Chapter 3

Flat Eventworlds

In this chapter, we introduce another structure important to the theory of special relativity – the structure of a flat space. We assume some familiarity on the part of the reader with linear spaces; for reference, a brief summary of important results is included in Appendix D.

We begin with introductory concepts in §**3.1**, and proceed immediately to discuss eventworlds in the context of flat spaces in §**3.2** and timed eventworlds in the context of flat spaces in §**3.3**. We conclude in §**3.4** with some topological considerations. (This last section may be omitted without loss by the reader unfamiliar with topological concepts.)

3.1 Flat Spaces

—Translation Groups

Two concepts that are often confused in mathematics are "point" and "vector". We often see a pair of real numbers (x, y) representing both a "point in the plane with coordinates x and y", and the "vector" which represents, roughly, the "translation necessary to take a point with coordinates a and b to a point with coordinates $a + x$ and $b + y$". In the plane, we often label p and $\mathbf{v}$ with the same pair; namely, $(3, 2)$ (see Figure 31a).

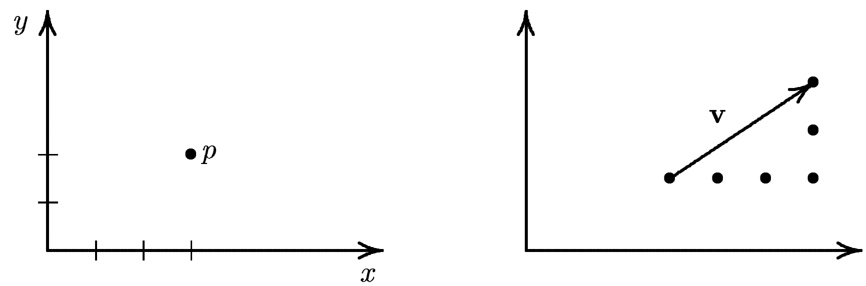

Figure 31a

In applying abstract linear algebra to special relativity, it is critical that these ideas be conceptually distinguished.

Illustration: Suppose $\mathcal{P}$ is the set of all points in the plane. How might we distinguish points from vectors? One way is to make *explicit* the idea that vectors represent, roughly, translations.

Imagine that you are sitting at your desk, and on your desk are marked two points, say x and y (see Figure 31b(1)). (Points on your desk are "•"'s in the figures.) You then square a small sheet of clear plastic on your desk, and mark the point x on this sheet (see Figure 31b(2)). (Points on the plastic sheet are "+"'s in the figures.) Now imagine sliding the plastic sheet so that the point x on your sheet ends up directly over the point y on your desk (see Figures 31b(3),(4)). This "sliding" corresponds, roughly, to a translation. In this context, a translation may be considered as a mapping from the plane into itself; that is, a mapping which assigns to each point x in the plane the point where x "ends up" after the translation. In the previous example, if the translation given is represented by the mapping $\mathbf{v} : \mathcal{P} \to \mathcal{P}$, we would have $\mathbf{v}(x) = y$; *i.e.*, "x ends up at y after $\mathbf{v}$".

In considering that vectors represent translations, and that translations may be represented as mappings from the plane into itself, the set of all vectors may be considered a subset of $\mathsf{Map}(\mathcal{P}, \mathcal{P})$; *i.e.*, the set of all mappings from $\mathcal{P}$ into itself. But how can we describe this subset? How can we state in a precise mathematical way what distinguishes this set of mappings from any other set of mappings?

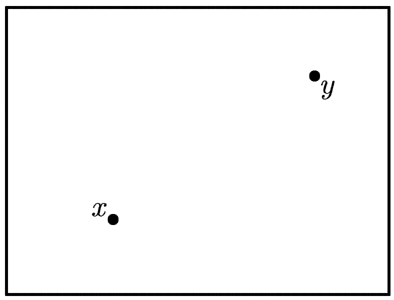

Figure 31b(1)

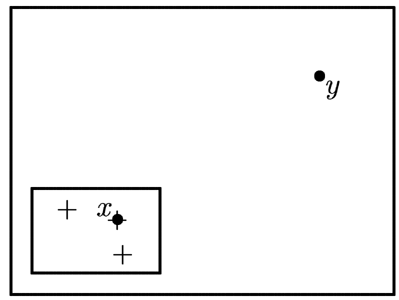

Figure 31b(2)

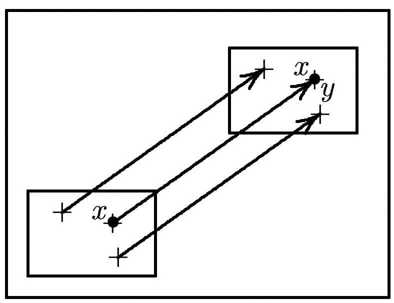

Figure 31b(3)

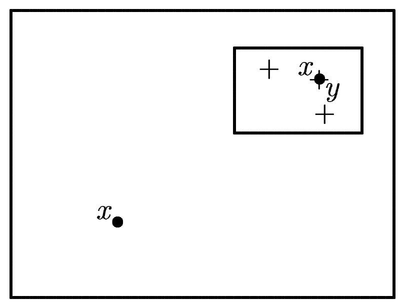

Figure 31b(4)

We now investigate a structure which allows us to describe mathematically the intuitive ideas of points and translations. This can be efficiently accomplished by considering certain properties of the set of *all* translations rather than describing individual translations. (For the purposes of the following examples, it is useful to imagine sitting at an "infinitely large desk" with an "infinitely large sheet of plastic" on top of it.)

Let $\mathcal{E}$ be a nonempty set. Roughly, $\mathcal{E}$ may be considered to be the set of all "points in the plane", or "points in space". As mentioned earlier, a translation $\mathbf{v}$ may be considered as a mapping from $\mathcal{E}$ into $\mathcal{E}$. If we denote by $\mathcal{V}$ the set of all translations, we see that $\mathcal{V} \subset \mathsf{Map}(\mathcal{E}, \mathcal{E})$. In what follows, we will describe properties that a subset $\mathcal{V}$ of $\mathsf{Map}(\mathcal{E}, \mathcal{E})$ must have in order that it corresponds, roughly, to the set of translations as described above. Again, note the distinction between the set of points, $\mathcal{E}$, and the set of translations, $\mathcal{V}$.

3100 Definition: *Let a nonempty set $\mathcal{E}$ be given. A subset $\mathcal{V}$ of* $\mathsf{Map}(\mathcal{E},\mathcal{E})$ *is called a* **translation group of** $\mathcal{E}$ *if the following axioms are satisfied:*

(V_1) *For all $\mathbf{u},\mathbf{v} \in \mathcal{V}$, we have $\mathbf{u} \circ \mathbf{v} \in \mathcal{V}$. ($\mathcal{V}$ is stable under composition.)*

(V_2) *For all $\mathbf{u},\mathbf{v} \in \mathcal{V}$, $\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$. ($\mathcal{V}$ is commutative.)*

(V_3) *For all $x, y \in \mathcal{E}$, there is some $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v}(x) = y$. ($\mathcal{V}$ acts transitively on $\mathcal{E}$.)*

What does this all mean in the context of the plastic sheet analogy? (V_1) means that the result of performing two successive translations is again a translation (see Figure 31c(1)).

(V_2) says that in performing one translation after another, the order in which they are performed is irrelevant; the end result is the same (see Figure 31c(3)).

(V_3) says, roughly, that given any two points, there is some translation that sends one to the other. In other words, given two points $x, y \in \mathcal{E}$, there is some translation that fills in the following blank: "x ends up at y after __" (see Figure 31c(4)).

Remark: The knowledgeable reader will notice the use of the term "group" in the previous definition. He or she might have expected axioms such as "the identity of $\mathcal{E}$ belongs to $\mathcal{V}$" and "$\mathcal{V}$ is stable under inversion". Such "axioms" actually follow from (V_1)–(V_3). The former is given in **Prop. 3102**, while the fact that $\mathcal{V}$ is stable under inversion is given in **Prop. 3103** and illustrated in Figure 31c(2).

Recall the plastic sheet analogy and the type of "sliding" that corresponds to a translation. Given this, think for a moment about a translation which has a *fixed point*; that is, a point x for which "x ends up at x after the translation". Such a translation must, in fact, leave *every* point fixed; this is verified in the following Proposition.

3101 Proposition: *If $\mathbf{v} \in \mathcal{V}$ is such that $\mathbf{v}$ has a fixed point; that is, $\mathbf{v}(x) = x$ for some $x \in \mathcal{E}$, then $\mathbf{v} = 1_{\mathcal{E}}$; that is, $\mathbf{v}(y) = y$ for all $y \in \mathcal{E}$.*

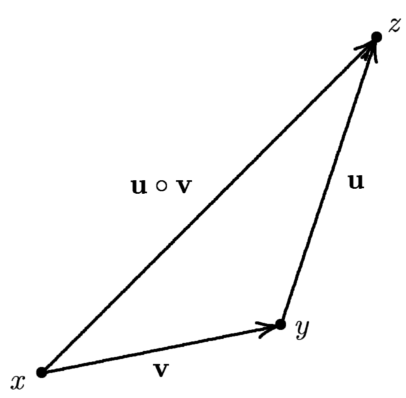

Figure 31c(1)

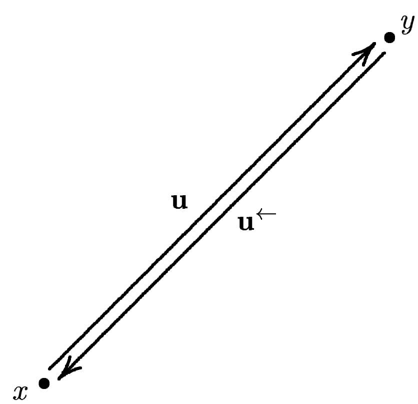

Figure 31c(2)

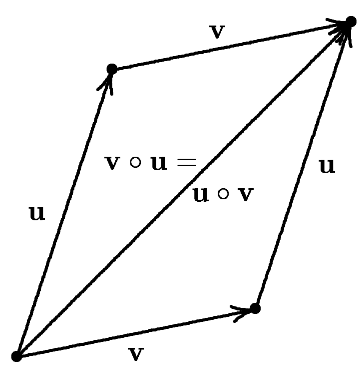

Figure 31c(3)

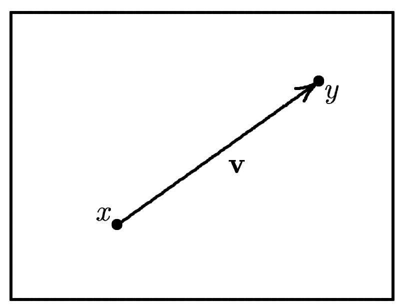

Figure 31c(4)

Proof: Let $\mathbf{v} \in \mathcal{V}$ be given, and suppose that $\mathbf{v}$ has a fixed point, say $x \in \mathcal{E}$. We must show that $\mathbf{v}(y) = y$ for every point $y \in \mathcal{E}$. To this end, let $y \in \mathcal{E}$ be given. Then by (V_3), we may choose a translation $\mathbf{u} \in \mathcal{V}$ such that $y = \mathbf{u}(x)$. So

$$\begin{aligned}
\mathbf{v}(y) &= \mathbf{v}(\mathbf{u}(x)) \\
&= (\mathbf{v} \circ \mathbf{u})(x) \\
&= (\mathbf{u} \circ \mathbf{v})(x) \text{ by } (V_2) \\
&= \mathbf{u}(\mathbf{v}(x)) \\
&= \mathbf{u}(x) \qquad \text{since } \mathbf{v}(x) = x \\
&= y.
\end{aligned}$$

Since y was arbitrary in $\mathcal{E}$, the Proposition is proved. $\Diamond$

A second Proposition follows easily from the first.

3102 Proposition: $1_{\mathcal{E}} \in \mathcal{V}$.

Proof: Since $\mathcal{E}$ is not empty, choose $x \in \mathcal{E}$. By (V_3), we may choose $\mathbf{v} \in \mathcal{V}$ such that $\mathbf{v}(x) = x$. (Note that (V_3) does not require that x and y be distinct.) By the previous Proposition, it follows that $\mathbf{v} = 1_{\mathcal{E}}$. Hence, $1_{\mathcal{E}} = \mathbf{v} \in \mathcal{V}$. $\Diamond$

The following Propositions are stated without proof; the reader is urged to construct the proofs so as to gain familiarity with the above concepts and notations.

3103 Proposition: *For all* $\mathbf{v} \in \mathcal{V}$, $\mathbf{v}$ *is invertible and* $\mathbf{v}^{\leftarrow} \in \mathcal{V}$.

3104 Proposition: *For all* $x, y \in \mathcal{E}$, *there is exactly one* $\mathbf{v} \in \mathcal{V}$ *such that* $\mathbf{v}(x) = y$.

Notation: This Proposition essentially states that given $x, y \in \mathcal{E}$, there is exactly one translation from x to y. Hence, we may speak of *the* translation from x to y. We write $y - x$ for the translation from x to y; hence, $(y - x)(x) = y$. This notation will facilitate the description of many properties of vectors; this will be seen a little later on in this section.

We now introduce notation which is commonly used in discussing translations.

Notation:

(N_1)	$\mathbf{0} := 1_{\mathcal{E}}$	when regarded as an element of $\mathcal{V}$.
(N_2)	$\mathbf{v} + \mathbf{u} := \mathbf{u} \circ \mathbf{v}$	for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.
(N_3)	$-\mathbf{u} := \mathbf{u}^{\leftarrow}$	for all $\mathbf{u} \in \mathcal{V}$.
(N_4)	$\mathbf{v} - \mathbf{u} := \mathbf{v} + (-\mathbf{u})$	for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.
(N_5)	$x + \mathbf{v} := \mathbf{v}(x)$	for all $x \in \mathcal{E}$, $\mathbf{v} \in \mathcal{V}$.
(N_6)	$x - \mathbf{v} := x + (-\mathbf{v})$	for all $x \in \mathcal{E}$, $\mathbf{v} \in \mathcal{V}$.

This notation corresponds to the usual usage of the term "vector". With this in mind, we use "vector" as a synonym for "translation". Geometrically, we have:

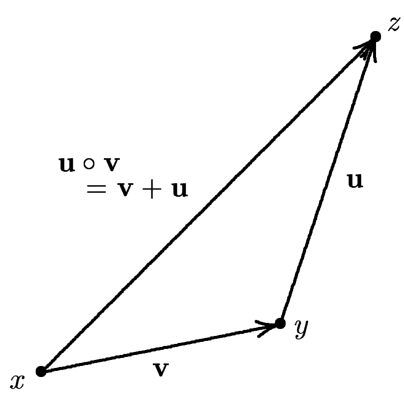

Figure 31d(1)

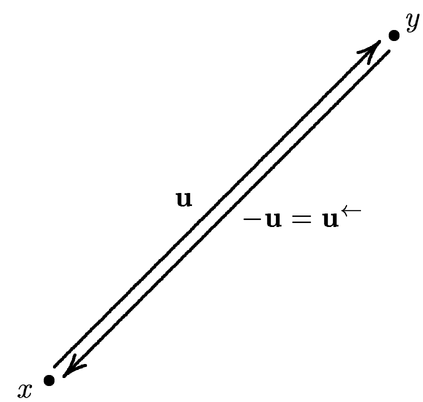

Figure 31d(2)

If $\mathbf{v}$ is the translation from x to y and $\mathbf{u}$ is the translation from y to z, then since $(\mathbf{u} \circ \mathbf{v})(x) = \mathbf{u}(\mathbf{v}(x)) = \mathbf{u}(y) = z$, $\mathbf{u} \circ \mathbf{v}$ is the translation from x to z. In considering translations as vectors, we see that $\mathbf{u} \circ \mathbf{v}$ corresponds to what we usually call "the sum of the vectors $\mathbf{u}$ and $\mathbf{v}$", and hence the notation is consistent with our experience of the addition of vectors (see Figure 31d(1)).

Similarly, if $\mathbf{u} = y - x$, *i.e.*, $\mathbf{u}$ is the translation from x to y, then $-\mathbf{u} = \mathbf{u}^{\leftarrow}$ is the inverse translation, *i.e.*, the translation from y to x. Hence, $-\mathbf{u} = x - y$ (see Figure 31d(2)).

$\mathbf{0}$ corresponds to the translation which leaves every point fixed; since for every translation $\mathbf{v}$ we have $1_{\mathcal{E}} \circ \mathbf{v} = \mathbf{v} \circ 1_{\mathcal{E}} = \mathbf{v}$, it follows from (N_1) and (N_2) that $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$, as we would expect with vector addition.

The notation (N_5) must be used with great care; a second use of the symbol "+" is introduced. In this context, we write $x + \mathbf{v}$ for "the place that x ends up after $\mathbf{v}$". Note that an expression like $\mathbf{v} + x$ is not meaningful; the second usage of "+" requires a point before the "+" symbol and a vector afterwards. Thus, although $x + \mathbf{0} = x$, it makes no sense to say that $\mathbf{0} + x = x$.

Why bother introducing all these notations? As it happens, many of the rules that we usually associate with the symbols "$\mathbf{0}$", "$+$", and "$-$" remain valid when dealing with meaningful expressions. Several examples follow.

For all $x, y \in \mathcal{E}$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}$, we have

$$\begin{array}{lrcl}
(R_1) & x + (y - x) & = & y \\
(R_2) & (x + \mathbf{u}) - x & = & \mathbf{u} \\
(R_3) & x - x & = & \mathbf{0} \\
(R_4) & \mathbf{u} - \mathbf{u} & = & \mathbf{0} \\
(R_5) & x + (\mathbf{u} + \mathbf{v}) & = & (x + \mathbf{u}) + \mathbf{v} \\
(R_6) & x + \mathbf{u} = x + \mathbf{v} & \Longrightarrow & \mathbf{u} = \mathbf{v} \\
(R_7) & x + \mathbf{u} = y + \mathbf{u} & \Longrightarrow & x = y \\
(R_8) & (x - y) + \mathbf{v} & = & (x + \mathbf{v}) - y \\
(R_9) & (x - y) + (y - z) & = & (x - z).
\end{array}$$

(R_1) is essentially the restatement of a previous remark on notation.

How may we interpret (R_2)? We know that $(x + \mathbf{u}) - x$ is the translation from x to $x + \mathbf{u}$. We also know that $\mathbf{u}$ takes the point x to the point $x + \mathbf{u}$. Hence, $(x + \mathbf{u}) - x$ and $\mathbf{u}$ are the same translation.

We illustrate a proof of (R_9); the reader is encouraged to attempt the proofs of other rules in order to become familiar with the notations and definitions.

To see (R_9), we first note that

$$(R_{10}) \quad \text{For all } \mathbf{u}, \mathbf{v} \in \mathcal{V}, \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}.$$

This follows from (N_2) and (V_2).

Now $(x - z)$ is the translation which takes z to x. $(y - z) + (x - y)$ is also a translation which takes z to x (z is taken to y by $(y - z)$, and then y is taken to x by $(x - y)$). Hence, by **Prop. 3104**, they are the same translation, *i.e.*, $(y - z) + (x - y) = (z - x)$. (R_9) follows easily upon applying (R_{10}).

Other rules may be validated similarly. Although detailed, step-by-step proofs may be given to justify each rule, all that is usually required is to return to the notational definitions and supply an informal argument. The following proof is a formal verification of (R_9); however, it masks the essentially intuitive ideas of the proof.

Verification of (R_9): Let $x, y, z \in \mathcal{E}$ be given. Our strategy is as follows: we show that

$$z + ((x - y) + (y - z)) = z + (x - z),$$

and then use (R_6). To this end,

$$\begin{aligned} z + ((x - y) + (y - z)) &= z + ((y - z) + (x - y)) && \text{by } (R_{10}) \\ &= (z + (y - z)) + (x - y) && \text{by } (R_5) \\ &= y + (x - y) && \text{by } (R_1) \\ &= x && \text{by } (R_1) \\ &= z + (x - z) && \text{by } (R_1). \end{aligned}$$

Hence, we may apply (R_6) to conclude that $(x - y) + (y - z) = (x - z)$. As $x, y, z \in \mathcal{E}$ were arbitrary, we conclude that (R_9) holds.

Remark: It is worth noting that using the notations (N_1)–(N_6), it follows that (A_1)–(A_4) as described in Appendix D are also valid results.

—Flat Spaces

The concept of a translation group, as it happens, is insufficient for the purpose of developing the concepts needed both in classical geometry and the theory of special relativity. A "richer" structure is needed; that is, we need more mathematical structure so that we may easily express subtler and more varied distinctions. (Analogously, one may prepare a greater variety of dishes with a more fully stocked pantry.)

One may then ask, "What ideas must we be capable of expressing that can't be described by the theory of translation groups?" One answer to the question is "the idea of a *multiple* of a vector". But one might contend, "We may easily describe multiples of vectors. For example, 3 times a given vector $\mathbf{v}$ is just $\mathbf{v} + \mathbf{v} + \mathbf{v}$. Or -2 times a vector $\mathbf{v}$ is $-(\mathbf{v} + \mathbf{v})$. Any integral multiple of a vector may likewise be expressed. Even the idea of a fractional multiple of a vector is possible. For example, $\frac{1}{5}$ of $\mathbf{v}$ is the only vector $\mathbf{u}$ such that 5 times $\mathbf{u}$ gives the vector $\mathbf{v}$. Other fractional multiples may be handled similarly." And indeed, that would be correct (as long as you assumed that the notion of fractional multiples makes sense; *i.e.*, that given a vector $\mathbf{v}$, there is exactly one vector $\mathbf{u}$ such that 5 times $\mathbf{u}$ gives $\mathbf{v}$). If such assumptions are made, we might construct a theory of *rational* multiples of vectors. But how does one describe arbitrary real multiples of vectors, like π times $\mathbf{v}$? These questions must be answerable so that physical phenomena germane to the theory of special relativity may be easily described. This is a motivation for the theory of flat spaces.

3105 Definition: *A* **flat space** *is a nonempty set $\mathcal{E}$ structured by prescribing two ingredients. The first ingredient is a subset $\mathcal{V}$ of $\mathsf{Map}(\mathcal{E},\mathcal{E})$ which is a translation group of $\mathcal{E}$. The second is a mapping*

$$\mathsf{sm} : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$$

satisfying axioms (S_1)–(S_4) as given in Appendix D. In other words, with the additive structure as described in the previous section, sm *gives $\mathcal{V}$ the structure of a linear space. $\mathcal{V}$ is called the* **translation space of** $\mathcal{E}$. *If $\mathcal{V}$ is finite-dimensional, then the* **dimension of** $\mathcal{E}$, *denoted by* $\dim \mathcal{E}$, *is given by* $\dim \mathcal{E} := \dim \mathcal{V}$. *We put* $\mathcal{V}^{\times} := \mathcal{V} \setminus \{\mathbf{0}\}$.

Notation: We introduce the following notations, which are essentially extensions of (N_2), (N_3), (N_5), and (N_6). Here, $x \in \mathcal{E}$, $\mathbf{v} \in \mathcal{V}$, $\mathcal{G}, \mathcal{H} \subset \mathcal{E}$, and $\mathcal{U}, \mathcal{W} \subset \mathcal{V}$.

$$\begin{array}{lrl}
(N_7) & \mathcal{H} + \mathcal{U} & := \{x + \mathbf{u} \,|\, x \in \mathcal{H}, \mathbf{u} \in \mathcal{U}\}, \\
(N_8) & \mathcal{W} + \mathcal{U} & := \{\mathbf{w} + \mathbf{u} \,|\, \mathbf{w} \in \mathcal{W}, \mathbf{u} \in \mathcal{U}\}, \\
(N_9) & x + \mathcal{U} & := \{x\} + \mathcal{U}, \\
(N_{10}) & \mathcal{H} + \mathbf{v} & := \mathcal{H} + \{\mathbf{v}\}, \\
(N_{11}) & \mathcal{H} - \mathcal{G} & := \{y - x \,|\, x \in \mathcal{G}, y \in \mathcal{H}\}, \\
(N_{12}) & -\mathcal{U} & := \{-\mathbf{u} \,|\, \mathbf{u} \in \mathcal{U}\}.
\end{array}$$

We may define in the obvious way analogues to (N_7)–(N_{10}) where "+" is replaced by "−".

3106 Definition: *Let a subspace $\mathcal{U}$ of $\mathcal{V}$ be given. We say that a nonempty subset $\mathcal{H}$ of $\mathcal{E}$ is a* **flat in** $\mathcal{E}$ **with direction space** $\mathcal{U}$ *if $\mathcal{H} + \mathcal{U} = \mathcal{H}$ and $\mathcal{H} - \mathcal{H} = \mathcal{U}$. When $\mathcal{U}$ is finite-dimensional, we define* $\dim \mathcal{H} := \dim \mathcal{U}$. *If $\mathcal{H}$ is a flat in $\mathcal{E}$ such that* $\dim \mathcal{H} = 1$, *then we say that $\mathcal{H}$ is a* **straight line**.

In addition to defining straight lines in $\mathcal{E}$, we may define line segments in $\mathcal{E}$.

3107 Definition: *For all $x, y \in \mathcal{E}$, we define*

$$[x, y] := \{x + \alpha(y - x) \mid \alpha \in [0, 1]\}.$$

$[x, y]$ is called the **line segment from x to y**. *We also define $]x, y[:= [x, y] \setminus \{x, y\}$, $[x, y[:= [x, y] \setminus \{y\}$, and $]x, y] := [x, y] \setminus \{x\}$.*

We now introduce a Theorem which will show how, in many situations, a set can be endowed with the structure of a flat space in a natural way.

3108 Theorem: *Let a nonempty set $\mathcal{E}$, a linear space $\mathcal{W}$, and a mapping $\mathsf{D} : \mathcal{E} \times \mathcal{E} \to \mathcal{W}$ be given which satisfy*

(1) $\mathsf{D}(x, z) = \mathsf{D}(x, y) + \mathsf{D}(y, z)$ *for all $x, y, z \in \mathcal{E}$, and*

(2) *For all $x \in \mathcal{E}$ and $\mathbf{w} \in \mathcal{W}$, there is exactly one $y \in \mathcal{E}$ such that $\mathsf{D}(x, y) = \mathbf{w}$.*

Then $\mathcal{E}$ may be given in exactly one way the structure of a flat space with translation space $\mathcal{V}$ such that there is a mapping $\varphi : \mathcal{W} \to \mathcal{V}$ with the following properties:

(i) $\varphi(\mathsf{D}(x, y)) = y - x$ *for all $x, y \in \mathcal{E}$, and*

(ii) $\varphi(\alpha\mathbf{w}) = \alpha\varphi(\mathbf{w})$ *for all $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathcal{W}$.*

The mapping φ is linear and invertible.

Proof: The condition (2) above can be expressed as follows:

($2'$) *For every $\mathbf{w} \in \mathcal{W}$, there is exactly one $\psi_{\mathbf{w}} \in \mathsf{Map}(\mathcal{E}, \mathcal{E})$ such that $\mathsf{D}(x, \psi_{\mathbf{w}}(x)) = \mathbf{w}$ for all $x \in \mathcal{E}$.*

Now assume that a flat space structure on $\mathcal{E}$ with translation space $\mathcal{V}$ and a mapping $\varphi : \mathcal{W} \to \mathcal{V}$ satisfying (i) and (ii) are given, and choose $q \in \mathcal{E}$. It follows from (i) that $\varphi(\mathsf{D}(q, q + \mathbf{v})) = \mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$, showing that φ is surjective. It follows from ($2'$) and (i) that $\varphi(\mathbf{w}) = \psi_{\mathbf{w}}(x) - x$, and hence

$$\varphi(\mathbf{w})(x) = x + \varphi(\mathbf{w}) = \psi_{\mathbf{w}}(x)$$

for all $\mathbf{w} \in \mathcal{W}$ and $x \in \mathcal{E}$. Hence we have $\varphi(\mathbf{w}) = \psi_{\mathbf{w}}$ for all $\mathbf{w} \in \mathcal{W}$. Using the surjectivity of φ, we conclude that

$$\mathcal{V} = \{\psi_{\mathbf{w}} \,|\, \mathbf{w} \in \mathcal{W}\} \tag{31.1}$$

and that φ is uniquely determined. It can also be shown that the linear-space structure on $\mathcal{V}$ is uniquely determined by the linear-space structure on $\mathcal{W}$.

To prove the existence of the flat space structure on $\mathcal{E}$ with the prescribed properties, we *define* $\mathcal{V}$ by (31.1). Let $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$ and $x \in \mathcal{E}$ be given. Using (1) with $y := \psi_{\mathbf{w}}(x)$ and $z := \psi_{\mathbf{w}'}(y) = (\psi_{\mathbf{w}'} \circ \psi_{\mathbf{w}})(x)$, we see that

$$\mathsf{D}(x, (\psi_{\mathbf{w}'} \circ \psi_{\mathbf{w}})(x)) = \mathsf{D}(x, \psi_{\mathbf{w}}(x)) + \mathsf{D}(y, \psi_{\mathbf{w}'}(y)),$$

and hence, by (2′), that

$$\mathsf{D}(x, (\psi_{\mathbf{w}'} \circ \psi_{\mathbf{w}})(x)) = \mathbf{w} + \mathbf{w}' = \mathsf{D}(x, \psi_{\mathbf{w}+\mathbf{w}'}(x)).$$

Since $x \in \mathcal{E}$ was arbitrary, it follows from the uniqueness of $\psi_{\mathbf{w}+\mathbf{w}'}$ that

$$\psi_{\mathbf{w}+\mathbf{w}'} = \psi_{\mathbf{w}'} \circ \psi_{\mathbf{w}}. \tag{31.2}$$

Since (31.2) holds for all $\mathbf{w}, \mathbf{w}' \in \mathcal{W}$, it follows from the definition of $\mathcal{V}$ (see (31.1)) that $\mathcal{V}$ satisfies the conditions (V_1) and (V_2) for a translation group (see **Def. 3100**). The condition (V_3) is satisfied because, given $x, y \in \mathcal{E}$, we have from (2′) that $\psi_{\mathbf{w}}(x) = y$ when $\mathbf{w} = \mathsf{D}(x, y)$. We now define $\varphi : \mathcal{W} \to \mathcal{V}$ by $\varphi(\mathbf{w}) := \psi_{\mathbf{w}}$ for all $\mathbf{w} \in \mathcal{W}$, and the scalar multiplication on $\mathcal{V}$ by $\alpha\psi_{\mathbf{w}} := \psi_{\alpha\mathbf{w}}$ for all $\alpha \in \mathbb{R}$ and $\mathbf{w} \in \mathcal{W}$. It is clear that φ has the properties (i) and (ii) and is surjective and linear. The proof of invertibility is left as an Exercise. $\Diamond$

It very often happens that a nonempty set $\mathcal{E}$, a linear space $\mathcal{W}$, and a mapping $\mathsf{D} : \mathcal{E} \times \mathcal{E} \to \mathcal{W}$ satisfying conditions (1) and (2) of **Thm. 3108** are given. The previous Theorem then tells us that $\mathcal{E}$ has the natural structure of a flat space whose translation space is naturally isomorphic to $\mathcal{W}$ *via* the mapping φ. If this is the case, we normally use φ to identify $\mathcal{W}$ with the translation space of $\mathcal{E}$, except that we use the term **external translation space** for $\mathcal{W}$ to indicate that it is obtained in this way. In many cases, the mapping D is described by the notation $y - x := \mathsf{D}(x, y)$ in the first place, and the condition (i) of **Thm. 3108** is merely a reflection of the identification of $\mathcal{W}$ with the translation space of $\mathcal{E}$.

The previous Theorem has many applications. We discuss a few in the context of familiar concepts.

—–**Examples**

1. Upon examining **Prop. 2403**, it is not difficult to see that the mapping $\mathsf{t}^* : \Gamma \times \Gamma \to \mathbb{R}$ is a mapping as described in **Thm. 3108**. Hence, Γ has the structure of a flat space with external translation space $\mathbb{R}$ (see the discussion following **Not. 2404**).

2. If $\mathcal{V}$ is a linear space, then the subtraction operation is a mapping as described above. Hence, $\mathcal{V}$ may be considered as a flat space which is its own external translation space.

3. Let a flat space $\mathcal{E}$ be given, and let $\mathcal{H}$ be a flat in $\mathcal{E}$ with direction space $\mathcal{U}$. Then the mapping $\mathsf{D} : \mathcal{H} \times \mathcal{H} \to \mathcal{U}$ given by

   $$\mathsf{D}(x, y) := y - x$$

   for all $x, y \in \mathcal{H}$ is a mapping as described in **Thm. 3108**. Hence $\mathcal{H}$ may be considered as a flat space with external translation space $\mathcal{U}$, the flat space structure of which is "inherited" from $\mathcal{E}$.

4. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be flat spaces with translation spaces $\mathcal{V}_1$ and $\mathcal{V}_2$, respectively. (Either $\mathcal{V}_1$ or $\mathcal{V}_2$ (or both) may be external translation spaces.) Then the mapping

   $$\mathsf{D} : (\mathcal{E}_1 \times \mathcal{E}_2) \times (\mathcal{E}_1 \times \mathcal{E}_2) \to \mathcal{V}_1 \times \mathcal{V}_2$$

   given by

   $$\mathsf{D}((x_1, x_2), (y_1, y_2)) := (y_1 - x_1, y_2 - x_2)$$

   for all $(x_1, x_2), (y_1, y_2) \in \mathcal{E}_1 \times \mathcal{E}_2$ is a mapping as described above. (Here, the symbol "$-$" is used with two possibly different meanings; the meaning of each occurrence is clear from the context.) Hence, we may consider $\mathcal{E}_1 \times \mathcal{E}_2$ as a flat space with external translation space $\mathcal{V}_1 \times \mathcal{V}_2$.

3.2 Flat Eventworlds

The structure we have developed thus far has enabled us to find time-parameterizations of worldpaths where the "time" parameter takes on real values. However, we are as yet unable to say what it means for a worldpath to be "continuous" or "smooth". In particular, we should at least be able to ask whether or not a worldpath is differentiable with respect to its time parameter.

The structure of a flat space is well-suited as a context in which to investigate questions of differentiability. But it makes no sense to consider a flat-space structure in a timed eventworld unless that structure is related to the precedence relation and the timelapse function in a reasonable way. We first discuss the relationship between the precedence relation and the flat-space structure. The analogous relationship between the timelapse and the flat-space structure is given in **Def. 3300**.

Let $\mathcal{E}$ be a finite-dimensional flat space with translation space $\mathcal{V}$.

3200 Definition: *We say that a relation* $\triangleleft$ *on* $\mathcal{E}$ *is* **translation-invariant** *if*

$$x \triangleleft y \implies x + \mathbf{v} \triangleleft y + \mathbf{v}$$

for all $x, y \in \mathcal{E}$ *and* $\mathbf{v} \in \mathcal{V}$. *We say that* $\triangleleft$ *is* **connected** *if* $[x, y] \subset [\![x, y]\!]_{\triangleleft}$ *for all* $x, y \in \mathcal{E}$ *such that* $x \triangleleft y$.

3201 Definition: *A subset* $\mathcal{K}$ *of* $\mathcal{V}$ *is said to be a* **linear cone** *if* $\mathcal{K}+\mathcal{K} \subset \mathcal{K}$ *and* $\mathbb{P}^{\times}\mathcal{K} \subset \mathcal{K}$. *That is, for all* $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ *and* $\lambda \in \mathbb{P}^{\times}$, *both* $\mathbf{u} + \mathbf{v} \in \mathcal{K}$ *and* $\lambda\mathbf{u} \in \mathcal{K}$.

Relations on $\mathcal{E}$ which have the properties given in **Def. 3200** are easily described, as in the following.

3202 Theorem: *If* $\triangleleft$ *is a transitive, translation-invariant, and connected relation on* $\mathcal{E}$, *then the subset* $\mathcal{K}$ *of* $\mathcal{V}$ *given by*

$$\mathcal{K} := \{y - x \mid (x, y) \in \mathsf{Gr}(\triangleleft)\}$$

is a linear cone; this cone is called the **direction cone of** $\triangleleft$.

Conversely, if $\mathcal{K} \subset \mathcal{V}$ is a linear cone, then the relation $\lhd$ on $\mathcal{E}$ defined by

$$x \lhd y \;:\Longleftrightarrow\; y - x \in \mathcal{K}$$

for all $x, y \in \mathcal{E}$ is transitive, translation-invariant, and connected, and $\mathcal{K}$ is the direction cone of $\lhd$.

Moreover, given the above notation, $\mathcal{K} \cap (-\mathcal{K}) \subset \{\mathbf{0}\}$ if and only if $\lhd$ is antisymmetric.

Proof: Let $\lhd$ be a transitive, translation-invariant, and connected relation on $\mathcal{E}$, and suppose that $\mathcal{K}$ is given by

$$\mathcal{K} := \{y - x \mid (x, y) \in \mathsf{Gr}(\lhd)\}.$$

To see that $\mathcal{K}$ is a linear cone, let $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ be given. Given $x \in \mathcal{E}$, we have $x \lhd x + \mathbf{u}$ as $\mathbf{u} \in \mathcal{K}$; we also have $x + \mathbf{u} \lhd x + (\mathbf{u} + \mathbf{v})$ since $\mathbf{v} \in \mathcal{K}$. So $x \lhd x + (\mathbf{u} + \mathbf{v})$, and hence $\mathbf{u} + \mathbf{v} \in \mathcal{K}$. Since $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ were arbitrary, we see that $\mathcal{K}$ is stable under addition.

Now let $\lambda \in \mathbb{P}^\times$ and $\mathbf{u} \in \mathcal{K}$ be given. Then we may choose $n \in \mathbb{N}$ such that $\lambda \leq n$. Since $\mathcal{K}$ is stable under addition, it follows readily by induction that $n\mathbf{u} \in \mathcal{K}$. Since $\lhd$ is connected, we have

$$x + \lambda\mathbf{u} \in [x, x + n\mathbf{u}] \subset [\![x, x + n\mathbf{u}]\!]_\lhd,$$

and therefore $x \lhd x + \lambda\mathbf{u}$. Thus, $\lambda\mathbf{u} \in \mathcal{K}$. As $\lambda \in \mathbb{P}^\times$ and $\mathbf{u} \in \mathcal{K}$ were arbitrary, we see that $\mathcal{K}$ is stable under scalar multiplication by strictly positive numbers. It follows that $\mathcal{K}$ is a linear cone.

The remainder of the proof is analogous and is left as an Exercise. $\Diamond$

Remark: If $\lhd$ is a connected relation on $\mathcal{E}$, then given $x \in \mathcal{E}$, we see that x being related to some $y \in \mathcal{E}$ implies that $x \lhd x$. Hence, if $\lhd$ is also translation-invariant, then $\lhd$ fails to be reflexive if and only if $\mathsf{Gr}(\lhd) = \emptyset$ (equivalently, $\mathcal{K} = \emptyset$).

As it happens, some important precedence relations are translation-invariant and connected relations in a flat space. Thus, the previous Theorem gives us a useful means of relating the precedence relation to the flat-space structure of an eventworld. This motivates the following.

3203 Definition: *A* **flat eventworld** *is an eventworld $\mathcal{E}$ (with precedence $\prec$) such that $\mathcal{E}$ has the structure of a flat space (with translation space $\mathcal{V}$) and $\prec$ is translation-invariant and connected. We denote the direction cone of $\prec$ by $\mathcal{F}$, and call $\mathcal{F}$ the* **future cone**.

In light of **Thm. 3202**, we are tempted to conjecture that any linear cone containing $\mathbf{0}$ can induce the structure of a flat eventworld on $\mathcal{E}$. This, however, is not the case; the following Proposition indicates precisely when such a structure may be induced on $\mathcal{E}$.

3204 Proposition: *Let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ be given. If $\mathcal{E}$ is a flat eventworld with future cone $\mathcal{F}$, then $\mathcal{V} = \mathcal{F} - \mathcal{F}$.*

Conversely, if $\mathcal{F}$ is a linear cone satisfying $\mathcal{V} = \mathcal{F} - \mathcal{F}$, then the relation $\prec$ on $\mathcal{E}$ defined by

$$x \prec y \;:\Longleftrightarrow\; x = y \text{ or } y - x \in \mathcal{F}$$

for all $x, y \in \mathcal{E}$ gives $\mathcal{E}$ the structure of a flat eventworld.

Proof: Suppose that $\mathcal{E}$ is a flat eventworld with future cone $\mathcal{F}$. Choose $x \in \mathcal{E}$, and let $\mathbf{v} \in \mathcal{V}$ be given. By **Def. 1200**(2), we may choose $z \in \mathcal{E}$ such that $z \prec x$ and $z \prec x + \mathbf{v}$. Then we have

$$\mathbf{v} = ((x + \mathbf{v}) - z) - (x - z) \in \mathcal{F} - \mathcal{F}.$$

As $\mathbf{v} \in \mathcal{V}$ was arbitrary, the first half of the Proposition is proved.

On the other hand, suppose that $\mathcal{F}$ if a linear cone which satisfies $\mathcal{V} = \mathcal{F} - \mathcal{F}$, and suppose that $\prec$ is defined as above. As a result of **Thm. 3202**, it remains to show that **Def. 1200**(2) is satisfied. To this end, let $x, y \in \mathcal{E}$ be given. Since $\mathcal{V} = \mathcal{F} - \mathcal{F}$, we may determine $\mathbf{u}, \mathbf{v} \in \mathcal{F}$ such that $y - x = \mathbf{v} - \mathbf{u}$; put $z := x - \mathbf{u}$. Clearly, $z \prec x$ since $x - z = \mathbf{u} \in \mathcal{F}$. Moreover, $z \prec y$ since

$$y - z = y - (x - \mathbf{u}) = \mathbf{v} \in \mathcal{F}.$$

As $x, y \in \mathcal{E}$ were arbitrary, we see that **Def. 1200**(2) is satisfied. $\Diamond$

In the context of a flat eventworld, it makes sense to consider the role of straight lines. As it happens, worldpaths which are also subsets of straight lines are very important in special relativity as well as in Galilean spacetimes (see §**4.3**), since they may be used to describe worldpaths of particles that are free from outside influences. Such applications are described in §**6.1**.

However, not every flat eventworld allows for a trouble-free discussion of worldpaths which happen to be subsets of straight lines. Examples of such "troublesome" flat eventworlds will be discussed in the Exercises. We will avoid such difficulties by introducing a subclass of flat eventworlds which will be suitable for our purposes.

3205 Definition: *A flat eventworld is said to be* **genuine** *if whenever* $x, y \in \mathcal{E}$ *are such that* $x \prec y$*, then* $[x, y]$ *is a worldpath from* x *to* y*.*

Let a genuine flat eventworld $\mathcal{E}$ be given.

We are now able to provide a definition and a useful characterization of straight worldpaths.

3206 Definition: *A worldpath* $\mathcal{L}$ *is said to be* **straight** *if for all* $x, y \in \mathcal{L}$*, we have* $[x, y] \subset \mathcal{L}$*.*

3207 Theorem: *Let* $\mathcal{L}$ *be a worldpath. Then* $\mathcal{L}$ *is straight if and only if we may determine a straight line* $\mathcal{S} \subset \mathcal{E}$ *such that* $\mathcal{L} \subset \mathcal{S}$*.*

Proof: Suppose that $\mathcal{L}$ is straight. Choose distinct elements $x, y \in \mathcal{L}$ such that $x \prec y$, and put $\mathcal{S} := x + \mathbb{R}(y - x)$. Clearly, $\mathcal{S}$ is a straight line. We claim that $\mathcal{L} \subset \mathcal{S}$.

To see this, let $q \in \mathcal{L}$ be given. Since $\mathcal{L}$ is totally ordered, at least one of the following cases must apply: $q \prec x$, $x \prec q \prec y$, or $y \prec q$.

Suppose that $q \prec x$. Since $\mathcal{L}$ is straight, we have $[q, y] \subset \mathcal{L}$. It follows from $\prec$ being connected that

$$[q, y] \subset [\![q, y]\!] \cap \mathcal{L}.$$

Since x and y are distinct and $\mathcal{L}$ is ordered, we must have $x \prec y$ and hence $q \prec y$; we then know from **Def. 3205** that $[q, y]$ is a worldpath and hence maximally totally ordered in $[\![q, y]\!]$. Since $\mathcal{L}$ is a worldpath, $[\![q, y]\!] \cap \mathcal{L}$ is also maximally totally ordered in $[\![q, y]\!]$. Hence, the above inclusion must in fact be an equality, yielding $[q, y] = [\![q, y]\!] \cap \mathcal{L}$. Since $[\![x, y]\!] \subset [\![q, y]\!]$ and $\prec$ is connected, we have

$$[x, y] \subset [\![x, y]\!] \cap \mathcal{L} \subset [\![q, y]\!] \cap \mathcal{L} = [q, y].$$

It is easy to see that $y - q$ must be a positive scalar multiple of $y - x$, and hence $q \in \mathcal{S}$.

If either $x \prec q \prec y$ or $y \prec q$, we may argue similarly to conclude that $q \in \mathcal{S}$. Since $q \in \mathcal{L}$ was arbitrary, the forward implication of the Theorem is proved.

The reverse implication is left as an Exercise. $\Diamond$

3208 Corollary: *Let $\mathcal{L}$ be worldpath. Then $\mathcal{L}$ is straight if and only if there is a straight line $\mathcal{H} \subset \mathcal{E}$ such that $\mathcal{L}$ is a genuine interval in $\mathcal{H}$ with respect to $\prec|_{\mathcal{H}\times\mathcal{H}}$ (see* **Def. 1404***). $\mathcal{L}$ is a straight worldline if and only if $\mathcal{L}$ is a straight line in $\mathcal{E}$.*

3.3 Timed Flat Eventworlds

In the previous section, we saw how the precedence could be related to the flat-space structure of an eventworld. It remains to be seen how the timelapse function should be related to this flat-space structure. This, however, is a simple task; the result is as follows.

3300 Definition: *A* **timed flat eventworld** *$\mathcal{E}$ is a genuine flat eventworld (with future cone $\mathcal{F}$) which is also a timed eventworld whose timelapse t satisfies the following two conditions:*

(1) *t is translation-invariant; that is,*

$$\mathsf{t}(x, y) = \mathsf{t}(x + \mathbf{v}, y + \mathbf{v})$$

for all $(x, y) \in \mathsf{Gr}(\prec)$ and all $\mathbf{v} \in \mathcal{V}$, and

(2) *For all $x, y \in \mathcal{E}$ with $x \prec y$, we have*

$$\mathsf{t}_{[x,y]}(x,y) = \mathsf{t}(x,y);$$

i.e., the timelapse from x to y coincides with the timelapse from x to y along the worldpath $[x,y]$ (which is indeed a worldpath since $\mathcal{E}$ is genuine; see **Def. 3205***).*

Remark: Recall that in §**2.2**, we remarked that for $x, y \in \mathcal{E}$ which satisfy $x \prec y$, $\mathsf{t}(x,y)$ is an upper bound for possible timelapses along worldpaths from x to y. The condition (2) above guarantees that this upper bound is actually achieved when $x \prec y$.

Let a timed flat eventworld $\mathcal{E}$ be given with future cone $\mathcal{F}$. As a result of condition (1) in the above definition, we see that the timelapse $\mathsf{t}(x,y)$ depends not on the particular choice of events x and y, but only on the vector $y - x$. With this in mind, we offer **Thm. 3302**, which will allow for a simpler expression of some important ideas. But first, we state and prove a useful preliminary result.

3301 Lemma: *Suppose that a given $f : \mathbb{P} \to \mathbb{P}$ satisfies $f(\alpha + \beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in \mathbb{P}$. Then $f(\alpha) = \alpha f(1)$ for all $\alpha \in \mathbb{P}$.*

Proof: It follows easily by induction that $f(p\alpha) = pf(\alpha)$ for all $p \in \mathbb{N}$ and $\alpha \in \mathbb{P}$. This in turn implies that for all $p \in \mathbb{N}$ and $q \in \mathbb{N}^{\times}$, we have

$$qf\left(\frac{p}{q}\right) = f(p) = pf(1),$$

and hence

$$f\left(\frac{p}{q}\right) = \frac{p}{q}f(1).$$

Now let $\alpha \in \mathbb{P}$ be given, and suppose that $p \in \mathbb{N}$ and $q \in \mathbb{N}^{\times}$ are such that $p/q \leq \alpha$. We see from the preceding argument and the assumption about the additivity of f that

$$f(\alpha) = f\left(\frac{p}{q}\right) + f\left(\alpha - \frac{p}{q}\right) \geq f\left(\frac{p}{q}\right) = \frac{p}{q}f(1).$$

Since this inequality holds for all $p \in \mathbb{N}$ and $q \in \mathbb{N}^{\times}$ with $p/q \leq \alpha$, we must have $f(\alpha) \geq \alpha f(1)$. A similar argument shows that we must also have $f(\alpha) \leq \alpha f(1)$. Combining these inequalities yields the desired conclusion. Since $\alpha \in \mathbb{P}$ was arbitrary, the Lemma is proved. $\Diamond$

Remark: The statement in the previous Lemma is no longer valid if "$\mathbb{P}$" is replaced by "$\mathbb{R}$". In other words, one may show the existence of a mapping $f : \mathbb{R} \to \mathbb{R}$ which satisfies $f(\alpha + \beta) = f(\alpha) + f(\beta)$ for all $\alpha, \beta \in \mathbb{R}$, but which *fails* to satisfy $f(\alpha) = \alpha f(1)$ for all $\alpha \in \mathbb{R}$. Such a demonstration involves the Axiom of Choice.

3302 Theorem: *Let a timed flat eventworld $\mathcal{E}$ be given with future cone $\mathcal{F}$. There is exactly one $\tau : \mathcal{F} \to \mathbb{P}$ such that*

(1) $\mathsf{t}(x, y) = \tau(y - x)$ *for all* $(x, y) \in \mathsf{Gr}(\prec)$.

This τ satisfies

(2) $\tau(\mathbf{u}) + \tau(\mathbf{v}) \leq \tau(\mathbf{u} + \mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in \mathcal{F}$, *and*

(3) $\tau(\alpha \mathbf{u}) = \alpha \tau(\mathbf{u})$ *for all* $\mathbf{u} \in \mathcal{F}$ *and* $\alpha \in \mathbb{P}$.

Conversely, let $\mathcal{E}$ be a genuine flat eventworld, and let $\tau : \mathcal{F} \to \mathbb{P}$ be a function satisfying (2) *and* (3). *If $\mathsf{t} : \mathsf{Gr}(\prec) \to \mathbb{P}$ is defined by* (1), *then t endows $\mathcal{E}$ with the structure of a timed flat eventworld.*

We call τ the **time-span function**.

Proof: We see from **Def. 3300**(1) that for a given $\mathbf{v} \in \mathcal{F}$, the value $\mathsf{t}(x, x + \mathbf{v})$ is independent of $x \in \mathcal{E}$. We may define $\tau(\mathbf{v})$ to be this value, yielding (1). Condition (2) is then merely a restatement of the Intermediate Event Inequality (**Def. 2100**).

To see (3), let $\mathbf{u} \in \mathcal{F}$ be given. If $\mathbf{u} \in -\mathcal{F}$, then it easy to see (as a result of **Prop. 2101**) that $\tau(\alpha \mathbf{u}) = 0$ for all $\alpha \in \mathbb{P}^{\times}$, and hence (3) is valid. Otherwise, we have $\mathbf{u} \in \mathcal{F} \setminus (-\mathcal{F})$. Now let $\alpha, \beta \in \mathbb{P}^{\times}$ be given and put

$$y := x + \alpha \mathbf{u}, \quad z := y + \beta \mathbf{u} = x + (\alpha + \beta)\mathbf{u}.$$

Since $\mathbf{u} \in \mathcal{F} \setminus (-\mathcal{F})$, we have $x \prec z$, and hence $[x, z]$ is worldpath from x to z since $\mathcal{E}$ is genuine (see **Def. 3205**). Moreover, we have $\{x, y, z\} \in \mathsf{Fto}[x, z]$ (see **Def. 2200**). Thus, we see from (1), **Def. 2202**, and **Def. 3300**(2) that

$$\begin{aligned} \tau(\alpha\mathbf{u}) + \tau(\beta\mathbf{u}) &= \mathsf{t}(x, y) + \mathsf{t}(y, z) \\ &= \Sigma(\{x, y, z\}) \\ &\geq \mathsf{t}|_{[x,z]}(x, z) \\ &= \mathsf{t}(x, z) \\ &= \tau((\alpha + \beta)\mathbf{u}). \end{aligned}$$

This, with (2), implies that

$$\tau(\alpha\mathbf{u}) + \tau(\beta\mathbf{u}) = \tau((\alpha + \beta)\mathbf{u}). \tag{33.1}$$

It is also easy to see that (33.1) is valid when either $\alpha = 0$ or $\beta = 0$. Hence, we see that (33.1) is valid for all $\alpha, \beta \in \mathbb{P}$.

Now consider the function $f : \mathbb{P} \to \mathbb{P}$ given by

$$f(\alpha) := \tau(\alpha\mathbf{u})$$

for all $\alpha \in \mathbb{P}$. It follows from (33.1) and **Lemma 3301** that

$$\begin{aligned} \tau(\alpha\mathbf{u}) &= f(\alpha) \\ &= \alpha f(1) \\ &= \alpha\tau(\mathbf{u}) \end{aligned}$$

for all $\alpha \in \mathbb{P}$. Since $\mathbf{u} \in \mathcal{F} \setminus (-\mathcal{F})$ was arbitrary, we see that (3) is valid.

The proof of the converse is left as an Exercise. ◊

Remark: Let $\mathcal{E}$ be a genuine flat eventworld which is also a timed eventworld whose timelapse t is translation-invariant. Then the condition **Def. 3300**(2) above is equivalent to

(2′) *For all* $\mathbf{v} \in \mathcal{F} \setminus (-\mathcal{F})$ *and* $p \in \mathbb{N}^{\times}$, *we have*

$$\tau(p\mathbf{v}) = p\tau(\mathbf{v}).$$

One may use this condition, along with **Thm. 3302**(2), to give an alternative proof of **Thm. 3302**(3).

—Topological Considerations

In the remainder of this section and the next, we discuss issues which arise when a timed flat eventworld is considered as a topological space with the usual flat-space topology (see Chapter 5 of [**7**]). The reader who is unfamiliar with topology may skip this discussion without loss. The only results which will explicitly be used later are **Thm. 3407**, **Prop. 3409**, and **Cor. 3410**.

For this section and the next, let a timed flat eventworld $\mathcal{E}$ be given, whose precedence is denoted by $\prec$ and whose direction cone is denoted by $\mathcal{F}$. We consider $\mathcal{E}$ as being endowed with the structure of a topological space by the prescription of the usual flat-space topology. Since we may view the translation space $\mathcal{V}$ as a flat space (see Example 2 in §**3.1**), we may analogously consider $\mathcal{V}$ as being endowed with the usual flat-space topology.

Recall (see **Prop. 3204**) that $\mathcal{V} = \mathcal{F} - \mathcal{F}$. This is equivalent (since $\mathcal{E}$ is finite-dimensional) to the statement that $\mathcal{F}$ has a nonempty interior. For $\mathcal{S} \subset \mathcal{E}$ (or $\mathcal{V}$), we denote by $\mathsf{Int}\ \mathcal{S}$ the topological interior of $\mathcal{S}$, and by $\mathsf{Clo}\ \mathcal{S}$ the topological closure of $\mathcal{S}$.

In the following Theorem, we use τ as described in **Thm. 3302**.

3303 Theorem: *τ is continuous at $\mathbf{0}$. Hence, t is continuous at (x,x) for all $x \in \mathcal{E}$.*

Proof: We must show that for all $\varepsilon \in \mathbb{P}^{\times}$, there is some neighborhood $\mathcal{O}$ of $\mathbf{0}$ in $\mathcal{V}$ such that $\tau_{>}(\mathcal{O} \cap \mathcal{F}) \subset [0, \varepsilon[$.

Let $\varepsilon \in \mathbb{P}^{\times}$ be given, and choose $\mathbf{u} \in \mathsf{Int}\ \mathcal{F}$. Determine $\gamma \in \mathbb{P}^{\times}$ such that $\gamma\tau(\mathbf{u}) < \varepsilon$, and put $\mathcal{O} := \gamma\mathbf{u} - \mathcal{F}$. Since $\mathbf{u} \in \mathsf{Int}\ \mathcal{F}$, $\mathcal{O}$ is a neighborhood of $\mathbf{0}$ in $\mathcal{V}$.

Now let $\mathbf{v} \in \mathcal{O} \cap \mathcal{F}$ be given. Then we may choose $\mathbf{f} \in \mathcal{F}$ such that $\mathbf{v} = \gamma\mathbf{u} - \mathbf{f}$. By **Thm. 3302**(2) and (3), we have

$$0 \leq \tau(\mathbf{v}) + \tau(\mathbf{f}) \leq \tau(\gamma\mathbf{u}) = \gamma\tau(\mathbf{u}) < \varepsilon.$$

Hence $\tau(\mathbf{v}) < \varepsilon$. Since $\mathbf{v} \in \mathcal{O} \cap \mathcal{F}$ was arbitrary, we have $\tau_{>}(\mathcal{O} \cap \mathcal{F}) \subset [0, \varepsilon[$. Since $\varepsilon \in \mathbb{P}^{\times}$ was arbitrary, the first statement of the Theorem is proved. The second statement follows immediately from the first. $\Diamond$

Remark: As it happens, the condition **Thm. 3302**(3) is not needed in the foregoing proof. Instead of selecting $\gamma \in \mathbb{P}^{\times}$ such that $\gamma\tau(\mathbf{u}) < \varepsilon$, one may determine $q \in \mathbb{N}^{\times}$ such that $\tau(\mathbf{u}) < q\varepsilon$. With such a determination of q, one may dispense with **Thm. 3302**(3).

Moreover, using **Thm. 3302**(3), one may show that τ is continuous on $\mathsf{Int}\ \mathcal{F}$. One may also construct examples of functions $\tau : \mathcal{F} \to \mathbb{R}$ satisfying **Thm. 3302**(2) but not **Thm. 3302**(3) and failing to be continuous at some $\mathbf{v} \in \mathsf{Int}\ \mathcal{F}$. Details are left to the Exercises.

3.4 Parameterizations

Let a flat eventworld $\mathcal{E}$ with translation space $\mathcal{V}$ and future cone $\mathcal{F}$ be given.

We wish to impose further requirements on $\mathcal{E}$ which, while not being too restrictive, still allow for a discussion of some important results which will be useful later. As a result, we confine our attention to the case when $\mathcal{F}$ is closed because a discussion of parameterizations becomes simpler with this assumption.

Hence, for the remainder of this section, we assume that $\mathcal{F}$ is closed. We note the following consequence of this assumption. The proof is left as an Exercise.

3400 Proposition: *$\mathcal{E}$ is genuine.*

The following two Propositions give conditions sufficient to ensure that a given mapping is the parameterization of a worldpath.

3401 Proposition: *Let I be a genuine interval in $\mathbb{R}$, and $q : I \to \mathcal{E}$ be continuous and strictly isotone; i.e., for all $t_1, t_2 \in I$, $t_1 < t_2 \implies q(t_1) \prec\!\!\!\cdot\ q(t_2)$. Then $\mathcal{L} := \mathsf{Rng}\ q$ is a worldpath and q is a parameterization of $\mathcal{L}$ (see* **Def. 2305***).*

Proof: Since q is strictly isotone, then $\mathcal{L}$ is totally ordered by $\prec\!\!\!\cdot$, and hence by $\prec$. To show that $\mathcal{L}$ is l.m.t.o. (see **Def. 1300**), let $x_1, x_2 \in \mathcal{L}$ be given such that $x_1 \prec\!\!\!\cdot\ x_2$ and let $z \in [\![x_1, x_2]\!]$ be given such that

$\{z\} \cup (\mathcal{L} \cap [\![x_1, x_2]\!])$ is totally ordered. Determine $t_1, t_2 \in I$ such that $x_1 = q(t_1)$ and $x_2 = q(t_2)$. Then $t_1 < t_2$ and

$$\mathcal{L} \cap [\![x_1, x_2]\!] = q_>([t_1, t_2]).$$

The assumptions that $z \in [\![x_1, x_2]\!]$ and that $\{z\} \cup q_>([t_1, t_2])$ is totally ordered imply that the sets

$$J^+ := \{t \in [t_1, t_2] \mid q(t) - z \in \mathcal{F}\} = q^<(z + \mathcal{F})$$

and

$$J^- := \{s \in [t_1, t_2] \mid z - q(s) \in \mathcal{F}\} = q^<(z - \mathcal{F})$$

are nonempty and satisfy $J^+ \cup J^- = [t_1, t_2]$. Since q is continuous, $\mathcal{F}$ is closed, and hence $J^+ = q^<(z+\mathcal{F})$ is closed, it follows that J^- is also closed (in $\mathbb{R}$). Moreover, since q is isotone, we have $s \leq t$ for all $s \in J^-$ and $t \in J^+$. These observations imply that $\sigma := \inf J^+ = \sup J^-$ satisfies $\{\sigma\} = J^+ \cap J^-$. Therefore, we have $q(\sigma) \prec z$ and $z \prec q(\sigma)$. Since the restriction of $\prec$ to $\{z\} \cup q_>([t_1, t_2])$ is antisymmetric, we conclude that $z = q(\sigma)$ and hence that $z \in \mathcal{L} \cap [\![x_1, x_2]\!]$. Since z was arbitrary, it follows that $\mathcal{L}$ is l.m.t.o. Since I is genuine and q is strictly isotone, $\mathcal{L}$ must contain at least two events, and hence $\mathcal{L}$ is a worldpath.

That q is a parameterization of $\mathcal{L}$ follows immediately upon inspection of **Def. 2305**. ◊

3402 Proposition: *Let I be a genuine interval in $\mathbb{R}$, and let $q : I \to \mathcal{E}$ be differentiable with* $\mathsf{Rng}\, q^\bullet \subset \mathcal{F} \setminus (-\mathcal{F})$. *Then* $\mathcal{L} := \mathsf{Rng}\, q$ *is a worldpath, and q is a parameterization of $\mathcal{L}$.*

Proof: Since $\mathcal{F}$ is closed and convex and $\mathsf{Rng}\, q^\bullet \subset \mathcal{F}$, we have by the Difference-Quotient Theorem (see [**7**], §**61**) that

$$q(t_2) - q(t_1) \in (t_2 - t_1)\mathcal{F} \subset \mathcal{F}$$

for all $t_1, t_2 \in I$ such that $t_1 \leq t_2$. Hence q is isotone.

Now assume that q is not strictly isotone. Then we may find $t_1, t_2 \in I$ with $t_1 < t_2$ such that $z := q(t_1) = q(t_2)$. Let $s_1, s_2 \in [t_1, t_2]$ be given such that $s_1 \leq s_2$. Since $t_1 \leq s_1 \leq s_2 \leq t_2$, the isotonicity of q implies that $z \prec q(s_1) \prec q(s_2) \prec z$, and hence $q(s_2) - q(s_1) \in \mathcal{F} \cap (-\mathcal{F})$. Since $s_1, s_2 \in [t_1, t_2]$ were arbitrary and $\mathcal{F} \cap (-\mathcal{F})$ is closed, it follows that $q^{\bullet}_{>}(]t_1, t_2[) \subset \mathcal{F} \cap (-\mathcal{F})$, which is inconsistent with the assumption that $\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F} \setminus (-\mathcal{F})$. Hence q must be strictly isotone. The assertion of the Proposition then follows from **Prop. 3401**. ◊

Remark: When the precedence $\prec$ is relativistic (that is, antisymmetric), the condition "$\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F} \setminus (-\mathcal{F})$" in the previous Proposition is equivalent to the condition "$\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F}^{\times}$".

When the precedence is classical (that is, total), the condition "$\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F} \setminus (-\mathcal{F})$" is equivalent to "$\mathsf{Rng}\, q^{\bullet} \subset \mathsf{Int}\, \mathcal{F}$".

We now discuss a few results which are valid when $\prec$ is relativistic. The proof of the first Proposition is left as an Exercise.

3403 Proposition: *The precedence $\prec$ is relativistic (i.e., $\mathcal{F} \cap (-\mathcal{F}) = \{\mathbf{0}\}$) and $\mathcal{F}$ is closed if and only if for all $x, y \in \mathcal{E}$ such that $x \prec y$, $[\![x, y]\!]$ is closed and bounded.*

3404 Proposition: *Assume that $\prec$ is relativistic. Then every closed, totally ordered, nonempty set of $\mathcal{E}$ bounded below [above] has a mimimum [maximum].*

Proof: Suppose that $\mathcal{S}$ is a closed, totally ordered, nonempty set bounded below by $a \in \mathcal{E}$. For each $q \in \mathcal{S}$, define $\mathcal{T}_q := [\![a, q]\!] \cap \mathcal{S}$, and put $\mathsf{T} := \{\mathcal{T}_q \mid q \in \mathcal{S}\}$. By **Prop. 3403**, T is a collection of closed, bounded, nonempty subsets of $\mathcal{E}$ and is easily seen to be totally ordered by set inclusion. Hence, by a familiar topological result, $\bigcap \mathsf{T} \neq \emptyset$.

Choose $m \in \bigcap \mathsf{T}$. By the definition of T, we must have $m \in \mathcal{S}$, as well as $m \in [\![a, q]\!]$ and hence $m \prec q$ for all $q \in \mathcal{S}$. Hence, since $\prec|_{\mathcal{S}}$ is an order, m must be the minimum of $\mathcal{S}$.

The proof regarding the maximum is analogous. ◊

We introduce the following convenient notation.

Notation: $\mathcal{N} := \mathcal{F} \cup (-\mathcal{F})$.

We remark that for all $x, y \in \mathcal{E}$, x is related to y if and only if $y - x \in \mathcal{N}$. In addition, since $\mathcal{F}$ is closed, it immediately follows that $\mathcal{N}$ is closed.

3405 Proposition: *Assume that $\prec$ is relativistic. Let a totally ordered, nonempty set $\mathcal{S} \subset \mathcal{E}$ be given such that $\mathcal{S}$ is bounded below [above]. Then $\mathcal{S}$ has an infimum [supremum] which belongs to* $\mathsf{Clo}\ \mathcal{S}$.

Proof: We first show that $\mathsf{Clo}\ \mathcal{S}$ is totally ordered, nonempty, and bounded below. To see that $\prec$ restricted to $\mathsf{Clo}\ \mathcal{S}$ is total, let $x, y \in \mathsf{Clo}\ \mathcal{S}$ and $q \in \mathcal{S}$ be given. Since $\mathcal{S}$ is totally ordered, then $\mathcal{S} \subset q + \mathcal{N}$. Since $q + \mathcal{N}$ is closed, it follows that $x \in \mathsf{Clo}\ \mathcal{S} \subset q + \mathcal{N}$. As $\mathcal{N} = -\mathcal{N}$, we see that $q \in x + \mathcal{N}$. This relationship is valid for all $q \in \mathcal{S}$, so that $\mathcal{S} \subset x + \mathcal{N}$. Again, since $x + \mathcal{N}$ is closed, we have that $y \in \mathsf{Clo}\ \mathcal{S} \subset x + \mathcal{N}$. Thus $y \in x + \mathcal{N}$, and hence x is related to y. As $x, y \in \mathsf{Clo}\ \mathcal{S}$ were arbitrary, it follows that $\prec$ on $\mathsf{Clo}\ \mathcal{S}$ is total. Since $\prec$ on $\mathcal{E}$ is an order, so is $\prec$ restricted to $\mathsf{Clo}\ \mathcal{S}$, and hence we see that $\mathsf{Clo}\ \mathcal{S}$ is totally ordered.

We now show that $\mathsf{Clo}\ \mathcal{S}$ is bounded below. To this end, let a be a lower bound for $\mathcal{S}$. Then $\mathcal{S} \subset a + \mathcal{F}$. Since $a + \mathcal{F}$ is closed, we must have $\mathsf{Clo}\ \mathcal{S} \subset a + \mathcal{F}$, and hence a is a lower bound for $\mathsf{Clo}\ \mathcal{S}$. Thus, $\mathsf{Clo}\ \mathcal{S}$ is bounded below. Finally, since $\mathcal{S}$ is nonempty, so is $\mathsf{Clo}\ \mathcal{S}$.

These observations allow us to apply the previous Proposition to $\mathsf{Clo}\ \mathcal{S}$; we denote by m the minimum of $\mathsf{Clo}\ \mathcal{S}$. Clearly, $m \prec q$ for all $q \in \mathcal{S}$ since $\mathcal{S} \subset \mathsf{Clo}\ \mathcal{S}$. To see that m is the greatest lower bound for $\mathcal{S}$, suppose that $m' \in \mathcal{E}$ is such that $m \prec m' \prec q$ for all $q \in \mathcal{S}$. Then we have $\mathcal{S} \subset m' + \mathcal{F}$. Since $m' + \mathcal{F}$ is closed, this results in $m \in \mathsf{Clo}\ \mathcal{S} \subset m' + \mathcal{F}$, and hence $m' \prec m$. The antisymmetry of $\prec$ implies that $m = m'$. Thus, $m \in \mathsf{Clo}\ \mathcal{S}$ is seen to be the infimum of $\mathcal{S}$.

The proof regarding the supremum is analogous. ◊

We assume now and for the remainder of this section that $\mathcal{E}$ has the structure of a timed flat eventworld (see **Def. 3300**). As a result, we are in a position to prove some interesting results about timelapses along worldpaths and time-parameterizations of material worldpaths.

We first prove a result about time-parameterizations of material worldpaths in the case when the precedence is relativistic.

3406 Theorem: *Assume that the precedence is relativistic. Suppose that a material worldpath $\mathcal{L}$, $I \subset \mathbb{R}$, and a time-parameterization $p : I \to \mathcal{E}$ of $\mathcal{L}$ are given. Then I is a genuine interval in $\mathbb{R}$.*

Proof: To see that I is an interval, let $a, b \in I$ be given and suppose that $c \in \mathbb{R}$ is such that $a < c < b$. Define

$$\begin{aligned} J^+ &:= \{r \in I \,|\, c < r \leq b\} \cap [a, b], \\ J^- &:= \{r \in I \,|\, a \leq r \leq c\} \cap [a, b] = (I \cap [a, b]) \setminus J^+, \\ \mathcal{L}^+ &:= p_>(J^+), \\ \mathcal{L}^- &:= p_>(J^-) = \mathcal{L} \setminus \mathcal{L}^+. \end{aligned}$$

Now $\mathcal{L}^+$ is totally ordered and bounded below by $p(a)$, and $\mathcal{L}^-$ is totally ordered and bounded above by $p(b)$. Hence, by **Prop. 3405**, we may define both $e^+ := \inf \mathcal{L}^+$ and $e^- := \sup \mathcal{L}^-$. In addition, we must have $e^+ = e^- \in \mathcal{L}$, otherwise $(\mathcal{L} \cap [\![p(a), p(b)]\!]) \cup [e^-, e^+]$ would be a totally ordered subset of $[\![p(a), p(b)]\!]$ strictly including $\mathcal{L} \cap [\![p(a), p(b)]\!]$, which is impossible as $\mathcal{L}$ is l.m.t.o.

As a result, we may choose $c' \in I$ such that $p(c') = e^- = e^+$. Since $e^+ = \inf \mathcal{L}^+$, it follows from the continuity of $\mathtt{t}$ at (e^+, e^+) (see **Thm. 3303**) that

$$0 = \inf\{\mathtt{t}(e^+, p(t)) \,|\, t \in J^+\}.$$

Now $\mathtt{t}(e^+, p(t)) \geq \mathtt{t}_{\mathcal{L}}(e^+, p(t))$ for all $t \in J^+$, so we have, since $c \leq \inf J^+$,

$$\begin{aligned} 0 &= \inf\{\mathtt{t}(e^+, p(t)) \,|\, t \in J^+\} \\ &\geq \inf\{\mathtt{t}_{\mathcal{L}}(p(c'), p(t)) \,|\, t \in J^+\} \\ &= \inf\{t - c' \,|\, t \in J^+\} \\ &= \inf J^+ - c' \\ &\geq c - c'. \end{aligned}$$

This implies that $c \leq c'$. We may analogously show that $c' \leq c$ by applying the same analysis to $\mathcal{L}^-$; from this we may conclude that $c = c' \in I$. Since $a, b \in I$ and $c \in \mathbb{R}$ were arbitrary, and since $\mathcal{L}$ being a worldpath implies that I contains at least two members, we see that I is a genuine interval in $\mathbb{R}$. ◊

We now assume that τ is continuous. This will allow for a simple description of the timelapse function along material worldpaths.

Although the following result is often used to *define* the timelapse along a material worldpath, it follows from the more general consideration of timelapses discussed in Chapter 2. We leave the technical proof as an Exercise. It is directly analogous to the derivation of the formula for arc length in a Euclidean space.

3407 Theorem: *Let I be a genuine interval in $\mathbb{R}$, and let $q : I \to \mathcal{E}$ be a smooth*[1] *mapping such that $\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F} \setminus (-\mathcal{F})$. Then $\mathcal{L} := \mathsf{Rng}\, q$ is a worldpath, and for all $a, b \in I$, we have*

$$\bar{\mathfrak{t}}_{\mathcal{L}}(q(a), q(b)) = \int_a^b \tau \circ q^{\bullet}. \tag{34.1}$$

For convenience, we introduce the following notation.

3408 Notation: $\mathcal{F}_1 := \{\mathbf{v} \in \mathcal{F} \,|\, \tau(\mathbf{v}) = 1\}$.

3409 Proposition: *Let a genuine interval I in $\mathbb{R}$ and a smooth mapping $q : I \to \mathcal{E}$ such that $\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F}$ be given. If $\tau(q^{\bullet}(t)) > 0$ for all $t \in I$, then $\mathcal{L}$ is a material worldpath. In this case, $\mathcal{L}$ has a smooth time-parameterization.*

Moreover, q is a time-parameterization of $\mathcal{L}$ if and only if $\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F}_1$.

Proof: It is easy to show that $\mathbf{v} \in \mathcal{F} \cap (-\mathcal{F})$ implies that $\tau(\mathbf{v}) = 0$. Hence the assumption that $\tau(q^{\bullet}(t)) > 0$ for all $t \in I$ implies that $\mathsf{Rng}\, q^{\bullet} \subset \mathcal{F} \setminus (-\mathcal{F})$. It follows from **Prop. 3402** that $\mathcal{L}$ is a worldpath. That $\mathcal{L}$ is material follows immediately from **Def. 2300** and (34.1). That $\mathcal{L}$ has a smooth time-parameterization is an application of **Thm. 2306**.

[1] More precisely, we assume that q is continuously differentiable, and that $q^{\bullet}$ is differentiable except at a discrete set of points, at which the derivative of $q^{\bullet}$ has both a left and right limit.

Now suppose that q is a time-parameterization of $\mathcal{L}$. Then it follows from (34.1) that for all $a, b \in I$, we have

$$\int_a^b \tau \circ q^\bullet = b - a.$$

We see from the Fundamental Theorem of Calculus that $\tau \circ q^\bullet = 1$, and hence $\mathsf{Rng}\ q^\bullet \subset \mathcal{F}_1$.

If $\mathsf{Rng}\ q^\bullet \subset \mathcal{F}_1$, then the fact that q is a time-parameterzation of $\mathcal{L}$ follows directly from **Def. 2305** and (34.1). ◊

Remark: The converse of the first assertion in the previous Proposition is not necessarily true. One may, for example, exhibit material worldpaths for which there is $t \in I$ such that $\tau(q^\bullet(t)) = 0$.

3410 Corollary: *Let I be a genuine interval in $\mathbb{R}$ and let $\mathbf{d} : I \to \mathcal{F}$ be a mapping which is piecewise continuously differentiable and satisfies $\mathsf{Rng}\ \mathbf{d} \subset \mathcal{F}_1$ for all $t \in I$. Given $c \in I$ and $q \in \mathcal{E}$, the mapping $p : I \to \mathcal{E}$ defined by*

$$p(t) := q + \int_c^t \mathbf{d}$$

for all $t \in I$ is a smooth time-parameterization of a material worldpath.

Exercises

EXERCISES, I

1. Prove **Prop. 3103**.
2. Prove **Prop. 3104**.
3. Complete the proof of **Thm. 3108**.
4. Complete the proof of **Thm. 3202**.
5. Complete the proof of **Thm. 3207**.

6. Complete the proof of **Thm. 3302**.

7. As alluded to in the Remark following the proof of **Thm. 3303**, prove **Thm. 3303** without using the assumption **Thm. 3302**(3).

8. Let $\mathcal{E}$ be a flat timed eventworld (see **Def. 3300**). Show that τ is continuous on $\mathsf{Int}\ \mathcal{F}$ (see the Remark following the proof of **Thm. 3303**).

9. As alluded to in the Remark following the proof of **Thm. 3303**, produce an example of a timed flat eventworld $\mathcal{E}$ with future cone $\mathcal{F}$, a mapping $\tau : \mathcal{F} \to \mathbb{P}$ which satisfies **Thm. 3302**(1)–(2), fails to satisfy **Thm. 3302**(3), and fails to be continuous at some $\mathbf{v} \in \mathsf{Int}\ \mathcal{F}$.

10. Prove **Prop. 3400**.

11. Prove **Prop. 3403**.

12. Prove **Thm. 3407**.

13. Prove **Cor. 3410**.

EXERCISES, II

In Exercises 1–3, let a flat space $\mathcal{E}$ with translation space $\mathcal{V}$ be given. In these Exercises, the results to be shown are theorems of *affine geometry* (*i.e.,* the theory of flat spaces). That is, they are results about parallelism of lines, concurrence of lines, or ratios of line segments. In affine geometry, there is no length or angle measurement – these subjects belong to the realm of Euclidean geometry. Nonetheless, many interesting theorems can be proved.

1. Let $x, y \in \mathcal{E}$ and $\lambda, \mu \in \mathbb{P}^{\times}$ be given.

   (a) Show that there is exactly one point $z \in \mathcal{E}$ such that

   $$\mu(x - z) + \lambda(y - z) = 0.$$

   We say that z is the point that **divides the pair** (x, y) **in the ratio** $\lambda : \mu$. The point that divides (x, y) in the ratio $1 : 1$ is called the **midpoint of** (x, y).

(b) Show that the point z described in (a) satisfies

$$z = q + \frac{1}{\lambda + \mu}(\lambda(x-q) + \mu(y-q))$$

for all $q \in \mathcal{E}$.

2. (a) Let $p, q, r, s \in \mathcal{E}$ be given. Show that the following are equivalent:

   i. The midpoint of (p, r) coincides with the midpoint of (q, s),

   ii. $q - p = r - s$,

   iii. $r - q = s - p$.

   If any one (and hence all three) of these conditions is satisfied, then we say that p, q, r, and s are the **vertices of a parallelogram**.

   (b) Show that the midpoints of the sides of a quadrilateral (not necessarily plane) in a flat space are the vertices of a parallelogram. In other words, given $a, b, c, d \in \mathcal{E}$, show that the midpoints p, q, r, and s of the pairs (a, b), (b, c), (c, d), and (d, a), respectively, are the vertices of a parallelogram.

3. We may interpret three points $a, b, c \in \mathcal{E}$ which do not all lie in the same straight line as the vertices of a **triangle**. We call (a, b), (b, c), and (c, a) the **sides** of the triangle. Given a vertex, then the pair whose terms are the other two vertices of the triangle is called the **side opposite** the vertex. For example, (b, c) is the side opposite a.

   (a) A **median** of a triangle is a line segment (see **Def. 3107**) from a vertex to the midpoint of the side opposite that vertex. Show that the three medians of a triangle are concurrent; that is, they intersect in exactly one point. This intersection is called the **centroid** of the triangle. Show also that the centroid divides a vertex and the midpoint of the side opposite this vertex in the ratio 1 : 2.

   (b) Consider three points $x_1, x_2, x_3 \in \mathcal{E}$ which do not all lie on the same line. Assume that y_1 divides (x_2, x_3) in the ratio $\lambda_1 : \mu_1$, and y_2 divides (x_1, x_3) in the ratio $\lambda_2 : \mu_2$. Determine the ratios in which the point c of intersection of $[x_1, y_1]$ and $[x_2, y_2]$ divides (x_1, y_1) and (x_2, y_2).

4. (a) Let x_1, x_2, x_3 be three points not on a line. Suppose that y_1 divides (x_2, x_3) in the ratio $\lambda_1 : 1$, y_2 divides (x_3, x_1) in the ratio $\lambda_2 : 1$, and y_3 divides (x_1, x_2) in the ratio $\lambda_3 : 1$. Show that the line segments $[x_1, y_1]$, $[x_2, y_2]$, and $[x_3, y_3]$ are concurrent if and only if $\lambda_1\lambda_2\lambda_3 = 1$. This result is often referred to as *Ceva's theorem.*

   (b) Use the result of (a) to obtain a possibly different proof for Exercise 3(a).

5. Let $\mathcal{E}$ be a flat space with translation space $\mathcal{V}$, and let $\mathcal{K}$ be a linear cone in $\mathcal{V}$. Show that $\mathcal{K}$ satisfies both

$$\mathcal{K} \cap (-\mathcal{K}) = \{\mathbf{0}\} \quad \text{and} \quad \mathcal{K} \cup (-\mathcal{K}) = \mathcal{V}$$

   if and only if $\mathcal{K}$ is the direction cone of a total order.

6. Let a flat eventworld $\mathcal{E}$ be given. Show that the subsets $\mathsf{Pres}(x) - x$ and $\mathsf{Past}(x) - x$ of $\mathcal{V}$ do not depend on the choice of $x \in \mathcal{E}$.

Exercises, III

1. Let $\prec$ and t be as in Exercise II,2 of Chapter 2. We see that $\mathbb{R}^2$ has the structure of a flat space with translation space $\mathbb{R}^2$. Show that $\mathbb{R}^2$ is in fact a timed flat eventworld, and find the direction cone of $\prec$. Is this eventworld genuine (in the sense of **Def. 3205**)?

2. Consider the example in Exercise II,5 of Chapter 1, with $I := \mathbb{R}$ and $\mathcal{S} := \mathbb{R}$. $\mathcal{E} = \mathbb{R}^2$ has a natural flat-space structure. Show that $\prec$ is translation-invariant and connected, and find the direction cone of $\prec$. Is $\mathcal{E}$ genuine (in the sense of **Def. 3205**)?

Exercises, IV

1. Give an example of a flat, non-genuine (in the sense of **Def. 3205**), relativistic eventworld of dimension two.

2. Give an example of a flat space $\mathcal{E}$ and a reflexive, transitive relation $\lhd$ on $\mathcal{E}$ such that:

   (a) $\lhd$ is translation-invariant but not connected,

(b) $\triangleleft$ is connected but not translation-invariant.

3. Give an example of a genuine flat eventworld $\mathcal{E}$ and $x, y \in \mathcal{E}$ with $x \neq y$ such that $x \prec y$ but $[x, y]$ is not a worldpath from x to y.

4. Produce an example of a genuine flat timed eventworld $\mathcal{E}$ with time-lapse t such that

   (a) $\mathcal{G} := \{(x, y) \in \mathsf{Gr}(\prec) \mid x \neq y \text{ and } \mathsf{t}(x, y) = 0\} \neq \emptyset$, and

   (b) At every pair in $\mathcal{G}$, t fails to be continuous.

5. Produce an example of a flat eventworld such that there is a totally ordered subset that is bounded below but has no infimum. (Hint: Is this possible if the precedence is relativistic and the future cone is closed?)