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On the concept of force
by Walter Noll, April 2007

This essay is a reaction to an article by Frank Wilczek entitled "Whence the Force $\mathbf{F} = m\mathbf{a}$? " in the October 2004 issue of Physics Today and to the letters referring to this article in the August 2005 issue. The letter by Ramamutri Avi refers to a paper of mine, published in 1959, but few people in the physics community seem to be aware of the work on the conceptual mathematical foundations of classical mechanics that I and others have done since then.

1) Force in classical mechanics.

When most physicists today hear the term "classical mechanics" , they tend to think of Newtonian particle mechanics. In this context, Wilczek's statement "Newton's second law of motion, $\mathbf{F} = m\mathbf{a}$, is the soul of classical mechanics", may have some merit. However, Newtonian particle mechanics is only a very small and relatively trivial part of classical mechanics. When applied to the rest of classical mechanics, Wilczek's statement is absurd. For example, engineering students often take a course called "Statics", which deals with forces in systems having no moving parts at all, and hence accelerations are completely absent. The beginning of a textbook on statics often contains a statement of Newton's laws, but this functions like a prayer before a business meeting; it is almost totally irrelevant to the substance of the subject. The substance of statics consists in singling out parts of the system under consideration by drawing "free-body diagrams". For sufficiently many of such parts, one writes down two equations: The first states that the sum of all the forces acting on the part is zero, and the second that all the torques acting on the part is zero. In this way, one obtains sufficiently many linear equations to determine the force acting on each structural member of the system. This information is then used to decide whether the system may or may not collapse.

Engineering students often also take a course called "Dynamics". Its basic structure differs from the course in statics only by including the inertial forces among the forces considered. (I have taught courses on Statics and Dynamics in the late 1950s, and this experience has influenced my analysis of the foundations of mechanics.)

The two basic principles of classical mechanics are these:

1) Balance of forces: The total force acting on a physical system and each of its parts is zero.

2) Balance of torques: The total torque acting on a physical system and each of its parts is zero.

These principles, in some form or other, precede Newton. A form of the law of balance of torques goes back to Archimedes. In order to give a precise

formulation of these principles, an axiomatic mathematical description of the concepts of *physical systems, force-systems*, and *systems of torques* must be supplied. Such a description can be found in [FMT], [BI] or in Sect.4 of [MBM]. Newton's third law, the law of action and reaction, is a logical consequence of the law of balance of forces stated above. The proof can be found in all of the papers just quoted.

Inertial forces must be included when the two balance laws are applied. The inertial force on a single particle is given by the constitutive law of inertia: $\mathbf{i} = -m\mathbf{a}$. If the total non-inertial force acting on the particle is denoted by $\mathbf{F}$, then the law of balance of forces gives $\underline{0} = \mathbf{F} + \mathbf{i} = \mathbf{F} + (-m\mathbf{a})$ and hence Newton's second law. Now, it makes no sense to talk about position, velocity, and acceleration unless a frame of reference is specified. The law of inertia is valid only if the acceleration $\mathbf{a}$ is taken relative to an *inertial frame of reference*. If the frame used is arbitrary, not necessarily inertial, the constitutive law of inertia takes the form

$$\mathbf{i} = -m(\mathbf{u}^{\bullet\bullet} + 2\mathbf{A}\mathbf{u}^{\bullet} + (\mathbf{A}^{\bullet} - \mathbf{A}^2)\mathbf{u}). \tag{1}$$

Here, the value $\mathbf{u}(t)$ of the function $\mathbf{u}$ at time t denotes the position vector of the particle relative to a reference point (often called "Origin") which is at rest in some inertial frame, although not necessarily in the frame used. The value $\mathbf{A}(t)$ of the function $\mathbf{A}$ at time t is a skew linear transformation; it measures the rate of rotation of the given frame relative to some inertial frame. Dots denote time-derivatives. If the reference point is at rest not only in some inertial frame but also in the frame used and if $\mathbf{A}$ is constant, i.e. if $\mathbf{A}^{\bullet}$ is zero, then the contributions to the inertial force given by the second and third term on the right of (1) are called *Coriolis* force and *centrifugal* force, respectively.

The frame of reference determined by the earth is approximately inertial for small-scale phenomena, but the contribution of the Coriolis force can be decisive when large-scale wind or ocean-current phenomena are analyzed. In his famous pendulum experiment in 1851, Foucault demonstrated that the Coriolis force can be detected even on a small scale. This effect is nowadays used in inertial guidance systems.

The concept of a *frame of reference* is not the same as the concept of a *coordinate system*, as some people seem to believe. For a precise definition and discussion, see [1] and [2]. I believe that the use of coordinate systems is an impediment to insight in conceptual consideration in all of physics, classical or modern. Some discussion of this matter is contained in Sect. 1.2. of [MTW], entitled "Spacetime with and without coordinates". It contains the following quote by Einstein: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning." Unfortunately, most physicists (and many mathematicians) are still stuck with outdated mathematical infrastructures, which make it difficult to get away from coordinate systems. Better mathematical infrastructures can be found in [FDS] and [GDM]. As far as basic concepts is concerned, these

treatments are completely coordinate-free and $\mathbf{R}^n$-free. Sect.55 of [GDM] contains a coordinate-free definition of the Einstein tensor field of general relativity.

In practice, the frame of reference determined by the fixed stars and the sun at rest is very close to being an inertial frame and hence can serve, with Newton's laws, to determine the orbits of planets. I recall that Ernst Mach has suggested that inertia may be considered to be an action of the far away parts of the universe on the bodies in our nearby environment, including the solar system. I believe that, in classical mechanics, it is always useful to consider inertial forces to be exerted by the far outside world on whatever system is considered.

The less trivial parts of classical mechanics are *rigid body mechanics* and, above all, *continuum mechanics.* Since about 1900, this field has been largely abandoned by physicists in favor of non-classical physics. However, there has been a lot of research on continuum mechanics and thermodynamics in the past 60 years or so by mathematicians like me and by engineering scientists.

In classical particle mechanics, inertial forces are paramount and the subject would collapse if they were neglected. In continuum mechanics, inertial forces are sometimes unimportant and can be neglected, for example when considering the motion of toothpaste as it is extruded from a tube. The soul of continuum mechanics is the analysis of contact forces. It requires the use of very sophisticated mathematics. (See, for example, [TSI].)

2) Force in non-classical physics.

In non-classical physics, the term "force" is used when such things as gravitational force , electromagnetic force, weak force, and strong force are considered. The term cannot be given an interpretation similar to the one used in classical physics as described in Sect. 1 above. It seems that in these cases forces are the result of action of fields, such as gravitational and electromagnetic fields. The fields themselves become quantized and give rise to particles such as gravitons, photons, and gluons. I am not aware of any general axiomatic mathematical framework that would encompass all of these kinds of non-classical forces as special cases.

References:

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[MBM] Noll, W. and B. Seguin: Monoids: Boolean Algebras, and Materially Ordered Sets, available on the website www.math.cmu.edu wn0g/noll .

[1] Noll, W.: On the Illusion of Physical Space, first part of *Five Contributions to Natural Philosophy*, available on the website www.math.cmu.edu wn0g/noll .

[2] Noll, W.: On Material Frame-Indifference, second part of *Five Contributions to Natural Philosophy*, available on the website www.math.cmu.edu wn0g/noll . .

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[GDM] Noll, W. and Sea-Mean Chiou: *The Geometry of Differentiable Manifolds*, preliminary manuscript, available on the website www.math.cmu.edu wn0g/noll .

[TSI] Noll, W.: On the Theory of Surface Interactions, available on the website www.math.cmu.edu wn0g/noll .