the proofs are valid for any ordered field. The Theorem is also equivalent to what is often called “Sylvester’s Law of Inertia”.

Corollary : *A number* k *is the index of some* $\mathbf{G} \in \text{Fmt}\,\mathcal{T}$ *if and only if* $k \in \{0..\frac{n}{2}\}$ if n is even or $k \in \{0..\frac{n-1}{2}\}$ if n is odd .

The subset

$$\mathrm{Pos}^+(\mathcal{T}) := \{\mathbf{G} \in \mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*) \mid (\mathbf{Gv})\mathbf{v} = \mathrm{qu}\mathbf{G}(\mathbf{v}) > 0 \quad \text{for all} \quad \mathbf{v} \in \mathcal{T}^\times\} \tag{3.5}$$

of $\mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*)$ will be identified with the set of all strictly positive bilinear forms. It is a **linear cone**[6] in $\mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*)$, but not a subspace.

Proposition 3: *the cone* $\mathrm{Pos}^+(\mathcal{T})$ *coincides with the set of all genuine formats in* $\mathrm{Fmt}\,(\mathcal{T})$.

Proof: It is easily seen that $\mathrm{Pos}^+(\mathcal{T}) \subset \mathrm{Lis}\,(\mathcal{T},\mathcal{T}^*)$ and hence, by (2.1), that $\mathrm{Pos}^+(\mathcal{T}) \subset \mathrm{Fmt}\,\mathcal{T}$. Let $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$ be given. By (3.1) $\mathbf{G}$ is positive regular. Hence, by Thm.1 above , we have $\mathrm{sig}^-\mathbf{G} = 0$, and the minus-signature of $\mathbf{G}$ is zero, which means that it is genuine. ∎

The following result is an immediate consequence of Props.2 and 3.

Proposition 4: *Given any* $\mathbf{G}, \mathbf{G}' \in \mathrm{Pos}^+(\mathcal{T})$ *there is an* $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ *such that* $\mathbf{G}' = \gamma_{\mathcal{A}}(\mathbf{G})$.

Now let $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$ be given. We use the notation

$$\mathrm{Pos}^+_{\mathbf{G}}\mathcal{T} = \{\mathbf{G}^{-1}\mathbf{H} \mid \mathbf{H} \in \mathrm{Pos}^+(\mathcal{T})\} = \{\mathbf{S} \in \mathrm{Sym}_{\,\mathbf{G}}\mathcal{T} \mid \mathbf{GS} \in \mathrm{Pos}^+(\mathcal{T})\} \tag{3.6}$$

for the set of all lineons that are **symmetric and strictly positive** relative to $\mathbf{G}$.

We now assume that $\mathcal{T}$ is a linear space over the real field $\mathbf{R}$. All the considerations of Chap.8 of [FDSI] can then be relativised to genuine formats. An example is the following

Theorem 2 (Spectral Theorem): *A lineon is symmetric relative to a given format* $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$ *if and only if the family of its spectral spaces is a decomposition of* $\mathcal{T}$ *that is orthogonal relative to* $\mathbf{G}$.

Now we no longer assume that a genuine format is given, but find how such a format can be obtained to satisfy given conditions.

From Prop.1 of Sect.2 it follow that for any basis $(\mathbf{b}_i \mid i \in \{1..n\})$ of $\mathcal{T}$, there is a genuine format $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$ such that $(\mathbf{b}_i \mid i \in \{1..n\})$ is genuinely $\mathbf{G}$ -orthonormal.

Theorem 3 (Inverse Spectral Theorem): *The family of spectral spaces of a given lineon is a decomposition of* $\mathcal{T}$ *if and only if the lineon is symmetric relative to some format* $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$.

We say that a lineon is **diagonable** if its matrix relative to some basis is diagonal.

Corollary: *A lineon is diagonable if and only if it is symmetric relative to some* $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$.

Theorem 4: *A lineon* $\mathbf{P}$ *belongs to* $\mathrm{Pos}^+_{\mathbf{G}}\mathcal{T}$ *for some configuration* $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$ *if and only if* $\mathrm{Spec}\,\mathbf{P} \subset \mathbb{P}^\times$ *and the family of its spectral spaces is a decomposition of* $\mathcal{T}$.

Remark: In most of the literature, an **inner-product space** $\mathcal{V}$ is defined to be a linear space endowed with additional structure by singling out a specific element $\mathrm{ip} \in \mathrm{Pos}^+(\mathcal{V},\mathcal{V}^*)$, called the **inner-product**. This inner-product is used to identify the linear space $\mathcal{V}$ with its dual $\mathcal{V}^*$. It is then customary to use the notation

$$\mathbf{v}\cdot\mathbf{u} := (\mathrm{ip}\mathbf{v})\mathbf{u} \quad \text{for all} \;\; \mathbf{v},\mathbf{u} \in \mathcal{V}.$$

The **magnitude** $|\mathbf{u}|$ of an element $\mathbf{u} \in \mathcal{V}$ is defined by $|\mathbf{u}| := \sqrt{\mathbf{u}\cdot\mathbf{u}}$.

If $\mathcal{T}$ is just a linear space over $\mathbf{R}$ without inner product then the entire theory of inner-product spaces can be applied relative to any $\mathbf{G} \in \mathrm{Pos}^+(\mathcal{T})$.

In [FDSI], the term *inner product* is also used when $\mathcal{V}$ is a linear space endowed with additional structure by singling out a specific element $\mathrm{ip} \in \mathrm{Fmt}\,\mathcal{V}$, not necessarily genuine. If not genuine, it is called a *double-signed* inner product. ∎

[6]For an analysis of linear cones in general, and of the present ones in particular, see the paper [NS] by Noll and Schäffer.

4. Orthogonal Groups

Let a linear space $\mathcal{T}$ over a field $\mathbb{F}$ and a format $\mathbf{G} \in \mathrm{Fmt}\ \mathcal{T}$ be given. We define the **orthogonal group** of $\mathbf{G}$ by

$$\mathrm{Orth}\ \mathbf{G} := \{\mathbf{R} \in \mathrm{Lis}\ \mathcal{T} \mid \gamma_{\mathbf{R}}(\mathbf{G}) = \mathbf{R}^{\top}\mathbf{G}\mathbf{R} = \mathbf{G}\}. \qquad (4.1)\ ,$$

The following result is an immediate consequence of the definition and properties of determinants.

Proposition 5: Orth $\mathbf{G}$ *is a subgroup of the unimodular group* Unim $\mathcal{T}$. *We have*

$$\mathrm{Orth}\ \mathbf{G} = \{\mathbf{R} \in \mathrm{Lis}\ \mathcal{T} \mid (\mathrm{qu}\mathbf{G}) \circ \mathbf{R} = \mathbf{G}\}\ . \qquad (4,2)$$

and

$$\mathrm{Orth}\ c\mathbf{G} = \mathrm{Orth}\ \mathbf{G} \quad \text{ for all } \quad c \in \mathbb{F}^{\times}\ . \qquad (4.3)$$

The group

$$\mathrm{Orth}^{+}\mathbf{G} := \mathrm{Orth}\ \mathbf{G}\ \cap \mathrm{Unim}^{+}\ \mathcal{T} \qquad (4.4)$$

is called the **proper orthogonal group**[7] of $\mathbf{G}$.

We say that the orthogonal group Orth $\mathbf{G}$ is **genuine** if $\mathbf{G}$ is genuine.

Proposition 6: *For every* $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$, *the orthogonal group of* $\gamma_{\mathbf{A}}(\mathbf{G})$ *is conjugate to the orthogonal group of* $\mathbf{G}$. *More precisely, we have*

$$\mathrm{Orth}\ (\gamma_{\mathbf{A}}(\mathbf{G})) = \mathbf{A}^{-1}(\mathrm{Orth}\ \mathbf{G})\mathbf{A} := \{\mathbf{A}^{-1}\mathbf{R}\mathbf{A} \mid \mathbf{R} \in \mathrm{Orth}\ \mathbf{G}\}\ . \qquad (4.5)$$

Proof: Let $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ and $\mathbf{R} \in \mathrm{Orth}\ \mathbf{G}$ be given. Then, observing (2.3) and (4.1), we obtain

$$(\mathbf{A}^{-1}\mathbf{R}\mathbf{A})^{\top}(\gamma_{\mathcal{A}}(\mathbf{G}))(\mathbf{A}^{-1}\mathbf{R}\mathbf{A}) = (\mathbf{A}^{-1}\mathbf{R}\mathbf{A})^{\top}(\mathbf{A}^{\top}\mathbf{G}\mathbf{A})(\mathbf{A}^{-1}\mathbf{R}\mathbf{A}) =$$

$$\mathbf{A}^{\top}\mathbf{R}^{\top}(\mathbf{A}^{-\top}\mathbf{A}^{\top})\mathbf{G}(\mathbf{A}\mathbf{A}^{-1})\mathbf{R}\mathbf{A} = \mathbf{A}^{\top}\mathbf{R}^{\top}\mathbf{G}\mathbf{R}\mathbf{A} = \mathbf{A}^{\top}\mathbf{G}\mathbf{A} = \gamma_{\mathcal{A}}(\mathbf{G})$$

Since $\mathbf{R} \in \mathrm{Orth}\ \mathbf{G}$ was arbitrary, this shows that

$$\mathrm{Orth}\ (\gamma_{\mathbf{A}}(\mathbf{G})) \supset \mathbf{A}^{-1}(\mathrm{Orth}\ \mathbf{G})\mathbf{A}\ .$$

The reverse inclusion follows by interchanging the roles of $\mathbf{G}$ and $\gamma_{\mathbf{A}}(\mathbf{G})$. ∎

Remark 1: If $\dim \mathcal{T} = 1$ then $\mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*) = \mathrm{Lin}\,(\mathcal{T},\mathcal{T}^*)$, $\dim \mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*) = 1$ and $\mathrm{Fmt}\ \mathcal{T} = (\mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*))^{\times}$. All formats are regular and any two of them differ only by a non-zero factor. Lin $\mathcal{T}$ can be identified with the field $\mathbb{F}$ and the linear group Lis $\mathcal{T}$ with multiplicative group $\mathbb{F}^{\times}$. In this case, the unimodular group Unim$\mathcal{T}$ is identified with the two-element subgroup $\{1,-1\}$ of $\mathbb{F}^{\times}$ and the proper unimodular group $\mathrm{Unim}^{+}\mathcal{T}$ with the trivial group $\{1\}$. There is only one orthogonal group and it coincides with the unimodular group $\{1,-1\}$. The only proper orthogonal group is the trivial group $\{1\}$. Caution: The spaces $\mathcal{T}$ and $\mathrm{Sym}\,(\mathcal{T},\mathcal{T}^*)$, although one-dimensional, should <u>not</u> be identified with $\mathbb{F}$ because they do not have distinctive elements corresponding to 1 and -1. ∎

Remark 2: Note that $\mathrm{Skew}\,(\mathcal{T},\mathcal{T}^*) \cap \mathrm{Lis}\,(\mathcal{T},\mathcal{T}^*)$ is not empty if and only if the dimension of $\mathcal{T}$ is even. Let $\mathbf{W} \in \mathrm{Skew}\,(\mathcal{T},\mathcal{T}^*) \cap \mathrm{Lis}\,(\mathcal{T},\mathcal{T}^*)$ be given, which can be identified with a non-degerate alternating bilinear form. The corresponding **symplectic group** is defined by

$$\mathrm{Sp}\,\mathbf{W} := \{\mathbf{S} \in \mathrm{Lis}\ \mathcal{T} \mid \mathbf{S}^{\top}\mathbf{W}\mathbf{S} = \mathbf{W}\}\ ,$$

in analogy for (4.1). ∎

[7] a.k.a. *special orthogonal group*

Theorem 5 (Witt's Extension theorem): *Given subspaces* $\mathcal{U}$ *and* $\mathcal{U}'$ *of* $\mathcal{T}$ *and* $\mathbf{L} \in \mathrm{Lis}\,(\mathcal{U},\mathcal{U}')$ *such that*

$$(\mathrm{qu}\mathbf{G}|_{\mathcal{U}'}) \circ \mathbf{L} = \mathrm{qu}\mathbf{G}|_{\mathcal{U}} \ , \tag{4.6}$$

there exists an $\mathbf{R} \in \mathrm{Orth}\ \mathbf{G}$ *such that* $\mathbf{R}_{>}(\mathcal{U}) = \mathcal{U}'$ *and* $\mathbf{R}|_{\mathcal{U}}^{\mathcal{U}'} = \mathbf{L}$.

A proof of this Theorem can be found in Sect.6.5 of the book [J] by Jacobson.

From now on we assume that there is a $c \in \mathbb{F}$ such that $c^2 \neq 1$, i.e that the characteristic of $\mathbb{F}$ is neither 2 nor 3.

Theorem 6: *A given subspace* $\mathcal{U}$ *of* $\mathcal{T}$ *is totally singular relative to* $\mathbf{G}$ *if and only if*

$$\mathrm{Lis}\,\mathcal{U} = \{\mathbf{R}|_{\mathcal{U}}^{\mathcal{U}} \mid \mathbf{R} \in \mathrm{Orth}\ \mathbf{G} \ \text{ and } \ \mathbf{R}_{>}(\mathcal{U}) = \mathcal{U}\} \ . \tag{4.7}$$

Proof: It is clear that the right side of (4.7) is included in the left side even if the subspace $\mathcal{U}$ of $\mathcal{T}$ is not totally singular relative to $\mathbf{G}$.

Assume that $\mathcal{U}$ is totally singular. It follows from (2.4) that (4.6) holds with $\mathcal{U}' = \mathcal{U}$ because both sides are zero. Hence, by Theorem 5, with $\mathcal{U}' = \mathcal{U}$, the left side of (4.7) is included in the right side.

Now assume that the left side of (4.7) is included in the right side. Choose $c \in \mathbb{F}$ such that $c^2 \neq 1$. Of course, $c\mathbf{1}_{\mathcal{U}} \in \mathrm{Lis}\,\mathcal{U}$. By Theorem 5 again, we can then choose $\mathbf{R} \in \mathrm{Orth}\ \mathbf{G}$ such that $\mathbf{R}_{>}(\mathcal{U}) = \mathcal{U}$ and $\mathbf{R}|_{\mathcal{U}}^{\mathcal{U}} = c\mathbf{1}_{\mathcal{U}}$. Let $\mathbf{u} \in \mathcal{U}$ be given. Since $\mathbf{R} \in \mathrm{Orth}\ \mathbf{G}$ it follows from (4.2) that

$$(c^2\mathrm{qu}\mathbf{G})(\mathbf{u}) = (\mathrm{qu}\mathbf{G})(c\mathbf{u}) = (\mathrm{qu}\mathbf{G})(\mathbf{R}\mathbf{u}) = (\mathrm{qu}\mathbf{G})(\mathbf{u}) \ .$$

Since $\mathbf{u} \in \mathcal{U}$ was arbitrary, we conclude that $(c^2 - 1)\mathrm{qu}\mathbf{G})|_{\mathcal{U}} = 0$. Since $(c^2 - 1) \neq 0$, it follows that $\mathrm{qu}\mathbf{G}|_{\mathcal{U}} = 0$, which means that $\mathcal{U}$ is totally singular. ▮

Corollary: *The index of the format* $\mathbf{G}$ *is the maximum of the dimensions of subspaces of* $\mathcal{T}$ *for which (4.7) holds. Hence if the orthogonal groups of two formats are the same, they must have the same index.*

From now on we assume, as Sect.3, that $\mathcal{T}$ is a linear space over an ordered field $\mathbb{F}$, for example the field of rational numbers or real numbers.

Theorem 7: *Let* $\mathbf{G}_1, \mathbf{G}_2 \in \mathrm{Fmt}\ \mathcal{T}$ *be given. Then* $\mathrm{Orth}\ \mathbf{G}_1$ *is conjugate to* $\mathrm{Orth}\ \mathbf{G}_2$ *if and only if* $\mathrm{ind}(\mathbf{G}_1) = \mathrm{ind}(\mathbf{G}_2)$.

Proof: Assume that $\mathrm{ind}(\mathbf{G}_1) = \mathrm{ind}(\mathbf{G}_2)$. Using Thm.1, we conclude that the minus-signature of $\mathbf{G}_2$ is equal to the minus-signature of $\mathbf{G}_1$ or the minus-signature of $-\mathbf{G}_1$. Hence, by Prop.2, we can choose $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ such that $\mathbf{G}_2 = \pm\gamma_{\mathcal{A}}(\mathbf{G})$. It follows from Prop.5, (4.3), and from Prop.6 that $\mathrm{Orth}\ \mathbf{G}_2 = \mathbf{A}^{-1}(\mathrm{Orth}\ \mathbf{G}_1)\mathbf{A}$ and hence that $\mathrm{Orth}\,\mathbf{G}_1$ is conjugate to $\mathrm{Orth}\,\mathbf{G}_2$.

Assume now that $\mathrm{Orth}\,\mathbf{G}_1$ is conjugate to $\mathrm{Orth}\,\mathbf{G}_2$ and choose $\mathbf{A} \in \mathrm{Lis}\,\mathcal{T}$ such that $\mathrm{Orth}\ \mathbf{G}_2 = \mathbf{A}^{-1}(\mathrm{Orth}\ \mathbf{G}_1)\mathbf{A}$. By Prop.6 it follows that $\mathrm{Orth}\ \mathbf{G}_2 = \mathrm{Orth}\ (\gamma_{\mathbf{A}}\mathbf{G})$. In view of (2.7), it follow from the Corollary to Thm.6 that $\mathrm{ind}(\mathbf{G}_1) = \mathrm{ind}(\mathbf{G}_2)$. ▮

The following result is a consequence of the Theorem above and the Corollary to Thm.1.

Corrollary: If n is even, there are $1 + \frac{n}{2}$ conjugacy classes of orthogonal groups, one for each index in $\{0..(\frac{n}{2})\}$. If n is odd, there are $1 + (\frac{n-1}{2})$ such classes , one for each index in $\{0..(\frac{n-1}{2})\}$.

From now on we assume that $\mathcal{T}$ is a linear space over the field $\mathbf{R}$ of real numbers, so that, by Prop.3, the set of genuine formats is the cone $\mathrm{Pos}^{+}(\mathcal{T})$.

Remark: If $\dim\mathcal{T} \geq 2$, then every genuine orthogonal group is a maximal subgroup of $\mathrm{Unim}\mathcal{T}$ and every genuine proper orthogonal group is a maximal subgroup of $\mathrm{Unim}^{+}\mathcal{T}$. The first of these results is highly non-trivial, and a proof is given in my paper [N3]. The second result is an easy consequence of the first. It is not known to me whether these results remain valid for non-genuine orthogonal groups or for the case when the field $\mathbb{F}$ is not the real field. ▮

Theorem 8: *Let a pair* $(\mathbf{G}_1, \mathbf{G}_2)$ *of genuine formats be given. Then there is exactly one* $\mathbf{U} \in \mathrm{Pos}^+_{\mathbf{G}_1}\mathcal{T} \cap \mathrm{Pos}^+_{\mathbf{G}_2}\mathcal{T}$ *such that*

$$\mathbf{G}_2 = \mathbf{G}_1\mathbf{U}^2 = \mathbf{U}^\top\mathbf{G}_1\mathbf{U} = \gamma_{\mathbf{U}}(\mathbf{G}_1) \ . \tag{4.8}$$

We have

$$\mathrm{Orth}\ \mathbf{G}_2 = \mathbf{U}^{-1}(\mathrm{Orth}\ \mathbf{G}_1)\mathbf{U} \ . \tag{4.9}$$

Proof: It is easily seen that

$$(\mathbf{G}_1)^{-1}\mathbf{G}_2 \in \mathrm{Pos}^+_{\mathbf{G}_1}\mathcal{T} \cap \mathrm{Pos}^+_{\mathbf{G}_2}\mathcal{T} \ , \tag{4.10}$$

i.e., that $(\mathbf{G}_1)^{-1}\mathbf{G}_2$ is symmetric and strictly positive relative to both $\mathbf{G}_1$ and $\mathbf{G}_2$. It follows from the Lineonic Square Root Theorem (see Sect.85 of [FDS]) that there is exactly one $\mathbf{U} \in \mathrm{Pos}^+_{\mathbf{G}_1}\mathcal{T} \cap \mathrm{Pos}^+_{\mathbf{G}_2}\mathcal{T}$ such that $\mathbf{U}^2 = (\mathbf{G}_1)^{-1}\mathbf{G}_2$ and hence $\mathbf{G}_2 = \mathbf{G}_1\mathbf{U}^2$. Since $\mathbf{U} \in \mathrm{Sym}_{\mathbf{G}_1}\mathcal{T}$, it follows from (2.8) that $\mathbf{U}^\top\mathbf{G}_1 = \mathbf{G}_1\mathbf{U}$, which shows that (4.8) holds. (4.9) follows from Prop.6 above. ▮

The following Theorem will be stated without proof even though the proof is not very hard.

Theorem 9: *Consider the family of spectral spaces of* $\mathbf{U}$, *which is an orthogonal decomposition of* $\mathcal{T}$ *relative to both* $\mathbf{G}_1$ *and* $\mathbf{G}_2$. *Then the intersection* $(\mathrm{Orth}\ \mathbf{G}_2) \cap (\mathrm{Orth}\ \mathbf{G}_1)$ *consists of all lineons in* $\mathrm{Orth}\ \mathbf{G}_1$ *that leave each of these spectral spaces invariant.*

Corollary 1: *The intersection* $(\mathrm{Orth}\ \mathbf{G}_2) \cap (\mathrm{Orth}\ \mathbf{G}_1)$ *is a finite group if and only if all the spectral spaces of* $\mathbf{U}$ *are one-dimensional. In this case the intersection has* $2^{\dim \mathcal{T}}$ *elements.*

Corollary 2: *We have*

$$\mathrm{Orth}\ \mathbf{G}_1 = \mathrm{Orth}\ \mathbf{G}_2 \quad \Longleftrightarrow \quad \mathbf{G}_1 = c\mathbf{G}_2 \quad \text{for some} \ \ c \in \mathbb{P}^\times \ . \tag{4.11}$$

Remark: Let dist : $\mathrm{Pos}^+(\mathcal{T}) \times \mathrm{Pos}^+(\mathcal{T}) :\longrightarrow \mathbf{R}$ be defined by

$$\mathrm{dist}(\mathbf{G}_1, \mathbf{G}_2) := \max\{|\lambda \mid |\lambda \in \mathrm{Spec}\,\mathbf{U}\}, \tag{4.12}$$

where $\mathbf{U}$ is determined according to Thm.9. This function turns out to be a natural metric on the cone $\mathrm{Pos}^+(\mathcal{T})$. The metric given by the formula (5.11) in the paper [NS] by Noll and Schäffer is just twice the metric defined here by (4.12). ▮

5. Applications

In continuum physics, a continuous body system $\mathcal{B}$ is described by a mathematical structure as defined in part 3 of my book *Five Contributions to Natural Philosophy* [FC]. The axioms for this structure endow $\mathcal{B}$ with the structure of a differentiable manifold. Hence at each point $X \in \mathcal{B}$ there is a tangent space $\mathcal{T}_X$, which is a real three-dimensional linear space. It is called the *infinitesimal body element* of $\mathcal{B}$ at X, since it is the mathematical representation of what many engineers refer to as an "infinitesimal element" of the body. The genuine formats of $\mathcal{T}_X$ are called *configurations* and a pair of such configuration is called a *deformation.* The material properties of the element, as described by a constitutive law, assign to each configuration $\mathbf{G}$ a *symmetry group*, which is a subgroup of Orth $\mathbf{G}$. For an example see my paper *A Frame-Free Formulation of Elasticity* [N4].

The event-world of the theory of Special Relativity is described by a four-dimensional non-genuine Euclidean space, called *Minkowskian Spacetime*, whose translation space is endowed with additional structure by singling out a format of index one, as defined by (2.6) above. The orthogonal group of this format is called the *Lorentz Group*. A detailed analysis is given in the book *Mathematical Structures of Special Relativity* [MN] by V. Matsko and me.

It is conceivable that a future theory for reconciling quantum mechanics with relativity might involve varying the the format of index one for spacetime and hence the Lorentz group.

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