Chapter 2

Timed Eventworlds

In this chapter, we introduce timelapse functions: the next important mathematical structure we will discuss in our presentation of a theory of relativity (the first two being the precedence relation and worldpaths, discussed in the previous chapter).

The inequality in **Def. 2100** is perhaps the most counter-intuitive concept in relativity, but perhaps the most important. Through a study of the consequences of this inequality, one may gradually develop an intuitive grasp of it. In §**2.2**, we extend the idea of timelapse in a natural way to timelapses along worldpaths. For certain worldpaths, such timelapses lead naturally to parameterizations of these worldpaths as discussed in §**2.3**. We then consider the timelapse function in a classical context in §**2.4**.

2.1 Timed Eventworlds

It appears that in everyday parlance that there is a notion of "absolute time". We ask, "What time is it?" or say that "It is exactly 7:52 p.m. EST on 21 March 1992", as if we could specify time precisely. Upon closer scrutiny, we see that what we are asking for or making statements about is a *timelapse.* In other words, when we say "It is exactly 7:52 p.m. EST on 21 March 1992", we mean that so many years, months, days, hours, and minutes have elapsed since some reference event, such as the alleged birth of Christ. Even though it appears that we refer to a specific "time", in fact we refer to a certain span of time measured by a particular clock originally based on the sun, and now based on atomic clocks using certain spectral frequencies.

What is our intuition about timelapses? Consider the following example. Suppose that you are timing a relay race with four runners. If four people had stopwatches and recorded how long each of the four runners held the baton (beginning when the gun was fired and ending when the last runner crossed the finish line) and you recorded the duration of the entire race on your stopwatch, you would expect the sum of the readings from the four others to equal your reading. In other words, we expect timelapses to be *additive.* Thus, we expect the sum of the timelapses from an event a to an event b and from b to a third event c to be the same as the timelapse from a to c.

Pitfall: Be careful not to confuse "event" with "location" (more will be said about "locations" in Chapter 4). For example, suppose that you and a friend depart at the same time from Pittsburgh for a mutual friend's house in Los Angeles – you are flying directly but your friend is flying through Dallas to visit relatives. Wouldn't the timelapse between the departure from Pittsburgh and arrival in Los Angeles be *greater* (and not equal to, as suggested above) for your friend's trip than for yours?

The answer is "yes", but this does not contradict the expectation that timelapses be additive. The event of you and your friend's departures is the same, but your arrivals are two *distinct* events (although they may occur at the same "location"; see **Def. 4200**). Clearly, your arrival precedes your friend's (assuming that flights depart and arrive on schedule). Thus, we may say that the sum of the timelapses from a (your departure) to b (your friend's arrival in Dallas) and from b to c (your friend's arrival in Los Angeles) is greater that the timelapse from a to d (your arrival in Los Angeles) without contradicting our expectation that timelapses be additive.

However, circumstances are not so simple in a relativistic world, where addition of timelapses is *superadditive.* For example, in a relativistic world, it is possible that a situation similar to the relay race example above might result in the sum of the four individual readings actually being *less* than your reading! Although this phenomenon occurs regularly on atomic levels and when considering models of interstellar travel, it is not so prevalent in our everyday lives. This fact, the superadditivity of timelapses in a relativistic world, is perhaps the most counter-intuitive of all ideas relevant to relativity. But it is also the most important idea, playing a significant role in determining the structure of a relativistic world. We will see later an important consequence of the superadditivity of the timelapse function: the timelapse along a worldpath from x to y depends upon the worldpath and is usually different for different worldpaths.

2100 Definition: *A* **timed eventworld** *is an eventworld $\mathcal{E}$ (with precedence $\prec$) endowed with additional structure by specifying a mapping*

$$\mathsf{t} : \mathsf{Gr}(\prec) \to \mathbb{P}$$

satisfying

$$\mathsf{t}(x,z) \geq \mathsf{t}(x,y) + \mathsf{t}(y,z)$$

for all $x, y, z \in \mathcal{E}$ such that $x \prec y \prec z$. This relation is called the **Intermediate Event Inequality**. *The function* t *is called a* **timelapse function for $\mathcal{E}$**.

Remark: The above inequality reminds one of a geometric inequality relating the lengths of sides of a triangle, except that the inequality sign is reversed. As a result, the above inequality is sometimes referred to as the "Reverse Triangle Inequality".

We use the simulaneity relation, $\sim$, as given in **Def. 1201**.

2101 Proposition: *If $x, y \in \mathcal{E}$ satisfy $x \sim y$, then* $\mathsf{t}(x,y) = \mathsf{t}(y,x) = 0$.

Proof: Let $x \in \mathcal{E}$ be given. Since $x \prec x \prec x$, it follows from the Intermediate Event Inequality that

$$\mathsf{t}(x,x) + \mathsf{t}(x,x) \leq \mathsf{t}(x,x).$$

This in turn implies that $\mathsf{t}(x,x) \leq 0$, and thus we must have $\mathsf{t}(x,x) = 0$ since timelapses are positive numbers.

Now let $x, y \in \mathcal{E}$ be given such that $x \sim y$; *i.e.*, $x \prec y \prec x$. By the Intermediate Event Inequality and the preceding result,

$$0 \leq \mathsf{t}(x,y) + \mathsf{t}(y,x) \leq \mathsf{t}(x,x) = 0.$$

Hence, we must have $\mathsf{t}(x,y) = \mathsf{t}(y,x) = 0$. Since $x, y \in \mathcal{E}$ were arbitrary, the proof is complete. ◊

We introduce the following related concept, which will prove to be very useful later on.

2102 Definition: *Let $\mathcal{E}$ be a timed eventworld with precedence $\prec$ and timelapse* t. *We define a* **signed timelapse function**,

$$\bar{\mathsf{t}} : \mathsf{Gr}(\prec) \cup \mathsf{Gr}(\succ) \to \mathbb{R}$$

(where $\succ$ denotes the reverse of the precedence relation; see **Def. C11** *of Appendix C), by*

$$\bar{\mathsf{t}}(x,y) := \begin{cases} \mathsf{t}(x,y), & \text{if } x \prec y \\ -\mathsf{t}(y,x), & \text{if } y \not\prec x \end{cases}$$

for all $(x,y) \in \mathsf{Gr}(\prec) \cup \mathsf{Gr}(\succ)$.

2103 Proposition: *For all $x, y \in \mathcal{E}$ such that $x \prec y$, we have*

$$\bar{\mathsf{t}}(x,y) = -\bar{\mathsf{t}}(y,x).$$

Proof: Let $x, y \in \mathcal{E}$ be given such that $x \prec y$. Then by the definition of $\bar{\mathsf{t}}$, we have $\bar{\mathsf{t}}(x,y) = \mathsf{t}(x,y)$. Since x and y are related, we must have either $y \prec x$ or $x \not\prec y$. In the former case, **Prop. 2101** implies that $\mathsf{t}(x,y) = \mathsf{t}(y,x) = 0$, and hence $\bar{\mathsf{t}}(y,x) = 0 = -\mathsf{t}(x,y)$. In the latter, we have from the preceding definition that $\bar{\mathsf{t}}(y,x) = -\mathsf{t}(x,y)$. In either case, we have $\bar{\mathsf{t}}(y,x) = -\mathsf{t}(x,y)$, and thus $\bar{\mathsf{t}}(x,y) = -\bar{\mathsf{t}}(y,x)$. Since $x, y \in \mathcal{E}$ were arbitrary, the proof is complete. ◊

2.2 Timelapses Along Worldpaths

Let a timed eventworld $\mathcal{E}$ with precedence $\prec$ and timelapse $\mathtt{t}$ be given. We will introduce the idea of a timelapse along a worldpath in such a way that for events $x, y \in \mathcal{E}$ such that $x \prec y$, the timelapse along any worldpath from x to y does not exceed $\mathtt{t}(x, y)$. Timelapses along worldpaths will be defined in a manner similar to that of defining the arc length along a "smooth" curve in Euclidean space. Before presenting this definition, however, we review the analogous problem in a Euclidean plane.

So assume that we know the distance between any two points a and b, which we also think of as the length of the shortest path between them; namely, a line segment. Now suppose we draw an arbitrary "smooth" curve from a to b (see Figure 22a(1)). Can we calculate the length of this curve from a to b with only our knowledge of lengths of line segments?

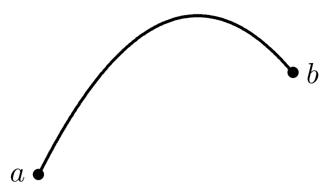

Figure 22a(1)

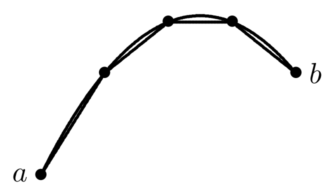

Figure 22a(2)

As the reader may recall from calculus, this is certainly possible, and can be done as follows. We pick some finite number of points along the curve, and find the length of the polygonal approximation to the curve by these points (see Figure 22a(2)). This result is certainly not greater that the answer sought; however, the more points appropriately chosen along the curve, the better our approximation. In fact, we may define the length of this curve as the *supremum* of the set of lengths of all possible polygonal approximations to the curve.

We may, in an analogous fashion, define the timelapse from x to y along a given worldpath $\mathcal{L}$ which contains the events x and y. The important differences are, in our case, that we are dealing with events in an eventworld, not points in a plane, and we have the Intermediate Event Inequality at our disposal, not the familiar Triangle Inequality. This section is concerned with making precise the details of this procedure.

2200 Definition: *Let $\mathcal{S}$ be a nonempty subset of $\mathcal{E}$. Then* $\mathsf{Fto}\ \mathcal{S}$ *is defined to be the set of all nonempty finite totally ordered subsets of $\mathcal{S}$. For all $x, y \in \mathcal{S}$ such that $x \prec y$, we define (see Appendix C for definitions of* min *and* max*)*

$$\mathsf{Fto}_{x,y}\mathcal{S} := \{\lambda \in \mathsf{Fto}\ \mathcal{S} \mid \min\lambda = x,\ \max\lambda = y\}.$$

Let $\lambda \in \mathsf{Fto}\ \mathcal{E}$ be given, and put $m := \#\lambda$. Since λ is finite and totally ordered, the elements of λ can be listed in increasing order; that is, there is exactly one list $\bar{\lambda} := (\bar{\lambda}_k \mid k \in 1..m)$ such that $\bar{\lambda}$ is strictly isotone (that is, $\bar{\lambda}_k \prec \bar{\lambda}_{k+1}$ for all $k \in 1..(m-1)$) and $\mathsf{Rng}\ \bar{\lambda} = \lambda$. We have $\bar{\lambda}_1 = \min\lambda$ and $\bar{\lambda}_m = \max\lambda$.

2201 Definition: *For each $\lambda \in \mathsf{Fto}\ \mathcal{E}$, we define $\Sigma(\lambda)$ by*

$$\Sigma(\lambda) := \sum_{k \in 1..(m-1)} \mathsf{t}(\bar{\lambda}_k, \bar{\lambda}_{k+1}), \qquad \textit{where } m = \#\lambda.$$

Remark: Note that if $x \in \mathcal{E}$ is given and we put $\lambda := \{x\}$, then we have $\Sigma(\lambda) = 0$.

To illustrate the above concepts, we consider the eventworld $\mathcal{E} = I \times \mathcal{S}$ as described in Exercise II,5 of Chapter 1, equipped with a prescribed timelapse function, t.

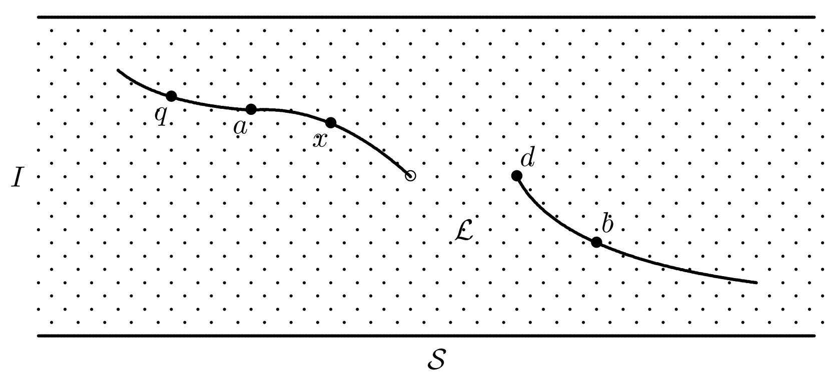

Figure 22b

Let $\mathcal{L}$ be represented graphically as in Figure 22b. We assume that values in I increase with height in the diagram; thus, we would have $a \prec q$. Put $\lambda := \{a, b, d, q, x\}$. Then $\lambda \in \mathsf{Fto}_{b,q}\mathcal{L} \subset \mathsf{Fto}\ \mathcal{L} \subset \mathsf{Fto}\ \mathcal{E}$ and $\{a, d, x\} \in \mathsf{Fto}_{d,a}\mathcal{L} \subset \mathsf{Fto}_{d,a}\mathcal{E} \subset \mathsf{Fto}\ \mathcal{E}$. Moreover, $\bar{\lambda} = (b, d, x, a, q), \bar{\lambda}_1 = b, \bar{\lambda}_2 = d, \bar{\lambda}_3 = x, \bar{\lambda}_4 = a$, and $\bar{\lambda}_5 = q$. Finally, we have $\Sigma(\lambda) = \mathsf{t}(b, d) + \mathsf{t}(d, x) + \mathsf{t}(x, a) + \mathsf{t}(a, q)$.

Having introduced some preliminary definitions, we are ready to give the definition of timelapse along a worldpath.

2202 Definition: *Let $\mathcal{L}$ be a worldpath. We define*

$$\mathsf{t}_{\mathcal{L}} : \mathsf{Gr}(\prec|_{\mathcal{L}}) \to \mathbb{P}$$

by

$$\mathsf{t}_{\mathcal{L}}(x, y) := \inf \{\Sigma(\lambda) \,|\, \lambda \in \mathsf{Fto}_{x,y}\,\mathcal{L}\}$$

for all $(x, y) \in \mathsf{Gr}(\prec|_{\mathcal{L}})$, and we say that $\mathsf{t}_{\mathcal{L}}(x, y)$ is the **timelapse from** x **to** y **along** $\mathcal{L}$. *$\mathsf{t}_{\mathcal{L}}$ is called the* **timelapse along** $\mathcal{L}$.

This, then, is the analogue of the definition of arc length along a curve. Note, however, that we have used “infimum” where we would have used “supremum” in defining arc length. This difference is due to the fact that timelapses are superadditive, whereas distances are subadditive.

Now suppose that $\mathcal{L}$ is a worldpath, and $x, y \in \mathcal{L}$ are such that $x \prec y$. Since $\{x, y\} \in \mathsf{Fto}_{x,y}\mathcal{L}$, we see from the previous definition that

$$\mathsf{t}_{\mathcal{L}}(x, y) \leq \Sigma(\{x, y\}) = \mathsf{t}(x, y).$$

This proves our previous remark that the timelapse along any path from x to y cannot exceed $\mathsf{t}(x, y)$. We also have the following easy Proposition.

2203 Proposition: *Let $\mathcal{L}$ be a worldpath. Then $\mathsf{t}_{\mathcal{L}}(x, x) = 0$ for all $x \in \mathcal{L}$.*

Proof: Since $0 \leq \mathsf{t}_{\mathcal{L}}(x, y) \leq \mathsf{t}(x, y)$ for all $x, y \in \mathcal{L}$ such that $x \prec y$, the result follows immediately from **Prop. 2101**. $\Diamond$

Now consider the following situation. Let a worldpath $\mathcal{L}$ be given, which we will interpret as the worldpath of some space traveller. Suppose that this space traveller has a clock, and that the function $\mathsf{t}_{\mathcal{L}}$ as decribed in **Def. 2202** may be determined by recording elapsed time using this clock. Finally, suppose that $x, y, z \in \mathcal{L}$ are such that $x \prec y \prec z$. Now if the traveller looks at the clock at the event x, and again at the event y, then the elapsed time would be $\mathsf{t}_{\mathcal{L}}(x, y)$. With this method of determining timelapses, it seems intuitive that we should have $\mathsf{t}_{\mathcal{L}}(x, z) = \mathsf{t}_{\mathcal{L}}(x, y) + \mathsf{t}_{\mathcal{L}}(y, z)$. It is indeed true that $\mathsf{t}_{\mathcal{L}}$ is additive in this sense, but the proof is somewhat involved and requires knowledge of basic real analysis.

First, however, two important Propositions must be introduced. The reader uninterested in the details may, without loss, skip to the end of the proof of **Thm. 2207**.

2204 Proposition: *Let $x, y, z \in \mathcal{E}$ be given such that $x \prec y \prec z$, and let $\lambda \in \mathsf{Fto}_{x,y}\mathcal{E}$ and $\lambda' \in \mathsf{Fto}_{y,z}\mathcal{E}$ be given. Then $\lambda \cup \lambda' \in \mathsf{Fto}_{x,z}\mathcal{E}$, and*

$$\Sigma(\lambda \cup \lambda') = \Sigma(\lambda) + \Sigma(\lambda').$$

Proof: The idea of the proof is fairly simple: the terms which lead to the sums $\Sigma(\lambda)$ and $\Sigma(\lambda')$ together comprise the terms which lead to the sum $\Sigma(\lambda \cup \lambda')$. The proof is essentially one of "index shuffling"; details are left to the Exercises. $\Diamond$

We are now in a position to prove the following Proposition, which is a generalization of the Intermediate Event Inequality.

2205 Proposition: *Let $x, y \in \mathcal{E}$ be given such that $x \prec y$, and let $\lambda, \lambda' \in \mathsf{Fto}_{x,y}\mathcal{E}$ be given. Then*

$$\lambda \subset \lambda' \implies \Sigma(\lambda') \leq \Sigma(\lambda).$$

Proof: We proceed by mathematical induction. For all $n \in \mathbb{N}$, define the statement P_n by

$$P_n \;:\Longleftrightarrow\; \text{For all } x, y \in \mathcal{E} \text{ such that } x \prec y, \text{ and for all } \lambda, \lambda' \in \mathsf{Fto}_{x,y}\mathcal{E} \text{ such that } \#\lambda' = n, \text{ we have } \lambda \subset \lambda' \implies \Sigma(\lambda') \leq \Sigma(\lambda).$$

We see that P_0 is vacuously valid, since if $x, y \in \mathcal{E}$ are such that $x \prec y$, then $\lambda' \in \mathsf{Fto}_{x,y}\mathcal{E}$ cannot be empty (see **Def. 2200**), *i.e.,* $\#\lambda' \neq 0$.

Now assume that $n \in \mathbb{N}$ is given and that P_n is valid. Let $x, y \in \mathcal{E}$ be such that $x \prec y$, and $\lambda, \lambda' \in \mathsf{Fto}_{x,y}\mathcal{E}$ be given such that $\#\lambda' = n + 1$ and $\lambda \subset \lambda'$. If in fact $\lambda = \lambda'$ (which must be the case if $n = 0$ or $n = 1$), then the desired conclusion, *i.e.,* $\Sigma(\lambda') \leq \Sigma(\lambda)$, is immediate. If not, then we must have $n \geq 2$, and hence we may choose $z \in \lambda' \setminus \lambda$ and $k \in 2..n$ such that $z = \bar{\lambda}'_k$. We then have $\lambda \subset \lambda' \setminus \{z\}$ and $\#\lambda' \setminus \{z\} = n$, so that P_n implies

$$\Sigma(\lambda' \setminus \{z\}) \leq \Sigma(\lambda).$$

We also have from the Intermediate Event Inequality that

$$\mathsf{t}(\bar{\lambda}'_{k-1}, z) + \mathsf{t}(z, \bar{\lambda}'_{k+1}) \leq \mathsf{t}(\bar{\lambda}'_{k-1}, \bar{\lambda}'_{k+1}).$$

The previous two inequalities result in

$$\begin{aligned}
\Sigma(\lambda') &= \sum_{i \in 1..n} \mathsf{t}(\bar{\lambda}'_i, \bar{\lambda}'_{i+1}) \\
&= \sum_{i \in 1..(k-2)} \mathsf{t}(\bar{\lambda}'_i, \bar{\lambda}'_{i+1}) + \mathsf{t}(\bar{\lambda}'_{k-1}, z) + \mathsf{t}(z, \bar{\lambda}'_{k+1}) + \sum_{i \in (k+1)..n} \mathsf{t}(\bar{\lambda}'_i, \bar{\lambda}'_{i+1}) \\
&\leq \sum_{i \in 1..(k-2)} \mathsf{t}(\bar{\lambda}'_i, \bar{\lambda}'_{i+1}) + \mathsf{t}(\bar{\lambda}'_{k-1}, \bar{\lambda}'_{k+1}) + \sum_{i \in (k+1)..n} \mathsf{t}(\bar{\lambda}'_i, \bar{\lambda}'_{i+1}) \\
&= \Sigma(\lambda' \setminus \{z\}) \\
&\leq \Sigma(\lambda).
\end{aligned}$$

Since $x, y \in \mathcal{E}$ and $\lambda, \lambda' \in \mathsf{Fto}_{x,y}\mathcal{E}$ were arbitrary, we see that P_{n+1} is valid. By the principle of mathematical induction, the Proposition is proved. $\diamondsuit$

2206 Corollary: *Let $x, y \in \mathcal{E}$ be given such that $x \prec y$. If $\lambda \in \mathsf{Fto}_{x,y}\mathcal{E}$, then $\Sigma(\lambda) \leq \mathsf{t}(x, y)$.*

Proof: Let $\lambda \in \mathsf{Fto}_{x,y}\mathcal{E}$ be given. Then $\{x, y\} \in \mathsf{Fto}_{x,y}\mathcal{E}$, and $\{x, y\} \subset \lambda$. The desired conclusion follows immediately upon applying the previous Proposition. ◊

2207 Theorem: *Let a worldpath $\mathcal{L}$ and $x, y, z \in \mathcal{L}$ such that $x \prec y \prec z$ be given. Then*

$$\mathsf{t}_{\mathcal{L}}(x, z) = \mathsf{t}_{\mathcal{L}}(x, y) + \mathsf{t}_{\mathcal{L}}(y, z).$$

Proof: We begin by showing that for all $\lambda \in \mathsf{Fto}_{x,z}\,\mathcal{L}$, $\mu \in \mathsf{Fto}_{x,y}\,\mathcal{L}$, and $\mu' \in \mathsf{Fto}_{y,z}\,\mathcal{L}$, we have the following two inequalities:

$$\mathsf{t}_{\mathcal{L}}(x, z) \leq \Sigma(\mu) + \Sigma(\mu') \tag{22.1}$$

$$\Sigma(\lambda) \geq \mathsf{t}_{\mathcal{L}}(x, y) + \mathsf{t}_{\mathcal{L}}(y, z). \tag{22.2}$$

To see (22.1), let $\mu \in \mathsf{Fto}_{x,y}\,\mathcal{L}$ and $\mu' \in \mathsf{Fto}_{y,z}\,\mathcal{L}$ be given. It follows from **Prop. 2204** that $\mu \cup \mu' \in \mathsf{Fto}_{x,z}\,\mathcal{L}$, and hence from the definition of $\mathsf{t}_{\mathcal{L}}$ that $\mathsf{t}_{\mathcal{L}}(x, z) \leq \Sigma(\mu \cup \mu')$. Also, from **Prop. 2204**, we have that $\Sigma(\mu \cup \mu') = \Sigma(\mu) + \Sigma(\mu')$. Thus, (22.1) follows.

To see (22.2), let $\lambda \in \mathsf{Fto}_{x,z}\,\mathcal{L}$ be given. Put

$$\mu := (\lambda \cap [\![x, y]\!]) \cup \{y\},$$

$$\mu' := (\lambda \cap [\![y, z]\!]) \cup \{y\}.$$

It can easily be shown that $\lambda \cup \{y\} = \mu \cup \mu'$, $\mu \in \mathsf{Fto}_{x,y}\,\mathcal{L}$, and $\mu' \in \mathsf{Fto}_{y,z}\,\mathcal{L}$. From **Prop. 2204**, we have that $\Sigma(\lambda \cup \{y\}) = \Sigma(\mu) + \Sigma(\mu')$. We also know from the definition of $\mathsf{t}_{\mathcal{L}}$ that $\Sigma(\mu) \geq \mathsf{t}_{\mathcal{L}}(x, y)$ and $\Sigma(\mu') \geq \mathsf{t}_{\mathcal{L}}(y, z)$. Thus,

$$\Sigma(\lambda \cup \{y\}) \geq \mathsf{t}_{\mathcal{L}}(x, y) + \mathsf{t}_{\mathcal{L}}(y, z).$$

But we know from **Prop. 2205** that $\Sigma(\lambda) \geq \Sigma(\lambda \cup \{y\})$, and thus (22.2) follows.

Let $\mu' \in \mathsf{Fto}_{y,z}\,\mathcal{L}$ be given. From (22.1), we have

$$\mathsf{t}_{\mathcal{L}}(x, z) \leq \inf \{\Sigma(\mu) \mid \mu \in \mathsf{Fto}_{x,y}\,\mathcal{L}\} + \Sigma(\mu') = \mathsf{t}_{\mathcal{L}}(x, y) + \Sigma(\mu').$$

Since μ' was arbitrary, we conclude that

$$\mathsf{t}_{\mathcal{L}}(x, z) \leq \mathsf{t}_{\mathcal{L}}(x, y) + \inf \{\Sigma(\mu') \mid \mu' \in \mathsf{Fto}_{y,z}\,\mathcal{L}\},$$

and thus

$$\mathsf{t}_{\mathcal{L}}(x,z) \le \mathsf{t}_{\mathcal{L}}(x,y) + \mathsf{t}_{\mathcal{L}}(y,z). \tag{22.3}$$

Since λ was arbitrary in (22.2), we have

$$\inf\{\Sigma(\lambda) \mid \lambda \in \mathsf{Fto}_{x,z}\,\mathcal{L}\} \ge \mathsf{t}_{\mathcal{L}}(x,y) + \mathsf{t}_{\mathcal{L}}(y,z),$$

and hence

$$\mathsf{t}_{\mathcal{L}}(x,z) \ge \mathsf{t}_{\mathcal{L}}(x,y) + \mathsf{t}_{\mathcal{L}}(y,z). \tag{22.4}$$

From (22.3) and (22.4), it follows that

$$\mathsf{t}_{\mathcal{L}}(x,z) = \mathsf{t}_{\mathcal{L}}(x,y) + \mathsf{t}_{\mathcal{L}}(y,z).$$

$\diamondsuit$

This Theorem is perhaps one of the most important results about worldpaths to this point. Even though the Intermediate Event Inequality may at times be a strict inequality, timelapses along worldpaths are *always* additive.

We now introduce an extension of **Thm. 1309**.

2208 Theorem: *Let two worldpaths $\mathcal{L}$ and $\mathcal{M}$ be given such that $\mathcal{L}$ has an end and $\mathcal{M}$ has a beginning which satisfy $\mathsf{end}\,\mathcal{L} = \mathsf{beg}\,\mathcal{M}$. Put $h := \mathsf{end}\,\mathcal{L}$ and $\mathcal{Q} := \mathcal{L} \cup \mathcal{M}$. Then if $x, y \in \mathcal{Q}$ are such that $x \prec y$, we have*

$$\mathsf{t}_{\mathcal{Q}}(x,y) = \begin{cases} \mathsf{t}_{\mathcal{L}}(x,y) & \textit{if } x,y \in \mathcal{L}, \\ \mathsf{t}_{\mathcal{M}}(x,y) & \textit{if } x,y \in \mathcal{M}, \textit{ and} \\ \mathsf{t}_{\mathcal{L}}(x,h) + \mathsf{t}_{\mathcal{M}}(h,y) & \textit{if } x \in \mathcal{L} \textit{ and } y \in \mathcal{M}. \end{cases}$$

Proof: Suppose that $x, y \in \mathcal{Q}$ are such that $x \prec y$. Now if $x, y \in \mathcal{L}$, then it is easy to see that $\mathsf{Fto}_{x,y}\mathcal{Q} = \mathsf{Fto}_{x,y}\mathcal{L}$, and hence $\mathsf{t}_{\mathcal{Q}}(x,y) = \mathsf{t}_{\mathcal{L}}(x,y)$ (see **Def. 2202**). We argue similarly if $x, y \in \mathcal{M}$.

Now suppose that $x \in \mathcal{L}$ and $y \in \mathcal{M}$. From **Thm. 2207**, we know that

$$\mathsf{t}_{\mathcal{Q}}(x,y) = \mathsf{t}_{\mathcal{Q}}(x,h) + \mathsf{t}_{\mathcal{Q}}(h,y).$$

From the preceding argument, we see that $\mathsf{t}_{\mathcal{Q}}(x,h) = \mathsf{t}_{\mathcal{L}}(x,h)$ and $\mathsf{t}_{\mathcal{Q}}(h,y) = \mathsf{t}_{\mathcal{M}}(h,y)$. Hence

$$\mathsf{t}_{\mathcal{Q}}(x,y) = \mathsf{t}_{\mathcal{L}}(x,h) + \mathsf{t}_{\mathcal{M}}(h,y).$$

Since $x, y \in \mathcal{Q}$ were arbitrary, the Theorem is proved. $\diamondsuit$

We will later find it convenient to consider a timelapse function whose domain is $\mathcal{L} \times \mathcal{L}$ rather than $\mathsf{Gr}(\prec|_{\mathcal{L}})$. The following definition gives the obvious extension. Subsequent Propositions are straightforward consequences of this definition; these will be used in §**2.3**.

2209 Definition: *Let $\mathcal{L}$ be a worldpath. We define the* **signed timelapse along** $\mathcal{L}$,

$$\bar{\mathsf{t}}_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbb{R},$$

by

$$\bar{\mathsf{t}}_{\mathcal{L}}(x, y) := \begin{cases} \mathsf{t}_{\mathcal{L}}(x, y) & \text{if } x \prec y \\ -\mathsf{t}_{\mathcal{L}}(y, x) & \text{if } y \preccurlyeq x. \end{cases}$$

Remark: Note that the above definition makes sense since for all members x and y of a $\preccurlyeq$-totally ordered set $\mathcal{S}$, exactly one of the conditions $x \prec y$ and $y \preccurlyeq x$ must obtain.

2210 Proposition: *Let a worldpath $\mathcal{L}$ be given. Then for all $x, y \in \mathcal{L}$, $\bar{\mathsf{t}}_{\mathcal{L}}(x, y) = -\bar{\mathsf{t}}_{\mathcal{L}}(y, x)$.*

Proof: Let $x, y \in \mathcal{L}$ be given. If $x = y$, then the conclusion is immediate. Otherwise, we must have $x \preccurlyeq y$ or $y \preccurlyeq x$. Since the desired result is symmetric in "x" and "y", we assume without loss that $x \preccurlyeq y$. It follows from **Def. 2209** that $\bar{\mathsf{t}}_{\mathcal{L}}(x, y) = \mathsf{t}_{\mathcal{L}}(x, y)$ and $\bar{\mathsf{t}}_{\mathcal{L}}(y, x) = -\mathsf{t}_{\mathcal{L}}(x, y)$. Hence, $\bar{\mathsf{t}}_{\mathcal{L}}(x, y) = -\bar{\mathsf{t}}_{\mathcal{L}}(y, x)$. Since $x, y \in \mathcal{L}$ were arbitrary, the proof is complete. ◊

2211 Theorem: *Let $\mathcal{L}$ be a worldpath. Then for all $x, y, z \in \mathcal{L}$,*

$$\bar{\mathsf{t}}_{\mathcal{L}}(x, y) + \bar{\mathsf{t}}_{\mathcal{L}}(y, z) = \bar{\mathsf{t}}_{\mathcal{L}}(x, z).$$

Proof: Let $x, y, z \in \mathcal{L}$ be given. Then at least one of the following six cases must occur since $\mathcal{L}$ is $\prec$-totally ordered: $x \prec y \prec z$, $x \prec z \prec y$, $y \prec x \prec z$, $y \prec z \prec x$, $z \prec x \prec y$, or $z \prec y \prec x$.

Assume first that $x \prec y \prec z$. It follows from **Thm. 2207** that $\mathsf{t}_{\mathcal{L}}(x,y)+\mathsf{t}_{\mathcal{L}}(y,z) = \mathsf{t}_{\mathcal{L}}(x,z)$. Hence, by **Def. 2209**, we have $\bar{\mathsf{t}}_{\mathcal{L}}(x,y)+\bar{\mathsf{t}}_{\mathcal{L}}(y,z) = \bar{\mathsf{t}}_{\mathcal{L}}(x,z)$. Now **Prop. 2210** yields that $\bar{\mathsf{t}}_{\mathcal{L}}(x,z) = -\bar{\mathsf{t}}_{\mathcal{L}}(z,x)$, so that

$$\bar{\mathsf{t}}_{\mathcal{L}}(x,y) + \bar{\mathsf{t}}_{\mathcal{L}}(y,z) + \bar{\mathsf{t}}_{\mathcal{L}}(z,x) = 0. \tag{22.5}$$

Because of the symmetry in (22.5), we see that this equation is also valid when $y \prec z \prec x$ or $z \prec x \prec y$. Thus, in these three cases, we again use **Prop. 2210** to obtain the desired result.

Interchanging the roles of "x" and "z" in the preceding argument yields

$$\bar{\mathsf{t}}_{\mathcal{L}}(z,y) + \bar{\mathsf{t}}_{\mathcal{L}}(y,x) + \bar{\mathsf{t}}_{\mathcal{L}}(x,z) = 0$$

whenever $z \prec y \prec x$, $y \prec x \prec z$, or $x \prec z \prec y$. Using **Prop. 2210** and rearranging terms yields the desired result. Since $x, y, z \in \mathcal{L}$ were arbitrary, the Theorem is proved. $\Diamond$

2.3 Material Worldpaths

In this section, we wish to engage in a brief exploration of worldpaths which may be used to represent space travellers or particles such as protons or electrons. Such worldpaths differ markedly from those which might represent an electromagnetic signal or a photon. Much more will be said about this difference in §**6.1**–§**6.3**. At this point, however, we may study the consequences of an important property of the worldpaths which we wish to explore.

Let a timed eventworld $\mathcal{E}$ and a worldpath $\mathcal{L}$ be given. Recall that the signed timelapse function for $\mathcal{L}$ is denoted by $\bar{\mathsf{t}}_{\mathcal{L}}$ (see **Def. 2209**).

2300 Definition: *$\mathcal{L}$ is said to be* **material** *if for all $x, y \in \mathcal{L}$, we have $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = 0 \implies x = y$.*

Remark: As we will see in §**6.1**, if $\mathcal{L}$ were to represent an electromagnetic signal, then we would have $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = 0$ for all $x, y \in \mathcal{L}$.

2301 Proposition: *If $\mathcal{L}$ is material, then for all $x, y \in \mathcal{L}$, $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) \geq 0 \iff x \prec y$ and $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) > 0 \iff x \precdot y$.*

Proof: Let $x, y \in \mathcal{L}$ be given. To see the first half of the Proposition, suppose that $x \prec y$. Then by the definition of $\bar{\mathsf{t}}_{\mathcal{L}}$, we have

$$\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = \mathsf{t}_{\mathcal{L}}(x,y) \geq 0.$$

On the other hand, suppose that $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) \geq 0$. Now exactly one of the following two cases must occur: either $x \prec y$ or $y \precdot x$. Suppose that $y \precdot x$. Then we have $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = -\mathsf{t}_{\mathcal{L}}(y,x) \leq 0$. Since $\mathcal{L}$ is material, this inequality must be strict, implying that $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) < 0$. As this is impossible, $y \precdot x$ can not occur. Hence, we must have $x \prec y$. Thus, the first half of the Proposition is proved.

To see the second half, we have from the first half of the Proposition that

$$(\text{not } \bar{\mathsf{t}}_{\mathcal{L}}(y,x) \geq 0) \iff (\text{not } y \prec x).$$

Since exactly one of $y \prec x$ and $x \precdot y$ must occur, this implies that

$$\bar{\mathsf{t}}_{\mathcal{L}}(y,x) < 0 \iff x \precdot y.$$

Prop. 2210 yields $\bar{\mathsf{t}}_{\mathcal{L}}(y,x) = -\bar{\mathsf{t}}_{\mathcal{L}}(x,y)$, resulting in

$$\bar{\mathsf{t}}_{\mathcal{L}}(x,y) > 0 \iff x \precdot y.$$

Since $x, y \in \mathcal{L}$ were arbitrary, the proof is complete. $\Diamond$

The main result of this section, **Thm. 2306**, describes a procedure by which we may parameterize a material worldpath in a useful way. Before we state and prove this Theorem, however, we set the stage with some notation and a few helpful Propositions.

2302 Notation: *Given $q \in \mathcal{L}$, we define the mapping*

$$\bar{\mathsf{t}}^{q}_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$$

by

$$\bar{\mathsf{t}}^{q}_{\mathcal{L}}(x) := \bar{\mathsf{t}}_{\mathcal{L}}(q,x)$$

for all $x \in \mathcal{L}$.

2303 Proposition: *For all $x, y \in \mathcal{L}$, we have*

$$\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = \bar{\mathsf{t}}^q_{\mathcal{L}}(y) - \bar{\mathsf{t}}^q_{\mathcal{L}}(x).$$

Proof: Let $x, y \in \mathcal{L}$ be given. By **Thm. 2211**, we have

$$\bar{\mathsf{t}}_{\mathcal{L}}(q,x) + \bar{\mathsf{t}}_{\mathcal{L}}(x,y) = \bar{\mathsf{t}}_{\mathcal{L}}(q,y).$$

The result follows easily by applying the definition of $\bar{\mathsf{t}}^q_{\mathcal{L}}$. $\Diamond$

2304 Proposition: *Let $q \in \mathcal{L}$ be given. Then $\mathcal{L}$ is material if and only if $\bar{\mathsf{t}}^q_{\mathcal{L}}$ is injective.*

Proof: By **Def. 2300**, we see that $\mathcal{L}$ is material if and only if for all $x, y \in \mathcal{L}$, we have $\bar{\mathsf{t}}_{\mathcal{L}}(x,y) = 0 \implies x = y$. From **Prop. 2303**, we see that this condition is equivalent to stating that for all $x, y \in \mathcal{L}$, we have $\bar{\mathsf{t}}^q_{\mathcal{L}}(x) = \bar{\mathsf{t}}^q_{\mathcal{L}}(y) \implies x = y$. But this is precisely the statement that $\bar{\mathsf{t}}^q_{\mathcal{L}}$ is injective. $\Diamond$

2305 Definition: *Let a material worldpath $\mathcal{L}$ be given. Then a mapping $p : I \to \mathcal{E}$, where $I \subset \mathbb{R}$, is said to be a* **parameterization of $\mathcal{L}$** *if*

(1) *For all $s, s' \in I$, $s < s' \implies p(s) \prec p(s')$; that is, p is strictly isotone, and*

(2) $\mathsf{Rng}\, p = \mathcal{L}$.

If, in addition, p satisfies

(3) *For all $s, s' \in I$, we have $\bar{\mathsf{t}}_{\mathcal{L}}(p(s), p(s')) = s' - s$,*

then p is said to be a **time-parameterization of $\mathcal{L}$**.

2306 Theorem: *Every material worldpath has a time-parameterization.*

Proof: Let a material worldpath $\mathcal{L}$ be given, and choose $q \in \mathcal{L}$. Define $I \subset \mathbb{R}$ by

$$I := \mathsf{Rng}\ \bar{\mathsf{t}}^q_{\mathcal{L}}.$$

Since $\mathcal{L}$ is material, it follows from **Prop. 2304** that $\bar{\mathsf{t}}^q_{\mathcal{L}}$ is injective. Thus, it makes sense to define

$$p : I \to \mathcal{E}$$

as follows: for each $s \in I$, $p(s)$ is that event in $\mathcal{L}$ satisfying

$$\bar{\mathsf{t}}^q_{\mathcal{L}}(p(s)) = s.$$

It is immediate that $\mathsf{Rng}\ p = \mathcal{L}$, and hence **Def. 2305**(2) is satisfied.

To see **Def. 2305**(1) and **Def. 2305**(3), let $s, s' \in I$ be given. Then by **Prop. 2303**, we have

$$\bar{\mathsf{t}}_{\mathcal{L}}(p(s), p(s')) = \bar{\mathsf{t}}^q_{\mathcal{L}}(p(s')) - \bar{\mathsf{t}}^q_{\mathcal{L}}(p(s)) = s' - s.$$

Moreover, this equality along with **Prop. 2301** yields

$$\begin{aligned} s < s' &\implies \bar{\mathsf{t}}_{\mathcal{L}}(p(s), p(s')) > 0 \\ &\implies p(s) \prec p(s'). \end{aligned}$$

Since $s, s' \in I$ were arbitrary, **Def. 2305**(1) and **Def. 2305**(3) follow. $\Diamond$

One reason why this Theorem is so useful is that once we have found *one* time-parameterization of a material worldpath, we have essentially found *all* of them. As it happens, any two time-parameterizations of a material worldpath differ only by an additive shift as follows.

2307 Theorem: *Suppose that I and I' are subsets of $\mathbb{R}$ and $p : I \to \mathcal{E}$ and $p' : I' \to \mathcal{E}$ are time-parameterizations of a given material worldpath, $\mathcal{L}$. Then there is some $a \in \mathbb{R}$ such that $I' = I + a$ and $p'(t + a) = p(t)$ for all $t \in I$.*

Proof: Let $q \in \mathcal{L}$ be given. Then we may choose $c \in I$ and $c' \in I'$ such that $p(c) = p'(c') = q$. Now let $t \in I$ be given. We may choose $t' \in I'$ such that $p(t) = p'(t')$. Then

$$\begin{aligned} t' - c' &= \bar{\mathsf{t}}_{\mathcal{L}}(p'(c'), p'(t')) \\ &= \bar{\mathsf{t}}_{\mathcal{L}}(p(c), p(t)) \\ &= t - c, \end{aligned}$$

and hence $t' = t + c' - c$. As $t \in I$ was arbitrary, we see that $p(t) = p'(t + c' - c)$ for all $t \in I$. Hence, it follows easily that $a := c' - c$ satisfies the conditions of the statement of the Theorem. $\Diamond$

As we have seen, there is (up to a parameter shift) essentially one time-parameterization of a given material worldpath. Yet given a material world-path, a time-parameterization for that worldpath is usually not obvious. The following Theorem guarantees us that if we find *any* parameterization of a material worldpath, then we may derive from it a time-parameterization of the same worldpath. The proof of this Theorem closely parallels the proof of **Thm. 2306**, and is therefore left as an Exercise.

2308 Theorem: *Suppose that $p : I \to \mathcal{E}$ is a parameterization of some material worldpath $\mathcal{L}$, and let $c \in I$ be given. Since $\bar{\mathsf{t}}_{\mathcal{L}}^{p(c)}$ is injective, it makes sense to define*

$$\bar{p} : \mathsf{Rng}\ \bar{\mathsf{t}}_{\mathcal{L}}^{p(c)} \to \mathcal{E}$$

so that $\bar{p}(s)$ is that element of $\mathcal{L}$ satisfying $\bar{\mathsf{t}}_{\mathcal{L}}^{p(c)}(\bar{p}(s)) = s$ for all $s \in \mathsf{Rng}\ \bar{\mathsf{t}}_{\mathcal{L}}^{p(c)}$. Then $\bar{p}$ is a time-parameterization of $\mathcal{L}$.

2.4 Classical Timed Eventworlds

In this section, we freely use all notations introduced in §**1.4**.

Consider the relay race scenario as described in the beginning of §**2.1**. In a day-to-day context, we expect timelapses to add (as indicated in that example). This motivates, in part, the following definition. We remark that in a classical eventworld $\mathcal{E}$, the totality of the precedence relation implies that the domain of $\bar{\mathsf{t}}$ (see **Def. 2102**) is $\mathcal{E} \times \mathcal{E}$.

2400 Definition: *A* **classical timed eventworld** *is a timed eventworld $\mathcal{E}$ (with precedence $\prec$ and signed timelapse $\bar{\mathsf{t}}$) such that the precedence is classical (i.e., total) and the following additional conditions are satisfied:*

(1) $\bar{\mathsf{t}}(x, z) = \bar{\mathsf{t}}(x, y) + \bar{\mathsf{t}}(y, z)$ *for all* $x, y, z \in \mathcal{E}$,

(2) *For all* $x, y \in \mathcal{E}$, $\bar{\mathsf{t}}(x, y) = 0 \implies x \sim y$, *and*

(3) *For all $x \in \mathcal{E}$ and $s \in \mathbb{R}$, there is some event $y \in \mathcal{E}$ such that $\bar{\mathsf{t}}(x, y) = s$.*

For the remainder of this section, we assume that a classical timed eventworld $\mathcal{E}$ is given.

Note that (1) requires that timelapses be additive. (2) is a non-degeneracy condition which essentially states that a zero timelapse is possible *only* between simultaneous events. Condition (3) states, in particular, that given an event x in $\mathcal{E}$, there are events occurring arbitrarily far in its future and arbitrarily distantly in its past. This is perhaps an idealization – that "time is unbounded" – but it is an idealization which simplifies the exposition of the theory. Little would be gained or lost were we to limit the future or past in this sense (for example, by insisting that no event occurred before some primary event, such as a "big bang"), except that the presentation would become unnecessarily cumbersome.

Conditions (1) and (2) are fairly restrictive; some consequences of these requirements as regards worldpaths and their description are outlined in the following Theorem.

2401 Theorem: *Let $\mathcal{L}$ be a worldpath. Then $\bar{\mathsf{t}}_{\mathcal{L}}(x, y) = \bar{\mathsf{t}}(x, y)$ for all $x, y \in \mathcal{L}$. Moreover, $\mathcal{L}$ is necessarily material.*

Proof: Let $x, y \in \mathcal{L}$ be given such that $x \prec y$. As a consequence of the additivity of $\bar{\mathsf{t}}$, it follows readily by induction that $\Sigma(\lambda) = \mathsf{t}(x, y)$ for all $\lambda \in \mathsf{Fto}_{x,y}\mathcal{L}$. Hence, we have from **Def. 2202** that

$$\mathsf{t}_{\mathcal{L}}(x, y) = \inf\{\Sigma(\lambda) \mid \lambda \in \mathsf{Fto}_{x,y}\mathcal{L}\} = \mathsf{t}(x, y).$$

As $x, y \in \mathcal{L}$ were arbitrary, the first half of the Theorem follows from **Defs. 2102** and **2209**.

To see that $\mathcal{L}$ is material, let $x, y \in \mathcal{L}$ be given, and assume that $\bar{\mathsf{t}}_{\mathcal{L}}(x, y) = 0$. Then we see from above that $\bar{\mathsf{t}}(x, y) = 0$. Since $\mathcal{E}$ is a classical timed eventworld, we have from **Def. 2400**(2) that $x \sim y$; that is, $x \prec y \prec x$. But $\prec$ is antisymmetric on $\mathcal{L}$ since $\mathcal{L}$ is a worldpath, and hence $x = y$. As $x, y \in \mathcal{L}$ were arbitrary, we see from **Def. 2300** that $\mathcal{L}$ is material. $\Diamond$

Moreover, as a corollary of this Theorem, all of the results from §**2.3** about material worldpaths are valid for *every* worldpath in $\mathcal{E}$. In addition, we may reformulate **Thm. 2306**; but before doing so, we must investigate an adaptation of $\bar{\mathsf{t}}$ to $\Gamma \times \Gamma$.

It is not difficult to show that given $x, x', y, y' \in \mathcal{E}$ such that $x \sim x'$ and $y \sim y'$, we have $\bar{\mathsf{t}}(x, y) = \bar{\mathsf{t}}(x', y')$. As one might expect, then, the signed timelapse between two events depends only upon the instants to which the events belong. As a result, we may derive in a natural way a signed timelapse function on $\Gamma \times \Gamma$ as seen in the following Proposition.

2402 Proposition: *For $x, y \in \mathcal{E}$, the value $\bar{\mathsf{t}}(x, y)$ depends only on the instants to which x and y belong. More precisely, for all $\sigma, \tau \in \Gamma$, $x, x' \in \sigma$, and $y, y' \in \tau$, we have $\bar{\mathsf{t}}(x, y) = \bar{\mathsf{t}}(x', y')$. Hence, there is exactly one function*

$$\mathsf{t}^* : \Gamma \times \Gamma \to \mathbb{R}$$

such that for each $\sigma, \tau \in \Gamma$ and $x \in \sigma$, $y \in \tau$, we have

$$\mathsf{t}^*(\sigma, \tau) = \bar{\mathsf{t}}(x, y).$$

Proof: Let $\sigma, \tau \in \Gamma$ be given, along with $x, x' \in \sigma$ and $y, y' \in \tau$. Since $\tilde{\preceq}$ is total on Γ (see **Prop. 1403**), we must have either $\sigma \tilde{\preceq} \tau$ or $\tau \tilde{\preceq} \sigma$ (or perhaps both); assume without loss that $\sigma \tilde{\preceq} \tau$. Then $x \prec x' \prec y' \prec y$, and thus the additivity of $\bar{\mathsf{t}}$ implies that

$$\bar{\mathsf{t}}(x, y) = \bar{\mathsf{t}}(x, x') + \bar{\mathsf{t}}(x', y') + \bar{\mathsf{t}}(y', y).$$

Since $x \sim x'$ and $y' \sim y$, it follows from **Prop. 2101** that $\bar{\mathsf{t}}(x, x') = \bar{\mathsf{t}}(y', y) = 0$, and thus $\bar{\mathsf{t}}(x, y) = \bar{\mathsf{t}}(x', y')$. As all choices of instants and events were arbitrary, the first part of the Proposition is proved. The second part follows easily and is left as an Exercise. $\diamondsuit$

We now provide an analogue of **Def. 2400** for Γ. The proof is left as an Exercise.

2403 Proposition: t^* *satisfies the following:*

(1) $\mathsf{t}^*(\pi, \tau) = \mathsf{t}^*(\pi, \sigma) + \mathsf{t}^*(\sigma, \tau)$ *for all* $\pi, \sigma, \tau \in \Gamma$,

(2) $\mathsf{t}^*(\sigma,\tau) = 0 \Longrightarrow \sigma = \tau$ *for all* $\sigma,\tau \in \Gamma$, *and*

(3) *For all* $\sigma \in \Gamma$ *and* $s \in \mathbb{R}$, *there is exactly one* $\tau \in \Gamma$ *such that* $\mathsf{t}^*(\sigma,\tau) = s$.

We now introduce some suggestive notation which will make the statement and discussion of various concepts simpler and more concise.

2404 Notation: *We introduce the notation*

$$\tau - \sigma := \mathsf{t}^*(\sigma,\tau)$$

for all $\sigma,\tau \in \Gamma$, *and*

$$\sigma + s := \tau$$

for all $\sigma \in \Gamma$ *and* $s \in \mathbb{R}$, *where* τ *is the instant in* Γ *described in* (3) *of the previous Proposition.*

With these notations, the three statements in the previous Proposition become

(1) $\tau - \pi = (\sigma - \pi) + (\tau - \sigma)$ *for all* $\pi,\sigma,\tau \in \Gamma$,

(2) $\tau - \sigma = 0 \Longrightarrow \sigma = \tau$ *for all* $\sigma,\tau \in \Gamma$, *and*

(3) $(\sigma + s) - \sigma = s$ *for all* $\sigma \in \Gamma$ *and* $s \in \mathbb{R}$.

Note that the rules of addition and subtraction are valid, except that one cannot add instants. This notation is consistent with considering Γ as a flat space with external translation space $\mathbb{R}$ (see §**3.1**, Example 1).

2405 Definition: *Given* $\gamma \in \Gamma$, *we define the mapping*

$$\mathsf{t}^*_\gamma : \Gamma \to \mathbb{R}$$

by

$$\mathsf{t}^*_\gamma(\tau) := \tau - \gamma$$

for all $\tau \in \Gamma$.

Now let $\gamma \in \Gamma$ be given. We note that the mapping t^*_γ is invertible, with $(\mathsf{t}^*_\gamma)^{\leftarrow}(s) = \gamma + s$ for all $s \in \mathbb{R}$. Moreover, we have $\sigma \widetilde{\prec} \tau$ if and only if

$\mathsf{t}^*_\gamma(\sigma) \le \mathsf{t}^*_\gamma(\tau)$; that is, t^*_γ and its inverse preserve order. It is in this sense that Γ and $\mathbb{R}$ are isomorphic; in the literature, instants are often labeled by real numbers *via* the mapping t^*_γ.

With the preceding remark in mind, we may state properties of the natural parameterization $w_{\mathcal{L}}$ (see **Def. 1406**) which are analogous to characteristic properties of time-parameterizations. We reformulate **Thm. 2306** as follows. The proof is left as an Exercise.

2406 Proposition: *Let $\mathcal{L}$ be a worldpath, with $w_{\mathcal{L}} : \Lambda_{\mathcal{L}} \to \mathcal{L}$ as described in* **Def. 1406**. *Then*

(1) *For all $\sigma, \tau \in \Lambda_{\mathcal{L}}$, $\sigma \tilde{\prec} \tau \implies w_{\mathcal{L}}(\sigma) \prec w_{\mathcal{L}}(\tau)$,*

(2) $\mathsf{Rng}\, w_{\mathcal{L}} = \mathcal{L}$, *and*

(3) *For all $\sigma, \tau \in \Lambda_{\mathcal{L}}$, we have $\mathsf{t}_{\mathcal{L}}(w_{\mathcal{L}}(\sigma), w_{\mathcal{L}}(\tau)) = \tau - \sigma$.*

These properties of $w_{\mathcal{L}}$ allow us to easily obtain necessary and sufficient conditions that a mapping be a time-parameterization of a worldpath. Such conditions are given in the following Proposition.

2407 Proposition: *Let a worldpath $\mathcal{L}$ and a subset I of $\mathbb{R}$ be given. Then a mapping $p : I \to \mathcal{E}$ is a time-parameterization of $\mathcal{L}$ if and only if there is some $\gamma \in \Gamma$ such that*

(1) $I = (\mathsf{t}^*_\gamma)_>(\Lambda_{\mathcal{L}})$, *and*

(2) $p \circ \mathsf{t}^*_\gamma|_{\Lambda_{\mathcal{L}}} = w_{\mathcal{L}}$; *i.e., $p(\tau - \gamma) = w_{\mathcal{L}}(\tau)$ for all $\tau \in \Lambda_{\mathcal{L}}$.*

Proof: Suppose that $p : I \to \mathcal{E}$ is a time-parameterization of $\mathcal{L}$, and let $c \in I$ be given. Let σ be the instant to which $p(c)$ belongs, and define $\gamma := \sigma - c$.

To see (1), we show the validity of

$$\tau \in \Lambda_{\mathcal{L}} \iff \tau - \gamma \in I. \tag{24.1}$$

To this end, let $\tau \in \Lambda_{\mathcal{L}}$ be given. Since p is a time-parameterization of $\mathcal{L}$, we may choose $t \in I$ such that $p(t) \in \tau$. Then, since $p(c) = w_{\mathcal{L}}(\sigma)$ and $p(t) = w_{\mathcal{L}}(\tau)$, it follows from **Prop. 2406**(3) that

$$\tau - \sigma = \mathsf{t}_{\mathcal{L}}(w_{\mathcal{L}}(\sigma), w_{\mathcal{L}}(\tau)) = \mathsf{t}_{\mathcal{L}}(p(c), p(t)) = t - c.$$

Thus, we see that

$$t = \tau - (\sigma - c) = \tau - \gamma.$$

Hence, $\tau - \gamma \in I$. As $\tau \in \Lambda_{\mathcal{L}}$ was arbitrary, the forward implication of (24.1) is proved. The reverse implication is proved similarly.

We now proceed to (2). Since $\{w_{\mathcal{L}}(\tau)\} = \mathcal{L} \cap \tau$ for all $\tau \in \Lambda_{\mathcal{L}}$ (see **Def. 1406**), we see that (2) is equivalent to demonstrating that $p(\tau - \gamma) \in \tau$ for all $\tau \in \Lambda_{\mathcal{L}}$. To this end, let $\tau \in \Lambda_{\mathcal{L}}$ be given, and choose $t \in I$ such that $p(t) \in \tau$. We see from the above that $t = \tau - \gamma$, and hence $p(\tau - \gamma) = p(t) \in \tau$. As $\tau \in \Lambda_{\mathcal{L}}$ was arbitrary, (2) is proved.

On the other hand, suppose that there is $\gamma \in \Gamma$ such that (1) and (2) are valid. To show that p is a time-parameterization of $\mathcal{L}$, we must demonstrate that (1)–(3) of **Def. 2305** are valid.

That (1) and (2) are valid is left to the reader.

To see **Def. 2305**(3), let $s, s' \in I$ be given. Then we see from **Prop. 2407**(2) and **Prop. 2406**(3) that

$$\begin{aligned} \bar{\mathsf{t}}_{\mathcal{L}}(p(s), p(s')) &= \bar{\mathsf{t}}_{\mathcal{L}}(w_{\mathcal{L}}(\gamma + s), w_{\mathcal{L}}(\gamma + s')) \\ &= (\gamma + s') - (\gamma - s) \\ &= s' - s. \end{aligned}$$

Since $s, s' \in I$ were arbitrary, **Def. 2305**(3) follows. $\Diamond$

Exercises

Exercises, I

1. Complete the proof of **Prop. 2204**.
2. Prove **Thm. 2308**.
3. Complete the proof of **Prop. 2402**.

4. Prove **Prop. 2403**.

5. Prove **Prop. 2406**.

6. Complete the proof of **Prop. 2407**.

EXERCISES, II

1. Let a timed eventworld $\mathcal{E}$ with precedence $\prec$ and timelapse t be given. Also, let a material worldpath $\mathcal{L}$ and an event y not in $\mathcal{L}$ be given. Moreover, assume that $x, z \in \mathcal{L}$ are given such that $x \prec y \prec z$ and $\mathsf{t}(x, y) = \mathsf{t}(y, z) = 0$. Prove that
$$\mathsf{Fut}(y) \cap \mathcal{L} \cap [\![x, z]\!] = \{z\}$$
and
$$\mathsf{Past}(y) \cap \mathcal{L} \cap [\![x, z]\!] = \{x\}.$$

2. Define the relation $\prec$ on $\mathbb{R}^2$ as in **Def. 1502**. Hence, if $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathbb{R}^2$ are such that $(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2)$, it follows that $(\beta_2 - \alpha_2)^2 \geq k^2(\beta_1 - \alpha_1)^2$. Thus, it makes sense to define $\mathsf{t} : \mathsf{Gr}(\prec) \to \mathbb{P}$ by
$$\mathsf{t}((\alpha_1, \alpha_2), (\beta_1, \beta_2)) := \sqrt{(\beta_2 - \alpha_2)^2 - k^2(\beta_1 - \alpha_1)^2}$$
for all $((\alpha_1, \alpha_2), (\beta_1, \beta_2)) \in \mathsf{Gr}(\prec)$.

   (a) Show that $\prec$ and t are translation-invariant; in other words, show that if $x, y \in \mathbb{R}^2$ are such that $x \prec y$ and $\mathbf{v} \in \mathbb{R}^2$, then
   $$x + \mathbf{v} \prec y + \mathbf{v} \quad \text{and} \quad \mathsf{t}(x + \mathbf{v}, y + \mathbf{v}) = \mathsf{t}(x, y).$$

   (b) Put $\underline{0} := (0, 0)$. Show that if $x, y \in \mathbb{R}^2$ are such that $\underline{0} \prec x$ and $x \prec y$, then $\mathsf{t}(\underline{0}, y) \geq \mathsf{t}(\underline{0}, x) + \mathsf{t}(x, y)$.

   (c) Use (a) and (b) to show that t gives $\mathbb{R}^2$ the structure of a timed eventworld. In other words, show that
   $$\mathsf{t}(x, z) \geq \mathsf{t}(x, y) + \mathsf{t}(y, z)$$
   for all $x, y, z \in \mathbb{R}^2$ such that $x \prec y \prec z$.

   (d) Show that the Intermediate Event Inequality does *not* reduce to equality; *i.e.,* produce $x, y, z \in \mathbb{R}^2$ such that $x \prec y \prec z$ and $\mathsf{t}(x, z) > \mathsf{t}(x, y) + \mathsf{t}(y, z)$.

For Exercises 3 and 4, assume that $\prec$ and $\mathtt{t}$ are given as in the previous Exercise.

3. Let I be an open interval in $\mathbb{R}$ and let continuous and differentiable functions $f, g : I \to \mathbb{R}$ be given such that $g^\bullet \geq k|f^\bullet|$. Show that $t \mapsto (f(t), g(t)) : I \to \mathbb{R}^2$ is the parameterization of some worldpath $\mathcal{L}$, and that

$$\bar{\mathtt{t}}_{\mathcal{L}}((f(a), g(a)), (f(b), g(b))) = \int_a^b \sqrt{(g^\bullet)^2 - k^2(f^\bullet)^2}$$

for all $a, b \in I$.[1] Moreover, show that if $g^\bullet > k|f^\bullet|$, then $\mathcal{L}$ is material.

4. Show that the parameterization $t \mapsto (f(t), g(t))$ of the previous Exercise is a time-parameterization if and only if $k^2(f^\bullet)^2 - (g^\bullet)^2 = -1$. (Hint: Use **Thm. 2308**.)

Exercises, III

For Exercises 1–4, we assume that $\prec$ and $\mathtt{t}$ are defined as in Exercise II,2 above.

1. Suppose that $\kappa \in \mathbb{R}$ is given such that $|\kappa| \geq k$. Show that

$$\begin{aligned} \mathcal{L} &:= \mathbb{R}\{(1, \kappa)\} \\ &= \{(\alpha, \alpha\kappa) \mid \alpha \in \mathbb{R}\} \end{aligned}$$

is a worldline, and that

$$\bar{\mathtt{t}}_{\mathcal{L}}((\alpha, \alpha\kappa), (\beta, \beta\kappa)) = \bar{\mathtt{t}}((\alpha, \alpha\kappa), (\beta, \beta\kappa))$$

for all $\alpha, \beta \in \mathbb{R}$. Finally, show that $\mathcal{L}$ is material if and only if $|\kappa| > k$.

2. Use the results of Exercise II,3 with $I := \mathbb{R}$ to show that

$$t \mapsto (\cosh t, k \sinh t)$$

is a parameterization of some worldline in $\mathbb{R}^2$, and describe this worldline geometrically.

[1] Here, we use the "dummy-free" notation "$\int_a^b h$" instead of the notation "$\int_a^b h(t)\, dt$". See, for example, [**7**], §**08**.

3. Find a time-parameterization of the worldline described in the previous Exercise. Use the results of Exercise II,3 and **Thm. 2308**.

4. Suppose $\kappa \in \mathbb{R}$ and $\mathcal{L}$ are given as in Exercise III,1 above. Use the results of Exercise II,3 and **Thm. 2308** to find a time-parameterization of $\mathcal{L}$.

5. Let I be a genuine interval in $\mathbb{R}$, and let $\mathcal{S}$ be any nonempty set. Define a precedence on $\mathcal{E} := I \times \mathcal{S}$ as in Exercise II,5 of Chapter 1. Thus, if $(a, s), (b, t) \in \mathcal{E}$ are are such that $(a, s) \prec (b, t)$, it follows that $b \geq a$, and hence it makes sense to define $\mathsf{t} : \mathsf{Gr}(\prec) \to \mathbb{P}$ by
$$\mathsf{t}((a, s), (b, t)) := b - a$$
for all $(a, s), (b, t) \in \mathsf{Gr}(\prec)$.
   (a) Show that t gives $\mathcal{E}$ the structure of a timed eventworld, and, moreover, that the Intermediate Event Inequality reduces to equality.
   (b) Show that with $I := \mathbb{R}$, t gives $\mathcal{E}$ the structure of a classical timed eventworld.

6. Let $\mathcal{E} := \mathbb{R}^2$ be the classical eventworld with precedence $\prec$ as described in Exercise III,4 of Chapter 1.
   (a) Describe the function t^* of **Prop. 2402**, when $\bar{\mathsf{t}} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$ is given by
$$\bar{\mathsf{t}}((a, b), (c, d)) := c - a$$
for all $(a, b), (c, d) \in \mathcal{E}$.
   (b) Show that $q : \mathbb{R} \to \mathcal{E}$ given by
$$q(t) := (t^3, t)$$
for all $t \in \mathbb{R}$ is a parameterization of some material worldpath, $\mathcal{L}$, and determine this worldpath. Also, find a time-parameterization $p : \mathbb{R} \to \mathcal{E}$ of $\mathcal{L}$ as described in **Prop. 2407**.

Exercises, IV

1. Notice the requirement in **Thm. 2308** that $\mathcal{L}$ be material. Give an example to show that the assertion of this Theorem is no longer valid if this condition is omitted. For example, produce a worldpath $\mathcal{Q}$ (which is not material) for which there exists *no* time-parameterization. (Hint: Choose $\mathcal{E}$ as in Exercise II,2 above.)