Chapter 6

World-Momentum

This chapter deals with the important concept of world-momentum. Basic definitions are introduced in §**6.1**, with the Conservation Laws following in §**6.2**. A further application to interstellar travel is offered in §**6.3**.

6.1 Free Particles

We are now ready to begin a study of basic relativistic dynamics. We commence with a definition of a free particle; as one might expect, the worldpath of such a particle is straight. However, we must also specify the momentum of the particle, as this cannot be determined solely from the worldpath of the particle.

Let a Minkowskian spacetime $\mathcal{E}$ be given.

6100 Definition: *A* **free particle** *is defined to be a pair* $(\mathcal{L}, \mathbf{p})$*, where* $\mathcal{L}$ *is a straight worldpath and* $\mathbf{p} \in \mathcal{F}^{\times}$ *is a non-zero future-directed world-vector in the direction of the worldpath.* $\mathbf{p}$ *is called the* **world-momentum** *of the particle.*

Let a free particle $(\mathcal{L}, \mathbf{p})$ be given. For brevity, we will use "particle" for "free particle" in this section and in §**6.2**.

Note that since $\mathbf{p} \in \mathcal{F}^{\times}$, we have $\mathbf{p} \cdot \mathbf{p} \leq 0$. We call

$$m := \sqrt{-\mathbf{p} \cdot \mathbf{p}} \tag{61.1}$$

the **mass** of the particle.

Note that $m > 0$ if and only if $\mathcal{L}$ is a material worldpath. In this case, we say that the particle is **material**.

Let a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given. Put

$$m_{\mathbf{d}} := -\mathbf{p} \cdot \mathbf{d}. \tag{61.2}$$

Since $\mathbf{p} \in \mathcal{F}^{\times}$ and $\mathbf{d} \in \mathcal{F}_1$, it follows from **Cor. 5205** that $m_{\mathbf{d}} \in \mathbb{P}^{\times}$. $m_{\mathbf{d}}$ is called the **apparent mass of the particle relative to d**.

Since $\mathbb{R}\mathbf{d} + \{\mathbf{d}\}^{\perp} = \mathcal{V}$, there is exactly one $\mathbf{p}_{\mathbf{d}} \in \{\mathbf{d}\}^{\perp}$ such that

$$\mathbf{p} = m_{\mathbf{d}}\mathbf{d} + \mathbf{p}_{\mathbf{d}}. \tag{61.3}$$

$\mathbf{p}_{\mathbf{d}}$ is called the **momentum of the particle relative to d**.

It follows from (61.3) that

$$m^2 = -\mathbf{p} \cdot \mathbf{p} = m_{\mathbf{d}}^2 - |\mathbf{p}_{\mathbf{d}}|^2. \tag{61.4}$$

Hence, $m_{\mathbf{d}} \geq m$. We call

$$k_{\mathbf{d}} := m_{\mathbf{d}} - m \geq 0 \tag{61.5}$$

the **kinetic energy of the particle relative to d**.

We illustrate these definitions in the following diagram, where $\mathbf{e} \in \{\mathbf{d}\}^{\perp}$ is chosen such that $|\mathbf{e}| = 1$ and $\mathbf{p} \in \mathsf{Lsp}\,\{\mathbf{d}, \mathbf{e}\}$.

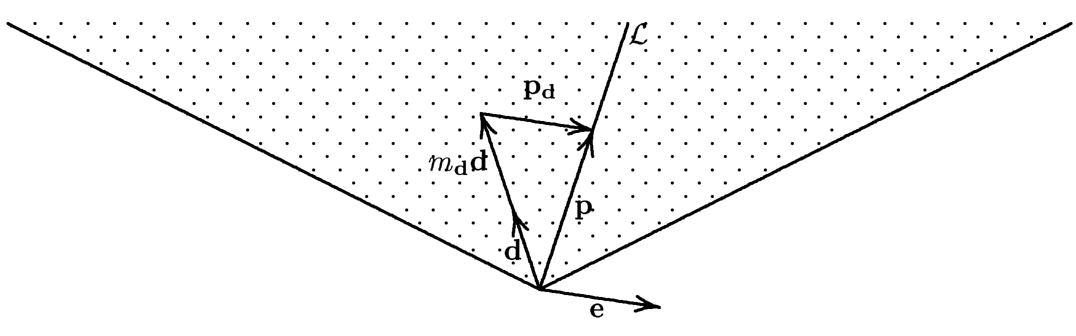

Figure 61a

A particle need not have a strictly positive mass. We first examine the case of a material particle; that is, a particle with a strictly positive mass. After that discussion, we will turn our attention to particles with zero mass. The following Proposition is a straightforward consequence of the definition of a material particle.

6101 Proposition: *Let a material particle* $(\mathcal{L}, \mathbf{p})$ *be given. Then there is* $\mathbf{w} \in \mathcal{F}_1$ *such that* $\mathcal{L}$ *has world-direction* $\mathbf{w}$ *and* $\mathbf{p} = m\mathbf{w}$.

Let a material particle $(\mathcal{L}, \mathbf{p})$ be given. By the preceding Proposition, we may choose $\mathbf{w} \in \mathcal{F}_1$ such that $\mathbf{p} = m\mathbf{w}$. Let $\mathbf{d} \in \mathcal{F}_1$ be given such that $\mathbf{d} \neq \mathbf{w}$.

By **Thm. 5401**, we may choose $\nu \in]0,1[$, $\mu \in 1 + \mathbb{P}^\times$, and $\mathbf{e} \in \{\mathbf{d}\}^\perp$ with $|\mathbf{e}| = 1$ such that

$$\mathbf{w} = \mu(\mathbf{d} + \nu\mathbf{e}). \tag{61.6}$$

Then

$$\begin{aligned} \mathbf{p} &= m\mathbf{w} \\ &= m(\mu(\mathbf{d} + \nu\mathbf{e})) \\ &= (m\mu)\mathbf{d} + (m\mu\nu)\mathbf{e}. \end{aligned} \tag{61.7}$$

Since $\mathbb{R}\mathbf{d} + \{\mathbf{d}\}^\perp = \mathcal{V}$, it follows from (61.3) that

$$m_\mathbf{d} = m\mu.$$

Hence,

$$k_\mathbf{d} = m\mu - m = m(\mu - 1).$$

It follows easily from the relationships between μ and ν as given in **Thm. 5401** that

$$\mu - 1 = \nu^2 \frac{\mu^2}{1 + \mu},$$

and hence

$$k_\mathbf{d} = m\nu^2 \frac{\mu^2}{1 + \mu}. \tag{61.8}$$

Moreover, it follows from (61.7) and (61.3) that

$$\mathbf{p}_\mathbf{d} = (m\mu\nu)\mathbf{e},$$

and hence, since $|\mathbf{e}| = 1$, that

$$|\mathbf{p_d}| = m\mu\nu. \tag{61.9}$$

Remark: If ν is small, then $\mu \approx 1$, and

$$k_{\mathbf{d}} \approx \frac{1}{2}m\nu^2, \qquad |\mathbf{p_d}| \approx m\nu.$$

> Thus, if the speed of the particle relative to **d** is small, then the classical formulas give close approximations to the relativistic formulas.

Thus, we see that the apparent mass, momentum, and kinetic energy with respect to an observer are all dependent on the speed of the particle relative to the observer.

Of course, it may be the case that a particle is not material; that is, the particle has zero mass. Although there are a number of particles with zero mass described by physicists, we use poetic license and call all of these particles "photons".

6102 Definition: *We say that a free particle* $(\mathcal{L}, \mathbf{p})$ *is a* **photon** *if its mass is zero; i.e., if* $\mathbf{p} \cdot \mathbf{p} = 0$.

Let a photon $(\mathcal{L}, \mathbf{p})$ be given. Since $\mathbf{p} \cdot \mathbf{p} = 0$, we have $m = 0$.

Let a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given. Since $m^2 = 0$ and $m_{\mathbf{d}} \in \mathbb{P}^{\times}$, it follows from (61.4) that

$$m_{\mathbf{d}} = |\mathbf{p_d}| \neq 0. \tag{61.10}$$

This means that every observer perceives the photon as having a non-zero apparent mass even though its true mass is zero. Since $\mathbf{p_d} \in \{\mathbf{d}\}^{\perp}$ (see (61.3)) and $|\mathbf{p_d}| \neq 0$, we may determine $\mathbf{e} \in \{\mathbf{d}\}^{\perp}$ such that

$$\mathbf{p_d} = |\mathbf{p_d}|\mathbf{e}.$$

It follows from (61.3) and (61.10) that

$$\mathbf{p} = m_{\mathbf{d}}(\mathbf{d} + \mathbf{e}). \tag{61.11}$$

Since $m = 0$, it follows from (61.5) that

$$k_{\mathbf{d}} = m_{\mathbf{d}}.$$

Thus, the kinetic energy of the photon relative to **d** coincides with the mass of the photon relative to **d**. We know from quantum mechanics that the wave-like aspects of a photon are related to its particle-like aspects. In particular, if we denote by $f_{\mathbf{d}}$ the **frequency of the photon relative to d**, then we have

$$m_{\mathbf{d}} = f_{\mathbf{d}} h,$$

where h is Planck's constant. If we choose units so that $h = 1$, we have

$$f_{\mathbf{d}} = m_{\mathbf{d}} = k_{\mathbf{d}} = -\mathbf{p} \cdot \mathbf{d}. \tag{61.12}$$

Thus, we see that the apparent mass, kinetic energy, and frequency of a photon describe different aspects of essentially the same phenomenon. The following diagram serves as an illustration, where $\mathbf{e} \in \{\mathbf{d}\}^{\perp}$ is such that $\mathbf{p}$, $\mathbf{d}$, and $\mathbf{e}$ are all in the same two-dimensional subspace of $\mathcal{V}$.

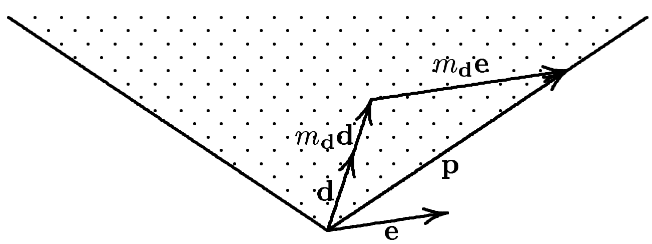

Figure 61b

—The Doppler Effect

As an application of these ideas, we consider the Doppler effect. We experience such an effect in our day-to-day lives with the passing of a train. We hear the pitch of the rumblings of the train rise steadily until it is upon us, and then lower again as the train passes. A similar phenomenon occurs with light; stated in its simplest form, light emitted at one frequency by an observer will be received at a different frequency by a second observer.

So let $(\mathcal{P}, \mathbf{p})$ be a photon emitted by an emitter with straight worldpath $\mathcal{L}_1$ and absorbed by a receiver with straight worldpath $\mathcal{L}_2$. Let $\mathbf{d}_1$, $\mathbf{d}_2$ be the world-directions of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively, and assume that $\mathbf{d}_1 \neq \mathbf{d}_2$.

By **Thm. 5401**, we may choose $\nu \in]0,1[$, $\mu \in 1+\mathbb{P}^\times$, and $\mathbf{e}_1 \in \{\mathbf{d}_1\}^\perp$ with $|\mathbf{e}_1| = 1$ such that

$$\mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1).$$

Here, ν is the relative speed of the receiver as seen by the emitter. Put

$$f_1 := f_{\mathbf{d}_1} = -\mathbf{p}\cdot\mathbf{d}_1, \quad f_2 := f_{\mathbf{d}_2} = -\mathbf{p}\cdot\mathbf{d}_2.$$

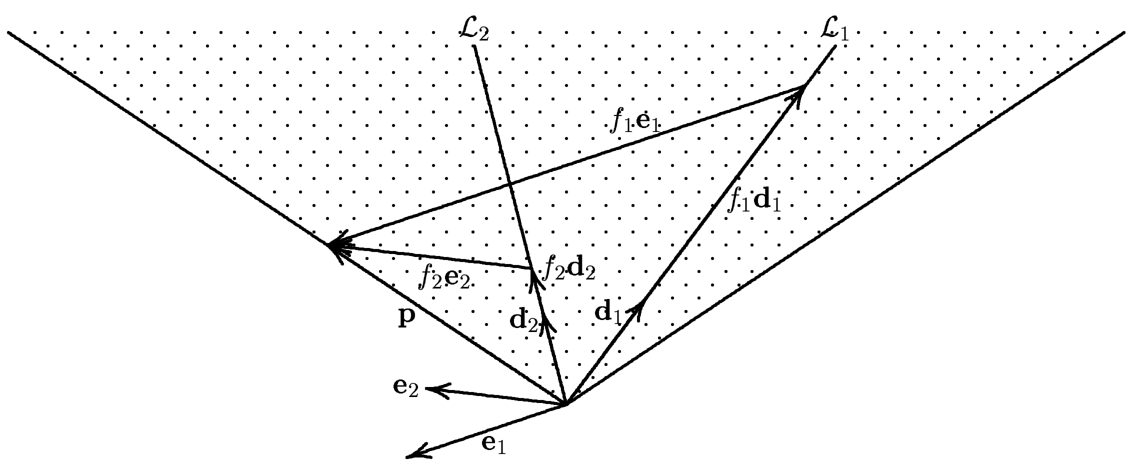

Figure 61c

f_1 is the frequency of emission; that is, the frequency of the photon relative to the emitter (see Figure 61(c)). Similarly, f_2 is the frequency of reception. From (61.11) and (61.12), we may determine $\mathbf{e} \in \{\mathbf{d}_1\}^\perp$ with $|\mathbf{e}| = 1$ such that

$$\mathbf{p} = f_1(\mathbf{d}_1 + \mathbf{e}),$$

and hence

$$\begin{aligned} f_2 &= -\mathbf{p}\cdot\mathbf{d}_2 \\ &= -f_1(\mathbf{d}_1+\mathbf{e})\cdot\mu(\mathbf{d}_1+\nu\mathbf{e}_1) \\ &= f_1\mu(1-\nu(\mathbf{e}\cdot\mathbf{e}_1)). \end{aligned}$$

Since $\mathbf{e}, \mathbf{e}_1 \in \{\mathbf{d}_1\}^\perp$ and $|\mathbf{e}| = |\mathbf{e}_1| = 1$, we may determine $\theta \in [0,\pi]$ such that $\mathbf{e}\cdot\mathbf{e}_1 = \cos\theta$. We may interpret θ is the angle between the direction of the photon and the direction of motion of the receiver, both relative to the emitter. We then see that

$$f_2 = f_1\mu(1 - \nu\cos\theta).$$

Since $f_1 \neq 0$, this is equivalent to

$$\begin{aligned} \frac{f_2}{f_1} &= \mu(1 - \nu\cos\theta) \\ &= \frac{1 - \nu\cos\theta}{\sqrt{1 - \nu^2}}. \end{aligned} \tag{61.13}$$

We examine three particular cases.

1. If $\theta = 0$, we have

$$\frac{f_2}{f_1} = \frac{1 - \nu}{\sqrt{1 - \nu^2}} = \sqrt{\frac{1 - \nu}{1 + \nu}} < 1.$$

   This corresponds to a red-shift, which occurs when the receiver moves directly away from the emitter.

2. If $\theta = \pi$, we have

$$\frac{f_2}{f_1} = \frac{1 + \nu}{\sqrt{1 - \nu^2}} = \sqrt{\frac{1 + \nu}{1 - \nu}} > 1.$$

   This corresponds to a blue-shift, which occurs when the receiver moves directly towards the emitter. Note that in the case that $\mathbf{p}$, $\mathbf{d}_1$, and $\mathbf{d}_2$ all lie in the same plane, the only possible values of θ are 0 and π.

3. If $\nu \ll 1$, we have

$$\frac{f_2}{f_1} \approx 1 - \nu\cos\theta.$$

   Thus, for small relative speeds, the formula for the classical Doppler shift is a close approximation to the relativistic formula.

We may apply these results to a problem of our space travellers as discussed in §**5.8**. Suppose that Dick sends to Jane a birthday greeting whose signal has a given frequency. At what frequency must Jane tune her receiver so as to be able to receive the message? Using the notations in §**5.8**, if Dick sends a message at frequency f_D at time t_e, at what frequency f_J will Jane receive the message at time s?

We know that the world-direction of the emitter (Dick) is $\mathbf{d}_0$; we also know that the world-direction of the receiver (Jane) is $\mathbf{d}(s) = \cosh(\gamma s)\mathbf{d}_0 + \sinh(\gamma s)\mathbf{e}$ (see (58.4)). Also recall that the worldpath of the signal being sent is $\mathcal{M} = p_D(t_e) + \mathbb{P}(\mathbf{d}_0 + \mathbf{e})$, and hence we may determine $f_D \in \mathbb{P}^\times$ such that the signal has world-momentum $\mathbf{p} = f_D(\mathbf{d}_0 + \mathbf{e})$. If, in the previous calculations, we put

$$\begin{aligned} \mathbf{d}_1 &:= \mathbf{d}_0, & \mathbf{d}_2 &:= \mathbf{d}(s), \\ f_1 &:= f_D, & f_2 &:= f_J, \end{aligned}$$

we find that

$$\mu = -\mathbf{d}_0 \cdot \mathbf{d}(s) = \cosh(\gamma s),$$

that $\mu\nu = \mathbf{d}(s) \cdot \mathbf{e} = \sinh(\gamma s)$ (see **Thm. 5401**), and hence

$$\nu = \tanh(\gamma s).$$

In our case, both the signal and the receiver (Jane) move directly away from the emitter (Dick), and hence $\theta = 0$. Thus, (61.13) becomes

$$\begin{aligned} \frac{f_J}{f_D} &= \cosh(\gamma s)(1 - \tanh(\gamma s)) \\ &= \cosh(\gamma s) - \sinh(\gamma s) \\ &= e^{-\gamma s}. \end{aligned}$$

If we denote the red-shift by

$$\sigma := \frac{f_D}{f_J} = e^{\gamma s},$$

we have the following chart, assuming $\gamma = 1$:

s	1	2	4	5
σ	2.72	7.39	54.60	148.41

Thus, if Dick sent a message at a frequency of 2.72 megahertz (MHz) wishing Jane a happy 21^{st} birthday, then Jane would need to tune her receiver to 1 MHz in order to receive the message.

6.2 Interactions

Let a Minkowskian spacetime $\mathcal{E}$ be given.

—–Basic Definitions

6200 Definition: *An* **interaction** *is given by two disjoint finite sets of worldpaths* A, B, *and a family* $(\mathbf{p}^{\mathcal{L}} \mid \mathcal{L} \in \mathsf{A} \cup \mathsf{B})$ *in* $\mathcal{F}$ *such that:*

(1) $(\mathcal{L}, \mathbf{p}^{\mathcal{L}})$ *is a free particle for all* $\mathcal{L} \in \mathsf{A} \cup \mathsf{B}$,

(2) *All worldpaths in* A *have an end, and all worldpaths in* B *have a beginning; moreover, we have* $\mathsf{end}\ \mathcal{L} = \mathsf{beg}\ \mathcal{M}$ *for all* $\mathcal{L} \in \mathsf{A}$ *and* $\mathcal{M} \in \mathsf{B}$, *and*

(3) *The Law of Conservation of World-Momentum is satisfied; i.e.,*

$$\sum_{\mathcal{L}\in\mathsf{A}} \mathbf{p}^{\mathcal{L}} = \sum_{\mathcal{L}\in\mathsf{B}} \mathbf{p}^{\mathcal{L}}.$$

Roughly, A is the set of worldpaths of "incoming" particles and B is the set of worldpaths of "outgoing" particles. As an example, a case where $\mathsf{A} = \{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3\}$, $\mathsf{B} = \{\mathcal{M}_1, \mathcal{M}_2\}$, and $q = \mathsf{end}\ \mathcal{L} = \mathsf{beg}\ \mathcal{M}$ for all $\mathcal{L} \in \mathsf{A}$ and $\mathcal{M} \in \mathsf{B}$ is illustrated below.

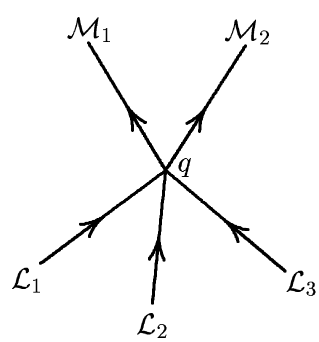

Figure 62a

Remark: Note that the definition of an interaction precludes two different particles from being described with the same worldpath. Thus, were two distinct particles to collide and "stick together", we would, in the context of **Def. 6200**, consider the resulting phenomenon as a single particle, rather that two distinct particles described with the same worldpath.

We assume that an interaction is given by A, B, and a family $(\mathbf{p}^{\mathcal{L}} \mid \mathcal{L} \in \mathsf{A} \cup \mathsf{B})$ in $\mathcal{F}$ as in **Def. 6200**; let a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given. For each material worldpath $\mathcal{L} \in \mathsf{A} \cup \mathsf{B}$, we can determine $m_{\mathbf{d}}^{\mathcal{L}} \in \mathbb{P}^{\times}$ and $\mathbf{p}_{\mathbf{d}}^{\mathcal{L}} \in \{\mathbf{d}\}^{\perp}$ (see (61.3)) such that

$$\mathbf{p}^{\mathcal{L}} = m_{\mathbf{d}}^{\mathcal{L}}\mathbf{d} + \mathbf{p}_{\mathbf{d}}^{\mathcal{L}}.$$

We denote the mass of the particle $(\mathcal{L}, \mathbf{p}^{\mathcal{L}})$ by $m^{\mathcal{L}}$, so that $m^{\mathcal{L}} = \sqrt{-\mathbf{p}^{\mathcal{L}} \cdot \mathbf{p}^{\mathcal{L}}}$.

6201 Proposition: *Let $\mathbf{d} \in \mathcal{F}_1$ be given. The Law of Conservation of World-Momentum* (**Def. 6200**(3)) *is satisfied if and only if both*

$$(1) \quad \sum_{\mathcal{L} \in \mathsf{A}} m_{\mathbf{d}}^{\mathcal{L}} = \sum_{\mathcal{L} \in \mathsf{B}} m_{\mathbf{d}}^{\mathcal{L}}$$

and

$$(2) \quad \sum_{\mathcal{L} \in \mathsf{A}} \mathbf{p}_{\mathbf{d}}^{\mathcal{L}} = \sum_{\mathcal{L} \in \mathsf{B}} \mathbf{p}_{\mathbf{d}}^{\mathcal{L}}$$

are valid. (1) *is called the Law of Conservation of Relative Mass, and* (2) *is called the Law of Conservation of Relative Momentum.*

We leave the proof of this Proposition as an Exercise. Note that in general, the sum of the masses of particles is not conserved.

We now proceed to give two examples which illustrate the above concepts.

—**Particle Decay**

We suppose that A is a singleton whose only member is $\mathcal{A}$. In other words, some particle spontaneously "splits" or "decays" into many particles. Conservation of World-Momentum yields

$$\mathbf{p}^{\mathcal{A}} = \sum_{\mathcal{L} \in \mathsf{B}} \mathbf{p}^{\mathcal{L}}.$$

For each material worldpath $\mathcal{L} \in \mathsf{B}$, we may determine (by **Prop. 6101**) $\mathbf{d}^{\mathcal{L}} \in \mathcal{F}_1$ such that $\mathbf{p}^{\mathcal{L}} = m^{\mathcal{L}}\mathbf{d}^{\mathcal{L}}$. Then given material worldpaths $\mathcal{L}, \mathcal{M} \in \mathsf{B}$ such that $\mathcal{L} \neq \mathcal{M}$, we have

$$\mathbf{p}^{\mathcal{L}} \cdot \mathbf{p}^{\mathcal{M}} = m^{\mathcal{L}}\mathbf{d}^{\mathcal{L}} \cdot m^{\mathcal{M}}\mathbf{d}^{\mathcal{M}} = m^{\mathcal{L}}m^{\mathcal{M}}(\mathbf{d}^{\mathcal{L}} \cdot \mathbf{d}^{\mathcal{M}}).$$

Since $\mathbf{d}^{\mathcal{L}}, \mathbf{d}^{\mathcal{M}} \in \mathcal{F}_1$, it easily follows from **Thm. 5201** and **Cor. 5205** that $\mathbf{d}^{\mathcal{L}} \cdot \mathbf{d}^{\mathcal{M}} \leq -1$, and hence

$$-\mathbf{p}^{\mathcal{L}} \cdot \mathbf{p}^{\mathcal{M}} \geq m^{\mathcal{L}}m^{\mathcal{M}}.$$

That this inequality remains valid when at least one of $\mathcal{L}$ and $\mathcal{M}$ is a photon follows immediately from **Cor. 5205** and the fact that at least one of $m^{\mathcal{L}}$ and $m^{\mathcal{M}}$ is zero. Hence, it follows that

$$\begin{aligned}
(m^{\mathcal{A}})^2 &= -\mathbf{p}^{\mathcal{A}} \cdot \mathbf{p}^{\mathcal{A}} \\
&= -\left(\sum_{\mathcal{L}\in\mathsf{B}} \mathbf{p}^{\mathcal{L}}\right) \cdot \left(\sum_{\mathcal{L}\in\mathsf{B}} \mathbf{p}^{\mathcal{L}}\right) \\
&= -\sum_{\mathcal{L}\in\mathsf{B}} \mathbf{p}^{\mathcal{L}} \cdot \mathbf{p}^{\mathcal{L}} - \sum_{\mathcal{L}\neq\mathcal{M}\in\mathsf{B}} \mathbf{p}^{\mathcal{L}} \cdot \mathbf{p}^{\mathcal{M}} \\
&\geq \sum_{\mathcal{L}\in\mathsf{B}} \left(m^{\mathcal{L}}\right)^2 + \sum_{\mathcal{L}\neq\mathcal{M}\in\mathsf{B}} m^{\mathcal{L}}m^{\mathcal{M}} \\
&= \left(\sum_{\mathcal{L}\in\mathsf{B}} m^{\mathcal{L}}\right)^2,
\end{aligned}$$

where by summing over "$\mathcal{L} \neq \mathcal{M} \in \mathsf{B}$" we mean summing over all pairs $(\mathcal{L}, \mathcal{M}) \in \mathsf{B} \times \mathsf{B}$ such that $\mathcal{L} \neq \mathcal{M}$. Since masses are positive, we conclude that

$$m^{\mathcal{A}} \geq \sum_{\mathcal{L}\in\mathsf{B}} m^{\mathcal{L}}. \tag{62.1}$$

We may interpret this result as follows: a particle can decay into more than one particle only if the mass of the initial particle is greater than the sum of the masses of the resultant particles. As an example, consider a hydrogen atom. The mass of a hydrogen atom is 1.00797amu (an *atomic mass unit* (amu) is the standard unit used in measuring the mass of atomic particles), while the masses of a proton and an electron are 1.007594amu and 0.000549amu, respectively. Since

$$1.00797 < 1.008143 = 1.007594 + 0.000549,$$

we see that a hydrogen atom cannot spontaneously decay into a proton and an electron. Also note that (62.1) is a necessary but not sufficient condition for particle decay. In other words, there may be particles which satisfy (62.1), but which fail to decay.

—The Compton Effect

6202 Definition: *We say that an interaction is* **elastic** *if the sum of the masses of the incoming particles is equal to the sum of the masses of the outgoing particles.*

We now consider the elastic interaction of a photon and an electron. We examine the Compton Effect: the change in the frequency of the photon relative to the electron following the interaction.

Let a photon and an electron (with world-momenta $\mathbf{p}$ and $\mathbf{q}$, respectively) interact elastically, and let $\mathbf{p}'$ and $\mathbf{q}'$ be their world-momenta, respectively, following the interaction. It follows from the Law of Conservation of World-Momentum that $\mathbf{p}+\mathbf{q}=\mathbf{p}'+\mathbf{q}'$, and hence $\mathbf{q}'=\mathbf{p}+\mathbf{q}-\mathbf{p}'$. Taking the inner product of each side of this equation with itself yields

$$\mathbf{q}'\cdot\mathbf{q}' = \mathbf{p}\cdot\mathbf{p}+\mathbf{q}\cdot\mathbf{q}+\mathbf{p}'\cdot\mathbf{p}'+2\mathbf{p}\cdot\mathbf{q}-2\mathbf{p}\cdot\mathbf{p}'-2\mathbf{q}\cdot\mathbf{p}'. \tag{62.2}$$

Put $m := \sqrt{-\mathbf{q}\cdot\mathbf{q}}$. Since the interaction is elastic and the mass of a photon is 0, m is the mass of the electron both before *and* after the interaction. Moreover, we must have $\mathbf{p}\cdot\mathbf{p}=0=\mathbf{p}'\cdot\mathbf{p}'$, and hence $-m^2=\mathbf{q}\cdot\mathbf{q}=\mathbf{q}'\cdot\mathbf{q}'$. Then (62.2) becomes

$$\mathbf{p}\cdot\mathbf{p}' = \mathbf{q}\cdot(\mathbf{p}-\mathbf{p}'). \tag{62.3}$$

Now put $\mathbf{d} := \frac{1}{m}\mathbf{q}$. From §**6.1** (see (61.11)), we may find $\mathbf{e},\mathbf{e}'\in\{\mathbf{d}\}^\perp$ with $|\mathbf{e}|=|\mathbf{e}'|=1$ and $\nu_\mathbf{d},\nu'_\mathbf{d}\in\mathbb{P}^\times$ such that

$$\mathbf{p}=\nu_\mathbf{d}(\mathbf{d}+\mathbf{e}),\quad \mathbf{p}'=\nu'_\mathbf{d}(\mathbf{d}+\mathbf{e}').$$

It is easy to see that $\mathbf{p}\cdot\mathbf{p}'=-\nu_\mathbf{d}\nu'_\mathbf{d}(1-\mathbf{e}\cdot\mathbf{e}')$. It also follows from the definition of $\mathbf{d}$ that $\mathbf{q}\cdot\mathbf{p}=-m\nu_\mathbf{d}$ and $\mathbf{q}\cdot\mathbf{p}'=-m\nu'_\mathbf{d}$. Then (62.3) becomes

$$-\nu_\mathbf{d}\nu'_\mathbf{d}(1-\mathbf{e}\cdot\mathbf{e}') = -m(\nu_\mathbf{d}-\nu'_\mathbf{d}). \tag{62.4}$$

Since $|\mathbf{e} \cdot \mathbf{e}'| \leq 1$, we may find $\theta \in [0, \pi]$ such that $\cos\theta = \mathbf{e} \cdot \mathbf{e}'$. Roughly speaking, θ is the angle between the directions of motion of the photon, relative to the electron, preceding and following the interaction. In addition, put

$$\lambda_{\mathbf{d}} := \frac{1}{\nu_{\mathbf{d}}}, \quad \lambda'_{\mathbf{d}} := \frac{1}{\nu'_{\mathbf{d}}}.$$

Then (62.4) becomes

$$-\frac{1}{\lambda_{\mathbf{d}}\lambda'_{\mathbf{d}}}(1 - \cos\theta) = -m\left(\frac{1}{\lambda_{\mathbf{d}}} - \frac{1}{\lambda'_{\mathbf{d}}}\right),$$

which may be rewritten as

$$\frac{2}{m}\sin^2\frac{\theta}{2} = \lambda'_{\mathbf{d}} - \lambda_{\mathbf{d}}.$$

$\nu_{\mathbf{d}}$ and $\nu'_{\mathbf{d}}$ are interpreted as the frequencies of the photons $\mathbf{p}$ and $\mathbf{p}'$ relative to $\mathbf{d}$; $\lambda_{\mathbf{d}}$ and $\lambda'_{\mathbf{d}}$ are analogously interpreted as wavelengths. Thus, interpreting θ as given above, we see that $\frac{2}{m}\sin^2\frac{\theta}{2}$ gives the difference between the wavelengths of the photon before and after the collision relative to the world-direction of the electron before the collision.

6.3 Interstellar Travel Revisited

Let a Minkowskian spacetime $\mathcal{E}$ be given.

We wish to continue the discussion of interstellar travel begun in §**5.8**. However, we will alter our perspective in view of §**6.1** and §**6.2**. Thus, we consider the worldpath $\mathcal{J}$ (see §**5.8**) as the worldpath of the rocket-propelled spaceship in which Jane travels, and assume that a time-parameterization p_J of $\mathcal{J}$ is given by (58.5).

We assume that the rocket propulsion is the result of the emission of free particles (which may result, when the particles are material, from the burning of fuel). Hence, the mass of the rocket (which includes the mass of the fuel) strictly decreases with time. We therefore assume that a smooth mapping $m : \mathbb{P} \to \mathbb{P}^\times$ is prescribed, and we interpret $m(s)$ as the mass of the spaceship (with fuel) at Jane's time s for all $s \in \mathbb{P}$. With $\mathbf{d} := p^\bullet$, we see that the mapping $\mathbf{p} : \mathbb{P} \to \mathcal{V}$ given by $\mathbf{p} := m\mathbf{d}$ describes the world-momentum of the spaceship.

In addition, we assume that we may describe the world-momentum of the emitted free particles by a smooth mapping $\mathbf{r} : \mathbb{P} \to \mathcal{V}$. In addition, we assume that the world-momentum of the emitted particles will differ only slightly over short time intervals. In particular, let $s \in \mathbb{P}$ be given; then for small $h \in \mathbb{P}^\times$, we assume that the sum of the world-momenta of the particles emitted between the events $p_J(s)$ and $p_J(s+h)$ is approximately $h\mathbf{r}(s)$. The change in world-momentum of the spaceship during this interval will be $\mathbf{p}(s+h) - \mathbf{p}(s)$. It follows from the Law of Conservation of World-Momentum (**Def. 6200**(3)) that

$$\mathbf{p}(s+h) - \mathbf{p}(s) \approx -h\mathbf{r}(s).$$

Taking the limit as h goes to zero, we find that

$$\mathbf{p}^\bullet = -\mathbf{r}. \tag{63.1}$$

We now examine Jane's interstellar journey with these ideas in mind. We will use the concepts developed above to determine how much fuel Jane would need to travel to α Centauri. We recall from §**5.8** that

$$\mathbf{d} \cdot \mathbf{d} = -1, \quad \mathbf{d} \cdot \mathbf{d}^\bullet = 0, \quad \text{and} \quad |\mathbf{d}^\bullet| = \gamma. \tag{63.2}$$

Differentiating $\mathbf{p} = m\mathbf{d}$ and using (63.1) yields

$$m\mathbf{d}^\bullet + m^\bullet\mathbf{d} = -\mathbf{r}. \tag{63.3}$$

Taking the inner product of each side of (63.3) with itself and using (63.2) yields

$$m^2\gamma^2 - (m^\bullet)^2 = \mathbf{r} \cdot \mathbf{r}. \tag{63.4}$$

We now proceed to examine two cases: the case when Jane's rockets emit material free particles, and the case when the rockets emit photons. We will subsequently examine the relative efficiencies of these two methods of propulsion.

At this point, we consider the case when Jane's rockets emit material free particles. We now assume that all such free particles have the same speed $\nu \in]0,1[$ relative to the rockets, which are attached to and therefore assumed to be at rest relative to the spaceship. In other words, if the direction of $\mathbf{p}(s)$ were decomposed relative to the direction of $\mathbf{r}(s)$ as in **Thm. 5401** for $s \in \mathbb{P}$, we would find that ν was independent of $s \in \mathbb{P}$.

Since the emitted free particles are material, it follows that there are mappings $\widehat{m} : \mathbb{P} \to \mathbb{P}^\times$ and $\widehat{\mathbf{d}} : \mathbb{P} \to \mathcal{V}$ such that $\mathsf{Rng}\ \widehat{\mathbf{d}} \subset \mathcal{F}_1$ and $\mathbf{r} = \widehat{m}\widehat{\mathbf{d}}$. With this representation of $\mathbf{r}$, (63.4) becomes

$$m^2\gamma^2 - (m^\bullet)^2 = -\widehat{m}^2. \tag{63.5}$$

Put $\mu := -\mathbf{d} \cdot \widehat{\mathbf{d}}$; taking the inner product of each side of (63.3) with $\mathbf{d}$ while keeping in mind that $\mathbf{r} = \widehat{m}\widehat{\mathbf{d}}$ yields

$$-m^\bullet = \widehat{m}\mu. \tag{63.6}$$

The previous two equations together yield

$$(m^\bullet)^2 = \frac{\gamma^2}{\nu^2} m^2,$$

where $\nu = \sqrt{1 - \mu^{-2}}$. Since $\mathsf{Rng}\ \widehat{m} \subset \mathbb{P}^\times$ and $\mu \in 1 + \mathbb{P}^\times$, it follows from (63.6) that $m^\bullet < 0$. Thus, we have

$$m^\bullet = -\frac{\gamma}{\nu} m.$$

Since we have assumed ν to be constant, we therefore see from elementary calculus that

$$m(s) = m(0)e^{-\frac{\gamma}{\nu}s}$$

for all $s \in \mathbb{P}$. So if Jane's trip takes S units of time, and she desires to burn the last particle of fuel as she lands at her destination so that the mass of the rocket without fuel is $m(S)$, the mass of the fuel necessary to make the trip is

$$m(0) - m(S) = m(S)(e^{\frac{\gamma}{\nu}S} - 1).$$

The ratio

$$\frac{m(S)}{m(0)} = e^{-\frac{\gamma}{\nu}S}$$

is called the **payload factor**.

For Jane's journey to α Centauri, recall that $\gamma = 1$ and $S \approx 7.3$. Thus, even for relative speeds of ν close to 1, we find that the payload factor is close to $e^{-7.3} \approx \frac{1}{1480}$; that is, the mass of the fuel that Jane would need for her trip would be approximately 1500 times the mass of her ship. The payload factors for various values of ν are included in the following chart.

ν	Payload factor
0.5	4.56×10^{-7}
0.7	2.96×10^{-5}
0.9	3.00×10^{-4}
0.95	4.60×10^{-4}
0.97	5.39×10^{-4}

What happens if Jane's rockets emit photons instead of material free particles? How does the payload factor change? Since the emitted free particles are not material, we cannot determine $\widehat{m}$ and $\widehat{\mathbf{d}}$ as before. However, as the particles are photons, we do know that $\mathbf{r} \cdot \mathbf{r} = 0$.

Hence (63.4) becomes

$$m^2\gamma^2 - (m^\bullet)^2 = 0,$$

and consequently

$$(m^\bullet)^2 = m^2\gamma^2.$$

Since $\mathbf{d} \in \mathcal{F}_1$ and $\mathbf{r} \in \mathcal{F}$, we take the inner product of each side of (63.3) with $\mathbf{d}$ and use (63.2) and **Cor. 5205** to see that

$$0 \leq -\mathbf{r} \cdot \mathbf{d} = -m^\bullet.$$

Hence $m^\bullet \leq 0$, and we therefore have

$$m^\bullet = -\gamma m.$$

It follows that

$$m(s) = m(0)e^{-\gamma s}$$

for all $s \in \mathbb{P}$. As before, the ratio

$$\frac{m(S)}{m(0)} = e^{-\gamma S}$$

is the payload factor when Jane is propelled by a photon rocket. In comparing the payload factor of a photon rocket with that of a rocket which emits material free particles, it is seen that the photon rocket is a more efficient means of propulsion. However, from an engineering point of view, much more is known about the design and construction of rockets which emit material free particles than those which emit photons. A photon rocket would need two tanks, one containing particles and the other anti-particles. In the rocket, photons would be generated by annihilating particles with anti-particles. The difficulty is to construct a tank that confines anti-particles; it could not have walls consisting of ordinary material. The "wall" would somehow consist of magnetic fields.

Exercises

Exercises, I

1. Prove **Prop. 6201**.

Exercises, II

In the following Exercises, suppose that a Minkowskian spacetime $\mathcal{E}$ is given.

1. Let $\mathcal{S}$ and $\mathcal{G}$ be two material worldpaths having the same beginning $q := \mathsf{beg}\ \mathcal{S} = \mathsf{beg}\ \mathcal{G}$. We think of $\mathcal{S}$ as the worldpath of our solar system, $\mathcal{G}$ as the worldpath of a distant galaxy, and q as the "birth of the universe", or "big bang". At some event $x \in \mathcal{G}$, a photon is emitted with frequency f_g relative to $\mathcal{G}$, and is received by $\mathcal{S}$ at $y \in \mathcal{S}$ with frequency f_s relative to $\mathcal{S}$. Put $t := \mathsf{t}_{\mathcal{S}}(q, y)$; *i.e.,* the "age of the solar system at the reception of the photon", and put $r := \mathsf{dst}(x, \mathcal{S})$.

(a) Show that

$$\frac{f_s}{f_g} = \sqrt{1 - \frac{2r}{t}}.$$

(b) With d as the *luminosity distance of $\mathcal{G}$ from $\mathcal{S}$; i.e.,*

$$d := r\left(\frac{f_g}{f_s}\right)^2,$$

and with *red-shift*

$$\sigma := \frac{f_g - f_s}{f_s},$$

show that

$$\sigma = \sqrt{1 + \frac{2d}{t}} - 1.$$

(σ and d may be determined by astronomical observation. The above formula may then be used to find the present age of the solar system. However, the analysis given here is primitive and unrealistic. More realistic analyses of the age of the universe involve general relativity.)

EXERCISES, III

1. A physicist is arrested for going through a red light. In court he pleads that he approached the intersection at such a speed that the red light looked green to him. The judge changes the charge to speeding and fines the defendant one dollar for every mile per hour he exceeded the speed limit of twenty miles per hour. What is the fine? (Take the wavelength of green [red] light to be 5300×10^{-10} [6500×10^{-10}] meters.)

2. Suppose that a small spaceship has a mass of 10,000 kg without fuel. Calculate the payload factor and the mass of the fuel necessary to reach a star D light-years away if

   (a) the fuel consists of material particles and is emitted at a speed of 0.01 relative to the spaceship,

(b) the fuel consists of photons.

3. Suppose that a material particle with world-momentum $\mathbf{p}$ splits into two particles with world-momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. It follows from the Law of Conservation of World-Momentum (see **Def. 6200**) that $\mathbf{p}$, $\mathbf{p}_1$, and $\mathbf{p}_2$ must all lie in the same plane. Put $m := \sqrt{-\mathbf{p} \cdot \mathbf{p}}$; since the original particle is material, we may determine $\mathbf{d} \in \mathcal{F}_1$ such that $\mathbf{p} = m\mathbf{d}$. Also, define $m_1 := \sqrt{-\mathbf{p}_1 \cdot \mathbf{p}_1}$ and $m_2 := \sqrt{-\mathbf{p}_2 \cdot \mathbf{p}_2}$.

   (a) Consider the case that $\mathbf{p}_1$ and $\mathbf{p}_2$ are the world-momenta of material particles. Then we may determine $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{F}_1$ such that $\mathbf{p}_1 = m_1\mathbf{d}_1$ and $\mathbf{p}_2 = m_2\mathbf{d}_2$.

      i. Show that we may decompose $\mathbf{d}_1$ and $\mathbf{d}_2$ according to the following formulae:

      $$\mathbf{d}_1 = \mu_1(\mathbf{d} + \nu_1\mathbf{e}), \quad \mathbf{d}_2 = \mu_2(\mathbf{d} - \nu_2\mathbf{e}),$$

      where $\mu_1, \mu_2, \nu_1, \nu_2 \in \mathbb{P}$ and $\mathbf{e} \in \{\mathbf{d}\}^\perp$ is such that $|\mathbf{e}| = 1$.

      ii. Show that $\nu_1\mu_1 m_1 = \nu_2\mu_2 m_2$.

      iii. Find formulae for μ_1 and μ_2 in terms of m, m_1, and m_2.

   (b) Consider the case that $\mathbf{p}_1$ is the world-momentum of a material particle and $\mathbf{p}_2$ is the world-momentum of a photon. Then we may determine $\mathbf{d}_1 \in \mathcal{F}_1$ such that $\mathbf{p}_1 = m_1\mathbf{d}_1$.

      i. Show that we may decompose $\mathbf{d}_1$ and $\mathbf{p}_2$ according to the following formulae:

      $$\mathbf{d}_1 = \mu_1(\mathbf{d} + \nu_1\mathbf{e}), \quad \mathbf{p}_2 = f_2(\mathbf{d} - \mathbf{e}),$$

      where $\mu_1, \nu_1 \in \mathbb{P}$, $\mathbf{e} \in \{\mathbf{d}\}^\perp$ is such that $|\mathbf{e}| = 1$, and f_2 is the frequency of the photon relative to $\mathbf{d}$.

      ii. Show that $f_2 = \nu_1\mu_1 m_1$.

      iii. Find formulae for μ_1 and f_2 in terms of m and m_1.

   (c) Consider the case that $\mathbf{p}_1$ and $\mathbf{p}_2$ are the world-momenta of two photons.

      i. Show that we may decompose $\mathbf{p}_1$ and $\mathbf{p}_2$ according to the following formulae:

      $$\mathbf{p}_1 = f_1(\mathbf{d} + \mathbf{e}), \quad \mathbf{p}_2 = f_2(\mathbf{d} - \mathbf{e}),$$

      where f_1 and f_2 are the frequencies of the photons relative to $\mathbf{d}$ and $\mathbf{e} \in \{\mathbf{d}\}^\perp$ is such that $|\mathbf{e}| = 1$.

ii. Show that

$$f_1 = f_2 = \frac{m}{2}.$$