LINEARLY INDUCED MAPPINGS BETWEEN CONES OF QUADRATIC FORMS

RAY E. ARTZ AND WALTER NOLL

ABSTRACT. This paper deals with mappings between cones of positive quadratic forms which are induced by linear mappings between the underling vector spaces, *i.e.*, the spaces which are the domains of the forms. Three fundamental results are proved, two of which were previously announced by the second author. The first result states that there is an inclusion-reversing one-to-one correspondence between the lattice of subspaces of a given space and the lattice of faces of the cone of quadratic forms on that space. The second result states that all cone-isomorphisms between cones of quadratic forms are induced by linear isomorphisms between the underlying spaces. The third result states that a given cone-linear mapping F between cones of quadratic forms is induced by some linear mapping between the underlying spaces if and only if both F and its transpose preserve faces.

CONTENTS

1. Introduction

Mappings between linear cones have been recently studied in [NS77], [NS78], [Sch79c], [Sch79b], [Sch81], and [Sch79a]. The purpose of the present paper is twofold. First, we give proofs of two theorems that were announced without proof in [NS78] (where they were designated as Theorems 1 and 3). These theorems deal with linear cones consisting of quadratic forms over finite-dimensional real-linear spaces. The first of these, called Theorem 1 here, states that there is an inclusion-reversing one-to-one correspondence between the lattice of subspaces of a given space and the lattice of faces of the cone of quadratic forms on that space. The second, called Theorem 2 here, states that cone-isomorphisms between cones of quadratic forms are induced by linear isomorphisms between the underlying spaces (*i.e.*, the linear spaces which are the domains of the forms).

The second purpose of the paper is to show (Theorem 3 here) that a cone-linear mapping F between cones of quadratic forms on two linear spaces is induced by a linear mapping between the underlying spaces if and only if both F and its cone-transpose are face-preserving mappings.[1]

One of the reasons for studying the mappings described above is the possibility of generalizations to include the case when the cones of quadratic forms are replaced by Hermitian forms over complex-linear spaces, which arise in the study of quantum mechanics (see [Art78]). These generalizations, as well as generalizations which apply to forms over quaternionic-linear spaces, are deferred to possible future papers.

The notation and terminology of [Nol87] is used in this paper. In particular, $\mathbb{N}$ denotes the set of all natural numbers and $\mathbb{P}$ the set of all *positive* real numbers (both including zero). A superscript $^\times$ indicates the removal of zero; in particular $\mathbb{P}^\times$ denotes the set of all *strictly positive* real numbers. Given $n \in \mathbb{N}^\times$, we denote by $n^]$ the set consisting of the first n non-zero natural numbers. The collection of all subsets of a given set $\mathcal{S}$ is denoted by $\mathrm{Sub}\,\mathcal{S}$. Given a mapping ϕ and subsets $\mathcal{A}$ of its *domain* $\mathrm{Dom}\,\phi$ and $\mathcal{B}$ of its *codomain* $\mathrm{Cod}\,\phi$, we denote the *image* of $\mathcal{A}$ under ϕ by $\phi_>(\mathcal{A}) := \{\phi(x) \mid x \in \mathcal{A}\}$ and the *pre-image* of $\mathcal{B}$ under ϕ by $\phi^<(\mathcal{B}) := \{x \in \mathrm{Dom}\,\phi \mid \phi(x) \in \mathcal{B}\}$. If $\phi_>(\mathcal{A}) \subset B$, we define the *adjustment* $\phi|_\mathcal{A}^\mathcal{B} \colon \mathcal{A} \to \mathcal{B}$ of ϕ by

$$\phi|_\mathcal{A}^\mathcal{B}(x) := \phi(x) \qquad \text{for all } x \in \mathcal{A}.$$

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[1]The proof of Thm. 2 presented here was found by R.A; it is different and more direct that the one W.N. had in mind when [NS77] was published. Thm. 3 was discovered by R.A in 1992.

If ϕ is bijective, we denote its *inverse* by $\phi^{\leftarrow}\colon \operatorname{Cod}\phi \to \operatorname{Dom}\phi$.

Let a set $\mathcal{A}$ and $\mathcal{S} \in \operatorname{Sub}\mathcal{A}$ be given. We denote by $\mathbf{1}_{\mathcal{S}\subset\mathcal{A}} \in \operatorname{Map}(\mathcal{S},\mathcal{A})$ the *inclusion mapping*, *i.e.*, the mapping which satisfies $\mathbf{1}_{\mathcal{S}\subset\mathcal{A}}(x) = x$ for all $x \in \mathcal{S}$. We abbreviate $\mathbf{1}_{\mathcal{A}} := \mathbf{1}_{\mathcal{A}\subset\mathcal{A}}$, so $\mathbf{1}_{\mathcal{A}}$ is the *identity* mapping on $\mathcal{A}$.

When we say "let a linear space be given" (or equivalent language), we mean "let a finite-dimensional real linear space be given". When dealing with linear spaces, we extensively use the notation, terminology, and results of Chapters 1 and 2 of [Nol87]. In particular, when a linear space $\mathcal{V}$ is given, we identify $\mathcal{V}^{**} \cong \mathcal{V}$.

Let a linear space $\mathcal{V}$ and a subset $\mathcal{S}$ of $\mathcal{V}$ be given. We note that the *closure*[2] of $\mathcal{S}$ remains unchanged if $\mathcal{V}$ is replaced by a subspace of $\mathcal{V}$ that includes $\mathcal{S}$. The *interior* of $\mathcal{S}$, however, depends not only on $\mathcal{S}$ but also on the imbedding space $\mathcal{V}$; we write $\operatorname{Int}_{\mathcal{V}}\mathcal{S}$ for the interior of $\mathcal{S}$ to make this dependence on $\mathcal{V}$ explicit (See [Nol87], Sect. 53).

The collection of all subspaces of $\mathcal{V}$ is denoted by $\operatorname{Subsp}\mathcal{V}$. For each $\mathcal{U} \in \operatorname{Subsp}\mathcal{V}$, we denote by $\Omega_{\mathcal{V}/\mathcal{U}} \in \operatorname{Lin}(\mathcal{V}, \mathcal{V}/\mathcal{U})$ the *quotient mapping*; , *i.e.*, the mapping which satisfies $\Omega_{\mathcal{V}/\mathcal{U}}\boldsymbol{v} := \boldsymbol{v} + \mathcal{U}$ for all $\boldsymbol{v} \in \mathcal{V}$.

Let a second linear space $\mathcal{W}$ and a (not necessarily linear) mapping $\phi\colon \mathcal{V} \to \mathcal{W}$ be given. We call $\operatorname{Null}\phi := \phi^{<}(\{\mathbf{0}\})$ the **nullset of** ϕ.[3] We record several elementary facts for later reference:

Proposition 1.1. *Let sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and a surjective mapping $\alpha\colon \mathcal{C} \to \mathcal{A}$ be given. Then the mapping*

$$(1.1) \qquad (\phi \mapsto \phi \circ \alpha)\colon \operatorname{Map}(\mathcal{A},\mathcal{B}) \to \operatorname{Map}(\mathcal{C},\mathcal{B})$$

is injective.

Proposition 1.2. *Let $\mathcal{U} \in \operatorname{Subsp}\mathcal{V}$ be given. Then $\boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}} \in \mathcal{U}^{\perp}$ for all $\boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^{*}$ and*

$$(1.2) \qquad (\boldsymbol{\mu} \mapsto \boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}})\colon (\mathcal{V}/\mathcal{U})^{*} \to \mathcal{U}^{\perp}$$

is a linear isomorphism.

Proposition 1.3. *Let L be a linear mapping. Then there is exactly one linear isomorphism $\tilde{L}\colon \operatorname{Dom}L/\operatorname{Null}L \to \operatorname{Rng}L$ such that*

$$(1.3) \qquad L = \Omega_{\operatorname{Dom}L/NullL}\tilde{L}\,\mathbf{1}_{\operatorname{Rng}L\subset\operatorname{Cod}L}\,.$$

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[2]All topological terms are to be understood in the context of the usual topologies for finite-dimensional spaces; see Chapter 5 of [Nol87].

[3]In [Nol87], this notation was used only for the case in which ϕ is a linear mapping; in that case $\operatorname{Null}\phi$ is a subspace and is called the null*space* of ϕ.

Corollary 1. *Let L and L' be surjective linear mappings such that* $\operatorname{Dom} L = \operatorname{Dom} L'$ *and* $\operatorname{Null} L = \operatorname{Null} L'$. *Then there is exactly one linear isomorphism*
$A\colon \operatorname{Cod} L \to \operatorname{Cod} L'$ *such that* $L' = AL$.

Corollary 2. *Let L and L' be injective linear mappings such that* $\operatorname{Rng} L = \operatorname{Rng} L'$. *Then there is exactly one linear isomorphism* $A\colon \operatorname{Dom} L \to \operatorname{Dom} L'$ *such that*
$L = L'A$.

2. Linear cones

Throughout this section, we assume that a linear space $\mathcal{W}$ is given. A subset $\mathcal{P}$ of $\mathcal{W}$ is called a **linear cone** in $\mathcal{W}$ if it is stable under addition and under scalar multiplication by *strictly positive* real numbers, *i.e.,* if $\mathcal{P} + \mathcal{P} \subset \mathcal{P}$ and $\mathbb{P}^\times \mathcal{P} \subset \mathcal{P}$. Linear cones are convex sets. The interior and the closure of a given linear cone are again linear cones. The intersection of a collection of linear cones is again a linear cone. Hence, for every subset $\mathcal{S}$ of $\mathcal{W}$, there is exactly one smallest linear cone that includes $\mathcal{S}$; we call this linear cone the **cone-span** of $\mathcal{S}$ and denote it by $\operatorname{Csp}\mathcal{S}$ (see [Nol87], Sect. 03)

For the remainder of this section, we assume that a linear cone $\mathcal{P} \in \mathcal{W}$ is given.

If $\mathcal{P}$ is not empty, then its *linear span* is given by $\operatorname{Lsp}\mathcal{P} = \mathcal{P} - \mathcal{P}$. Thus $\mathcal{P}$ *spans* $\mathcal{W}$ if and only if $\mathcal{P} - \mathcal{P} = \mathcal{W}$. The **dimension** of $\mathcal{P}$, denoted by $\dim\mathcal{P}$, is defined to be the dimension $\dim\operatorname{Lsp}\mathcal{P}$ of its linear span.

A linear cone that is included in $\mathcal{P}$ is called a **subcone** of $\mathcal{P}$. We say that a subcone $\mathcal{F}$ of $\mathcal{P}$ is a **face** of $\mathcal{P}$ if it includes $\mathcal{P} \cap \{\mathbf{0}\}$[4] and

$$(2.1) \qquad \mathbf{u} + \mathbf{v} \in \mathcal{F} \implies \mathbf{u}, \mathbf{v} \in \mathcal{F} \qquad \text{for all } \mathbf{u}, \mathbf{v} \in \mathcal{P}.$$

The intersection of a collection of faces of $\mathcal{P}$ is again a face of $\mathcal{P}$. Hence, for a given subset $\mathcal{S}$ of $\mathcal{P}$, there is exactly one smallest face of $\mathcal{P}$ that includes $\mathcal{S}$; we call this face the **facial span** of $\mathcal{S}$, and denote it by $\operatorname{Fcsp}_{\mathcal{P}}\mathcal{S}$. Of course, $\mathcal{P}$ is a face of itself. A face of $\mathcal{P}$ which is a proper subset of $\mathcal{P}$ is called a **proper face** of $\mathcal{P}$. A one-dimensional face of $\mathcal{P}$ is called an **extreme ray** of $\mathcal{P}$. The concepts of face, facial span, and extreme ray do not depend on the linear space in which $\mathcal{P}$ is considered to be a linear cone.

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[4]This definition is revised relative to the one presented in [NS78]; the first condition has been added to insure that a face of a cone with zero contains the zero and is thus non-empty. This revision is necessary for Thm. 1 of [NS78] (as well as Thm. 1 of this paper) to be correct as stated.

Proposition 2.1. *Assume that $\mathcal{P}$ is not empty. Then $\mathcal{P}$ spans $\mathcal{W}$ if and only if $\mathrm{Int}_{\mathcal{W}}\mathcal{P}$ is non-empty.*

Proof. Since $\mathcal{P}$ is convex, this result is an immediate consequence of Prop. 6 of Sect. 54 of [Nol87]. $\qquad\square$

Proposition 2.2. *Let $\boldsymbol{u} \in \mathrm{Int}_{\mathcal{W}}\mathcal{P}$ be given. Then $\mathrm{Fcsp}_{\mathcal{P}}\{\boldsymbol{u}\} = \mathcal{P}$.*

Proof. Put $\mathcal{F} := \mathrm{Fcsp}\{\boldsymbol{u}\}$ and let $\boldsymbol{v} \in \mathcal{P}$ be given. Since $\boldsymbol{u} \in \mathrm{Int}_{\mathcal{W}}\mathcal{P}$, we may and do choose $\epsilon \in \mathbb{P}^{\times}$ such that $\boldsymbol{w} := \boldsymbol{u} - \epsilon\boldsymbol{v} \in \mathcal{P}$ and hence $\boldsymbol{w} + \epsilon\boldsymbol{v} = \boldsymbol{u}$. Since $\boldsymbol{u} \in \mathcal{F}$ and since $\mathcal{F}$ is a face, it follows from (2.1) that $\epsilon\boldsymbol{v} \in \mathcal{F}$. Hence, since $\mathcal{F}$ is a linear cone, we have $\boldsymbol{v} = \frac{1}{\epsilon}(\epsilon\boldsymbol{v}) \in \mathcal{F}$. Since $\boldsymbol{v} \in \mathcal{P}$ was arbitrary, it follows that $\mathcal{F} = \mathcal{P}$. $\qquad\square$

Proposition 2.3. *Every non-empty face of $\mathcal{P}$ is the facial span of a singleton.*

Proof. Let a non-empty face $\mathcal{F}$ of $\mathcal{P}$ be given and put $\mathcal{U} := \mathrm{Lsp}\,\mathcal{F}$. Then $\mathcal{F}$ is a linear cone that spans $\mathcal{U}$. Hence, by Prop. 2.1, the interior $\mathrm{Int}_{\mathcal{U}}\mathcal{F}$ of $\mathcal{F}$ relative to $\mathcal{U}$ is not empty. Choose $\boldsymbol{u} \in \mathrm{Int}_{\mathcal{U}}\mathcal{F}$. By Prop. 2.2 we have $\mathrm{Fcsp}_{\mathcal{P}}\{\boldsymbol{u}\} = \mathcal{F}$. $\qquad\square$

We say that a family $(\mathcal{F}_i \,|\, i \in I)$ of faces of $\mathcal{P}$ is **facially independent** if the facial span of its union properly includes the facial span of the union of each of its proper subfamilies, *i.e.*, if
(2.2)
$$\mathrm{Fcsp}_{\mathcal{P}}\left(\bigcup_{j \in J}\mathcal{F}_j\right) = \mathrm{Fcsp}_{\mathcal{P}}\left(\bigcup_{i \in I}\mathcal{F}_i\right) \implies J = I \quad \text{for all } J \in \mathrm{Sub}\,I.$$

A facially-independent family of faces of $\mathcal{P}$ is called a **facial decomposition** of $\mathcal{P}$ if the facial span of its union is $\mathcal{P}$. We note that a family $(\mathcal{F}_i \,|\, i \in I)$ of faces of $\mathcal{P}$ is a facial decomposition of $\mathcal{P}$ if and only if

$$(2.3) \qquad \mathrm{Fcsp}_{\mathcal{P}}\left(\bigcup_{j \in J}\mathcal{F}_j\right) = \mathcal{P} \iff J = I \quad \text{for all } J \in \mathrm{Sub}\,I.$$

The **dual** of the linear cone $\mathcal{P}$ is defined by

$$(2.4) \qquad\qquad \mathcal{P}^{\star} := \{\boldsymbol{\lambda} \in \mathcal{W}^* \,|\, \boldsymbol{\lambda}_{>}(\mathcal{P}) \subset \mathbb{P}\}.$$

It is easily seen that $\mathcal{P}^{\star}$ is a closed linear cone in $\mathcal{W}^*$.

Now let, in addition to $\mathcal{W}$ and $\mathcal{P}$, a linear space $\mathcal{W}'$ and a linear cone $\mathcal{P}'$ be given. Also, let a linear mapping $Q \colon \mathcal{W} \to \mathcal{W}'$ be given. We say that Q is **cone-compatible** (relative to $\mathcal{P}$ and $\mathcal{P}'$) if $Q_{>}(\mathcal{P}) \subset \mathcal{P}'$ and **cone-preserving** if $Q_{>}(\mathcal{P}) = \mathcal{P}'$. Now let a mapping $P \colon \mathcal{P} \to \mathcal{P}'$ be given. We say that P is **cone-linear** if it preserves addition and scalar multiplication by strictly positive numbers. If P is also invertible, then its inverse is also cone-linear and P is called a **cone-isomorphism**. When dealing with cone-linear mappings, we adopt the

same "multiplicative" notation used in [Nol87] for linear mappings. In particular, if P is cone linear, then, given $\boldsymbol{x} \in \mathcal{P}$, we abbreviate $P\boldsymbol{x} := P(\boldsymbol{x})$; if P is also invertible we denote its inverse by $P^{-1} := P^{\leftarrow}$. We denote by $\mathrm{Lin}(\mathcal{P}, \mathcal{P}')$ the set of all cone-linear mappings from $\mathcal{P}$ to $\mathcal{P}'$.[5] If $Q\colon \mathcal{W} \to \mathcal{W}'$ is a cone-compatible linear mapping, then $Q|_{\mathcal{P}}^{\mathcal{P}'}\colon \mathcal{P} \to \mathcal{P}'$ is cone-linear. Conversely, if $P\colon \mathcal{P} \to \mathcal{P}'$ is cone-linear, then there is a cone-compatible linear mapping $Q\colon \mathcal{W} \to \mathcal{W}'$ such that $P = Q|_{\mathcal{P}}^{\mathcal{P}'}$; moreover, if $\mathcal{P}$ spans $\mathcal{W}$ then Q is uniquely determined by P; also, if $\mathcal{P}$ and $\mathcal{P}'$ span $\mathcal{W}$ and $\mathcal{W}'$ respectively, and P is a cone-isomorphism, then Q is a cone-preserving linear isomorphism.

Suppose that $\mathcal{P}$ spans $\mathcal{W}$ and that P is cone-linear, and denote the (cone-compatible) linear extension of P by $Q\colon \mathcal{W} \to \mathcal{W}'$. It is easy to see that $Q^{\mathrm{T}}\colon \mathcal{W}'^* \to \mathcal{W}^*$ is cone-compatible relative to the dual cones $\mathcal{P}'^{\bigstar}$ and $\mathcal{P}^{\bigstar}$ (defined by (2.4)). Thus we may and do define the **cone-transpose** of P by $P^{\mathsf{T}} := Q^{\mathrm{T}}|_{\mathcal{P}'^{\bigstar}}^{\mathcal{P}^{\bigstar}}$. Clearly, $P^{\mathsf{T}}\colon \mathcal{P}'^{\bigstar} \to \mathcal{P}^{\bigstar}$ is cone-linear.

Again, suppose that P is cone-linear. It is not hard to show that the pre-image under P of each face of $\mathcal{P}'$ is a face of $\mathcal{P}$. We say that P is **face-preserving** if the *image* under P of each face of $\mathcal{P}$ is a face of $\mathcal{P}'$. Of course, every cone-isomorphism is face-preserving and the range of every face-preserving cone-linear mapping is a face. An injective cone-linear mapping is face-preserving if and only if its range is a face. We note that the transpose of a face-preserving cone-linear mapping need not be face-preserving. (See Sect. 6.)

3. Spaces and cones of quadratic forms

Throughout this section, we assume that a linear space $\mathcal{V}$ is given.

Our treatment of quadratic forms is based on Sect. 27 of [Nol87]. In particular, we define the space $\mathrm{Qu}\,\mathcal{V}$ of **quadratic forms** on $\mathcal{V}$ as the range of the injective linear mapping

$$(3.1) \qquad S \mapsto S \circ (\mathbf{1}_{\mathcal{V}}, \mathbf{1}_{\mathcal{V}})\colon \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \to \mathrm{Map}(\mathcal{V}, \mathbb{R}),$$

and we shall make use of the natural isomorphism[6]

$$(3.2) \qquad \phi \mapsto \phi^{\sqcup}\colon \mathrm{Qu}\,\mathcal{V} \to \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \cong \mathrm{Sym}(\mathcal{V}, \mathcal{V}^*)$$

characterized by

$$(3.3) \qquad \phi^{\sqcup}(\boldsymbol{v}, \boldsymbol{v}) = \phi(\boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

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[5]We note that if $\mathcal{P}$ and $\mathcal{P}'$ equal their linear spans, then the set of cone-linear mappings from $\mathcal{P}$ to $\mathcal{P}'$ and the set of linear mappings from $\mathcal{P}$ to $\mathcal{P}'$ are one and the same. Thus $\mathrm{Lin}(\mathcal{P}, \mathcal{P}')$ is not ambiguous.

[6]Clearly, (3.2) is the inverse of the linear isomorphism obtained by adjusting the codomain of (3.1).

Let $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathcal{V}^*$ be given. The *tensor product* $\boldsymbol{\lambda} \otimes \boldsymbol{\mu} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}^*)$ is defined in Sect. 25 of [Nol87] by $(\boldsymbol{\lambda} \otimes \boldsymbol{\mu})\boldsymbol{v} := (\boldsymbol{\mu}\boldsymbol{v})\boldsymbol{\lambda}$ for all $\boldsymbol{v} \in \mathcal{V}$; it follows from Prop. 1 of the same section that the *symmetric tensor product* $\frac{1}{2}(\boldsymbol{\lambda} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\lambda})$ is a member of $\mathrm{Sym}(\mathcal{V}, \mathcal{V}^*)$. In this paper, we find it convenient to use the *value-wise product* $\boldsymbol{\lambda}\boldsymbol{\mu} \colon \mathcal{V} \to \mathbb{R}$ defined by

$$(\boldsymbol{\lambda}\boldsymbol{\mu})(\boldsymbol{v}) := (\boldsymbol{\lambda}\boldsymbol{v})(\boldsymbol{\mu}\boldsymbol{v}) \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

This function is the quadratic form on $\mathcal{V}$ which corresponds under the isomorphism (3.2) to the symmetric tensor product of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, *i.e.*,

$$(3.4) \qquad \boldsymbol{\lambda}\boldsymbol{\mu} \in \mathrm{Qu}\,\mathcal{V} \qquad \text{and} \qquad (\boldsymbol{\lambda}\boldsymbol{\mu})^{\sqcup} = \tfrac{1}{2}(\boldsymbol{\lambda} \otimes \boldsymbol{\mu} + \boldsymbol{\mu} \otimes \boldsymbol{\lambda}).$$

In particular, we have $(\boldsymbol{\lambda}^2)^{\sqcup} = \boldsymbol{\lambda} \otimes \boldsymbol{\lambda}$ when $\boldsymbol{\lambda}^2$ denotes the value-wise square of $\boldsymbol{\lambda}$. It is not hard to show that

$$(3.5) \qquad \mathrm{Qu}\,\mathcal{V} = \mathrm{Lsp}\{\boldsymbol{\lambda}\boldsymbol{\mu} \,|\, \boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathcal{V}^*\} = \mathrm{Lsp}\{\boldsymbol{\lambda}^2 \,|\, \boldsymbol{\lambda} \in \mathcal{V}^*\},$$

where the symbol Lsp denotes linear span in the space $\mathrm{Map}(\mathcal{V}, \mathbb{R})$ (see Problem 8, Chapter 2 of [Nol87]).

We denote by[7]

$$(3.6) \qquad \mathrm{Pqu}\,\mathcal{V} := \{\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V} \,|\, \mathrm{Rng}\,\boldsymbol{\phi} \subset \mathbb{P}\}$$

the set of all **positive** quadratic forms on $\mathcal{V}$. It is clear that $\mathrm{Pqu}\,\mathcal{V}$ is a closed linear cone in $\mathrm{Qu}\,\mathcal{V}$. Since $\boldsymbol{\lambda}^2 \in \mathrm{Pqu}\,\mathcal{V}$ for every $\boldsymbol{\lambda} \in \mathcal{V}^*$, it is clear from (3.5) that $\mathrm{Pqu}\,\mathcal{V}$ spans $\mathrm{Qu}\,\mathcal{V}$. Also, we have

$$(3.7) \qquad \mathrm{Pqu}\,\mathcal{V} = \mathrm{Csp}\{\boldsymbol{\lambda}^2 \,|\, \boldsymbol{\lambda} \in \mathcal{V}^*\}.$$

(This fact can easily be inferred from [Nol87], Prop. 2 of Sect. 85 by introducing a genuine inner product in $\mathcal{V}$, using the resulting natural isomorphism from $\mathrm{Pqu}\,\mathcal{V}$ to $\mathrm{Pos}\,\mathcal{V}$, and then using the Spectral Theorem.) The interior of $\mathrm{Pqu}\,\mathcal{V}$ is the (open) linear cone[8]

$$\mathrm{Pqu}^+\mathcal{V} := \{\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V} \,|\, \boldsymbol{\phi}_>(\mathcal{V}^\times) \subset \mathbb{P}^\times\}$$
$$= \{\boldsymbol{\phi} \in \mathrm{Pqu}\,\mathcal{V} \,|\, \mathrm{Null}\,\boldsymbol{\phi} = \{\boldsymbol{0}\}\}$$

of all **strictly positive** quadratic forms on $\mathcal{V}$. (This fact can easily be inferred from the first statement in the Theorem on the Smoothness of the Strict Lineonic Square Root in Sect. 85 of [Nol87].) In view of Prop. 2.3, we have

$$\mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}}\{\boldsymbol{\phi}\} = \mathrm{Pqu}\,\mathcal{V} \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Pqu}^+\,\mathcal{V}.$$

In view of [Nol87], Prop. 1 of Sect. 27, we have

$$(3.8) \qquad \dim \mathrm{Qu}\,\mathcal{V} = \dim \mathrm{Pqu}\,\mathcal{V} = \dim \mathrm{Pqu}^+\mathcal{V} = \tfrac{\dim \mathcal{V}(1+\dim \mathcal{V})}{2}.$$

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[7]In [NS78], the symbol $\overline{\mathrm{Qu}^+}(\mathcal{V})$ was used for our $\mathrm{Pqu}\,\mathcal{V}$.

[8]In [NS78], the symbol $\mathrm{Qu}^+(\mathcal{V})$ was used for our $\mathrm{Pqu}^+\mathcal{V}$.

Proposition 3.1. *Let $\boldsymbol{\phi} \in \mathrm{Pqu}\,\mathcal{V}$ be given. Then*[9]

(3.9) $$\left|\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})\right|^2 \le \boldsymbol{\phi}(\boldsymbol{u})\boldsymbol{\phi}(\boldsymbol{v}) \qquad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}.$$

Proof. Let $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ be given. Since $\mathrm{Rng}\,\boldsymbol{\phi} \subset \mathbb{P}$, we have

(3.10) $$0 \le \boldsymbol{\phi}(\alpha\boldsymbol{u} - \beta\boldsymbol{v}) = \alpha^2\boldsymbol{\phi}(\boldsymbol{u}) + \beta^2\boldsymbol{\phi}(\boldsymbol{v}) - 2\alpha\beta\,\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})$$

for all $\alpha, \beta \in \mathbb{R}$.

Suppose that $\boldsymbol{\phi}(\boldsymbol{u}) = \boldsymbol{\phi}(\boldsymbol{v}) = 0$. Then using (3.10) with $\alpha := \frac{1}{2}$, $\beta := -1$ yields $0 \le \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})$. Using (3.10) with $\alpha := \frac{1}{2}$, $\beta := 1$ yields $0 \le -\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})$, and we conclude that $\boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v}) = 0$, so that (3.9) holds because both sides are zero.

Suppose that one of $\boldsymbol{\phi}(\boldsymbol{u})$ and $\boldsymbol{\phi}(\boldsymbol{v})$, say $\boldsymbol{\phi}(\boldsymbol{v})$, is not zero. Using (3.10) with $\alpha := \boldsymbol{\phi}(\boldsymbol{v})^2$ and $\beta := \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})$ yields

$$0 \le \boldsymbol{\phi}(\boldsymbol{v})\left(\boldsymbol{\phi}(\boldsymbol{u})\boldsymbol{\phi}(\boldsymbol{v}) - \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v})^2\right).$$

Since $\boldsymbol{\phi}(\boldsymbol{v}) > 0$, and since $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$ were arbitrary, it follows that (3.9) holds. $\square$

For each $\boldsymbol{u} \in \mathcal{V}$, we define $\boldsymbol{u}+ \colon \mathcal{V} \to \mathcal{V}$ by

$$(\boldsymbol{u}+)(\boldsymbol{v}) := \boldsymbol{u} + \boldsymbol{v} \qquad \text{for all } \boldsymbol{v} \in \mathcal{V}.$$

Proposition 3.2. *Let $\boldsymbol{\phi} \in \mathrm{Pqu}\,\mathcal{V}$ be given. Then*[10]

(3.11) $$\mathrm{Null}\,\boldsymbol{\phi} = \mathrm{Null}\,\boldsymbol{\phi}^{\sqcup} = \{\boldsymbol{u} \in \mathcal{V} \,|\, \boldsymbol{\phi} \circ (\boldsymbol{u}+) = \boldsymbol{\phi}\}$$

when $\boldsymbol{\phi}^{\sqcup}$ is regarded as an element of $\mathrm{Sym}(\mathcal{V}, \mathcal{V}^*)$.

Proof. Let $\boldsymbol{u} \in \mathcal{V}$ be given.

On the one hand, suppose that $\boldsymbol{u} \in \mathrm{Null}\,\boldsymbol{\phi}$, so that that $\boldsymbol{\phi}(\boldsymbol{u}) = 0$. By Prop. 3.1 we then have $(\boldsymbol{\phi}^{\sqcup}\boldsymbol{u})\boldsymbol{v} = \boldsymbol{\phi}^{\sqcup}(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \mathcal{V}$ and hence $\boldsymbol{\phi}^{\sqcup}\boldsymbol{u} = \boldsymbol{0}$, showing that $\boldsymbol{u} \in \mathrm{Null}\,\boldsymbol{\phi}^{\sqcup}$. Since $\boldsymbol{u} \in \mathrm{Null}\,\boldsymbol{\phi}$ was arbitrary, it follows that $\mathrm{Null}\,\boldsymbol{\phi} \subset \mathrm{Null}\,\boldsymbol{\phi}^{\sqcup}$. On the other hand, suppose that $\boldsymbol{u} \in \mathrm{Null}\,\boldsymbol{\phi}^{\sqcup}$; then $0 = (\boldsymbol{\phi}^{\sqcup}\boldsymbol{u})\boldsymbol{u} = \boldsymbol{\phi}(\boldsymbol{u})$ and hence $\boldsymbol{u} \in \mathrm{Null}\,\boldsymbol{\phi}$. It follows that $\mathrm{Null}\,\boldsymbol{\phi}^{\sqcup} \subset \mathrm{Null}\,\boldsymbol{\phi}$. It also follows that

$$\boldsymbol{\phi}(\boldsymbol{u} + \boldsymbol{w}) = \boldsymbol{\phi}(\boldsymbol{u}) + \boldsymbol{\phi}(\boldsymbol{w}) + 2(\boldsymbol{\phi}^{\sqcup}\boldsymbol{u})\boldsymbol{w} = \boldsymbol{\phi}(\boldsymbol{w}) \qquad \text{for all } \boldsymbol{w} \in \mathcal{V},$$

so that

$$\mathrm{Null}\,\boldsymbol{\phi} \subset \{\boldsymbol{u} \in \mathcal{V} \,|\, \boldsymbol{\phi} \circ (\boldsymbol{u}+) = \boldsymbol{\phi}\}.$$

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[9]In the special case when $\boldsymbol{\phi}$ is strictly positive, it can be used to make $\mathcal{V}$ an inner-product space and (3.9) reduces to the Inner-Product Inequality (see Sect. 42 of [Nol87]).

[10]Indeed, the second equality in (3.11) holds for all $\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V}$; the first fails when $\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V}$ is *double-signed*, *i.e.*, when neither $\boldsymbol{\phi}$ nor $-\boldsymbol{\phi}$ is positive.

Finally, suppose, instead, that $\boldsymbol{\phi} \circ (\boldsymbol{u}+) = \boldsymbol{\phi}$. Then $\boldsymbol{\phi}(\boldsymbol{u}) = \boldsymbol{\phi}(\boldsymbol{u} + \boldsymbol{u}) = 4\boldsymbol{\phi}(\boldsymbol{u})$ and hence $\boldsymbol{\phi}(\boldsymbol{u}) = 0$, so $\boldsymbol{u} \in \operatorname{Null} \boldsymbol{\phi}$. It follows that

$$\operatorname{Null} \boldsymbol{\phi} \supset \{\boldsymbol{u} \in \mathcal{V} \,|\, \boldsymbol{\phi} \circ (\boldsymbol{u}+) = \boldsymbol{\phi}\}. \quad \square$$

As an immediate consequence of Prop. 3.2, the nullset $\operatorname{Null} \boldsymbol{\phi}$ is a sub*space* for all $\boldsymbol{\phi} \in \operatorname{Pqu} \mathcal{V}$.[11]

For every $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$, the set

$$(3.12) \quad \Phi_{\mathcal{V}}(\mathcal{U}) := \{\boldsymbol{\phi} \in \operatorname{Pqu} \mathcal{V} \,|\, \boldsymbol{\phi}|_{\mathcal{U}} = 0\} = \{\boldsymbol{\phi} \in \operatorname{Pqu} \mathcal{V} \,|\, \mathcal{U} \subset \operatorname{Null} \boldsymbol{\phi}\}$$

is clearly a subcone of $\operatorname{Pqu} \mathcal{V}$.

Proposition 3.3. *Let* $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$ *be given. Then the subcone* $\Phi_{\mathcal{V}}(\mathcal{U})$ *of* $\operatorname{Pqu} \mathcal{V}$ *defined by* (3.12) *is actually a face of* $\operatorname{Pqu} \mathcal{V}$.

Proof. Let $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \operatorname{Pqu} \mathcal{V}$ be given such that

$$\boldsymbol{\phi} := \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 \in \Phi_{\mathcal{V}}(\mathcal{U}).$$

By (3.12) this means that $\boldsymbol{\phi}_1(\boldsymbol{u}) + \boldsymbol{\phi}_2(\boldsymbol{u}) = 0$ for all $\boldsymbol{u} \in \mathcal{U}$. Since $\operatorname{Rng} \boldsymbol{\phi}_1 \subset \mathbb{P}$ and $\operatorname{Rng} \boldsymbol{\phi}_2 \subset \mathbb{P}$ by (3.6), we conclude that $\boldsymbol{\phi}_1(\boldsymbol{u}) = 0 = \boldsymbol{\phi}_2(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $\mathcal{U}$, *i.e.*, that $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2 \in \Phi_{\mathcal{V}}(\mathcal{U})$. $\square$

In view of Prop. 3.3, we may consider (3.12) to be the definition of a mapping

$$(3.13) \qquad\qquad \Phi_{\mathcal{V}} \colon \operatorname{Subsp} \mathcal{V} \to \operatorname{Face}(\operatorname{Pqu} \mathcal{V})$$

from $\operatorname{Subsp} \mathcal{V}$ to the lattice $\operatorname{Face}(\operatorname{Pqu} \mathcal{V})$ of all (nonempty) faces of $\operatorname{Pqu}(\mathcal{V})$.

Proposition 3.4. *Let* $\mathcal{U} \in \operatorname{Subsp} \mathcal{V}$ *be given. Then* $\boldsymbol{\rho} \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} \in \Phi_{\mathcal{V}}(\mathcal{U})$ *for all* $\boldsymbol{\rho} \in \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$, *and the mapping*

$$(3.14) \qquad\qquad \boldsymbol{\rho} \mapsto \boldsymbol{\rho} \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} \colon \operatorname{Pqu}(\mathcal{V}/\mathcal{U}) \to \Phi_{\mathcal{V}}(\mathcal{U})$$

is a cone-isomorphism.

Proof. Let $\boldsymbol{\rho} \in \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$ be given. Since the composite of a linear mapping with a positive quadratic form is again a positive quadratic form, it is clear that $\boldsymbol{\rho} \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} \in \operatorname{Pqu} \mathcal{V}$. Also, since $\mathcal{U}$ is the zero-element of $\mathcal{V}/\mathcal{U}$ and since $\boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}}\boldsymbol{u} = \mathcal{U}$ for all $\boldsymbol{u}$ in $\mathcal{U}$, it is clear that $(\boldsymbol{\rho} \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}})|_{\mathcal{U}} = \boldsymbol{0}$, so that $\boldsymbol{\rho} \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} \in \Phi_{\mathcal{V}}(\mathcal{U})$ in view of (3.12).

It is evident that the mapping (3.14) is cone-linear. Since $\boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}}$ is surjective, the mapping

$$\phi \mapsto \phi \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} \colon \operatorname{Map}(\mathcal{V}/\mathcal{U}, \mathbb{R}) \to \operatorname{Map}(\mathcal{V}, \mathbb{R})$$

is injective by Prop. 1.1; it follows that (3.14) is injective.

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[11]If $\boldsymbol{\phi} \in \operatorname{Qu} \mathcal{V}$ is double-signed, then $\operatorname{Null} \boldsymbol{\phi}$ is *not* a subspace.

Now let ϕ in $\Phi_{\mathcal{V}}(\mathcal{U})$ be given, so that $\mathcal{U} \subset \operatorname{Null}\phi$ by (3.12). It follows by Prop. 3.2 that

$$\phi_{>}(v + \mathcal{U}) = \phi_{>}(\{v\}) = \{\phi(v)\} \qquad \text{for all } v \in \mathcal{V}.$$

Thus we can determine $\rho \colon \mathcal{V}/\mathcal{U} \to \mathbb{R}$ by the condition $\rho(\mathcal{G}) \in \phi_{>}(\mathcal{G})$ for all $\mathcal{G}$ in $\mathcal{V}/\mathcal{U}$; it is clear that $\phi = \rho \circ \Omega_{\mathcal{V}/\mathcal{U}}$. Choosing a right inverse $L \in \operatorname{Lin}(\mathcal{V}/\mathcal{U}, \mathcal{V})$ for $\Omega_{\mathcal{V}/\mathcal{U}}$, we note that $\phi \circ L = \rho \circ \Omega_{\mathcal{V}/\mathcal{U}} \circ L = \rho$. It follows that $\rho \in \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$. Since $\phi \in \Phi_{\mathcal{V}}(\mathcal{U})$ was arbitrary, if follows that (3.14) is surjective. $\qquad\square$

It follows from Proposition 3.4 and (3.8) that

$$(3.15) \quad \dim \Phi_{\mathcal{V}}(\mathcal{U}) = \dim \operatorname{Pqu}(\mathcal{V}/\mathcal{U}) = \tfrac{(\dim \mathcal{V} - \dim \mathcal{U})(1 + \dim \mathcal{V} - \dim \mathcal{U})}{2}$$
$$\text{for all } \mathcal{U} \in \operatorname{Subsp}\mathcal{V}.$$

Proposition 3.5. *Let $\phi \in \operatorname{Pqu}\mathcal{V}$ be given. Then*

$$(3.16) \qquad\qquad \operatorname{Fcsp}_{\operatorname{Pqu}\mathcal{V}}\{\phi\} = \Phi_{\mathcal{V}}(\operatorname{Null}\phi).$$

Proof. We noted after proving Prop. 3.2 that $\operatorname{Null}\phi \in \operatorname{Subsp}\mathcal{V}$ for all $\phi \in \operatorname{Pqu}\mathcal{V}$. Let $\phi \in \operatorname{Pqu}\mathcal{V}$ be given and put $\mathcal{U} := \operatorname{Null}\phi$. It is clear from (3.12) and Prop. 3.3 that $\Phi_{\mathcal{V}}(\mathcal{U})$ is a face of $\operatorname{Pqu}\mathcal{V}$ that contains ϕ and hence that $\operatorname{Fcsp}_{\operatorname{Pqu}\mathcal{V}}\{\phi\} \subset \Phi_{\mathcal{V}}(\mathcal{U})$. In view of Prop. 3.4, we may determine ρ in $\operatorname{Pqu}(\mathcal{V}/\mathcal{U})$ such that $\rho \circ \Omega_{\mathcal{V}/\mathcal{U}} = \phi$, so $\rho(v + \mathcal{U}) = \phi(v)$ for all $v \in \mathcal{V}$. Since $\mathcal{U} = \operatorname{Null}\phi$, it follows that

$$\operatorname{Null}\rho = \{v + \mathcal{U} \mid v \in \operatorname{Null}\phi\} = \{\mathcal{U}\}.$$

Thus, since $\mathcal{U}$ is the zero-element of $\mathcal{V}/\mathcal{U}$, and since clearly $\rho \in \operatorname{Pqu}\mathcal{V}$, we have $\rho \in \operatorname{Pqu}^{+}(\mathcal{V}/\mathcal{U})$. Since $\operatorname{Pqu}^{+}(\mathcal{V}/\mathcal{U})$ is the interior of $\operatorname{Pqu}(\mathcal{V}/\mathcal{U})$, it follows by Prop. 2.2 that $\operatorname{Pqu}(\mathcal{V}/\mathcal{U}) = \operatorname{Fcsp}_{\operatorname{Pqu}\mathcal{V}}\{\rho\}$. Hence, by Prop. 3.4, we have $\Phi_{\mathcal{V}}(\mathcal{U}) = \operatorname{Fcsp}_{\operatorname{Pqu}\mathcal{V}}\{\phi\}$. $\qquad\square$

Theorem 1. *The mapping $\Phi_{\mathcal{V}}$ of (3.13), defined by (3.12), is an inclusion-reversing bijection.*

Proof. It is easily seen from (3.12) and (3.8) that $\Phi_{\mathcal{V}}$ is strictly inclusion-reversing in the sense that

$$(3.17) \qquad\qquad \mathcal{U} \subsetneqq \mathcal{U}' \quad \Longrightarrow \quad \Phi_{\mathcal{V}}(\mathcal{U}) \supsetneqq \Phi_{\mathcal{V}}(\mathcal{U}')$$

for all $\mathcal{U}, \mathcal{U}' \in \operatorname{Subsp}\mathcal{V}$. To show that $\Phi_{\mathcal{V}}$ is bijective, let a face $\mathcal{F}$ of the cone $\operatorname{Pqu}\mathcal{V}$ be given. By Prop. 2.3, we may choose $\phi \in \operatorname{Pqu}\mathcal{V}$ such that $\mathcal{F} = \operatorname{Fcsp}_{\operatorname{Pqu}\mathcal{V}}\{\phi\}$. By Prop. 3.5 above, we have $\mathcal{F} = \Phi_{\mathcal{V}}(\operatorname{Null}\phi)$ and hence $\mathcal{F} \in \operatorname{Rng}\Phi_{\mathcal{V}}$. In view of (3.12), we also have $\mathcal{U} = \bigcap_{\phi \in \mathcal{F}} \operatorname{Null}\phi$, showing that $\mathcal{U}$ is uniquely determined by $\mathcal{F}$. Since the face $\mathcal{F}$ was arbitrary, it follows that $\Phi_{\mathcal{V}}$ is surjective. $\qquad\square$

It follows from Theorem 1 that for every family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$, we have

(3.18) $$\Phi_\mathcal{V}\left(\bigcap_{i \in I} \mathcal{U}_i\right) = \mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}}\left(\bigcup_{i \in I} \Phi_\mathcal{V}(\mathcal{U}_i)\right),$$

(3.19) $$\Phi_\mathcal{V}\left(\mathrm{Lsp}\left(\bigcup_{i \in I} \mathcal{U}_i\right)\right) = \bigcap_{i \in I} \Phi_\mathcal{V}(\mathcal{U}_i);$$

these formulas relate greatest lower bounds in the lattice $\mathrm{Subsp}\,\mathcal{V}$ to least upper bounds in the lattice $\mathrm{Face}\,(\mathrm{Pqu}\,\mathcal{V})$ and *vice versa*.

4. DUALITY

Throughout this section, we assume that linear spaces $\mathcal{V}$ and $\mathcal{V}'$ are given. We note the identification $\mathcal{V}^{**} \cong \mathcal{V}$ which states that every vector $\boldsymbol{v} \in \mathcal{V}$ may be regarded as linear form on $\mathcal{V}^*$. Hence we consider the value-wise square $\boldsymbol{v}^2$ of a given $\boldsymbol{v} \in \mathcal{V}$ as a quadratic form on $\mathcal{V}^*$, so that $\boldsymbol{v}^2 \in \mathrm{Qu}\,\mathcal{V}^*$.

Proposition 4.1. *There is exactly one bilinear mapping*

(4.1) $$\Gamma\colon \mathrm{Qu}\,\mathcal{V} \times \mathrm{Qu}\,\mathcal{V}^* \to \mathbb{R}$$

such that

(4.2) $\quad \Gamma(\boldsymbol{\lambda}^2, \boldsymbol{v}^2) = (\boldsymbol{\lambda v})^2 \qquad$ *for all $\boldsymbol{\lambda}$ in $\mathcal{V}^*$, $\boldsymbol{v} \in \mathcal{V}$.*

This mapping satisfies

(4.3) $\quad \Gamma(\boldsymbol{\phi}, \boldsymbol{v}^2) = \boldsymbol{\phi}(\boldsymbol{v}) \qquad$ *for all $\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V}$, $\boldsymbol{v}$ in $\mathcal{V}$,*

(4.4) $\quad \Gamma(\boldsymbol{\lambda}^2, \boldsymbol{f}) = \boldsymbol{f}(\boldsymbol{\lambda}) \qquad$ *for all $\boldsymbol{\lambda} \in \mathcal{V}^*$, $\boldsymbol{f}$ in $\mathrm{Qu}\,\mathcal{V}^*$,*

and the mappings

(4.5) $$\boldsymbol{\phi} \mapsto \Gamma(\boldsymbol{\phi}, \cdot)\colon \mathrm{Qu}\,\mathcal{V} \to (\mathrm{Qu}\,\mathcal{V}^*)^*,$$

(4.6) $$\boldsymbol{f} \mapsto \Gamma(\cdot, \boldsymbol{f})\colon \mathrm{Qu}\,\mathcal{V}^* \to (\mathrm{Qu}\,\mathcal{V})^*$$

are cone-preserving linear isomorphisms relative to the cones $\mathrm{Pqu}\,\mathcal{V}$, $(\mathrm{Pqu}\,\mathcal{V}^*)^\star$ and $\mathrm{Pqu}\,\mathcal{V}^*$, $(\mathrm{Pqu}\,\mathcal{V})^\star$, respectively.

Proof. Using [Nol87], Prop. 6 of Chap. 2, with the choices $\mathcal{V}_1 := \mathcal{V}_2 := \mathcal{V}^*$, $\mathcal{W} := \mathbb{R}$, and using the identifications $\mathcal{V}^{**} \cong \mathcal{V}$ and $\mathrm{Lin}(\mathcal{V}, \mathcal{V}^*) \cong \mathrm{Lin}_2(\mathcal{V}^2, \mathbb{R})$, we see that there is exactly one linear isomorphism

$$\Lambda\colon \mathrm{Lin}_2(\mathcal{V}^{*2}, \mathbb{R}) \to \left(\mathrm{Lin}_2(\mathcal{V}^2, \mathbb{R})\right)^*$$

such that $B(\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2) = \Lambda(B)(\boldsymbol{\lambda}_1 \otimes \boldsymbol{\lambda}_2)$ for all $B \in \mathrm{Lin}_2(\mathcal{V}^{*2}, \mathbb{R})$, and all $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathcal{V}$.

Define

$$\overline{\Gamma}\colon \mathrm{Sym}_2(\mathcal{V}^{*2}, \mathbb{R}) \to \left(\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})\right)^*$$

by

$$\overline{\Gamma}(S) := \Lambda(S)|_{\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})} \qquad \text{for all } S \in \mathrm{Sym}_2(\mathcal{V}^{*2}, \mathbb{R}).$$

Then $\overline{\Gamma}$ is easily seen to be the only linear isomorphism from $\mathrm{Sym}_2(\mathcal{V}^{*2}, \mathbb{R})$ to $(\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}))^*$ which satisfies $S(\boldsymbol{\lambda}, \boldsymbol{\lambda}) = \overline{\Gamma}(S)(\boldsymbol{\lambda} \otimes \boldsymbol{\lambda})$ for all $S \in \mathrm{Sym}_2(\mathcal{V}^{*2}, \mathbb{R})$ and all $\boldsymbol{\lambda} \in \mathcal{V}^*$. Replacing the spaces $\mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})$ and $\mathrm{Sym}_2(\mathcal{V}^{*2}, \mathbb{R})$ of bilinear mappings by the corresponding spaces $\mathrm{Qu}\,\mathcal{V}$ and $\mathrm{Qu}\,\mathcal{V}^*$ of quadratic forms yields the first statement of the proposition.

In view of (3.5) and the linearity of Γ in its first argument, (4.3) follows directly from (4.2); statement (4.4) is established similarly.

It follows from (4.3) that

$$\boldsymbol{\phi} = \mathbf{0} \quad \Longleftrightarrow \quad \Gamma(\boldsymbol{\phi}, \cdot) = \mathbf{0} \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V},$$

so that (4.5) is injective and hence a linear isomorphism. In view of (3.7) and the linearity of Γ in its first argument, it also follows from (4.3) that

$$\boldsymbol{\phi} \in \mathrm{Pqu}\,\mathcal{V} \quad \Longleftrightarrow \quad \Gamma(\boldsymbol{\phi}, \cdot) \in (\mathrm{Pqu}\,\mathcal{V}^*)^{\bigstar} \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V};$$

thus (4.5) is cone-preserving. The assertions that (4.6) is a linear isomorphism and cone-preserving are established similarly. $\square$

The mappings (4.5) and (4.6) establish natural linear isomorphisms between $\mathrm{Qu}\,\mathcal{V}$ and $(\mathrm{Qu}\,\mathcal{V}^*)^*$ and between $\mathrm{Qu}\,\mathcal{V}^*$ and $(\mathrm{Qu}\,\mathcal{V})^*$ which are compatible with the identification $(\mathrm{Qu}\,\mathcal{V})^{**} \cong \mathrm{Qu}\,\mathcal{V}$. They also establish cone-isomorphisms between $\mathrm{Pqu}\,\mathcal{V}$ and $(\mathrm{Pqu}\,\mathcal{V}^*)^{\bigstar}$ and between $\mathrm{Pqu}\,\mathcal{V}^*$ and $(\mathrm{Pqu}\,\mathcal{V})^{\bigstar}$. We do not treat these natural isomorphisms as identifications *per se*[12], but we re-interpret some notation and terminology (insofar as it applies to cones and spaces of quadratic forms) in order to let the four spaces and cones on the left-hand sides of these isomorphisms conveniently "stand in" for those on the right. In particular,

(1) we consider *annihilators* of subsets of $\mathrm{Qu}\,\mathcal{V}$ and $\mathrm{Qu}\,\mathcal{V}^*$ to be subsets of $\mathrm{Qu}\,\mathcal{V}^*$ and $\mathrm{Qu}\,\mathcal{V}$, respectively; thus

$$(4.7) \quad \mathcal{A}^{\perp} = \{\boldsymbol{f} \in \mathrm{Qu}\,\mathcal{V}^* \,|\, \Gamma(\boldsymbol{\phi}, \boldsymbol{f}) = 0 \text{ for all } \boldsymbol{\phi} \in \mathcal{A}\}$$
$$\text{for all } \mathcal{A} \in \mathrm{Sub}(\mathrm{Qu}\,\mathcal{V}),$$

$$(4.8) \quad \mathcal{B}^{\perp} = \{\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V} \,|\, \Gamma(\boldsymbol{\phi}, \boldsymbol{f}) = 0 \text{ for all } \boldsymbol{f} \text{ in } \mathcal{B}\}$$
$$\text{for all } \mathcal{B} \text{ in } \mathrm{Sub}(\mathrm{Qu}\,\mathcal{V}^*);$$

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[12]To do so would lead to awkward ambiguities, *e.g.*, identifying $\boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V}$ with $\Gamma(\boldsymbol{\phi}, \cdot) \in (\mathrm{Qu}\,\mathcal{V}^*)^*$ would make "$\mathrm{Rng}\,\boldsymbol{\phi}$" ambiguous.

(2) we consider the *transposes* of linear mappings $\operatorname{Qu}\mathcal{V} \to \operatorname{Qu}\mathcal{V}'$ to be linear mappings $\operatorname{Qu}\mathcal{V}'^* \to \operatorname{Qu}\mathcal{V}^*$, and transposes of cone-linear mappings $\operatorname{Pqu}\mathcal{V} \to \operatorname{Pqu}\mathcal{V}'$ to be cone-linear mappings $\operatorname{Pqu}\mathcal{V}'^* \to \operatorname{Pqu}\mathcal{V}^*$; thus, for all $Q \in \operatorname{Lin}(\operatorname{Qu}\mathcal{V}, \operatorname{Qu}\mathcal{V}')$ and all $P \in \operatorname{Lin}(\operatorname{Pqu}\mathcal{V}, \operatorname{Pqu}\mathcal{V}')$, the transpose Q^{T} and the cone-transpose P^{T} are determined by the properties[13]

$$(4.9) \qquad \Gamma(Q\boldsymbol{\phi}, \boldsymbol{f}) = \Gamma(\boldsymbol{\phi}, Q^{\mathrm{T}}\boldsymbol{f}) \qquad \text{for all } \boldsymbol{\phi} \in \operatorname{Qu}\mathcal{V},\ \boldsymbol{f} \text{ in } \operatorname{Qu}\mathcal{V}'^*,$$

$$(4.10) \quad \Gamma(P\boldsymbol{\phi}, \boldsymbol{f}) = \Gamma(\boldsymbol{\phi}, P^{\mathrm{T}}\boldsymbol{f}) \qquad \text{for all } \boldsymbol{\phi} \in \operatorname{Pqu}\mathcal{V},\ \boldsymbol{f} \text{ in } \operatorname{Pqu}\mathcal{V}'^*;$$

(3) given a linear space $\mathcal{W}$, $\boldsymbol{w} \in \mathcal{W}$, $\boldsymbol{\phi} \in \operatorname{Qu}\mathcal{V}$ and $\boldsymbol{f} \in \operatorname{Qu}\mathcal{V}^*$, we interpret the *tensor products* $\boldsymbol{w} \otimes \boldsymbol{\phi}$ and $\boldsymbol{w} \otimes \boldsymbol{f}$ as members of $\operatorname{Lin}(\operatorname{Qu}\mathcal{V}^*, \mathcal{W})$ and $\operatorname{Lin}(\operatorname{Qu}\mathcal{V}, \mathcal{W})$, respectively.

In addition, we feel free to speak of the spaces $\operatorname{Qu}\mathcal{V}$ and $\operatorname{Qu}\mathcal{V}^*$ (and of the cones $\operatorname{Pqu}\mathcal{V}$ and $\operatorname{Pqu}\mathcal{V}^*$) as being *dual* to each other.

Proposition 4.2. *Let a subspace $\mathcal{U}$ of $\mathcal{V}$ be given. Then we have*

$$(4.11) \qquad\qquad \Phi_{\mathcal{V}}(\mathcal{U}) = \operatorname{Csp}\{\boldsymbol{\lambda}^2 \mid \boldsymbol{\lambda} \in \mathcal{U}^{\perp}\}$$

and

$$(4.12) \qquad\qquad (\Phi_{\mathcal{V}}(\mathcal{U}))^{\perp} = \{\boldsymbol{f} \in \operatorname{Qu}\mathcal{V}^* \mid \boldsymbol{f}|_{\mathcal{U}^{\perp}} = 0\},$$

where $\Phi_{\mathcal{V}}(\mathcal{U})$ is defined by (3.12).

Proof. Since

$$(4.13) \qquad \boldsymbol{\mu}^2 \circ \boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}} = (\boldsymbol{\mu}\boldsymbol{\Omega}_{\mathcal{V}/\mathcal{U}})^2 \qquad \text{for all } \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*,$$

it follows from Prop. 1.2 that

$$\operatorname{Csp}\{\boldsymbol{\mu}^2 \circ \Omega_{\mathcal{V}/\mathcal{U}} \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} = \operatorname{Csp}\{(\boldsymbol{\mu}\Omega_{\mathcal{V}/\mathcal{U}})^2 \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} =$$
$$\operatorname{Csp}\{\boldsymbol{\lambda}^2 \mid \boldsymbol{\lambda} \in \mathcal{U}^{\perp}\}.$$

Since

$$\operatorname{Csp}\{\boldsymbol{\mu}^2 \mid \boldsymbol{\mu} \in (\mathcal{V}/\mathcal{U})^*\} = \operatorname{Pqu}(\mathcal{V}/\mathcal{U})$$

by (3.7), the assertion (4.11) follows by Prop. 3.4.

It follows from (4.4) that $\boldsymbol{f}|_{\mathcal{U}^{\perp}} = 0$ if and only if $\Gamma(\boldsymbol{f}, \boldsymbol{\lambda}^2) = 0$ for all $\boldsymbol{\lambda} \in \mathcal{U}^{\perp}$. By (4.11), this is the case if and only if $\Gamma(\boldsymbol{f})\boldsymbol{\phi} = 0$ for all $\boldsymbol{\phi} \in \Phi_{\mathcal{V}}(\mathcal{U})$, *i.e.*, in view of (4.7), if and only if $\boldsymbol{f} \in (\Phi_{\mathcal{V}}(\mathcal{U}))^{\perp}$. Since $\boldsymbol{f}$ in $\operatorname{Qu}\mathcal{V}^*$ was arbitrary, (4.12) follows. $\qquad\square$

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[13]We note that the symbol Γ on the right side of the equalities denotes the bilinear mapping (4.1) with domain $\operatorname{Qu}\mathcal{V} \times \operatorname{Qu}\mathcal{V}^*$, while the same symbol Γ on the left side denotes the corresponding bilinear mapping with domain $\operatorname{Qu}\mathcal{V}' \times \operatorname{Qu}\mathcal{V}'^*$.

Proposition 4.3. *Let $\boldsymbol{\phi} \in (\mathrm{Pqu}\,\mathcal{V})^{\times}$ be given. Then $\boldsymbol{\phi}$ belongs to an extreme ray of $\mathrm{Qu}\,\mathcal{V}$ if and only if there is at least one $\boldsymbol{\lambda} \in (\mathcal{V}^{*})^{\times}$ such that $\boldsymbol{\phi} = \boldsymbol{\lambda}^{2}$.*

Proof. Saying that $\boldsymbol{\phi}$ belongs to an extreme ray means that $\mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}}\{\boldsymbol{\phi}\} = \mathbb{P}\,\boldsymbol{\phi}$. By Prop. 3.9, this is the case if and only if $\Phi_{\mathcal{V}}(\mathrm{Null}\,\boldsymbol{\phi}) = \mathbb{P}\,\boldsymbol{\phi}$. By Prop. 4.2, (4.11), this is the case if and only if $\mathbb{R}\,\boldsymbol{\phi} = \mathbb{R}\,\boldsymbol{\mu}^{2}$ for some $\boldsymbol{\mu} \in \mathcal{V}^{*}$ and hence $\boldsymbol{\phi} = \boldsymbol{\lambda}^{2}$ for some $\boldsymbol{\lambda} \in \mathcal{V}^{*}$. $\square$

Let $n \in \mathbb{N}$ be given. We denote the set of all isotone pairs in $(n^{]})^{2}$ by $\mathrm{Ip}(n)$, *i.e.*,

$$(4.14) \qquad\qquad \mathrm{Ip}(n) := \{(i, k) \in (n^{]})^{2} \mid i \leq k\}.$$

Given any list $\boldsymbol{\beta} := (\boldsymbol{\beta}_{i} \mid i \in n^{]})$ in $(\mathcal{V}^{*})^{n}$, we define the family $\boldsymbol{\beta}\square\boldsymbol{\beta} \in (\mathrm{Qu}\,\mathcal{V})^{\mathrm{Ip}(n)}$ by

$$(4.15) \qquad (\boldsymbol{\beta}\square\boldsymbol{\beta})_{(i,k)} := \boldsymbol{\beta}_{i}\boldsymbol{\beta}_{k} \qquad \text{for all } (i, k) \in \mathrm{Ip}(n).$$

Proposition 4.4. *Let $n \in \mathbb{N}$, a linear space $\mathcal{W}$, and $\boldsymbol{\gamma} \in (\mathcal{W}^{*})^{n}$ be given. Then the following are equivalent:*

  (i)   *$\boldsymbol{\gamma}$ is a basis of $\mathcal{W}^{*}$;*
  (ii) *$\boldsymbol{\gamma}\square\boldsymbol{\gamma}$ is a basis of $\mathrm{Qu}\,\mathcal{W}$;*
  (iii) *$(\mathbb{P}\,\boldsymbol{\gamma}_{i}^{2} \mid i \in n^{]})$ is a facial decomposition of $\mathrm{Pqu}\,\mathcal{W}$.*

Proof. We first prove the equivalence of (i) and (ii). On the one hand assume (i). Then

$$(\tfrac{1}{2}(\boldsymbol{\gamma}_{i} \otimes \boldsymbol{\gamma}_{j} + \boldsymbol{\gamma}_{j} \otimes \boldsymbol{\gamma}_{i}) \mid (i, j) \in \mathrm{Ip}(n))$$

is a basis for $\mathrm{Sym}_{2}(\mathcal{W}^{2}, \mathbb{R})$ by an argument very similar to the one used to prove Prop. 1 of Sect. 27 of [Nol87]. In view of (3.4) and the fact that (3.2) is a linear isomorphism, we have (ii).

On the other hand, assume (ii). Let a list $t := (t_{i} \mid i \in n^{]})$ be given such that

$$(4.16) \qquad\qquad\qquad \sum_{i \in n^{]}} t_{i}\boldsymbol{\gamma}_{i} = \mathbf{0}.$$

Then we have

$$\mathbf{0} = \boldsymbol{\gamma}_{1} \sum_{i \in n^{]}} t_{i}\boldsymbol{\gamma}_{i} = \sum_{i \in n^{]}} t_{i}(\boldsymbol{\gamma}_{1}\boldsymbol{\gamma}_{i}).$$

Since $(\boldsymbol{\gamma}_{1}\boldsymbol{\gamma}_{i} \mid i \in n^{]})$ is a subfamily of the basis $\boldsymbol{\gamma}\square\boldsymbol{\gamma}$ of $\mathrm{Qu}\,\mathcal{W}$, it is linearly independent. It follows that $t = 0$.

Since t was arbitrary, subject to (4.16), $\boldsymbol{\gamma}$ is linearly independent, and hence a basis.

We next prove the equivalence of (i) and (iii). In view of (2.3), (3.18), and the fact that $\text{Pqu}\,\mathcal{W} = \Phi_{\mathcal{W}}(\{\mathbf{0}\})$, (iii) holds if and only if

$$\Phi_{\mathcal{W}}\left(\bigcap_{i \in J}\Phi_{\mathcal{W}}^{\leftarrow}(\mathbb{P}\,\boldsymbol{\gamma}_i{}^2)\right) = \Phi_{\mathcal{W}}(\{\mathbf{0}\}) \quad \Longleftrightarrow \quad J = n^{]} \qquad \text{for all } J \in \text{Sub}\,n^{]}.$$

Since, in view (3.16), we have

$$\mathbb{P}\,\boldsymbol{\gamma}_i{}^2 = \Phi_{\mathcal{W}}(\text{Null}\,\boldsymbol{\gamma}_i{}^2) = \Phi_{\mathcal{W}}(\text{Null}\,\boldsymbol{\gamma}_i) \qquad \text{for all } i \in n^{]},$$

it follows that (iii) holds if and only if

$$\bigcap_{i \in J}\text{Null}\,\boldsymbol{\gamma}_i = \{\mathbf{0}\} \quad \Longleftrightarrow \quad J = n^{]} \quad \text{for all } J \in \text{Sub}\,n^{]}.$$

Since

$$\bigcap_{i \in J}\text{Null}\,\boldsymbol{\gamma}_i = \left(\sum_{i \in J}\mathbb{R}\,\boldsymbol{\gamma}_i\right)^{\perp} = (\text{Lsp}\{\boldsymbol{\gamma}_i \,|\, i \in J\})^{\perp} \quad \text{for all } J \in \text{Sub}\,n^{]}$$

by [Nol87], Prop. 5 of Sect. 21, it follows that (iii) holds if and only if

$$\text{Lsp}\{\boldsymbol{\gamma}_i \,|\, i \in J\} = \mathcal{V} \quad \Longleftrightarrow \quad J = n^{]} \qquad \text{for all } J \in \text{Sub}\,n^{]},$$

which is equivalent to (i) because bases can be characterized as minimal spanning families. $\qquad\square$

Corollary 1. *Let $n \in \mathbb{N}$, $\mathcal{U} \in \text{Subsp}\,\mathcal{V}$, and $\boldsymbol{\beta} \in (\mathcal{V}^*)^n$ be given. Then the following are equivalent:*

(i) *$\boldsymbol{\beta}$ is a basis of $\mathcal{U}^{\perp}$;*
(ii) *$\boldsymbol{\beta}\square\boldsymbol{\beta}$ is a basis of $\text{Lsp}\,\Phi_{\mathcal{V}}(\mathcal{U})$;*
(iii) *$(\mathbb{P}\,\boldsymbol{\beta}_i{}^2 \,|\, i \in n^{]})$ is a facial decomposition of $\Phi_{\mathcal{V}}(\mathcal{U})$.*

Proof. If $\text{Rng}\,\boldsymbol{\beta} \not\subset \mathcal{U}^{\perp}$, then none of (i), (ii), and (iii) are valid. Assume $\text{Rng}\,\boldsymbol{\beta} \subset \mathcal{U}^{\perp}$. In view of Prop. 1.2 we may determine $\boldsymbol{\gamma} \in ((\mathcal{V}/\mathcal{U})^*)^n$ such that $\boldsymbol{\beta}_i = \boldsymbol{\gamma}_i\Omega_{\mathcal{V}/\mathcal{U}}$ for all $i \in n^{]}$. We note that the conclusion of Prop. 4.4 holds for $\mathcal{W} := \mathcal{V}/\mathcal{U}$ and $\boldsymbol{\gamma}$ as just chosen. The equivalence of (i), (ii), and (iii) (of the present proposition) follows by appeal to the isomorphisms (1.2) and (3.14). $\qquad\square$

5. Cone-isomorphisms

Throughout this section, we assume that non-zero linear spaces $\mathcal{V}$ and $\mathcal{V}'$ are given. It is clear that if $L\colon \mathcal{V}' \to \mathcal{V}$ is a linear isomorphism, then the mapping $P\colon \text{Pqu}\,\mathcal{V} \to \text{Pqu}\,\mathcal{V}'$ defined by

$$(5.1) \qquad\qquad P(\boldsymbol{\phi}) := \boldsymbol{\phi} \circ L \qquad \text{for all } \boldsymbol{\phi} \text{ in } \text{Pqu}\,\mathcal{V}$$

is a cone-isomorphism. (Indeed, its inverse is given by $P^{-1}(\boldsymbol{\psi}) = \boldsymbol{\psi} \circ L^{-1}$ for all $\boldsymbol{\psi}$ in $\text{Pqu}\,\mathcal{V}'$.)

Theorem 2. *Let a cone-isomorphism* $P \colon \mathrm{Pqu}\,\mathcal{V} \to \mathrm{Pqu}\,\mathcal{V}'$ *be given. Then there are exactly two linear isomorphisms* $L \colon \mathcal{V}' \to \mathcal{V}$ *such that* (5.1) *holds. If* L *is one of them, then* $-L$ *is the other.*

Proof. Denote by $\overline{P} \colon \mathrm{Qu}\,\mathcal{V} \to \mathrm{Qu}\,\mathcal{V}'$ the linear isomorphism that extends P.

Lemma 5.1. *Suppose that* $\boldsymbol{\lambda}$, $\boldsymbol{\mu} \in \mathcal{V}^*$ *and* $\boldsymbol{\lambda}'$, $\boldsymbol{\mu}' \in \mathcal{V}'^*$ *satisfy*

$$(5.2) \qquad P(\boldsymbol{\lambda}^2) = \boldsymbol{\lambda}'^2 \quad and \quad P(\boldsymbol{\mu}^2) = \boldsymbol{\mu}'^2.$$

Then

$$(5.3) \qquad \overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = \boldsymbol{\lambda}'\boldsymbol{\mu}' \quad or \quad \overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = -\boldsymbol{\lambda}'\boldsymbol{\mu}'$$

Proof (Lemma 5.1). The assertion is easily seen to be valid if one of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ is a scalar multiple of the other. We may assume, therefore, that $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ is linearly independent. Put $\mathcal{U} := \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}^{\perp}$, $\mathcal{U}' := \{\boldsymbol{\lambda}', \boldsymbol{\mu}'\}^{\perp}$. It is clear that $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ and $(\boldsymbol{\mu}', \boldsymbol{\lambda}')$ are bases of $\mathcal{U}$ and $(\mathcal{U}')^{\perp}$, respectively. It follows by Cor. 1 of Prop. 4.4 that $\mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}}\{\boldsymbol{\lambda}^2, \boldsymbol{\mu}^2\} = \Phi_{\mathcal{V}}(\mathcal{U})$ and $\mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}'}\{\boldsymbol{\gamma}'^2, \boldsymbol{\mu}'^2\} = \Phi_{\mathcal{V}}(\mathcal{U}')$. Since P is a cone-isomorphism, the images under P of facial spans in $\mathrm{Pqu}\,\mathcal{V}$ are corresponding facial spans in $\mathrm{Pqu}\,\mathcal{V}'$. In particular, we have $P_{>}\big(\mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}}\{\boldsymbol{\lambda}^2, \boldsymbol{\mu}^2\}\big) = \mathrm{Fcsp}_{\mathrm{Pqu}\,\mathcal{V}'}\{\boldsymbol{\lambda}'^2, \boldsymbol{\mu}'^2\}$, *i.e.,*

$$(5.4) \qquad P_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \Phi_{\mathcal{V}'}(\mathcal{U}').$$

Hence, since $\overline{P}$ is linear

$$(5.5) \qquad \overline{P}_{>}\left(\mathrm{Lsp}\,\Phi_{\mathcal{V}}(\mathcal{U})\right) = \mathrm{Lsp}\,P_{>}\left(\Phi_{\mathcal{V}}(\mathcal{U})\right) = \mathrm{Lsp}\,\Phi_{\mathcal{V}'}(\mathcal{U}').$$

It follows from item (ii) of Cor. 1 of Prop. 4.4 that $\overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) \in \mathrm{Lsp}\,\Phi_{\mathcal{V}'}(\mathcal{U}')$ and that $(\boldsymbol{\lambda}'^2, \boldsymbol{\lambda}'\boldsymbol{\mu}', \boldsymbol{\mu}'^2)$ is a basis of $\mathrm{Lsp}\,\Phi_{\mathcal{V}'}(\mathcal{U}')$. Hence we can determine a, b, $c \in \mathbb{R}$ such that

$$(5.6) \qquad \overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) = a\boldsymbol{\lambda}'^2 + b\boldsymbol{\lambda}'\boldsymbol{\mu}' + c\boldsymbol{\mu}'^2$$

Now let $t \in \mathbb{R}$ be given and put $\boldsymbol{\gamma} := \boldsymbol{\lambda} + t\boldsymbol{\mu}$. On the one hand, since P is linear, it follows from (5.2) and (5.6) that

$$(5.7) \qquad \begin{aligned} P(\boldsymbol{\gamma}^2) &= \boldsymbol{\lambda}'^2 + 2t\overline{P}(\boldsymbol{\lambda}\boldsymbol{\mu}) + t^2\boldsymbol{\mu}'^2 \\ &= (1 + 2ta)\boldsymbol{\lambda}'^2 + 2tb\boldsymbol{\lambda}'\boldsymbol{\mu}' + (t^2 + 2tc)\boldsymbol{\mu}'^2. \end{aligned}$$

On the other hand, since $\boldsymbol{\gamma}^2$ belongs, by Prop. 4.3, to an extreme ray of $\mathrm{Pqu}\,\mathcal{V}$, its image $P(\boldsymbol{\gamma}^2)$ must belong to an extreme ray of $\mathrm{Pqu}\,\mathcal{V}'$, and hence we may choose $\boldsymbol{\gamma}' \in \mathcal{V}'^*$ such that

$$(5.8) \qquad P(\boldsymbol{\gamma}^2) = \boldsymbol{\gamma}'^2.$$

Since $\boldsymbol{\gamma}^2 \in \Phi_{\mathcal{V}}(\mathcal{U})$, it follows from (5.4) that $\boldsymbol{\gamma}'^2 \in \Phi_{\mathcal{V}'}(\mathcal{U}')$ and hence, by (4.11), that $\boldsymbol{\gamma}' \in \mathcal{U}'^{\perp} = \mathrm{Lsp}\,\{\boldsymbol{\lambda}', \boldsymbol{\mu}'\}$. Thus we may determine s and $r \in \mathbb{R}$ such that $\boldsymbol{\gamma}' = s\boldsymbol{\lambda}' + r\boldsymbol{\mu}'$; it follows by by (5.8) that

$$(5.9) \qquad\qquad P(\boldsymbol{\gamma}^2) = s^2\boldsymbol{\lambda}'^2 + 2sr\boldsymbol{\lambda}'\boldsymbol{\mu}' + r^2\boldsymbol{\mu}'^2.$$

Since $(\boldsymbol{\lambda}'^2, \boldsymbol{\mu}'^2, \boldsymbol{\lambda}'\boldsymbol{\mu}')$ is linearly independent, we conclude from (5.7) and (5.9) that

$$s^2 = (1 + 2ta), \qquad sr = tb, \qquad r^2 = (t^2 + 2tc),$$

and hence that

$$t^2b^2 \;=\; s^2r^2 \;=\; (1+2ta)(t^2 + 2tc) \;=\; 2t^3a + (1 + 4ac)t^2 + 2tc.$$

Since this equality must be valid for all $t \in \mathbb{R}$, it follows that $a = c = 0$ and $b^2 = 1$. Hence we must have $b = 1$ or $b = -1$ and (5.6) reduces to (5.7). $\qquad\square$

We now put $n := \dim \mathcal{V} = \dim \mathcal{V}^*$ and choose a list-basis $\boldsymbol{\beta}$ of $\mathcal{V}^*$. We note that the conclusions of Prop. 4.4 apply. Hence $(\mathbb{P}\,\boldsymbol{\beta}_i^2 \,|\, i \in n^]\,)$ is a facial decomposition for $\mathrm{Pqu}\,\mathcal{V}$. Since P preserves extreme rays, it follows from Prop. 4.3 and Prop. 4.4 that we may choose a list-basis $\boldsymbol{\beta}'$ of $\mathcal{V}'^*$ such that

$$(5.10) \qquad\qquad P(\boldsymbol{\beta}_i^2) = \boldsymbol{\beta}_i'^2 \qquad \text{for all } i \text{ in } n^].$$

Let $i \in n^]$ be given. By Lemma 5.1 applied to $\boldsymbol{\lambda} := \boldsymbol{\beta}_1$ and $\boldsymbol{\mu} := \boldsymbol{\beta}_i$, we see that we may choose $s_i \in \{1, -1\}$ such that

$$(5.11) \qquad\qquad \overline{P}(\boldsymbol{\beta}_1\boldsymbol{\beta}_i) = s_i\boldsymbol{\beta}_1'\boldsymbol{\beta}_i'.$$

Hence we obtain a list $(s_i \,|\, i \in n^]\,) \in \{1, -1\}^n$ with $s_1 = 1$.

Lemma 5.2. *We have*

$$(5.12) \qquad\qquad \overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k) = s_is_k\boldsymbol{\beta}_i'\boldsymbol{\beta}_k' \qquad \text{for all } i,\, k \in n^].$$

Proof (Lemma 5.2). Since $s_1 = 1$, it is clear from (5.10) and (5.11) that (5.12) holds when $i = 1$ or $k = 1$ or $i = k$. Suppose, then, that i and $k \in n^]$ are given such that $i \neq 1$, $k \neq 1$, and $i \neq k$. Then the triple $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_i, \boldsymbol{\beta}_k)$ is linearly independent and we may apply Cor. 1 of Prop. 4.4 to it. Using an argument similar to the one used in the proof of Lemma 5.1, we determine a, b, and $c \in \mathbb{R}$ such that

$$P\left((\boldsymbol{\beta}_1 + \boldsymbol{\beta}_i + \boldsymbol{\beta}_k)^2\right) \;=\; (a\boldsymbol{\beta}_1' + b\boldsymbol{\beta}_i' + c\boldsymbol{\beta}_k')^2.$$

Using the linearity of $\overline{P}$ and (5.10) and (5.11), we find that

$$\boldsymbol{\beta}_1'^2 + \boldsymbol{\beta}_i'^2 + \boldsymbol{\beta}_k'^2 + 2s_i\boldsymbol{\beta}_1'\boldsymbol{\beta}_i' + 2s_k\boldsymbol{\beta}_1'\boldsymbol{\beta}_k' + 2\overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k)$$
$$= a^2\boldsymbol{\beta}_1'^2 + b^2\boldsymbol{\beta}_i'^2 + c^2\boldsymbol{\beta}_k'^2 + 2ab\boldsymbol{\beta}_1'\boldsymbol{\beta}_i' + 2ac\boldsymbol{\beta}_1'\boldsymbol{\beta}_k' + 2bc\boldsymbol{\beta}_i'\boldsymbol{\beta}_k'.$$

Since $\overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k) \in \mathbb{R}\,\boldsymbol{\beta}'_i\boldsymbol{\beta}'_k$ by Lemma 5.1 and since the sextuple

$$(\boldsymbol{\beta}'^{\,2}_1, \boldsymbol{\beta}'^{\,2}_i, \boldsymbol{\beta}'^{\,2}_k, \boldsymbol{\beta}'_1\boldsymbol{\beta}'_i, \boldsymbol{\beta}'_1\boldsymbol{\beta}'_k, \boldsymbol{\beta}'_i\boldsymbol{\beta}'_k)$$

is linearly independent, we conclude that $a^2 = b^2 = c^2 = 1$, $ab = s_i$, $ac = s_k$, and $\overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k) = bc\boldsymbol{\beta}'_i\boldsymbol{\beta}'_k$. Hence $s_is_k = (ab)(ac) = a^2bc = bc$ and $\overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k) = s_is_k\boldsymbol{\beta}'_i\boldsymbol{\beta}'_k$. □

Since $\boldsymbol{\beta}$ is a basis of $\mathcal{V}^*$ and $(s_i\boldsymbol{\beta}'_i \,|\, i \in n^]）$ is a basis of $\mathcal{V}'^*$, we can determine a linear transformation $L \colon \mathcal{V}' \to \mathcal{V}$ whose transpose $L^{\mathrm{T}} \colon \mathcal{V}^* \to \mathcal{V}'^*$ satisfies

$$L^{\mathrm{T}}\boldsymbol{\beta}_i = s_i\boldsymbol{\beta}'_i \qquad \text{for all } i \text{ in } n^].$$

By Lemma 5.2 we have

$$\overline{P}(\boldsymbol{\beta}_i\boldsymbol{\beta}_k) = (L^{\mathrm{T}}\boldsymbol{\beta}_i)(L^{\mathrm{T}}\boldsymbol{\beta}_k) = (\boldsymbol{\beta}_i\boldsymbol{\beta}_k) \circ L \qquad \text{for all } i, k \in n^].$$

Since $\boldsymbol{\beta}\square\boldsymbol{\beta} := (\boldsymbol{\beta}_i\boldsymbol{\beta}_k \,|\, (i,k) \in \mathrm{Ip}(n))$ is a basis of $\mathrm{Qu}\,\mathcal{V}$, we conclude that (5.1) holds. It is easy to see that (5.1) also holds when L is replaced by $-L$.

Now let a linear transformation $M \colon \mathcal{V} \to \mathcal{V}'$ be given such that

$$(5.13) \qquad \boldsymbol{\phi} \circ L = P(\boldsymbol{\phi}) = \boldsymbol{\phi} \circ M \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Pqu}\,\mathcal{V}.$$

Using (5.13) with $\boldsymbol{\phi} := \boldsymbol{\beta}_1^{\,2}$, we find that $(M^{\mathrm{T}}\boldsymbol{\beta}_1)^2 = (L^{\mathrm{T}}\boldsymbol{\beta}_1)^2$. Hence we may choose $\epsilon \in \{1, -1\}$ such that $M^{\mathrm{T}}\boldsymbol{\beta}_1 = \epsilon L^{\mathrm{T}}\boldsymbol{\beta}_1$. Now let $k \in n^]$ be given. Using (5.13) with $\boldsymbol{\phi} := \boldsymbol{\beta}_k\boldsymbol{\beta}_1$ we find that

$$(L^{\mathrm{T}}\boldsymbol{\beta}_k)(L^{\mathrm{T}}\boldsymbol{\beta}_1) = (M^{\mathrm{T}}\boldsymbol{\beta}_k)(M^{\mathrm{T}}\boldsymbol{\beta}_1) = (\epsilon M^{\mathrm{T}}\boldsymbol{\beta}_k)(L^{\mathrm{T}}\boldsymbol{\beta}_1).$$

Since $L^{\mathrm{T}}\boldsymbol{\beta}_1 \neq 0$ and since $k \in n^]$ was arbitrary, it follows that

$$L^{\mathrm{T}}\boldsymbol{\beta}_k = \epsilon M^{\mathrm{T}}\boldsymbol{\beta}_k \qquad \text{for all } k \in n^].$$

Since $\boldsymbol{\beta}$ is a basis of $\mathcal{V}^*$, we conclude that $L^{\mathrm{T}} = \epsilon M^{\mathrm{T}}$ and hence $M = \epsilon L$. □

6. Linearly induced mappings

Throughout this section, $\mathcal{V}$, $\mathcal{V}'$, and $\mathcal{V}''$ are given linear spaces. For a given $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we define $\mathrm{qu}(L)$ in $\mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ and $\mathrm{pqu}(L) \in \mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ by

$$(6.1) \qquad \mathrm{qu}(L)\boldsymbol{\phi} := \boldsymbol{\phi} \circ L \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V};$$

$$(6.2) \qquad \mathrm{pqu}(L)\boldsymbol{\phi} := \boldsymbol{\phi} \circ L \qquad \text{for all } \boldsymbol{\phi} \text{ in } \mathrm{Pqu}\,\mathcal{V}.$$

It is clear that

$$(6.3) \qquad \mathrm{pqu}(L) = \mathrm{qu}(L)\big|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} \qquad \text{for all } L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V}).$$

If a given $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ equals $\mathrm{qu}(L)$ for some L in $\mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we say that Q is **linearly induced** (**by** L); similarly, if a given $P \in$

$\mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ equals $\mathrm{pqu}(L)$ for some $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$, we say that P is **linearly induced** (**by** L).

The following two results are immediate.

Proposition 6.1. *Let $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ and $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ be given. Then $Q = \mathrm{qu}(L)$ if and only if Q is cone-compatible and $Q|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} = \mathrm{pqu}(L)$.*

Proposition 6.2. *A composite of linearly induced mappings is itself linearly induced. Indeed, let $L_1 \in \mathrm{Lin}(\mathcal{V}'', \mathcal{V}')$ and $L_2 \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ be given; then*

$$(6.4) \quad \mathrm{qu}(L_1)\,\mathrm{qu}(L_2) = \mathrm{qu}(L_2 L_1), \qquad \mathrm{pqu}(L_1)\,\mathrm{pqu}(L_2) = \mathrm{pqu}(L_2 L_1).$$

Proposition 6.3. *The transpose of a linearly induced linear mapping and the cone-transpose of a linearly induced cone-linear mapping are both linearly induced. Indeed, let $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ be given; then*

$$(6.5) \qquad (\mathrm{qu}(L))^{\mathrm{T}} = \mathrm{qu}(L^{\mathrm{T}}), \qquad (\mathrm{pqu}(L))^{\mathsf{T}} = \mathrm{pqu}(L^{\mathrm{T}})$$

Proof. It is clear that $\mathrm{qu}(L^{\mathrm{T}})\boldsymbol{w}^2 = (\boldsymbol{w}L^{\mathrm{T}})^2 = (L\boldsymbol{w})^2$ for all $\boldsymbol{w} \in \mathcal{V}'$; it follows by (4.2) that

$$\Gamma\left(\boldsymbol{\mu}^2, \mathrm{qu}(L^{\mathrm{T}})\boldsymbol{w}^2\right) = \Gamma\left(\boldsymbol{\mu}^2, (L\boldsymbol{w})^2\right) = (\boldsymbol{\mu}L\boldsymbol{w})^2 =$$
$$\Gamma\left((\boldsymbol{\mu}L)^2, \boldsymbol{w}^2\right) = \Gamma\left(\mathrm{qu}(L)\boldsymbol{\mu}^2, \boldsymbol{w}^2\right) \text{ for all } \boldsymbol{\mu} \in \mathcal{V}^*,\ \boldsymbol{w} \in \mathcal{V}'.$$

In view of (3.5) and the bilinearity of Γ, it follows that

$$\Gamma\left(\mathrm{qu}(L)\boldsymbol{\phi}, \boldsymbol{g}\right) = \Gamma(\boldsymbol{\phi}, \mathrm{qu}(L^{\mathrm{T}})\boldsymbol{g}) \qquad \text{for all } \boldsymbol{\phi} \in \mathrm{Qu}\,\mathcal{V},\ \boldsymbol{g} \text{ in } \mathrm{Qu}\,\mathcal{V}'^*.$$

In view of the determining conditions (4.9) and (4.10) for the transposes, we have the first equality in (6.5). In view of (6.3), the second follows by adjustment. $\square$

Proposition 6.4. *Let $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ be given. If L is surjective, then $\mathrm{qu}(L)$ and $\mathrm{pqu}(L)$ are injective. If L is injective, then $\mathrm{qu}(L)$ and $\mathrm{pqu}(L)$ are surjective.*

Proof. First, suppose that L is surjective. Then $\mathrm{qu}(L)$ and $\mathrm{pqu}(L)$ are injective by Prop. 1.1.

Next, suppose that L is injective, so L^{T} is surjective. Then, we have

$$\mathrm{Rng}\,\mathrm{pqu}(L) \supset \mathrm{Csp}\{\mathrm{pqu}(L)(\boldsymbol{\lambda}^2) \mid \boldsymbol{\lambda} \in \mathcal{V}^*\} =$$
$$\mathrm{Csp}\{(L^{\mathrm{T}}\boldsymbol{\lambda})^2 \mid \boldsymbol{\lambda} \in \mathcal{V}^*\} = \mathrm{Csp}\{\boldsymbol{\mu}^2 \mid \boldsymbol{\mu} \in \mathcal{V}'^*\}.$$

It follows by (3.7) that $\mathrm{Rng}\,\mathrm{pqu}(L) = \mathrm{Pqu}\,\mathcal{V}'$, so that $\mathrm{pqu}(L)$ is surjective. Of course $\mathrm{Rng}\,\mathrm{qu}(L) = \mathrm{Lsp}\,(\mathrm{Rng}\,\mathrm{pqu}(L))$, so $\mathrm{qu}(L)$ is also surjective. $\square$

Proposition 6.5. *Let $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ and $\mathcal{U} \in \mathrm{Subsp}\,\mathcal{V}$ be given. Then*

$$(6.6) \qquad \mathrm{pqu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) = \mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) = \Phi_{\mathcal{V}'} \left(L^< (\mathcal{U}) \right).$$

Proof. Let $\boldsymbol{\phi} \in \Phi_{\mathcal{V}}(\mathcal{U})$ be given. By (3.12), $\boldsymbol{\phi}|_{\mathcal{U}} = 0$. It follows immediately that $(\boldsymbol{\phi} \circ L)|_{L^<(\mathcal{U})} = 0$. Of course $\boldsymbol{\phi} \circ L$ is positive, so it follows by (3.12) that $\mathrm{qu}(L)\boldsymbol{\phi} = \boldsymbol{\phi} \circ L \in \Phi_{\mathcal{V}'}(L^<(\mathcal{U}))$. Since $\boldsymbol{\phi}$ in $\Phi_{\mathcal{V}}(\mathcal{U})$ was arbitrary, it follows that

$$\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \subset \Phi_{\mathcal{V}'} \left(L^< (\mathcal{U}) \right).$$

We note that, by the Theorem on Annihilators and Transposes ([Nol87], Sect. 21) and by (6.5) and (4.12), we have

$$(6.7) \quad \left(\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \right)^{\perp} = \mathrm{qu}(L^{\mathrm{T}})^< \left(\left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right)^{\perp} \right) =$$

$$\left(\mathrm{qu} \left(L^{\mathrm{T}} \right) \right)^< \left(\{ \boldsymbol{g} \in \mathrm{Qu}\,\mathcal{V}^* \,|\, \boldsymbol{g}|_{\mathcal{U}^{\perp}} = \boldsymbol{0} \} \right) = \{ \boldsymbol{f} \in \mathrm{Qu}\,\mathcal{V} \,|\, \left(\boldsymbol{f} \circ L^{\mathrm{T}} \right)|_{\mathcal{U}^{\perp}} = \boldsymbol{0} \}.$$

Now let $\boldsymbol{f} \in \left(\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \right)^{\perp}$ be given. It follows by (6.7) that $\left(\boldsymbol{f} \circ L^{\mathrm{T}} \right)|_{\mathcal{U}^{\perp}} = \boldsymbol{0}$. Using the Theorem on Annihilators and Transposes again, we obtain

$$\{ \boldsymbol{0} \} = \left(\boldsymbol{f} \circ L^{\mathrm{T}} \right)_> \left(\mathcal{U}^{\perp} \right) = \boldsymbol{f}_> \left(L^{\mathrm{T}}_> \left(\mathcal{U}^{\perp} \right) \right) = \boldsymbol{f}_> \left(\left(L^< (\mathcal{U}) \right)^{\perp} \right),$$

so $\boldsymbol{f}|_{(L^<(\mathcal{U}))^{\perp}} = 0$. In view of (4.12) again, it follows that $\boldsymbol{f} \in \left(\Phi_{\mathcal{V}'} \left(L^< (\mathcal{U}) \right) \right)^{\perp}$. Since $\boldsymbol{f} \in \left(\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \right)^{\perp}$ was arbitrary, it follows that

$$\left(\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \right)^{\perp} \subset \left(\Phi_{\mathcal{V}'} \left(L^< (\mathcal{U}) \right) \right)^{\perp}$$

and hence

$$\left(\mathrm{qu}(L)_> \left(\Phi_{\mathcal{V}} \left(\mathcal{U} \right) \right) \right) \supset \left(\Phi_{\mathcal{V}'} \left(L^< (\mathcal{U}) \right) \right).$$

This establishes the second equality in (6.6); the first follows by Prop. 6.1. $\qquad \square$

Theorem 3. *Let $P \in \mathrm{Lin}(\mathrm{Pqu}\,\mathcal{V}, \mathrm{Pqu}\,\mathcal{V}')$ be given. Then P is linearly induced if and only if both P and P^{T} are face-preserving.*

Before proving this theorem, we note that the cone-transpose of a face-preserving cone-linear mapping need not be face-preserving. Indeed, suppose that $\dim \mathcal{V} \geq 2$ and that $\dim \mathcal{V}' \geq 1$, and choose $\boldsymbol{f} \in \mathrm{Pqu}^+ \mathcal{V}^*$ and $\boldsymbol{\mu} \in (\mathcal{V}'^*)^{\times}$. Then the mapping $(\boldsymbol{\mu}^2 \otimes \boldsymbol{f})|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'}$ is face-preserving because the image of every non-zero face of $\mathrm{Pqu}\,\mathcal{V}$ under this mapping is the extreme ray $\mathbb{P}\,\boldsymbol{\mu}^2$ of $\mathrm{Pqu}\,\mathcal{V}'$. However, the range of the cone-transpose $(\boldsymbol{f} \otimes \boldsymbol{\mu}^2)|_{\mathrm{Pqu}\,\mathcal{V}'^*}^{\mathrm{Pqu}\,\mathcal{V}^*}$ of the mapping above is $\mathbb{P}\,\boldsymbol{f}$, which is not a face.

Proof (Thm. 3). On the one hand, suppose that P is linearly induced and choose $L \in \mathrm{Lin}(\mathcal{V}', \mathcal{V})$ such that $P = \mathrm{qu}(L)$. Let a $\mathcal{F} \in \mathrm{Face}\,(\mathrm{Pqu}\,\mathcal{V})$ be given. In view of Thm. 1, we may determine $\mathcal{U} \in \mathrm{Subsp}\,\mathcal{V}$ such that $\mathcal{F} = \Phi_{\mathcal{V}}(\mathcal{U})$. Then $P_{>}(\mathcal{F}) = \Phi_{\mathcal{V}'}\left(L^{<}(\mathcal{U})\right)$ by Prop. 6.5, so $P_{>}(\mathcal{F})$ was a face of $\mathrm{Qu}\,\mathcal{V}'$ by Prop. 3.3. Since $\mathcal{F} \in \mathrm{Face}\,(\mathrm{Pqu}\,\mathcal{V})$ was arbitrary, it follows that the P is face-preserving. Indeed, since P was an arbitrary linearly induced cone-linear mapping, every linearly induced cone-linear mapping is face-preserving. In particular, in view of Prop. 6.3, P^{T} is face-preserving.

On the other hand, suppose that P and P^{T} preserve faces. Then their ranges are faces of $\mathrm{Pqu}\,\mathcal{V}'$ and $\mathrm{Pqu}\,\mathcal{V}^{*}$, respectively. It follows by Thm. 1 that subspaces $\mathcal{U}$ and $\mathcal{U}'$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively, can be determined such that

$$(6.8) \qquad\qquad \mathrm{Rng}\,P = \Phi_{\mathcal{V}'}\left(\mathcal{U}'\right);$$

$$(6.9) \qquad\qquad \mathrm{Rng}\,P^{\mathsf{T}} = \Phi_{\mathcal{V}^{*}}\left(\mathcal{U}^{\perp}\right).$$

Denote by $Q \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}\,\mathcal{V}')$ the linear mapping determined by P, so that $Q|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'} = P$. It follows from (6.8) that

$$(6.10) \qquad\qquad \mathrm{Rng}\,Q = \mathrm{Lsp}\,\mathrm{Rng}\,P = \mathrm{Lsp}\,\Phi_{\mathcal{V}'}\left(\mathcal{U}'\right),$$

and from the Theorem on Annihilators and Transposes ([Nol87], Sect. 21), (6.9), and (4.12) that

$$(6.11) \quad \mathrm{Null}\,Q = \left(\mathrm{Rng}\,Q^{\mathrm{T}}\right)^{\perp} = \left(\mathrm{Rng}\,P^{\mathsf{T}}\right)^{\perp} =$$
$$\left(\Phi_{\mathcal{V}^{*}}\left(\mathcal{U}^{\perp}\right)\right)^{\perp} = \{\phi \in \mathrm{Qu}\,\mathcal{V} \mid \phi|_{\mathcal{U}} = 0\}.$$

We will now describe and establish the validity of the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathrm{Dom}\,Q & \xrightarrow{\;\boldsymbol{\Omega}_{\mathrm{Dom}\,Q/\,\mathrm{Null}\,Q}\;} & \mathrm{Dom}\,Q/\,\mathrm{Null}\,Q & \xrightarrow{\;\tilde{Q}\;} & \mathrm{Rng}\,Q & \xrightarrow{\;\mathbf{1}_{\mathrm{Rng}\,Q \subset \mathrm{Dom}\,Q}\;} & \mathrm{Cod}\,Q \\
\Big\| & & \updownarrow & & \updownarrow & & \Big\| \\
\mathrm{Qu}\,\mathcal{V} & \xrightarrow{\;\mathrm{qu}(\mathbf{1}_{\mathcal{U} \subset \mathcal{V}})\;} & \mathrm{Qu}\,\mathcal{U} & \xrightarrow{\;\tilde{\tilde{Q}}\;} & \mathrm{Qu}\,(\mathcal{V}'/\mathcal{U}') & \xrightarrow{\;\mathrm{qu}(\boldsymbol{\Omega}_{\mathcal{V}'/\mathcal{U}'})\;} & \mathrm{Qu}\,\mathcal{V}'
\end{array}
$$

The upper line in the diagram represents Q as the composite of three linear mappings as described in Prop. 1.3. Two linear isomorphism-pairs, represented in the diagram by two-headed vertical arrows, will be determined below. Then, in turn, a linear isomorphism $\tilde{\tilde{Q}} \in \mathrm{Lin}(\mathrm{Qu}\,\mathcal{V}, \mathrm{Qu}(\mathcal{V}'/\mathcal{U}'))$ will be determined such that

$$(6.12) \qquad\qquad Q = \mathrm{qu}(\boldsymbol{\Omega}_{\mathcal{V}'/\mathcal{U}'})\,\tilde{\tilde{Q}}\,\mathrm{qu}(\mathbf{1}_{\mathcal{U} \subset \mathcal{V}}).$$

It will be shown that $\tilde{\tilde{Q}}$ is linearly induced, and hence that the lower line in the diagram represents Q as a composite of three linearly induced mappings.

The linear mappings $\mathbf{\Omega}_{\mathrm{Dom}\,Q/\,\mathrm{Null}\,Q}$ and $\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ on the left side of the diagram are both surjective: the first because it is a quotient mapping and the second, since $\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ is injective, by Prop. 6.4. Of course $\mathrm{Null}\,\mathbf{\Omega}_{\mathrm{Dom}\,Q/\,\mathrm{Null}\,Q} = \mathrm{Null}\,Q$, and, in view of (6.11),

$$\mathrm{Null}\,\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}) = \{\phi \in \mathrm{Qu}\,\mathcal{V} \,|\, \phi|_{\mathcal{U}} = 0\} = \mathrm{Null}\,Q.$$

Thus $\mathbf{\Omega}_{\mathrm{Dom}\,Q/\,\mathrm{Null}\,Q}$ and $\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ are linear surjections with common domain and nullspace. It follows by Cor. 1 of Prop. 1.3 that the leftmost square in the commutative diagram determines a linear isomorphism-pair $\mathrm{Dom}\,Q/\,\mathrm{Null}\,Q \leftrightarrow \mathrm{Qu}\,\mathcal{V}$ as indicated.

Similarly, the linear mappings $\mathbf{1}_{\mathrm{Rng}\,Q\subset\mathrm{Cod}\,Q}$ and $\mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'})$ on the right side of the diagram are both injective: the first because it is an inclusion mapping and the second, since $\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'}$ is surjective, by Prop. 6.4. Of course $\mathrm{Rng}\,\mathbf{1}_{\mathrm{Rng}\,Q\subset\mathrm{Cod}\,Q} = \mathrm{Rng}\,Q$ and, in view of Prop. 3.4 and (6.10),

$$\mathrm{Rng}\,\mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'}) = \mathrm{Lsp}\,\Phi_{\mathcal{V}'}(\mathcal{U}') = \mathrm{Rng}\,Q.$$

Thus $\mathbf{1}_{\mathrm{Rng}\,Q\subset\mathrm{Cod}\,Q}$ and $\mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'})$ are linear injections with common range. It follows by Cor. 2 of Prop. 1.3 that the rightmost square in the commutative diagram determines a linear isomorphism-pair $\mathrm{Rng}\,Q \leftrightarrow \mathrm{Qu}(\mathcal{V}'/\mathcal{U}')$ as indicated.

Since the double-headed vertical arrows in the diagram indeed represent linear isomorphism-pairs, and since $\tilde{Q}$ is a linear isomorphism by Prop. 1.3, the center square in the diagram determines a linear isomorphism $\tilde{\tilde{Q}}$ as indicated.

We shall next show that $\tilde{\tilde{Q}}$ is cone-preserving relative to the cones $\mathrm{Pqu}\,\mathcal{V}$ and $\mathrm{Pqu}\,\mathcal{V}'$.

In view of (6.12) and (6.8), we have

$$(6.13)\quad \left(\mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'})\,\tilde{\tilde{Q}}\,\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})\right)_{>}(\mathrm{Pqu}\,\mathcal{V}) = Q_{>}(\mathrm{Pqu}\,\mathcal{V}) = \Phi_{\mathcal{V}'}(\mathcal{U}').$$

Since $\mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'})$ is injective, it follows immediately that

$$(6.14)\qquad \tilde{\tilde{Q}}_{>}(\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}}))_{>}(\mathrm{Pqu}\,\mathcal{V}) = \mathrm{qu}(\mathbf{\Omega}_{\mathcal{V}'/\mathcal{U}'})^{<}(\Phi_{\mathcal{V}'}(\mathcal{U}')).$$

Since $\mathrm{pqu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ is linearly induced, it preserves faces (as shown in the first part of this proof). It follows that the image under $\mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})$ of the cone $\mathrm{Pqu}\,\mathcal{V}$, which is spanning in $\mathrm{Qu}\,\mathcal{V}$, is a face of $\mathrm{Pqu}\,\mathcal{U}$ which is spanning in $\mathrm{Rng}\,P = \mathrm{Qu}\,\mathcal{U}$; hence

$$(6.15)\qquad\qquad \mathrm{qu}(\mathbf{1}_{\mathcal{U}\subset\mathcal{V}})_{>}(\mathrm{Pqu}\,\mathcal{V}) = \mathrm{Pqu}\,\mathcal{U}.$$

By Prop. 3.4 we have

(6.16) $$\mathrm{qu}(\boldsymbol{\Omega}_{\mathcal{V}'/\mathcal{U}'})^< (\Phi_{\mathcal{V}'}(\mathcal{U}')) = \mathrm{Pqu}\,(\mathcal{V}'/\mathcal{U}') .$$

Substitution of (6.15) and (6.16) into (6.14) yields

$$\tilde{\tilde{Q}}_> (\mathrm{Pqu}\,\mathcal{U}) = \mathrm{Pqu}\,(\mathcal{V}'/\mathcal{U}') ;$$

so the linear isomorphism $\tilde{\tilde{Q}}$ is cone-preserving. It follows that $\tilde{\tilde{Q}}|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'}$ is a cone-isomorphism; hence,in view of Thm. 2, there is a linear isomorphism, say $\tilde{\tilde{L}}\colon \mathcal{V}'/\mathcal{U}' \to \mathcal{U}$, which induces $\tilde{\tilde{Q}}|_{\mathrm{Pqu}\,\mathcal{V}}^{\mathrm{Pqu}\,\mathcal{V}'}$ and hence $\tilde{\tilde{Q}}$ as well. Thus $\tilde{\tilde{Q}} = \mathrm{qu}(\tilde{\tilde{L}})$. Putting $L := \mathbf{1}_{U\subset\mathcal{V}}\,\tilde{\tilde{L}}\Omega_{\mathcal{V}'/\mathcal{U}'} \in \mathrm{Lin}(\mathcal{V}',\mathcal{V})$, it follows, in view of (6.12) and Prop. 6.2, that $Q = \mathrm{qu}(L)$ and hence, in view of Prop. 6.1, that $P = \mathrm{pqu}(L)$. $\qquad\square$

Proposition 6.6. *Let $L, M \in \mathrm{Lin}(\mathcal{V}',\mathcal{V})$ be given. Then $\mathrm{pqu}(L) = \mathrm{pqu}(M)$ (equivalently, $\mathrm{qu}(L) = \mathrm{qu}(M)$) if and only if $M = L$ or $M = -L$.*

Proof. It is easy to see that $\mathrm{pqu}(L) = \mathrm{pqu}(-L)$. It is also clear that the result holds if $L = \mathbf{0}$. Assume that $L \neq \mathbf{0}$, put $n := \dim\mathcal{V}$, and choose a basis $\boldsymbol{\beta} = (\boldsymbol{\beta}_i \,|\, i \in n^]) $ of $\mathcal{V}'$ such that $\boldsymbol{\beta}_1 L \neq \mathbf{0}$. By the same argument used in the last part of the proof of Thm. 2, we may determine $\epsilon \in \{1, -1\}$ such that $L^{\mathrm{T}}\boldsymbol{\beta}_k = M^{\mathrm{T}}\boldsymbol{\beta}_k$ for all $k \in n^]$. It follows that $L = \epsilon M$. $\qquad\square$

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(R. Artz) 13140 Doyle's Court, Apple Valley, MN 55124
E-mail address: ray.e.artz@gmail.com

(W. Noll) Department of Mathematics, Carnegie Mellon University, Pittsburgh, PA 15213