Introduction

A. Audience.

This book is primarily intended to be a textbook for undergraduate students majoring in mathematics or in physics. Most, if not all, of the existing textbooks on Relativity use verbal descriptions together with what one might call “the mathematics of variables and coordinates”. We believe that such an approach makes it difficult to convey a deeper understanding of the subject and an intuition for relativistic phenomena, an ability to “think relativistically” as it were. Such an understanding can be gained, we believe, by employing a more comtemporary type of mathematics: the mathematics of sets, mappings, and relations. Typically, physics majors rarely learn about this kind of mathematics, and mathematics majors, when they learn about it, are not told how useful it can be to gain a deeper understanding of the concepts of physics.

B. Prerequisites.

To a student who wants to study this book, it would be very helpful to have had, in addition to the usual calculus courses, a course in linear algebra and a course which deals with sets, mappings, and relations (“Introduction to Modern Mathematics” is a popular title). A brief overview of these topics and an explanation of the notation used here are given in the Appendices. It is not expected that the student knows any physics beyond what is taught in the usual introductory courses. Having had a physics course on relativity may be a hindrance rather than a help in that the student may need to “unlearn” the habit of doing mathematics in terms of “variables and coordinates”. However, we hope that taking physics courses or studying physics books on relativity *after* having understood the present book would be very fruitful.

C. Mathematical Structures.

The principal conceptual tool that is used in this book for the understanding of relativistic phenomena is the concept of a mathematical structure. Each structure is specified by a list of “ingredients” and a list of “axioms”. The types of structure studied here in detail include “eventworlds”, “timed eventworlds”, “flat eventworlds”, and “Minkowskian spacetimes”. The natural languages, such as English, implicitly presuppose a classical structure, and hence are inherently ill-suited for the description of relativistic phenomena. It is this fact that makes relativity so counter-intuitive and full of apparent paradoxes. We believe a paradox-free intuition for relativity can be obtained by gaining an understanding of the mathematical structures appropriate to relativity.

D. Organization.

For the student or physicist familiar with Special Relativity, an approach to the subject is offered which is not seen in the usual texts. The precedence relation is introduced straightaway. Armed only with this relation, the concept of a worldpath is developed quite thoroughly. Chapter 1 concludes with a discussion of classical and relativistic precedence relations. Thus, we have in this Chapter progressed as far as we were able with the precedence relation as our sole conceptual tool.

Chapter 2 is primarily concerned with the timelapse function and related concepts. With the introduction of a global timelapse function which indicates for any two events related by precedence the time necessary for a particle free of external influences to proceed from one event to the other, one may develop timelapses along worldpaths in a manner analogous to that used to define arc length along a curve in Euclidean geometry. Finally, we are in a position to parameterize material worldpaths (*i.e.,* those which do not represent paths of particles of zero mass).

Chapter 3 is concerned with flat eventworlds; the first section is entirely devoted to flat spaces, as many undergraduates are not exposed to this idea in introductory linear algebra courses. We then examine the relationship between the precedence relation and the flat-space structure of a flat eventworld; the concept of a *future cone* is first introduced here. We go further to analyze the relationship between the timelapse function and the flat space structure of a flat eventworld. We conclude this Chapter with a discussion of parameterizations in timed flat eventworlds; some proofs and intermediate results here involve topological considerations, and may harmlessly be omitted by the reader unfamiliar with these.

In Chapter 4, we show how spacetimes of classical physics can be treated in the context of this book. We incorporate the idea of *distance* into that of a classical eventworld by introducing pre-classical spacetimes. We are then in a position to introduce the important concept of a *reference frame*. The

Chapter comes to a close with a discussion of Galilean spacetimes, as well as a discussion of the hierarchy of classical structures introduced thus far.

In Chapter 5, attention is focused entirely on Minkowskian spacetimes. We open with a discussion of the relevant "non-genuine" inner product and the related concepts in linear algebra, as the typical undergraduate is usually exposed only to the "genuine" inner product. This is followed by a practical discussion of the geometry of spacetime diagrams in two dimensions and how this geometry is accurately represented. Then Minkowskian spacetimes are introduced, relating the non-genuine inner product to the precedence relation and the timelapse function in a relativistic eventworld. Spacetime decompositions are presented, followed by some applications. Finally, a discussion of the parameterizations of worldpaths follows, including a detailed example about space travel.

Chapter 6 introduces some new concepts and applications. First, the concepts of world-momentum and free particles are discussed, with applications such as the Doppler effect. Conservation of World-Momentum is offered next, and is applied to such phenomena as particle decay and the Compton effect. This Chapter ends with an entire section devoted to an application of these ideas to rockets and space travel.

Chapter 7 is concerned with a discussion of elementary electromagnetism (Maxwell's equations are not discussed). The relevant concepts in linear algebra (*i.e.,* skew lineons) are presented, followed by an investigation of the structure of skew lineons representing electromagnetic fields in a Minkowskian spacetime. We conclude with an important application of these concepts; namely, the description of worldpaths of charged particles in constant electromagnetic fields.

E. Exercises.

The Exercises come in four varieties:

I. Problems alluded to in the main text (which usually involve the completion of proofs),

II. The writing of simple to moderately difficult proofs,

III. Calculations which require an ability to use basic concepts in a practical context, and

IV. Problems which necessitate a deeper understanding of the concepts involved and the capacity to think about them creatively. These typically occur as one of three types:

   (a) Statements such as "Find a relation which is transitive and reflexive but not antisymmetric.",

   (b) Problems such as "Show the necessity of [Condition X] in **Theorem 1234**; that is, provide an example where [Condition X] fails to be true and the result of the Theorem is invalid.", and

   (c) Propositions for which the student must either provide a proof or a counterexample.

At the end of each Chapter, such Exercises are listed in Categories I, II, III, and IV, respectively, and are numbered sequentially in each category.

F. Notation.

Some of the notations used in this book are the usual ones, some are not. Appendix B includes a list of all notations used along with the page number on which they first appear. Appendix A contains many explanatory notes about the notations, and is organized according to page number. Upon encountering an unfamiliar notation, one may find the page number of its first occurrence in Appendix B, and then see if there is a corresponding explanation in Appendix A. Lack of an explanation in Appendix A usually means that the notation is defined within the text.

G. Genesis.

This book grew from a course taught by one of us (W.N.) every two or three years since 1961 and is based on handwritten notes for this course written over the years. The text was written by V.M., who had taken the course as a graduate student in 1987. Moreover, V.M. contributed many original ideas, especially for Chapter 3; he supplied details of proofs, and he created virtually all of the Exercises. A few of the ideas presented in this book were published previously in 1964 [**5**] and 1967 [**6**].

H. Acknowledgments.

Many students and colleagues, too numerous to mention, have contributed to the formulation of this manuscript through their participation in classes, stimulating conversations, and the reading of rough drafts of pieces of the book. One person, however, deserves an honorable mention. Professor Juan Schäffer of Carnegie Mellon University took special pains to thoroughly read a draft of the manuscript in its form as a doctoral thesis, and made many extremely valuable comments and suggestions. Without these, it is certain that the quality of the book would have noticeably suffered.

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Chapter 1

Eventworlds

In this chapter, we discuss precedence relations. A general discussion is offered in §**1.1**, with the corresponding formal definitions given in §**1.2**. In §**1.3**, we define the concept of a worldpath using only the precedence relation and provide some illustrations. The precedence relation as we experience it in our day-to-day lives (*i.e.,* in a classical context) is considered in §**1.4**, while a relativistic precedence is discussed in §**1.5**.

1.1 Events, Precedence, and Simultaneity

—–Events

The concept of "event" is fundamental and implicit in all of physics. The word "event", roughly, will refer to an occurrence, a happening. The usual connotation of grandeur or significance as in "the main event", or "a historic event", is not appropriate here. Examples of events include the birth of a baby, the popping of a champagne cork, a stoplight turning from red to green, and the collision of two subatomic particles.

Often, the meaning of a word depends upon the context in which it is used. Take, for example, the word "point". A dot of chalk on a chalkboard might be considered a point in the context of explaining basic ideas of elementary geometry. In another context, the same dot of chalk might be considered as a collection of many tiny grains of chalk dust, each of which may be considered a point. The sun might be considered a point for the purposes of astronomical calculations, but may also be considered as an aggregate of trillions of molecules of gas, each of which may be considered a point.

Similarly, a stoplight turning from red to green may be considered as a single event, or a collection of millions of individual electrical events, depending upon the context in which the changing stoplight is considered. Although it is a bit vague, we will choose the "physical world" as the context for events. Einstein offers an interpretation as follows [**1**, pp. 138-9]:

> What do we mean by rendering objective the concept of time? Let us consider an example. A person A ("I") has the experience "it is lightning."[1] ... For the person A the idea arises that other persons also participate in the experience "it is lightning." "It is lightning" is now no longer interpreted as an exclusively personal experience, but as an experience of other persons (or eventually only as a "potential experience"). In this way arises the interpretation that "it is lightning," which originally entered into the consciousness as an "experience," is now also interpreted as an (objective) "event." It is just the sum total of all events that we mean when we speak of the "real external world."

Minkowski [**3**] introduced the term "world-point" ("Welt-Punkt") for what is generally called an "event" in relativity. Actually, "world-point" is a more appropriate term for what is intended; namely, an "atom" of the "real external world". The concept of an "event" is a primitive concept; other definitions will include the term "event", but "event" itself will not be defined. Throughout this book, the concept of an event will be further clarified and refined, both through discussion and mathematical exposition.

[1]Einstein may have thought in terms of the German "es blitzt", which connotes a single stroke of lightning, and hence is more appropriate here than the English "it is lightning".

—Precedence

Traditionally, physicists have described two types of relationships among events; namely, *causal precedence* and *temporal precedence*. We will begin with a brief discussion of these relationships, and then propose a third relationship, simply called *precedence,* that is more fundamental in nature and that will be used in our discussion of relativistic phenomena.

To say that an event x causally precedes an event y means that x exerts a causal influence on y, or that x is *a* (but not necessarily *the*) cause of y. For example, we can say that the sun rising causally influences the release of oxygen into the air by trees since we may imagine a causal sequence of events precipitated by the sun rising which results in such a release of oxygen. It should be noted that other events, such as a spring rainfall, may also exert causal influences.

Yet this idea of precedence does not capture all subtleties which are desirable in a conceptualization of precedence. For example, suppose that George Washington had eggs for breakfast on the morning of his inauguration. This event precedes the event of your reading this sentence, yet it seems absurd to think that Washington having eggs for breakfast exerted a causal influence on your reading this book!

One might then propose another interpretation of precedence; namely, that an event x precedes y if x happens before y. This idea of precedence is called *temporal precedence.* But one difficulty with such a proposal stands out immediately: what does "before" mean? How do we know whether x happens before y or not? Such an idea of precedence depends on some concept of absolute time. Although we will discuss (in §**1.4** and §**2.4**) a context in which precedence can be interpreted in this way, it does not suffice in a more general context. Indeed, we will see later that in special relativity there is *no* concept of absolute time. Although there does exist the notion of time relative to an observer in special relativity, it is possible that there exist two events x and y such that x occurs before y with respect to one observer while y occurs before x with respect to another!

Hence, the idea of temporal precedence also fails to capture basic ideas about precedence. How, then, do we interpret precedence? We proceed in two stages. We say that for all events x and y, "x is in the history of y" if "at y, a record of the occurrence of x could be present". We then interpret "x precedes y" as meaning "every event in the history of x is also in the history of y". We will use this as an operational definition of precedence in the sequel.

—**Simultaneity**

It may happen that two events are equivalent with regard to the operational definition of precedence just presented. In other words, two events x and y may be such that for each event z, z is in the history of x if and only if z is in the history of y.

Let us consider a practical, day-to-day interpretation of the fact that two events x and y are equivalent in the sense just described. If it were the case, for example, that x occurred "before" y, at y there could be present records of events which would not be accessible at x; namely, those events occurring "after" x and "before" y; in this case, x and y would be distinguishable. Thus, we see that x and y must occur "at the same time"; *i.e.,* "simultaneously".

As a result, we say that events x and y are "simultaneous" if they are equivalent with regard to the operational definition of precedence given above. In this case, we also say that "x is simultaneous with y". We make the idea of "simultaneity" precise in the following section.

1.2 Eventworlds

Note: For all notations and concepts relevant to the theory of relations, the reader is referred to Appendix C.

In proposing a mathematical model of events and precedence among events, it is first necessary to describe the mathematical structures which will represent "events" and "precedence". We model "events" with a set $\mathcal{E}$. For any event x in $\mathcal{E}$, we denote by $\mathsf{H}(x)$ the set of all events in the history of x.

Recall from §**1.1** that we interpret "x precedes y" as meaning "any event in the history of x is also in the history of y". In view of the previous discussion, we see that a mathematical reformulation of this statement is "$\mathsf{H}(x) \subset \mathsf{H}(y)$". We leave as an Exercise the straightforward demonstration that the precedence relation, defined in this way, is reflexive and transitive.

It is also reasonable to assume that an event belonging to the history of an event actually precedes this event; in other words, given $x, y \in \mathcal{E}$, $y \in \mathsf{H}(x)$ should imply $\mathsf{H}(y) \subset \mathsf{H}(x)$.

This discussion of developing a model for events and precedence motivates the following definition.

1200 Definition: *An* **eventworld** *is a set $\mathcal{E}$ with structure given by specifying a relation $\prec$ on $\mathcal{E}$ which satisfies the following:*

(1) *$\prec$ is both reflexive and transitive, and*

(2) *For all $x, y \in \mathcal{E}$, there is some $z \in \mathcal{E}$ such that both $z \prec x$ and $z \prec y$.*

$\prec$ is called a **precedence relation on** $\mathcal{E}$. *If $x, y \in \mathcal{E}$ and $x \prec y$, we say that "x precedes y".*

Remark: Condition (2) states that the histories of every pair of events in $\mathcal{E}$ have some event in common. This condition disallows eventworlds which are, in some sense, comprised of a collection of "parallel" eventworlds. It is included here for completeness, but will not be discussed again until §**3.2**.

Notation: We will often consider events $x, y, z \in \mathcal{E}$ such that both $x \prec y$ and $y \prec z$. In this case, we write $x \prec y \prec z$ and say that y **is intermediate between** x **and** z. Thus, the transitivity of $\prec$ may be expressed as follows: for all $x, y, z \in \mathcal{E}$, $x \prec y \prec z \implies x \prec z$.

How might we express the concept of simultaneity in this context? Let $x, y \in \mathcal{E}$ be given. Recall that x is simultaneous with y if x and y are equivalent with respect to the operational definition of precedence presented in the previous section. In other words, x and y are simultaneous if for all events $z \in \mathcal{E}$, we have that z is in the history of x if and only if z is in the history of y. One may easily see this to be equivalent to the conjunction of "x precedes y" and "y precedes x". Hence, we are led to the following formal definition of simultaneity.

1201 Definition: *Let $\mathcal{E}$ be an eventworld with precedence $\prec$. We define the relation $\sim$ on $\mathcal{E}$, called the* **simultaneity relation on** $\mathcal{E}$, *by*

$$x \sim y :\Longleftrightarrow x \prec y \prec x$$

for all $x, y \in \mathcal{E}$. When $x \sim y$, we say that "x is simultaneous with y", or "x and y are simultaneous".

For the remainder of this section, let an eventworld $\mathcal{E}$ with precedence $\prec$ and simultaneity $\sim$ be given.

1202 Theorem: *Simultaneity is an equivalence relation (see* **Def. C12** *of Appendix C). That is, simultaneity is reflexive, symmetric, and transitive.*

Proof: Let an eventworld $\mathcal{E}$ with precedence $\prec$ and simultaneity $\sim$ be given. Since $\prec$ is reflexive, it immediately follows that $\sim$ is reflexive.

To see that $\sim$ is symmetric, suppose $x, y \in \mathcal{E}$ are such that $x \sim y$. Then we have both $x \prec y$ and $y \prec x$. It follows from the definition of simultaneity that $y \sim x$. Since $x, y \in \mathcal{E}$ were arbitrary, we see that $\sim$ is symmetric.

Finally, to see that $\sim$ is transitive, suppose $x, y, z \in \mathcal{E}$ are such that $x \sim y \sim z$. Then, by definition, we have $x \prec y$ and $y \prec x$, as well as $y \prec z$ and $z \prec y$. As $\prec$ is transitive, $x \prec y$ and $y \prec z$ together imply that $x \prec z$, while $z \prec y$ and $y \prec x$ together imply that $z \prec x$. Hence, we have $x \sim z$. Since $x, y, z \in \mathcal{E}$ were arbitrary, we see that $\sim$ is transitive. $\Diamond$

1203 Definition: *We define the relation $\precdot$ on $\mathcal{E}$ as follows:*

$$x \precdot y :\Longleftrightarrow x \prec y \text{ but not } x \sim y$$

for all $x, y \in \mathcal{E}$. The relation $\precdot$ is called **strict precedence**.

Remark: If $\prec$ is an order, then $\prec$ is merely $\precdot$ augmented by the equality relation; that is, the relation $\stackrel{\precdot}{=}$ (see **Def. C10** of Appendix C). However, if $\prec$ is not an order, $\stackrel{\precdot}{=}$ is strictly finer (see **Def. C09** of Appendix C) than $\prec$.

1204 Proposition: *The relation* $\precdot$ *as given in* **Def. 1203** *is a strict-order (see* **Def. C05** *of Appendix C); that is, it is strictly antisymmetric and transitive.*

Proof: To see that $\precdot$ is strictly antisymmetric, let $x, y \in \mathcal{E}$ be given such that $x \precdot y$. Then we have $x \prec y$ but not $x \sim y$, and hence $y \prec x$ must be false. As a result, $y \precdot x$ must also be false. Hence, we see that $x \precdot y$ implies that $y \precdot x$ is false. As $x, y \in \mathcal{E}$ were arbitrary, we see that $\precdot$ is strictly antisymmetric.

To see that $\precdot$ is transitive, let $x, y, z \in \mathcal{E}$ be given such that $x \precdot y \precdot z$. Then by **Def. 1203**, we must have both $x \prec y$ and $y \prec z$. Since $\prec$ is transitive, it follows that $x \prec z$. Now assume that $z \prec x$. Then since $x \prec y$ and $\prec$ is transitive, we would have $z \prec y$. But this is impossible, since $y \prec z$ and since $y \sim z$ is false by **Def. 1203**. Thus, we must not have $z \prec x$, and consequently $x \sim z$ is false. Hence, we have $x \precdot z$. Since $x, y, z \in \mathcal{E}$ were arbitrary, $\precdot$ is seen to be transitive. $\Diamond$

We introduce the following definition for completeness; we define the future, past, and present of an event the same way in both classical and relativistic eventworlds. More will be said in §**1.4** and §**1.5** about how these concepts, as well as the others presented in this section, are used in classical and relativistic worlds.

1205 Definition: *Given* $x \in \mathcal{E}$, *define*

$$\begin{aligned} \mathsf{Past}(x) &:= \{z \in \mathcal{E} \mid z \precdot x\}, \\ \mathsf{Pres}(x) &:= \{z \in \mathcal{E} \mid x \sim z\}, \\ \mathsf{Fut}(x) &:= \{z \in \mathcal{E} \mid x \precdot z\}. \end{aligned}$$

$\mathsf{Fut}(x)$, $\mathsf{Past}(x)$, *and* $\mathsf{Pres}(x)$ *are called the* **past**, **present**, *and* **future of** x, *respectively.*

Remark: It follows immediately from the definitions of strict precedence and simultaneity that for all $x \in \mathcal{E}$, $\mathsf{Past}(x)$, $\mathsf{Pres}(x)$, and $\mathsf{Fut}(x)$ are pairwise disjoint; that is,

$$\mathsf{Past}(x) \cap \mathsf{Pres}(x) = \mathsf{Pres}(x) \cap \mathsf{Fut}(x) = \mathsf{Fut}(x) \cap \mathsf{Past}(x) = \emptyset.$$

1.3 Worldpaths and Worldlines

We now introduce a concept very important to both classical and relativistic theories: the concept of a worldpath. This concept was introduced by Minkowski with the following words [**3**] (translation by W. N.):

> The entire world seems to be reducible to such worldpaths, and I would like to begin by saying that, in my opinion, all physical laws should find their most perfect expression as interactions between such worldpaths.

We begin with some definitions and follow with several examples. Apart from the concept of an eventworld, that of a worldpath is perhaps the most fundamental mathematical tool we will use to discuss relativistic theories.

Intuitively, we would like worldpaths to represent such phenomena as the voyage of a spaceship, the transmission of a signal, or the life of a particle. Thus, we need some way of describing a set of events which might represent various stages in an interstellar journey or different states in a particle's life. Mathematically, the concept of a *total order* is ideal for distinguishing such sets of events from arbitrary sets of events.

But not every totally ordered set of events in, for example, a particle's life suffices to adequately describe the life of a particle. Simply having a set of snapshots of the particle, one snapshot having been taken each hour, would not be adequate to describe the life of the particle, especially if the particle behaved in an irregular manner. Thus, we need some way of insuring that we have as complete a representation of the life of the particle as possible. This can be done by introducing the concept of a set of events being *locally maximally totally ordered.*

1300 Definition: *Let a set $\mathcal{D}$, a transitive and reflexive relation ρ on $\mathcal{D}$, and a subset $\mathcal{L}$ of $\mathcal{D}$ be given. We say that $\mathcal{L}$ is* ρ**-total** *if $\rho|_{\mathcal{L}}$ (see* **Def. C03** *of Appendix C) is total. We say that $\mathcal{L}$ is* ρ**-totally ordered** *if $\rho|_{\mathcal{L}}$ is a total order. We say that $\mathcal{L}$ is* **maximally totally ordered with respect to** ρ *(or* ρ**-m.t.o.***) if $\mathcal{L}$ is ρ-totally ordered and if there is no ρ-totally ordered subset of $\mathcal{D}$ that properly includes $\mathcal{L}$. In other words, $\mathcal{L}$ is ρ-m.t.o. if for every ρ-totally ordered subset $\mathcal{L}'$ of $\mathcal{D}$, $\mathcal{L} \subset \mathcal{L}'$ implies $\mathcal{L} = \mathcal{L}'$. Finally,* *we say that* $\mathcal{L}$ *is* **locally maximally totally ordered with respect to** ρ *(or* ρ**-l.m.t.o.***) if for all* $x, y \in \mathcal{L}$, $\mathcal{L} \cap [\![x, y]\!]_\rho$ *(see* **Def. C06** *of Appendix C) is maximally totally ordered with respect to* $\rho|_{[\![x,y]\!]_\rho}$.

We now wish to consider the concepts just defined in the context of an eventworld. To this end, let an eventworld $\mathcal{E}$ (with precedence $\prec$) be given.

1301 Notation: *We use the notation* $\preceq \; := \; \stackrel{\prec}{=}$ *(see* **Def. C10** *of Appendix C) so that*

$$x \preceq y \iff x \prec y \text{ or } x = y$$

for all $x, y \in \mathcal{E}$.

We see from the remark following **Def. 1203** and **Prop. 1204** that $\preceq$ is an order that is finer than the relation $\prec$.

1302 Definition: *A subset* $\mathcal{L}$ *of* $\mathcal{E}$ *with at least two members is called a* **worldpath** *if* $\mathcal{L}$ *is* $\preceq$*-total and locally maximally totally ordered with respect to* $\preceq$. *A worldpath* $\mathcal{L}$ *is said to be a* **worldline** *if it is maximally totally ordered with respect to* $\preceq$. *In other words,* $\mathcal{L}$ *is a worldline if it is a maximal worldpath; i.e., if for all worldpaths* $\mathcal{L}'$, $\mathcal{L} \subset \mathcal{L}' \implies \mathcal{L} = \mathcal{L}'$.

1303 Definition: *Let* $\mathcal{L}$ *be a worldpath. If* $\mathcal{L}$ *has a minimum [maximum] with respect to* $\prec$*, then this minimum [maximum] is called the* **beginning** *[***end***] of* $\mathcal{L}$*, and is denoted by* $\mathsf{beg}\ \mathcal{L}$ *[*$\mathsf{end}\ \mathcal{L}$*]. If* $\mathcal{L}$ *has both a beginning and an end, denoted by* x *and* y*, respectively, then we say that* $\mathcal{L}$ *is a* **worldpath from** x **to** y.

Before introducing several examples, we describe an alternative way of characterizing worldpaths which directly involves the precedence relation rather than $\preceq$. The main result is given in **Thm. 1308**. The reader who is not concerned with the details which justify this Theorem may skip ahead to the statement of this Theorem without loss of continuity.

1304 Proposition: *Let* $\mathcal{S} \subset \mathcal{E}$ *be given. Then* $\prec|_\mathcal{S}$ *is total if and only if* $\prec|_\mathcal{S}$ *is a total order.*

Proof: Suppose that $\prec|_\mathcal{S}$ is total. Then we must show that $\prec|_\mathcal{S}$ is both total and antisymmetric. To show totality, let $x, y \in \mathcal{S}$ be given. Since

$\prec\!\!\!\!\cdot|_{\mathcal{S}}$ is total, either $x \prec\!\!\!\!\cdot\; y$, $y \prec\!\!\!\!\cdot\; x$, or $x = y$ is true. Clearly, this implies that one of $x \prec y$, $y \prec x$, or $x = y$ is true. As $x, y \in \mathcal{S}$ were arbitrary, we see that $\prec|_{\mathcal{S}}$ is total.

To see the antisymmetry of $\prec|_{\mathcal{S}}$, let $x, y \in \mathcal{S}$ be given such that $x \prec y$ and $y \prec x$. By **Def. 1201** we then have $x \sim y$ and hence, by **Def. 1203**, neither $x \prec\!\!\!\!\cdot\; y$ nor $y \prec\!\!\!\!\cdot\; x$ can be valid. Since $\prec\!\!\!\!\cdot$ was assumed to be total, it follows that $x = y$. As $x, y \in \mathcal{S}$ were arbitrary, we see that $\prec|_{\mathcal{S}}$ is antisymmetric.

The proof of the converse is left as an Exercise. $\lozenge$

1305 Corollary: *Let $\mathcal{S} \subset \mathcal{E}$ be given. Then $\preceq\!\!\!\!\cdot|_{\mathcal{S}}$ is total if and only if $\prec|_{\mathcal{S}}$ is a total order. If this is the case, then $\preceq\!\!\!\!\cdot|_{\mathcal{S}} = \prec|_{\mathcal{S}}$.*

1306 Proposition: *Let $\mathcal{S} \subset \mathcal{E}$ be given. If $\mathcal{S}$ is $\prec$-totally ordered (equivalently, $\preceq\!\!\!\!\cdot$-total), then for all $a, b \in \mathcal{S}$ such that $a \prec b$ (equivalently, $a \preceq\!\!\!\!\cdot\; b$), we have*

$$\mathcal{S} \cap [\![a, b]\!]_{\prec} = \mathcal{S} \cap [\![a, b]\!]_{\preceq\!\!\!\!\cdot}\ .$$

Moreover, we have

$$\mathcal{S} \subset [\![a, b]\!]_{\prec} \iff \mathcal{S} \subset [\![a, b]\!]_{\preceq\!\!\!\!\cdot}\ .$$

Proof: Assume that $\mathcal{S} \subset \mathcal{E}$ is $\prec$-totally ordered, and let $a, b \in \mathcal{S}$ be given such that $a \prec b$. To see that $\mathcal{S} \cap [\![a, b]\!]_{\prec} \subset \mathcal{S} \cap [\![a, b]\!]_{\preceq\!\!\!\!\cdot}$, let $q \in \mathcal{S} \cap [\![a, b]\!]_{\prec}$ be given. Then $a \prec q$. Now if $q \prec a$, then we see from the antisymmetry of $\prec|_{\mathcal{S}}$ that $a = q$. On the other hand, if $q \prec a$ is false, then we have (by definition) that $a \prec\!\!\!\!\cdot\; q$. Thus, in either case, we see that $a \preceq\!\!\!\!\cdot\; q$. We may similarly show that $q \preceq\!\!\!\!\cdot\; b$. Thus, we have $q \in \mathcal{S} \cap [\![a, b]\!]_{\preceq\!\!\!\!\cdot}$. Since q was arbitrary, the forward inclusion is proved.

The reverse inclusion follows from the fact that $[\![a, b]\!]_{\preceq\!\!\!\!\cdot} \subset [\![a, b]\!]_{\prec}$, as $\preceq\!\!\!\!\cdot$ is a finer relation than $\prec$. Hence, the first part of the Proposition follows.

To see the second part, it follows from the first part just proved and the fact that for any sets A and B, $A \subset B \iff A = A \cap B$, that

$$\begin{aligned} \mathcal{S} \subset [\![a, b]\!]_{\prec} &\iff \mathcal{S} = \mathcal{S} \cap [\![a, b]\!]_{\prec} \\ &\iff \mathcal{S} = \mathcal{S} \cap [\![a, b]\!]_{\preceq\!\!\!\!\cdot} \\ &\iff \mathcal{S} \subset [\![a, b]\!]_{\preceq\!\!\!\!\cdot}\ . \end{aligned}$$

$\lozenge$

1307 Proposition: *Let $\mathcal{L} \subset \mathcal{E}$ be given, and suppose that $\mathcal{L}$ is $\prec$-totally ordered (equivalently, $\preceq$-total). Then $\mathcal{L}$ is l.m.t.o. with respect to $\prec$ if and only if $\mathcal{L}$ is l.m.t.o. with respect to $\preceq$.*

Proof: Assume that $\mathcal{L} \subset \mathcal{E}$ is $\prec$-l.m.t.o. Let $a, b \in \mathcal{L}$ be given such that $a \preceq b$, and let $\mathcal{S} \subset [\![a, b]\!]_{\preceq}$ be such that $\mathcal{L} \cap [\![a, b]\!]_{\preceq} \subset \mathcal{S}$. Since $\mathcal{L}$ is $\prec$-totally ordered and $a \preceq b$, the preceding Proposition yields that

$$\mathcal{L} \cap [\![a, b]\!]_{\prec} = \mathcal{L} \cap [\![a, b]\!]_{\preceq} \subset \mathcal{S} \subset [\![a, b]\!]_{\prec}.$$

Since $\mathcal{L}$ is $\prec$-l.m.t.o., we must have $\mathcal{S} = \mathcal{L} \cap [\![a, b]\!]_{\prec}$; by the preceding Proposition, we have $\mathcal{S} = \mathcal{L} \cap [\![a, b]\!]_{\preceq}$. As $a, b \in \mathcal{L}$ were arbitrary, we see that $\mathcal{L}$ is $\preceq$-l.m.t.o.

The reverse implication is proved similarly. $\Diamond$

1308 Theorem: *$\mathcal{L}$ is a worldpath if and only if it contains at least two events, is $\prec$-totally ordered, and is l.m.t.o. with respect to $\prec$.*

Proof: This follows immediately from the definition of a worldpath (**Def. 1302**), **Cor. 1305**, and **Prop. 1307**. $\Diamond$

Remark: In comparing **Def. 1302** and **Thm. 1308**, it might seem more natural to define worldpaths relative to $\prec$ rather than $\preceq$. However, the relation $\preceq$ has the advantage of being an order even if the relation $\prec$ is not, and so is used in **Def. 1302**. Incidentally, the proof of **Thm. 1309** is simpler when using worldpaths as described in **Def. 1302** rather than **Thm. 1308**.

In order to clarify the concept of a worldpath in an order-theoretic context, several detailed examples follow. In subsequent chapters, we will include useful analytic characterizations of worldpaths and worldlines, and, as a result, have less need for the order-theoretic definitions. The reason for the number of examples is that the above definitions are not the usual ones, and hence a thorough illustration of the use of the concepts may help in understanding them.

For some examples that follow, the notational device of a graph will be employed. As an example, consider the diagram below.

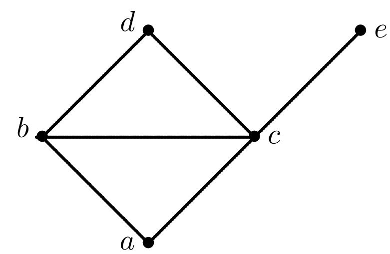


Figure 13a

This is a diagram of a relation, say $\prec$, on $\mathcal{E} := \{a, b, c, d, e\}$. The line segment going from a up to c represents the fact that $a \prec c$. In general, $x \prec y$ when x and y are opposite ends of a line segment and x is at the same level or below y in the diagram. Thus, that x and y are at opposite ends of a horizontal line segment indicates that both $x \prec y$ and $y \prec x$. Reflexivity and transitivity are tacitly assumed. For example, since $a \prec b$ and $b \prec d$, we assume that $a \prec d$ even though there is no line segment whose ends are a and d. In addition, we find that $[\![x, y]\!]$ contains x, y, and all points along any path from x to y in the diagram which does not "reverse direction" along the way (by "reversing direction" along a path, we mean travelling both up and down somewhere along the traversal, such as in going from b to d to c or from b to a to c); we may, however, traverse across horizontal segments, as in going from b to c to d. In particular, $[\![a, d]\!] = \{a, b, c, d\}$ and $[\![a, e]\!] = \{a, b, c, e\}$. We also see from the diagram that

$$\mathsf{Gr}(\prec) = \{(a,a), (b,b), (c,c), (d,d), (e,e), (a,b), (a,c), (b,c),\\ (c,b), (b,d), (c,d), (c,e), (a,d), (a,e), (b,e)\}.$$

Example 1: Let $\mathcal{E} := \{a, b, c, d, e, f\}$, and let $\prec$ on $\mathcal{E}$ be given by the following diagram.

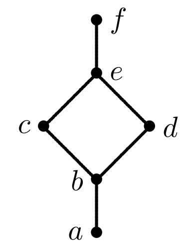


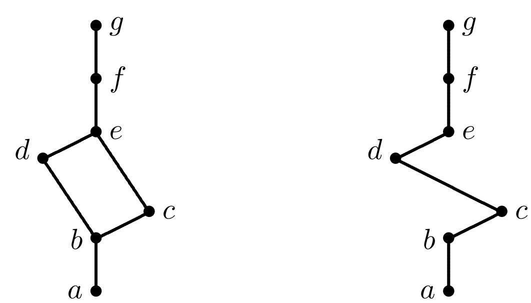
Figure 13b

What do worldpaths in $\mathcal{E}$ look like? One may verify that $\{b, c, e, f\}$, for example, is a worldpath in $\mathcal{E}$. Note that since c and d are incomparable, then no worldpath in $\mathcal{E}$ may contain both c and d.

The set $\mathcal{L} := \{a, b, f\}$ is *not* a worldpath, since the set $\{b, d, f\}$, which is a totally ordered subset of $[\![b, f]\!]$, strictly includes $\mathcal{L} \cap [\![b, f]\!]$. In other words, we may enlarge $\mathcal{L}$ by adding the event d without destroying the property of being totally ordered.

It is not difficult to see that the only worldlines in $\mathcal{E}$ are $\{a, b, c, e, f\}$ and $\{a, b, d, e, f\}$.

Example 2: Let $\mathcal{E} := \{a, b, c, d, e, f, g\}$, and suppose $\prec$ on $\mathcal{E}$ is described as in Figure 13c(1).



(1) (2)

Figure 13c

We see that $\mathcal{L} := \{a, b, c, e, f\}$ is a worldpath in $\mathcal{E}$ with respect to $\prec$. Now suppose that a precedence relation $\prec'$ on $\mathcal{E}$ is described as in Figure 13c(2). Note that $\mathsf{Gr}(\prec) \subset \mathsf{Gr}(\prec')$. Although $\mathcal{L}$ is totally ordered with respect to $\prec'$, it is *not* l.m.t.o. with respect to $\prec'$, since we may enlarge $\mathcal{L}$ by including $d \in [\![a, f]\!]$ without destroying the property of being totally ordered. Hence, $\mathcal{L}$ is not a worldpath with respect to $\prec'$. Thus, we see that a subset of $\mathcal{E}$ may be a worldpath with respect to one precedence on $\mathcal{E}$ but not another. This distinction will be very important in §**5.7**.

Example 3: Let $\mathcal{E}$ be the set of real numbers, $\mathbb{R}$, with precedence $\leq$. For each $a, b \in \mathbb{R}$, we put

$$[a, b] := \{a + \lambda(b - a) \mid \lambda \in \mathbb{R}, 0 \leq \lambda \leq 1\}.$$

A subset Λ of $\mathbb{R}$ is called an **interval** if

(I) *For all* $a, b \in \Lambda$, *we have* $[a, b] \subset \Lambda$.

In other words, an interval contains everything "between" any two of its members. Clearly, $[a, b]$ is an interval for all $a, b \in \mathbb{R}$. The empty set, singletons, half-lines, and all of $\mathbb{R}$ are also intervals. Notations for some of these and other types of intervals are given in Appendix A. We say that an interval is **genuine** if it contains at least two real numbers.

Remark: The notation "$[a, b]$" is usually used only when $a \leq b$. Our definition is consistent with the definition of line segments in flat spaces (see **Def. 3107**). We therefore have, for all $a, b \in \mathbb{R}$, that $[a, b] = [b, a]$.

We leave as an Exercise the proof that a genuine interval in $\mathbb{R}$ is a worldpath. Here, we show the converse: that a worldpath is a genuine interval.

To this end, suppose that we are given a worldpath $\mathcal{L}$ in $\mathbb{R}$. We first show that $\mathcal{L}$ is an interval. To begin, let $a, b \in \mathcal{L}$ and $c \in [a, b]$ be given. We must show that $c \in \mathcal{L}$. Consider the set $\mathcal{M} := (\mathcal{L} \cap [a, b]) \cup \{c\} \subset [a, b]$. Since $\mathbb{R}$ is totally ordered, $\mathcal{M}$ must be too; moreover, we have $\mathcal{L} \cap [a, b] \subset \mathcal{M}$. Since $\mathcal{L}$ is l.m.t.o., we must in fact have $\mathcal{L} \cap [a, b] = \mathcal{M}$; since $c \in \mathcal{M}$, we have $c \in \mathcal{L}$. As $c \in [a, b]$ was arbitrary, it follows that (I) holds, and hence $\mathcal{L}$ is an interval. Since by the definition of a worldpath $\mathcal{L}$ has at least two members, $\mathcal{L}$ is a genuine interval.

Thus, $\mathcal{L}$ is a genuine interval in $\mathbb{R}$ if and only if $\mathcal{L}$ is a worldpath in $\mathbb{R}$. Many intervals have a beginning or an end, many do not. For example, if $a, b \in \mathbb{R}$ and $a < b$, then $]a, b[$ has no beginning or end, but $[a, b]$

has both; $\mathsf{beg}\ [a, b] = a$ and $\mathsf{end}\ [a, b] = b$ (recall that the beginning or end of a worldpath must itself be a member of the worldpath; see **Def. 1303**). Moreover, we have $\mathsf{beg}\ [a, \infty[= a$, and hence an unbounded interval may have a beginning, while a bounded interval (such as $]a, b[$) may have neither a beginning nor an end. $\mathbb{R}$ itself is totally ordered and l.m.t.o.; in fact, $\mathbb{R}$ is a worldline. Moreover, $\mathbb{R}$ itself is the *only* worldline in $\mathbb{R}$.

The following Theorem is certainly one we would expect; informally, in certain circumstances we may join two worldpaths and thereby create a third. However, the proof of this fact is non-trivial. The reader is encouraged to carefully read the proof; all the mathematical properties of worldpaths are used.

Let an eventworld $\mathcal{E}$ (with precedence $\prec$) be given as before.

1309 Theorem: *Let two worldpaths $\mathcal{L}$ and $\mathcal{M}$ be given such that $\mathcal{L}$ has an end and $\mathcal{M}$ has a beginning that satisfy* $\mathsf{end}\ \mathcal{L} = \mathsf{beg}\ \mathcal{M}$. *Then $\mathcal{L} \cup \mathcal{M}$ is a worldpath.*

Proof: Put $\mathcal{Q} := \mathcal{L} \cup \mathcal{M}$, and $h := \mathsf{end}\ \mathcal{L}$, so that $x \prec h$ for all $x \in \mathcal{L}$ and $h \prec y$ for all $y \in \mathcal{M}$. It is clear that $\mathcal{Q}$ has at least two members. We must show that $\mathcal{Q}$ is total and locally maximally totally ordered with respect to $\preceq$.

For the remainder of this proof, intervals (such as $[\![x, y]\!]$) and the property of being l.m.t.o. are both understood to be with respect to $\preceq$.

It can easily be shown that $\preceq|_{\mathcal{Q}}$ is total. Now let $x, y \in \mathcal{Q}$ be given such that $x \preceq y$ and let $\mathcal{T} \subset [\![x, y]\!]$ be given such that $\mathcal{Q} \cap [\![x, y]\!] \subset \mathcal{T}$ and $\mathcal{T}$ is totally ordered. To show that $\mathcal{Q}$ is l.m.t.o., we must show that $\mathcal{Q} \cap [\![x, y]\!] = \mathcal{T}$.

Suppose x and y are both in $\mathcal{L}$. Then, since $\mathcal{L} \subset \mathcal{Q}$, we have $\mathcal{L} \cap [\![x, y]\!] \subset \mathcal{Q} \cap [\![x, y]\!] \subset \mathcal{T}$. Since $\mathcal{L}$ is l.m.t.o., we conclude that $\mathcal{L} \cap [\![x, y]\!] = \mathcal{T}$, and hence $\mathcal{Q} \cap [\![x, y]\!] = \mathcal{T}$. We apply a similar argument when x and y are both in $\mathcal{M}$.

Suppose $x \in \mathcal{L}$ and $y \in \mathcal{M}$. It is not difficult to show that $\mathcal{T} \cap [\![x, h]\!]$ is a totally ordered subset of $[\![x, h]\!]$ which includes $\mathcal{L} \cap [\![x, h]\!]$. As a result of $\mathcal{L}$ being a worldpath, we conclude that $\mathcal{T} \cap [\![x, h]\!] = \mathcal{L} \cap [\![x, h]\!]$. We may similarly argue that $\mathcal{T} \cap [\![h, y]\!] = \mathcal{M} \cap [\![h, y]\!]$.

Since $\mathcal{T}$ is totally ordered, these observations yield

$$\begin{aligned}\mathcal{T} &= \mathcal{T} \cap [\![x, y]\!] \\ &= (\mathcal{T} \cap [\![x, h]\!]) \cup (\mathcal{T} \cap [\![h, y]\!]) \\ &= (\mathcal{L} \cap [\![x, h]\!]) \cup (\mathcal{M} \cap [\![h, y]\!]).\end{aligned}$$

It is straightforward to show that both $\mathcal{L} \cap [\![x, h]\!] = \mathcal{L} \cap [\![x, y]\!]$ and $\mathcal{M} \cap [\![h, y]\!] = \mathcal{M} \cap [\![x, y]\!]$, so that

$$\begin{aligned}\mathcal{T} &= (\mathcal{L} \cap [\![x, y]\!]) \cup (\mathcal{M} \cap [\![x, y]\!]) \\ &= (\mathcal{L} \cup \mathcal{M}) \cap [\![x, y]\!] \\ &= \mathcal{Q} \cap [\![x, y]\!].\end{aligned}$$

We therefore obtain the desired conclusion that $\mathcal{Q} \cap [\![x, y]\!] = \mathcal{T}$. We cannot have $x \in \mathcal{M}$ and $y \in \mathcal{L}$ unless $x = y = h$, since this would imply that both $x \preceq y$ and $y \prec x$, which is impossible unless $x = y$. In this case, we have $\mathcal{Q} \cap [\![x, y]\!] = \{h\} = \mathcal{T}$.

Since the above cases are exhaustive and x and y were arbitrary, we see that $\mathcal{Q} = \mathcal{L} \cup \mathcal{M}$ is l.m.t.o. $\Diamond$

We now introduce a useful but subtle Proposition. Its proof involves a mathematical concept referred to as the "Axiom of Choice". The proof will be included for completeness, but can be omitted without loss. The only reference to this Proposition will be in the proof of **Thm. 1311**.

1310 Proposition: *Let $x, y \in \mathcal{E}$ be given such that $x \prec y$. Then there is a worldpath from x to y.*

Proof: We invoke a statement (see [**9**, pp. 395-6]) equivalent to the Axiom of Choice, known as

> **Hausdorff's Maximality Theorem:** Every nonempty ordered set includes a maximal totally ordered subset.

Since $x \prec y$, we have $\{x, y\} \subset [\![x, y]\!]$, and hence $[\![x, y]\!]$ is nonempty. By Hausdorff's Maximality Theorem, we may choose a maximal totally ordered subset $\mathcal{L}$ of $[\![x, y]\!]$. Since $\mathcal{L}$ contains at least two events, namely x and y, it is easily seen that $\mathcal{L}$ is a worldpath from x to y. $\Diamond$

We now present an interesting application of **Thm. 1309** and **Prop. 1310**. This Theorem will be of great importance when considering relativistic eventworlds in §**1.5**.

1311 Theorem: *Let $x, y \in \mathcal{E}$ be given such that $x \prec y$. Then there is exactly one worldpath from x to y if and only if $[\![x, y]\!]_{\preceq}$ is $\preceq$-total. If this is the case, then $[\![x, y]\!]_{\preceq}$ is that worldpath.*

Proof: We begin by stating a Lemma important to the proof of the Theorem; the proof is rather straightforward and is left as an Exercise.

1312 Lemma: *Suppose that ρ is a transitive relation on $\mathcal{D}$. Let $x, y \in \mathcal{D}$ be given such that $x \,\rho\, y$, and assume that $a, b \in [\![x, y]\!]_\rho$ are such that $a \,\rho\, b$. Then*

$$[\![a, b]\!]_\rho \cap [\![x, y]\!]_\rho = [\![a, b]\!]_\rho.$$

For the remainder of the proof, all intervals will be with respect to $\preceq$; that is, we write "$[\![x, y]\!]$" for "$[\![x, y]\!]_{\preceq}$" for brevity.

To see the forward implication of the Theorem, we proceed by contraposition; so suppose that $[\![x, y]\!]$ is not total with respect to $\preceq$. Then we may choose distinct events $a, b \in [\![x, y]\!]$ such that a and b are unrelated by $\preceq$. It is easy to see that $a \neq x$ and $a \neq y$. Hence, by **Prop. 1310**, we may choose a worldpath $\mathcal{A}'$ from x to a and a worldpath $\mathcal{A}''$ from a to y. By **Thm. 1309**, $\mathcal{A} := \mathcal{A}' \cup \mathcal{A}''$ is a worldpath from x to y. Clearly, $a \in \mathcal{A}$. In a similar way, we may find a worldpath, $\mathcal{B}$, from x to y which contains b. Since worldpaths are totally ordered, then we must have $a \notin \mathcal{B}$ since $b \in \mathcal{B}$ and a and b are unrelated by $\preceq$. Hence, $\mathcal{A}$ and $\mathcal{B}$ are distinct worldpaths from x to y.

To see the reverse implication, we assume that $[\![x, y]\!]$ is total with respect to $\preceq$. We first show that $[\![x, y]\!]$ is indeed a worldpath from x to y. As a result of our assumption, it only remains to show that $[\![x, y]\!]$ is $\preceq$ -l.m.t.o.

To this end, let $a, b \in [\![x, y]\!]$ be given, and assume without loss that $a \preceq b$. Further assume that $\mathcal{S}$ is a totally ordered subset of $[\![a, b]\!]$ satisfying $[\![x, y]\!] \cap [\![a, b]\!] \subset \mathcal{S}$. We must show that in fact, $\mathcal{S} = [\![x, y]\!] \cap [\![a, b]\!]$. But this follows from **Lemma 1312**, which, since $\preceq$ is transitive, results in the relations

$$[\![a, b]\!] = [\![x, y]\!] \cap [\![a, b]\!] \subset \mathcal{S} \subset [\![a, b]\!].$$

Since $a, b \in [\![x, y]\!]$ and $\mathcal{S} \subset [\![a, b]\!]$ were arbitrary, it follows that $[\![x, y]\!]$ is a worldpath from x to y.

To complete the proof, we must show that $[\![x, y]\!]$ is the only worldpath from x to y. To this end, let $\mathcal{L}$ be a worldpath from x to y. Now $[\![x, y]\!]$ itself is, by assumption, a totally ordered subset of $[\![x, y]\!]$ which includes $\mathcal{L} \cap [\![x, y]\!]$. Since $\mathcal{L}$ is a worldpath and hence l.m.t.o., it follows in fact that $\mathcal{L} \cap [\![x, y]\!] = [\![x, y]\!]$. Since $\mathcal{L}$ is a worldpath from x to y, it follows that $\mathcal{L} \subset [\![x, y]\!]$. Hence, we have $\mathcal{L} = [\![x, y]\!]$. Since $\mathcal{L}$ was arbitrary, we conclude that $[\![x, y]\!]$ is the only worldpath from x to y. $\Diamond$

1.4 Classical Eventworlds

Let an eventworld $\mathcal{E}$ (with precedence $\prec$) be given.

What are our common-sensical, day-to-day ideas regarding precedence? In our daily parlance, we often use phrases like "x happened before y" or "y happened after x". Usually, it is implicitly assumed that given two events x and y, either one happened "before" the other, or they are "simultaneous" happenings. Informally, events are "simultaneous" if they happen at "the same time", but possibly at different "places". Practically, this is decided by ascertaining "at what time" each event occurred (relative to a universal time standard) and comparing the respective results.

Remark: It is important to distinguish "coincidence" from "simultaneity". In our day-to-day experience, we would consider two happenings to be *coincident* if they happen at the same "time" *and* the same "place". We consider coincident happenings as merely different aspects of a single event.

We can make these assumptions explicit mathematically by saying that given distinct events x and y in $\mathcal{E}$, we must have either $x \prec y$ or $y \prec x$; that is, either "x happens before y" or "y happens before x". In other words, $\prec$ is total. We formalize this in the following.

1400 Definition: *A* **classical eventworld** *is an eventworld whose precedence is total.*

Let a classical eventworld $\mathcal{E}$ be given.

Simultaneity, as defined in **Def. 1201**, is an important concept in a classical eventworld. As a practical example, if the event of a news broadcast beginning at a residence in Pennsylvania and the event of a telephone ringing at a pay phone in Alabama both happen at 6:00, we would say that such events are simultaneous. In the vernacular, we would say that the events happened at the same "time" but at different "places". We will make such ideas more precise below; in this case, would would say that the events belonged to the same *instant* (see below) but different *locations* (see **Def. 4200**).

The following Proposition is a result of the fact that classical precedence is total. The proof is left as an Exercise.

1401 Proposition: *For all $x \in \mathcal{E}$, we have*

$$\mathcal{E} = \mathsf{Past}(x) \cup \mathsf{Pres}(x) \cup \mathsf{Fut}(x).$$

Hence, if $\mathsf{Past}(x) \neq \emptyset$ and $\mathsf{Fut}(x) \neq \emptyset$, then $\{\mathsf{Past}(x), \mathsf{Pres}(x), \mathsf{Fut}(x)\}$ is a partition of $\mathcal{E}$.

Since the simultaneity relation $\sim$ is an equivalence relation on $\mathcal{E}$ (see **Thm. 1202**), we may form the corresponding partition of $\mathcal{E}$ (see the discussion following **Def. C12** in Appendix C). We call this partition Γ; members of Γ are called **instants**. Thus,

$$\Gamma = \{\mathsf{Pres}(x) \,|\, x \in \mathcal{E}\}.$$

If $x \in \tau \in \Gamma$, we say that "x occurs at the instant τ", "x occurs at τ", or "τ is the time of x".

We may extend, in a natural way, the concept of precedence to instants, as described in the following definition.

1402 Definition: *For all $\sigma, \tau \in \Gamma$, we write $\sigma \,\tilde{\prec}\, \tau$, and we say that σ* **precedes** *τ, if $x \prec y$ for all $x \in \sigma$ and $y \in \tau$.*

1403 Proposition: *$\tilde{\prec}$ is a total order in Γ.*

Proof: We first show that $\tilde{\prec}$ is total. To this end, let $\sigma, \tau \in \Gamma$ be given and choose $\bar{x} \in \sigma$ and $\bar{y} \in \tau$. Since $\prec$ is total and reflexive, either $\bar{x} \prec \bar{y}$ or $\bar{y} \prec \bar{x}$ (or perhaps both). Assume first that $\bar{x} \prec \bar{y}$. Now for all $x \in \sigma$ and $y \in \tau$, we have $x \sim \bar{x} \prec \bar{y} \sim y$, and thus $x \prec y$. Hence, we conclude that $x \prec y$ for all $x \in \sigma$ and $y \in \tau$; that is, $\sigma \tilde{\prec} \tau$. If it were the case that $\bar{y} \prec \bar{x}$, we would in a similar manner argue that $\tau \tilde{\prec} \sigma$.

That $\tilde{\prec}$ is an order is left as an Exercise. $\Diamond$

1404 Definition: *We define intervals in Γ as for any totally ordered set; that is, we say that a subset Λ of Γ is an* **interval in** Γ *if for all $\pi, \tau \in \Lambda$ and $\sigma \in \Gamma$, it follows from $\pi \tilde{\prec} \sigma \tilde{\prec} \tau$ that $\sigma \in \Lambda$. If an interval Λ has at least two members, we say that Λ is* **genuine**.

In a classical eventworld, it is not difficult to characterize worldpaths in a fairly natural way, as shown in the following Theorem.

1405 Theorem: *A subset $\mathcal{L}$ of $\mathcal{E}$ is a worldpath if and only if*

(1) *For each $\tau \in \Gamma$, $\mathcal{L} \cap \tau$ contains at most one event, and*

(2) *$\Lambda := \{\tau \in \Gamma \,|\, \mathcal{L} \cap \tau \neq \emptyset\}$ is a genuine interval in Γ.*

Proof: In this proof, we use the characterization of worldpaths as given in **Thm. 1308**.

Suppose that $\mathcal{L} \subset \mathcal{E}$ is a worldpath. To see (1), let $\tau \in \Gamma$ and $x, y \in \mathcal{L} \cap \tau$ be given. Since $x, y \in \tau$, we have $x \sim y$, *i.e.*, $x \prec y \prec x$. Since $\mathcal{L}$ is ordered with respect to $\prec$, we must have $x = y$. Since x and y were arbitrary in $\mathcal{L} \cap \tau$, we see that (1) is valid.

Suppose that (2) were false. Then we could choose $\pi, \tau \in \Lambda$ and $\sigma \in \Gamma \setminus \Lambda$ such that $\pi \tilde{\prec} \sigma \tilde{\prec} \tau$. Now choose $x \in \mathcal{L} \cap \pi$ and $y \in \mathcal{L} \cap \tau$, and choose $z \in \sigma$. Then $\mathcal{L} \cap [\![x, y]\!]$ is not maximally totally ordered with respect to $\prec$, since for every $z \in \sigma$, we have that $(\mathcal{L} \cap [\![x, y]\!]) \cup \{z\}$ is totally ordered and

$$\mathcal{L} \cap [\![x, y]\!] \subsetneqq (\mathcal{L} \cap [\![x, y]\!]) \cup \{z\} \subset [\![x, y]\!].$$

Hence, $\mathcal{L}$ is not l.m.t.o., and is therefore not a worldpath.

On the other hand, let $\mathcal{L} \subset \mathcal{E}$ be given, and suppose that (1) and (2) are valid. Since Γ is totally ordered with respect to $\widetilde{\preceq}$ (see **Thm. 1403**), it is not difficult to show that a subset of $\mathcal{E}$ containing at most one event from each instant in Γ must be totally ordered with respect to $\prec$.

Since Λ is a genuine interval in Γ, $\mathcal{L}$ contains at least two events. It remains to show that $\mathcal{L}$ is l.m.t.o. with respect to $\prec$.

To this end, let $x, y \in \mathcal{L}$ be given, and let $\mathcal{S} \subset [\![x, y]\!]$ be totally ordered and such that $\mathcal{L} \cap [\![x, y]\!] \subset \mathcal{S}$. Let $z \in \mathcal{S}$ be given, and determine $\pi, \sigma, \tau \in \Gamma$ such that $x \in \pi$, $y \in \tau$, and $z \in \sigma$. Since $z \in [\![x, y]\!]$, we have $\pi \widetilde{\preceq} \sigma \widetilde{\preceq} \tau$, and since $\pi, \tau \in \Lambda$, we must have $\sigma \in \Lambda$ as a result of (2). Hence, we may choose $z' \in \mathcal{L} \cap \sigma \subset \mathcal{L} \cap [\![x, y]\!] \subset \mathcal{S}$. Since $z, z' \in \sigma \cap \mathcal{S}$ and $\mathcal{S}$ is totally ordered, we must have $z = z' \in \mathcal{L} \cap \sigma \subset \mathcal{L} \cap [\![x, y]\!]$. As $z \in \mathcal{S}$ was arbitrary, we conclude that $\mathcal{S} \subset \mathcal{L} \cap [\![x, y]\!]$, and hence $\mathcal{L} \cap [\![x, y]\!] = \mathcal{S}$. As x, y, and $\mathcal{S}$ were arbitrary, we see that $\mathcal{L}$ is l.m.t.o. with respect to $\prec$. $\Diamond$

In other words, $\mathcal{L}$ is a worldpath if and only if $\mathcal{L}$ is totally ordered and the instants in Γ to which events in $\mathcal{L}$ belong constitute a genuine interval in Γ. Thus, whether or not a totally ordered subset $\mathcal{L}$ of $\mathcal{E}$ is a worldpath depends not on the particular events in $\mathcal{L}$, but on the instants to which they belong. As a result, if Λ is a genuine interval in Γ, then *any* subset of $\mathcal{E}$ formed by choosing exactly one event from each instant in Λ is a worldpath. Moreover, every worldpath may be formed in precisely this fashion.

As a result of the previous Theorem, we may "parameterize" worldpaths as follows.

1406 Definition: *Let $\mathcal{L}$ be a worldpath, and let $\Lambda_{\mathcal{L}}$ be the interval as described in* (2) *of* **Thm. 1405**. *Then we may define the* **natural parameterization**

$$w_{\mathcal{L}} : \Lambda_{\mathcal{L}} \to \mathcal{E}$$

of $\mathcal{L}$ by requiring that for each $\tau \in \Lambda_{\mathcal{L}}$, $w_{\mathcal{L}}(\tau)$ be the event in the singleton $\mathcal{L} \cap \tau$.

Note that $\mathsf{Rng}\ w_{\mathcal{L}} = \mathcal{L}$. Also note that in the event that $\mathcal{L}$ is a worldline, we have $\Lambda_{\mathcal{L}} = \Gamma$.

1.5 Relativistic Eventworlds

We have just examined the idea of classical precedence; that is, the way that we experience precedence in our daily lives. In particular, we saw that classical precedence is total.

However, when we begin imagining situations which are outside of our usual daily routine – in particular, space travel and communication – we find that our classical conceptions of time are no longer adequate. Consider the following example in a given eventworld, $\mathcal{E}$. Here, we interpret "x precedes y" to mean, operationally, that "one can send a message from x, a record of the reception of which could be present at y".

Suppose that a manned spacecraft is experiencing some technical difficulty – a main power source will fail in five minutes (relative to the spacecraft) if the situaton is not remedied. So at an event x, the spacecraft sends a distress signal describing the problem to ground control, which is ten light-minutes away (see Figure 15a). At the event z, ground control receives the distress signal, and, being familiar with the situation, immediately sends a reply. The spacecraft receives the solution at the event y. Let z' be the event five minutes (relative to the spacecraft) after the distress signal was sent; that is, the anticipated failure of the power source.

We now pose the following question: in what manner are the events z and z' related? Well, it is clear that we cannot have $z \prec z'$; the message sent from z arrives at y, which occurs after z' relative to the spacecraft. Thus, we can have no record of the reception of the message sent from z at z'. Therefore, there is no way that ground control can prevent the failure of the power source. Likewise, we can not have a record of z' at z; were the power source to fail, a signal to that effect would be received after ground control was notified of the problem.

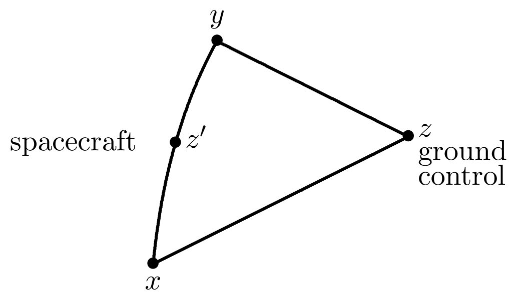


Figure 15a

Thus, we have two distinct events, z and z', neither of which is related to the other by the precedence relation. As a result, when we begin thinking outside of our daily routines and involve the emission and reception of electromagnetic signals over great distances, we find that precedence in such a context is not total and that our classical intuitions are insufficient to gain an understanding of such phenomena. This illustrates one of the greatest difficulties in developing an intuition about special relativity: one cannot rely on one's "classical" intuition, but must develop instead a "relativistic" intuition. A major goal of this book is to enable the reader to begin the process of developing such an intuition.

Let us consider further our current interpretation of precedence: that "x precedes y if one can send a message from x, a record of the reception of which could be present at y". To gain a fuller understanding of this interpretation, it is natural to inquire into the nature of the messages which may be sent from one event to another. We may send a message through the postal service, a radio signal, or an audio signal, for example. Yet in a real, practical sense, there is a limit to the "expeditiousness" with which a message can be sent; should one desire to send a message from one location to another (the term "location" is used here as in **Def. 4200**), there is no more expeditious way to send this message than by sending some sort of electromagnetic signal. In fact, the possibility of sending a more expeditious message leads to inconsistencies with the laws of electromagnetism (in a way which we cannot describe here). It is these inconsistencies which led Einstein to introduce the idea of relativistic precedence.

What might this imply about the precedence relation? Suppose we are given two distinct events x and y such that x precedes y. Is it then possible for the situation to be reversed, that is, for y to then send a message, a record of the reception of which could be present at x? If this were the case, then at x the record of a reply from a message sent from x would be present. Then as far as the event x is concerned, there must have been an *instantaneous* transmission of a message to y, and an *instantaneous* transmission of the reply from y. But this is impossible, since we have presumed that sending an electromagnetic signal is the most expeditious way to send a message, and we know from physical experiments that electromagnetic signals do not instantaneously propagate. Thus, it cannot then be the case that from y we can send a message, a record of the reception of which could be present at x. In other words, we cannot receive at x a reply to a message sent from x. Since for distinct events x and y we cannot have both x precedes y and y precedes x, then the precedence relation must be antisymmetric. Said another way, there are not two distinct simultaneous events in $\mathcal{E}$. When the precedence relation is interpreted in such a way that it is antisymmetric as well as reflexive and transitive (that is, the precedence relation is an order relation), we say that the precedence relation is *relativistic.* For completeness, we include the following.

1500 Definition: *We say that an eventworld $\mathcal{E}$ is a* **relativistic eventworld** *if the precedence relation is antisymmetric, and hence an order. In this case, we also say that the precedence relation itself is* **relativistic.**

We see, then, that electromagnetic signals (hereafter referred to simply as "signals") play a critical role in the development of the theory of special relativity. This, too, was borne out in the introductory example; because of physical limitations, it was not possible for the spacecraft to receive a reply to its distress signal any sooner (relative to the spacecraft) than y. This, in part, resulted in the events z and z' being unrelated.

Remark: Some authors [**10**, p. 7–10] have, for this reason, opted to begin their expositions of special relativity with the *signal relation;* that is, for two events x and y, x is *signal-related* to y if one may send a signal from x which can be received at y; that is, y is the reception of a signal sent from x.

Such an approach is logically equivalent to the present approach in terms of developing a theory of special relativity. Recall, however, that one of the goals of developing a general theory of precedence (as in §**1.1**-§**1.3**) was to provide a framework upon which *both* classical

> and relativistic theories may be built. Although the signal relation may well serve as a foundation for special relativity, it is not so well-suited to providing a basis for a classical theory.

Given the present approach, then, how might we obtain the signal relation from the precedence relation? Consider two events x and y such that y is the reception of a signal sent from x (and hence $x \prec y$). Let us imagine, for a moment, possible worldpaths from x to y. We must certainly include the worldpath of the signal itself. And since an electromagnetic signal is the optimal means of sending a message, any *other* worldpath from x to y must also be a signal. (Even though we may send a message *via* a signal of a different frequency, the worldpath of such a signal would be independent of this frequency.) In fact, we know from physical experiments that there is exactly one worldpath from x to y, and that this is the worldpath of a signal emitted from x and received at y.

Although not intended as a rigorous justification, the preceding discussion, along with **Thm. 1311**, serves as a motivation for the following. Note that since the precedence in a relativistic eventworld is antisymmetric, then in such an eventworld, the precedence $\prec$ coincides with the relation $\preceq$ as given in **Not. 1301**. Thus, the use of "$\prec$" in the following definition is justified.

1501 Definition: *Let $\mathcal{E}$ be a relativistic eventworld. For all $x, y \in \mathcal{E}$, we say that x* **is signal-related to** *y, and we write $x \rightharpoonup y$, if $x \prec y$ and if $[\![x, y]\!]_\prec$ is totally ordered. $\rightharpoonup$ is called the* **signal relation on $\mathcal{E}$**.

—Relativistic Eventworlds in $\mathbb{R}^2$

We introduce in this section some examples of relativistic eventworlds. Not only will these examples concretely involve several concepts presented in this chapter, but they will also serve as preparation for Chapter 5. The presentation in Chapter 5 will, however, be considerably more abstract; it is useful when learning abstract concepts to be able to refer to concrete instances of them.

Put $\mathcal{E} := \mathbb{R}^2$. We begin by defining two relations on $\mathcal{E}$. For the remainder of this section, assume that we have been given a strictly positive number $k \in \mathbb{P}^\times$.

1502 Definition: *We define the relations $\prec$ and $\triangleleft$ on $\mathcal{E}$ as follows:*

$$(\alpha_1, \alpha_2) \prec (\beta_1, \beta_2) \;:\Longleftrightarrow\; k|\beta_1 - \alpha_1| \leq \beta_2 - \alpha_2$$

and

$$(\alpha_1, \alpha_2) \lhd (\beta_1, \beta_2) \;:\Longleftrightarrow\; k|\beta_1 - \alpha_1| < \beta_2 - \alpha_2 \text{ or } (\alpha_1, \alpha_2) = (\beta_1, \beta_2)$$

for all $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in \mathcal{E}$.

We leave as an Exercise to show that both $\prec$ and $\lhd$ endow $\mathcal{E}$ with the structure of a relativistic eventworld; that is, that $\prec$ and $\lhd$ are order relations.

These relations appear very similar to each other, but the strict inequality in the definition of $\lhd$ has profound consequences. We look at these consequences by first examining intervals with respect to $\prec$ and $\lhd$.

We geometrically construct our diagrams as follows: we imagine a horizontal axis representing the abscissas of pairs in $\mathcal{E}$ and a vertical axis representing the ordinates; we label the origin (the intersection of the axes) as $(0, 0)$. This corresponds to the usual representation of $\mathbb{R}^2$ in the plane.

Many observations which follow will be described geometrically. Rigorous proofs of the validity of these observations are left to the reader; although they are not difficult, they involve some tedious algebra.

We denote by $\mathsf{Fut}_{\prec}(x)$ the future of $x \in \mathcal{E}$ relative to the precedence $\prec$; we similarly connote $\mathsf{Fut}_{\lhd}(x)$, as well as the analogues for the present and past. We see that for $x \in \mathcal{E}$, $\mathsf{Fut}_{\prec}(x)$ can be represented by an "angle" whose edges are half-lines with slope k or $-k$, relative to the coordinate system (see Figure 15b(1)). The circle around the vertex of the angle indicates that it is not an element of $\mathsf{Fut}_{\prec}(x)$; an event x is never a member of its own future. $\mathsf{Fut}_{\lhd}(x)$ can be represented by an angle similar to the one which represents $\mathsf{Fut}_{\prec}(x)$ except that the edges are absent; this is indicated by dashed lines along the edges of the angle (see Figure 15b(2)).

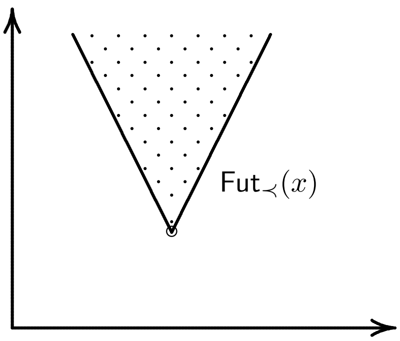


Figure 15b(1)

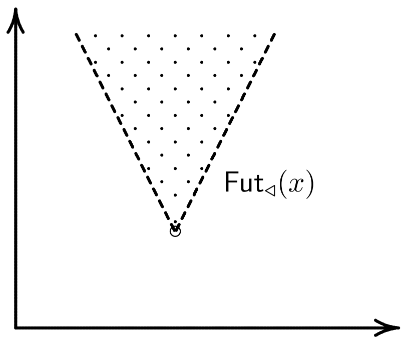


Figure 15b(2)

One may draw similar diagrams for $\mathsf{Past}_{\prec}(x)$ and $\mathsf{Past}_{\lhd}(x)$; the reader is encouraged to do so. With the aid of such diagrams, we are in a position to geometrically describe intervals with respect to $\prec$ and $\lhd$.

One can easily show that for $x, y \in \mathcal{E}$ such that $x \prec y$, we have

$$\begin{aligned} [\![x, y]\!]_{\prec} &= (\mathsf{Pres}_{\prec}(x) \cup \mathsf{Fut}_{\prec}(x)) \cap (\mathsf{Pres}_{\prec}(y) \cup \mathsf{Past}_{\prec}(y)) \\ &= \{x, y\} \cup (\mathsf{Fut}_{\prec}(x) \cap \mathsf{Past}_{\prec}(y)). \end{aligned}$$

We also see that for $x, y \in \mathcal{E}$ such that $x \lhd y$, we have

$$[\![x, y]\!]_{\lhd} = \{x, y\} \cup (\mathsf{Fut}_{\lhd}(x) \cap \mathsf{Past}_{\lhd}(y)).$$

Geometrically, we may represent such intervals as in the following two diagrams; we assume that $x, y \in \mathcal{E}$ are chosen such that both $x \prec y$ and $x \lhd y$.

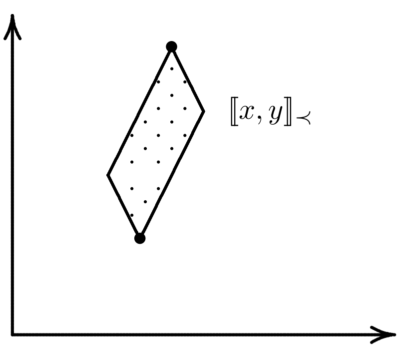


Figure 15c(1)

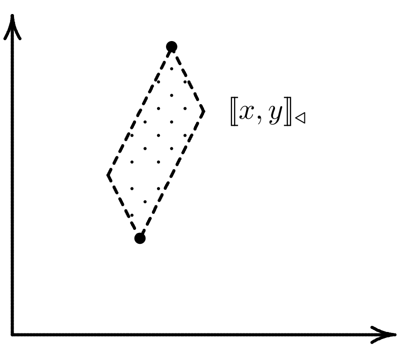


Figure 15c(2)

One critical difference between $\prec$ and $\lhd$ is their respective abilities to describe meaningful signal relations. One can easily see from the diagrams that if $x, y \in \mathcal{E}$ are such that $x \prec y$, then $[\![x, y]\!]_\prec$ is totally ordered if and only if either $x = y$ or y lies along an edge of $\mathsf{Fut}_\prec(x)$; however, the algebraic proof of this fact is not so straightforward. If $x, y \in \mathcal{E}$ are such that $x \prec y$ and $[\![x, y]\!]_\prec$ is totally ordered, we write $x \rightharpoonup y$ (see **Def. 1501**).

One may also verify geometrically that $[\![x, y]\!]_\lhd$ is *never* totally ordered unless $x = y$. In the literature, the relation $\lhd$ is often called *chronological precedence*, while $\prec$ is often referred to as *causal precedence*. We will use the former terminology, but refer to $\prec$ simply as *precedence*.

As it happens, the ability to create a signal relation (as described in **Def. 1501**) from the precedence relation is the crucial difference between $\prec$ and $\lhd$. In fact, we have the following: if $x, y \in \mathcal{E}$ are distinct, then $x \lhd y$ if and only if $x \prec y$ but not $x \rightharpoonup y$.

Thus, we have a purely order-theoretic way of deriving chronological precedence from the precedence relation. Roughly, chronological precedence is simply precedence "without" the signal relation. It is also possible to reconstruct the precedence relation from the chronological precedence relation solely in terms of the order-theoretic properties of chronological precedence. The details of this procedure, as well as other related issues, are fully described in the Exercises.

Exercises

Exercises, I

1. Complete the proof of **Prop. 1304**.
2. Complete the proof of **Prop. 1307**.
3. Consider $\mathbb{R}$ as an eventworld as in Example 3 of §**1.3**. Show that a genuine interval in $\mathbb{R}$ is a worldpath in $\mathbb{R}$.
4. Prove **Lemma 1312**.
5. Prove **Prop. 1401**.

6. Complete the proof of **Prop. 1403**.

7. Show that $\prec$ and $\lhd$, as given in **Def. 1502**, are order relations.

Exercises, II

1. Let a set $\mathcal{E}$ and a mapping $\mathsf{H} : \mathcal{E} \to \mathsf{Sub}\,\mathcal{E}$ be given (see the discussion at the beginning of §**1.2**). Define the relation $\prec$ on $\mathcal{E}$ by

$$x \prec y \;:\Longleftrightarrow\; \mathsf{H}(x) \subset \mathsf{H}(y)$$

for all $x, y \in \mathcal{E}$. Show that $\prec$ is reflexive and transitive.

In Exercises 2–4, we assume that an eventworld $\mathcal{E}$ with precedence $\prec$ is given.

2. Show that if a given worldpath $\mathcal{L}$ has a beginning, then $\mathcal{L} \setminus \{\mathsf{beg}\ \mathcal{L}\}$ is a worldpath. Show also that if $\mathcal{L}$ has an end, then $\mathcal{L} \setminus \{\mathsf{end}\ \mathcal{L}\}$ is a worldpath.

3. Let $\mathcal{L}$ be a worldpath, and let $x, y \in \mathcal{L}$ be given such that $x \prec\!\!\!\cdot\; y$. Show that the following are worldpaths:

   (a) $[\![x, y]\!] \cap \mathcal{L}$.
   (b) $\{z \in \mathcal{L} \,|\, z \prec x\}$.
   (c) $\{z \in \mathcal{L} \,|\, x \prec z\}$.
   (d) $\{z \in \mathcal{L} \,|\, z \prec\!\!\!\cdot\; x\}$.
   (e) $\{z \in \mathcal{L} \,|\, x \prec\!\!\!\cdot\; z\}$.

4. Let $\mathcal{L}$ be a worldpath. Let $x \in \mathcal{L}$ be given, and suppose that $y \in \mathcal{E}$ is such that $x \sim y$ and $y \notin \mathcal{L}$. Show that replacing x with y in $\mathcal{L}$ also results in a worldpath; that is, show that $(\mathcal{L}\setminus\{x\})\cup\{y\}$ is a worldpath. Illustrate this phenomenon by using a graph as in Examples 1 and 2. (Hint: Use the results of the previous Exercise.)

5. Let $\mathcal{S}$ be any set and I be a genuine interval in $\mathbb{R}$. Define a precedence $\prec$ on $\mathcal{E} := I \times \mathcal{S}$ by

$$(a, s) \prec (b, t) \;:\Longleftrightarrow\; a \leq b$$

for all $(a,s),(b,t) \in \mathcal{E}$. We call a subset $\mathcal{M}$ of $\mathcal{E}$ *functional* if no two distinct pairs of elements of $\mathcal{M}$ share the same first term. In other words, $\mathcal{M}$ is functional if

$$(a,s),(a,t) \in \mathcal{M} \implies s = t$$

for all $a \in I$ and $s,t \in \mathcal{S}$. Define $\pi : \mathsf{Sub}\ \mathcal{E} \to \mathsf{Sub}\ I$ by

$$\pi(\mathcal{M}) := \{a \in I \mid (a,s) \in \mathcal{M} \text{ for some } s \in \mathcal{S}\}$$

for all $\mathcal{M} \in \mathsf{Sub}\ \mathcal{E}$, so that $\pi(\mathcal{M})$ is the set of all first terms of pairs in $\mathcal{M}$.

Show that:

(a) A given $\mathcal{M} \in \mathsf{Sub}\ \mathcal{E}$ is totally ordered if and only if $\mathcal{M}$ is functional.

(b) A given $\mathcal{M} \in \mathsf{Sub}\ \mathcal{E}$ is a worldpath in $\mathcal{E}$ if and only if $\mathcal{M}$ is functional and $\pi(\mathcal{M})$ is a genuine interval in $I \subset \mathbb{R}$.

6. Let a relation ρ with domain $\mathcal{D}$ be given.

Definition: *For each $x \in \mathcal{D}$, define*

$$\mathsf{Pr}_\rho(x) := \{z \in \mathcal{D} \mid z \,\rho\, x\}.$$

We define the relation $\widehat{\rho}$ on $\mathcal{D}$ by

$$x \,\widehat{\rho}\, y :\Longleftrightarrow \mathsf{Pr}_\rho(x) \subset \mathsf{Pr}_\rho(y)$$

for each $x,y \in \mathcal{D}$.

(a) Show that $\widehat{\rho}$ is both reflexive and transitive.

(b) Show that $\widehat{\rho} = \widehat{\widehat{\rho}}$.

(c) Show that if ρ is reflexive and antisymmetric, then $\widehat{\rho}$ is antisymmetric.

(d) Show that if ρ is transitive and total, then $\widehat{\rho}$ is total.

(e) Show that if ρ is a total order, then $\rho = \widehat{\rho}$.

Exercises, III

1. Consider a relation ρ on $\mathcal{D} := \{a, b, c, d, e\}$ given by the graphical representation

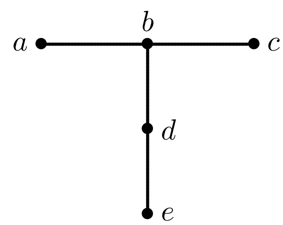


Figure Ex1(a)

as explained in §**1.3**. Find all subsets of $\mathcal{D}$ which are:

(a) worldpaths,

(b) worldlines,

(c) totally ordered but not l.m.t.o., and

(d) l.m.t.o. but not totally ordered.

2. Let $I, J \subset \mathbb{R}$ be genuine intervals in $\mathbb{R}$. Define a precedence $\prec$ on $\mathcal{E} := I \times J$ by

$$(a, b) \prec (c, d) \; :\Longleftrightarrow \; a \leq c \text{ and } b \geq d$$

for all $(a, b), (c, d) \in \mathcal{E}$.

(a) Show that $\prec$ is indeed a precedence on $\mathcal{E}$; that is, show that $\prec$ is transitive and reflexive.

(b) Find all minimal elements, maximal elements, minima, and maxima with respect to $\prec$.

(c) Describe the worldpaths and worldlines in $\mathcal{E}$ both mathematically and graphically.

Note: The answers to (b) and (c) may depend on what type the intervals I and J are (see note for pp. 14, 15 in Appendix A).

3. Proceed as in Exercise 2, except use the relation $\prec$ defined by

$$(a, b) \prec (c, d) \; :\Longleftrightarrow \; (a < c \text{ and } b > d) \text{ or } (a, b) = (c, d)$$

for all $(a, b), (c, d) \in \mathcal{E}$.

4. Define $\prec$ on $\mathcal{E} := \mathbb{R}^2$ by

$$(a,b) \prec (c,d) \;:\Longleftrightarrow\; a \leq c$$

for all $(a,b),(c,d) \in \mathcal{E}$.

 (a) Show that $\prec$ gives $\mathcal{E}$ the structure of a classical eventworld.
 (b) Determine Γ.
 (c) Determine $\{y - x \,|\, y \in \mathsf{Fut}(x)\}$ for each $x \in \mathcal{E}$.
 (d) Determine all worldlines in $\mathcal{E}$.

5. Proceed as in Exercise 4, except use the relation $\prec$ given by

$$(a,b) \prec (c,d) \;:\Longleftrightarrow\; a < c \text{ or } (a = c \text{ and } b \leq d)$$

for all $(a,b),(c,d) \in \mathcal{E}$.

6. We now address the issue of deriving the precedence relation $\prec$ from the chronological precedence relation $\triangleleft$ as given in **Def. 1502**. With the definition given in Exercise II,6 above in mind, show that

 (a) $\widehat{\triangleleft} = \prec$.
 (b) $\widehat{\precsim} = \prec$.

7. **Definition:** *Let ρ and τ be relations in a set $\mathcal{D}$. We say that τ is the* **transitive closure of** *ρ if τ is the finest transitive relation coarser than ρ (see* **Def. C09** *of Appendix C).*

 Now let an eventworld $\mathcal{E}$ with precedence $\prec$ be given as described in **Def. 1502**. Show that $\prec$ is the transitive closure of the corresponding signal relation, $\rightharpoonup$ (given in **Def. 1501**). In addition, for $(x,y) \in \mathsf{Gr}(\prec)$ such that $x \rightharpoonup y$ is false, find $z \in \mathcal{E}$ such that $x \rightharpoonup z \rightharpoonup y$.

Exercises, IV

1. Given a relation on $\mathbb{R}$, we may geometrically represent the relation by considering the graph of the relation as a subset of $\mathbb{R}^2$. As an example, consider the relation $\leq$ on $\mathbb{R}$. Then

$$\mathsf{Gr}(\leq) = \{(x,y) \in \mathbb{R}^2 \,|\, x \leq y\}.$$

We may graphically represent $\mathsf{Gr}(\leq)$ as the shaded portion of the following diagram along with the line forming its edge.

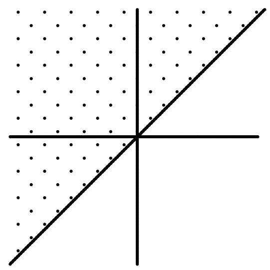


Figure Ex1(b)

If a relation on $\mathbb{R}$ is reflexive, we see that a graphical representation of the graph of the relation includes the line $y = x$ (the notation being the same as in standard analytic geometry). Describe geometrically the graphical representation of a relation that is:

(a) symmetric,

(b) antisymmetric,

(c) transitive,

(d) total,

(e) an order,

(f) a total order, and

(g) both symmetric and antisymmetric.

2. Do symmetry and antisymmetry necessarily imply reflexivity? Prove or provide a counterexample.

3. Let a relation ρ be given. Prove or provide a counterexample for the following: If δ is a reflexive, transitive relation which is coarser than ρ (see **Def. C09** of Appendix C), then it is also coarser than $\widehat{\rho}$ (see Exercise II,6). In other words, decide whether or not $\widehat{\rho}$ is the finest reflexive, transitive relation which is coarser than ρ.

4. Let a relation ρ with domain $\mathcal{D}$ be given.

Definition: *Define the relation* ρ_r *on* $\mathcal{D}$ *by*

$$x \, \rho_r \, y \;:\Longleftrightarrow\; x \, \rho \, y \text{ or } x = y$$

for each $x, y \in \mathcal{D}$. ρ_r is called the **reflexive closure of** *ρ.*

Of the two statements below, one is true and one is false. Provide a proof for the true statement and a counterexample for the false statement.

(a) If ρ is transitive, then ρ_r is transitive.

(b) If ρ_r is transitive, then ρ is transitive.

5. Suppose that ρ is a transitive and irreflexive relation on $\mathcal{D}$. Is $\rho_r = \widehat{\rho}$? (See Exercise II,6 and the previous Exercise.) Justify your answer.

6. Give an example of a relation which has exactly one minimal element which is not also a minimum.

7. Let an eventworld $\mathcal{E}$ with precedence $\prec$ be given. Suppose that $\mathcal{L}$ is a worldpath with a beginning, and $x \in \mathcal{E}$ is such that $x \prec \mathsf{beg}\ \mathcal{L}$. Also, suppose that $\mathcal{L} \cup \{x\}$ is a worldpath. What can you say about the interval $[\![x, \mathsf{beg}\ \mathcal{L}]\!]$?

8. Show the necessity of the assumption that ρ is reflexive in Exercise II,6(c) above; that is, exhibit an antisymmetric relation ρ which is *not* reflexive and for which $\widehat{\rho}$ is *not* antisymmetric.

9. In a manner analogous to that of the previous Exercise, show the necessity of the assumption that ρ is transitive in Exercise II,6(d) above.

10. Suppose that ρ is a relation on $\mathcal{D}$, and that $\rho = \widehat{\rho}$ (see Exercise II,6). Is ρ necessarily a total order? Prove or provide a counterexample.

11. Consider the following statement, which is a slight alteration of **Thm. 1311.** Is this statement true? Prove or provide a counterexample.

    Let $x, y \in \mathcal{E}$ be given such that $x \neq y$ and $x \prec y$. Then there is exactly one worldpath from x to y if and only if $[\![x, y]\!]_\prec$ is totally ordered. If this is the case, then $[\![x, y]\!]_\prec$ is that worldpath.