Finite Dimesional Spaces Vol.II
by Walter Noll (∼1990)

1 Invariants, Determinants, Covariants.

11. Skew forms, exterior products.

We assume that a linear space $\mathcal{V}$ is given and we put $n := \dim\mathcal{V}$. The members of the space $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ are called **skew** k-**forms**. In view of the remarks made after Prop.1 of Sect.02 and in view of Prop.4 of Sect.02, we have the identification

$$\mathrm{Skew}_0(\mathcal{V}^0, \mathbb{R}) \cong \mathbb{R}, \quad \mathrm{Skew}_1(\mathcal{V}^1, \mathbb{R}) \cong \mathcal{V}^*, \tag{11.1}$$

i.e., the skew 1-forms may be regarded as linear forms.

Let $k \in \mathbb{N}$, a list s of length $k+1$, and $j \in (k+1)^{]}$ be given. We then define the list $\mathrm{del}_j(s)$ of length k by

$$(\mathrm{del}_j(s))_i := \begin{cases} s_i & \text{if} \quad 1 \le i \le j-1 \\ s_{i+1} & \text{if} \quad j \le i \le k \end{cases} . \tag{11.2}$$

Intuitively, $\mathrm{del}_j(s)$ is obtained from s by deleting the j'th term.

Let $k \in \mathbb{N}$ and $\mathbf{A} \in \mathrm{Lin}(\mathcal{V}, \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ be given. We then define $\mathbf{\Lambda}(\mathbf{A}) \in \mathrm{Map}(\mathcal{V}^{k+1}, \mathbb{R})$ by

$$\mathbf{\Lambda}(\mathbf{A})(\mathbf{f}) := \sum_{j \in (k+1)^{]}} (-1)^{k+1-j} \mathbf{A}(\mathbf{f}_j)(\mathrm{del}_j(\mathbf{f})) \tag{11.3}$$

for all $\mathbf{f} \in \mathcal{V}_{k+1}$.

Lemma. *$\mathbf{\Lambda}(\mathbf{A})$ is a skew (k+1)-form, i.e.$\mathbf{\Lambda}(\mathbf{A}) \in \mathrm{Skew}_{k+1}(\mathcal{V}^{k+1}, \mathbb{R})$.*

Proof: It is immediate from the definition (11.3) that $\mathbf{\Lambda}(\mathbf{A})$ is multilinear. Assume now that a $\mathbf{f} \in \mathcal{V}^{k+1}$ with adjacent repeated terms is given. We may then choose $p \in k^{]}$ such that

$$\mathbf{u} := \mathbf{f}_p := \mathbf{f}_{(p+1)}. \tag{11.4}$$

Let $j \in (k+1)^{]}$ be given. If $j \notin \{p, p+1\}$, then $\mathrm{del}_j(\mathbf{f})$ has adjacent repeated terms and hence, by Prop.9 of Sect.11, $\mathbf{A}(\mathbf{f}_j)(\mathrm{del}_j(\mathbf{f})) = 0$. Therefore, (11.3) gives

$$\mathbf{\Lambda}(\mathbf{A})(\mathbf{f}) = (-1)^{k+1-p}\mathbf{A}(\mathbf{u})(\mathrm{del}_p(\mathbf{f})) + (-1)^{k+1-q}\mathbf{A}(\mathbf{u})(\mathrm{del}_q(\mathbf{f})). \tag{11.5}$$

It is clear from (11.4) and the definition (11.2) that $\mathrm{del}_p(\mathbf{f}) = \mathrm{del}_{(p+1)}(\mathbf{f})$. Hence the two terms on the right side of (11.5) cancel and we have $\mathbf{\Lambda}(\mathbf{A})(\mathbf{f}) = 0$. Since the non-injective $\mathbf{f} \in \mathcal{V}_{k+1}$ with adjacent repeated terms was arbitrary, it follows from Prop.9 of Sect.11 that $\mathbf{\Lambda}(\mathbf{A})$ is skew. ∎

In View of the Lemma, we may regard (11.3) as the definition of a mapping

$$\mathbf{\Lambda} : \mathrm{Lin}(\mathcal{V}, \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R}) \longrightarrow \mathrm{Skew}_{k+1}(\mathcal{V}^{k+1}, \mathbb{R}). \tag{11.6}$$

It is clear from (11.3) that $\mathbf{\Lambda}$ is linear. Given $\boldsymbol{\omega} \in \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ and $\boldsymbol{\lambda} \in \mathcal{V}^*$. we have $\mathbf{\Lambda}(\boldsymbol{\omega} \otimes \boldsymbol{\lambda}) \in \mathrm{Skew}_{k+1}(\mathcal{V}^{k+1}, \mathbb{R})$. Hence the following definition is meaningful.

Definition 1. *If $\boldsymbol{\omega}$ is a skew k-form and $\boldsymbol{\lambda}$ a linear form, then the skew (k+1)-form $\boldsymbol{\Lambda}(\boldsymbol{\omega} \otimes \boldsymbol{\lambda})$ is called the* **exterior product** *of $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$, and is denoted by*

$$\boldsymbol{\omega} \wedge \boldsymbol{\lambda} := \boldsymbol{\Lambda}(\boldsymbol{\omega} \otimes \boldsymbol{\lambda}). \tag{11.7}$$

In view of (11.3) we have

$$(\boldsymbol{\omega} \wedge \boldsymbol{\lambda})(\mathbf{f}) = \sum_{j \in (k+1)^{]}} (-1)^{k+1-j} (\boldsymbol{\lambda}\mathbf{f}_j) \boldsymbol{\omega}(\mathrm{del}_j(\mathbf{f})) \tag{11.8}$$

for all $\mathbf{f} \in \mathcal{V}^{k+1}$. It is clear that the mapping

$$((\boldsymbol{\omega}, \boldsymbol{\lambda}) \mapsto \boldsymbol{\omega} \wedge \boldsymbol{\lambda}) : \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R}) \times \mathcal{V}^* \longrightarrow \mathrm{Skew}_{k+1}(\mathcal{V}^{k+1}, \mathbb{R}) \tag{11.9}$$

is bilinear.

If $k = 0$ and $\omega \in \mathrm{Skew}_0(\mathcal{V}^0, \mathbb{R}) \cong \mathbb{R}$, then $\omega \wedge \boldsymbol{\lambda} = \omega\boldsymbol{\lambda}$.

Definition 2. *The* **exterior product** *$\bigwedge \boldsymbol{\phi}$ of a list $\boldsymbol{\phi} := (\boldsymbol{\phi}_i \in \mathrm{Lin}(\mathcal{V}, \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R}) \mid i \in k^{]})$ of length $k \in \mathbb{N}$ in $\mathcal{V}^*$ is defined recursively by $\bigwedge \boldsymbol{\phi} := \bigwedge \emptyset = 1$ if $k = 0$ and*

$$\bigwedge \boldsymbol{\phi} := (\bigwedge \boldsymbol{\phi}|_{(k-1)^{]}}) \wedge \boldsymbol{\phi}_k \quad \text{if} \quad k \geq 1, \tag{11.10}$$

where the right side is defined by Def.1.

Informally, Def.2 states that $\bigwedge \boldsymbol{\phi}$ is given by

$$\bigwedge \boldsymbol{\phi} := (\cdots(\boldsymbol{\phi}_1 \wedge \boldsymbol{\phi}_2) \wedge \boldsymbol{\phi}_3) \wedge \cdots \wedge \boldsymbol{\phi}_{k-1}) \wedge \boldsymbol{\phi}_k. \tag{11.11}$$

We assume now that $n \in \mathbb{N}$ and lists $\mathbf{f} \in \mathcal{V}^n$ and $\boldsymbol{\phi} \in \mathcal{V}^{*n}$ are given such that

$$\boldsymbol{\phi}_i \mathbf{f}_j = \delta_{i,j} = \begin{cases} 0 & \text{if} \quad i \neq j \\ 1 & \text{if} \quad i = j \end{cases} \tag{11.13}$$

for all $i, j \in n^{]}$. Given $k \in \mathbb{N}$, we denote the set of all strictly isotone lists of length k in $n^{]}$ by $\mathrm{Iso}(k, n)$. It is clear that the mapping

$$(s \mapsto \mathrm{Rng}s) : \mathrm{Iso}(k, n) \longrightarrow \mathrm{Fin}_k(n^{]}) \tag{11.14}$$

is invertible and hence that

$$\#\mathrm{Iso}(k, n) = \binom{n}{k} \tag{11.15}$$

(see Sect.05 of Vol.I). For every $s \in \mathrm{Iso}(k, n)$ we define $\boldsymbol{\sigma}_s \in \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ by

$$\boldsymbol{\sigma}_s := \bigwedge(\boldsymbol{\phi} \circ s). \tag{11.16}$$

Informally, by (11.11) we have

$$\boldsymbol{\sigma}_s := \boldsymbol{\phi}_{s_1} \wedge \boldsymbol{\phi}_{s_2} \wedge \cdots \wedge \boldsymbol{\phi}_{s_k}. \tag{11.17}$$

(parentheses are understood.)

Proposition 1. *We have*

$$\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = \delta_{s,t} = \begin{cases} 0 & \textit{if} \quad s \neq t \\ 1 & \textit{if} \quad s = t \end{cases} \tag{11.18}$$

for all $s, t \in \mathrm{Iso}(k, n)$.

In view of (11.17), we may read (11.18) informally as

$$(\boldsymbol{\phi}_{s_1} \wedge \boldsymbol{\phi}_{s_2} \wedge \cdots \wedge \boldsymbol{\phi}_{s_k})(\mathbf{f}_{t_1}, \mathbf{f}_{t_2}, \cdots, \mathbf{f}_{t_k}) = \delta_{s_1,t_1}\delta_{s_2,t_2}\cdots\delta_{s_k,t_k}. \tag{11.19}$$

Proof: We proceed by induction over $k \in \mathbb{N}$. If $k = 0$, then (11.18) is valid because $\boldsymbol{\sigma}_\emptyset = \bigwedge\emptyset = 1$. Assume, then, that $k \geq 1$, and let $s, t \in \mathrm{Iso}(k, n)$ be given. By (11.10) we have $\boldsymbol{\sigma}_s = (\boldsymbol{\sigma}_{s|_{(k-1)]}}) \wedge \boldsymbol{\sigma}_{s_k}$ and hence, by (11.8),

$$\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = \sum_{j\in k]} (-1)^{k-j}(\boldsymbol{\phi}_{s_k}\mathbf{f}_{t_j})\boldsymbol{\sigma}_{s|_{(k-1)]}}(\mathrm{del}_j(\mathbf{f}\circ t)).$$

Since $\boldsymbol{\phi}_{s_k}\mathbf{f}_{t_j} = 0$ when $s_k \neq t_j$, we obtain $\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = 0$ unless $s_k = t_j$ for some $j \in k]$. In this last case, we get

$$\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = (-1)^{k-j}\boldsymbol{\sigma}_{s|_{(k-1)]}}(\mathbf{f}\circ \mathrm{del}_j(t)). \tag{11.20}$$

By the induction hypothesis, $\boldsymbol{\sigma}_{s|_{(k-1)]}}(\mathbf{f}\circ \mathrm{del}_j(t))$ and hence $\boldsymbol{\sigma}_s(\mathbf{f}\circ t)$ is zero except when $s|_{(k-1)]} = \mathrm{del}_j(t)$.

We conclude that $\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = 0$ except when $s_k = t_j$ for some $j \in k]$ and $s|_{(k-1)]} = \mathrm{del}_j(t)$. Suppose that this is the case. Since s is strictly isotone, we then have $s_k > t_j$ for all $i \in k]\backslash\{j\}$ and hence $s_k = t_j = \max\{t_i|\ i \in k]\}$. Since t is strictly isotone, we conclude that $j = k$ and hence $s = t$. Thus, it follows that $\boldsymbol{\sigma}_s(\mathbf{f}\circ t) = 0$ except when $s = t$, in which case, by (11.20),

$$\boldsymbol{\sigma}_s(\mathbf{f}\circ s) = \boldsymbol{\sigma}_{s|_{(k-1)]}}(\mathbf{f}\circ \mathrm{del}_k(s)) = \boldsymbol{\sigma}_{s|_{(k-1)]}}((\mathbf{f}\circ s)|_{(k-1)]}),$$

which equals 1 by the induction hypothesis. ∎

12. Bases in spaces of skew forms.

We assume again that a linear space $\mathcal{V}$ is given and we put $n := \dim\mathcal{V}$. We assume, also, that a list-basis $\mathbf{b} := (\mathbf{b}_i \mid i \in n])$ of $\mathcal{V}$ is given. We also will make use of the dual basis $\mathbf{b}^* := ({\mathbf{b}^*}_i \mid i \in n])$ of $\mathbf{b}$, which is a basis of the dual space $\mathcal{V}^*$ (see Sect.23 of Vol.I.). We also let $k \in \mathbb{N}$ be given.

Theorem on Bases of $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$**.** *The family* $\boldsymbol{\sigma} := (\boldsymbol{\sigma}^s \mid s \in \mathrm{Iso}(k,n))$ *in* $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ *defined by*

$$\boldsymbol{\sigma}_s := \bigwedge(\mathbf{b}^*\circ s) \quad \text{for all} \quad s \in \mathrm{Iso}(k,n). \tag{13.1}$$

is a basis of $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$.

The proof will depend on the following

Lemma. *Let $\boldsymbol{\rho} \in \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ be given. If $\boldsymbol{\rho}(\mathbf{b} \circ s) = 0$ for all $s \in \mathrm{Iso}(k, n)$, then $\boldsymbol{\rho} = 0$.*

Proof: We proceed by induction over $k \in \mathbb{N}$. The assertion is trivial when $k = 0$. Assume, then, that $k \geq 1$, that $\boldsymbol{\rho}$ satisfies the condition, and that the assertion of the Lemma becomes valid after k has been replaced by $k - 1$.

Using the identification $\mathcal{V}^k \cong \mathcal{V}^{k-1} \times \mathcal{V}$ we define, for each $j \in n^{]}$, the (k-1)-form $\boldsymbol{\rho}_j$ by

$$\boldsymbol{\rho}_j := \boldsymbol{\rho}(\mathbf{f}, \mathbf{b}_j) \quad \text{for all} \quad \mathbf{f} \in \mathcal{V}^{k-1}.$$

Let $j \in n^{]}$ and $t \in \mathrm{Iso}(k-1, n)$ be given. If $j \in \mathrm{Rng}\, t$ then $\boldsymbol{\rho}_j(\mathbf{b} \circ t) = \boldsymbol{\rho}(\mathbf{b} \circ t, \mathbf{b}_j) = 0$ because the last term $\mathbf{b}_j$ of $(\mathbf{b} \circ t, \mathbf{b}_j) \in \mathcal{V}^k$ coincides with one of the other terms (see Prop.7 of Sect.11). If $j \notin \mathrm{Rng}\, t$ then

$$\boldsymbol{\rho}_j(\mathbf{b} \circ t) = \boldsymbol{\rho}(\mathbf{b} \circ t, \mathbf{b}_j) = \pm \boldsymbol{\rho}(\mathbf{b} \circ s) = 0,$$

where s is the strictly isotone list of length k obtained from the list t of length $k - 1$ by inserting j at the appropriate place. It follows that $\boldsymbol{\rho}_j(\mathbf{b} \circ t) = 0$ for all $j \in n^{]}$ and $t \in \mathrm{Iso}(k - 1, n)$. By the induction hypothesis, we conclude that $\boldsymbol{\rho}_j = 0$ for all $j \in n^{]}$.

Now let $\mathbf{g} \in \mathcal{V}^k$ be given. Since $\mathbf{b}$ spans $\mathcal{V}$, we may choose $\lambda \in \mathbb{R}^n$ such that

$$\mathbf{g}_k = \sum_{j \in n^{]}} \lambda_j \mathbf{b}_j.$$

Since $\boldsymbol{\rho}(\mathbf{g}.k) : \mathcal{V} \longrightarrow \mathbb{R}$ is linear, we obtain

$$\boldsymbol{\rho}(\mathbf{g}) = \boldsymbol{\rho}(\mathbf{g}.k)(\mathbf{g}_k) = \sum_{j \in n^{]}} \lambda_j \boldsymbol{\rho}(\mathbf{g}.k)(\mathbf{b}_j) =$$

$$\sum_{j \in n^{]}} \lambda_j \boldsymbol{\rho}(\mathbf{g}|_{(k-1)^{]}}, \mathbf{b}_j) = \sum_{j \in n^{]}} \lambda_j \boldsymbol{\rho}(\mathbf{g}|_{(k-1)^{]}}).$$

Since $\boldsymbol{\rho}_j = 0$ for all $j \in n^{]}$, as proved above, it follows that $\boldsymbol{\rho} = 0$. ∎

Proof of Theorem: Let $\boldsymbol{\omega} \in \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ be given. We must show that the equation

$$? \ \lambda \in \mathbb{R}^{\mathrm{Iso}(k,n)}, \qquad \sum_{s \in \mathrm{Iso}(k,n)} \lambda_s \boldsymbol{\sigma}_s = \boldsymbol{\omega} \tag{13.2}$$

has exactly one solution.

If λ is a solution of (13.2), it follows from Prop.1 of Sect.11 that

$$\boldsymbol{\omega}(\mathbf{b} \circ t) = \sum_{s \in \mathrm{Iso}(k,n)} \lambda_s \boldsymbol{\sigma}_s(\mathbf{b} \circ t) = \lambda_t$$

for all $t \in \mathrm{Iso}(k, n)$, which shows that we must have

$$\lambda = (\boldsymbol{\omega}(\mathbf{b} \circ t) \mid t \in \mathrm{Iso}(k, n)). \tag{13.3}$$

Now let $\lambda \in \mathbb{R}^{\mathrm{Iso}(k,n)}$ be *defined* by (13.3) and put

$$\boldsymbol{\rho} := \boldsymbol{\omega} - \sum_{s \in \mathrm{Iso}(k,n)} \lambda_s \boldsymbol{\sigma}_s \ \in \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R}).$$

Using Prop.1 of Sect.11 again we see that

$$\boldsymbol{\rho}(\mathbf{b} \circ t) := \lambda_t - \sum_{s \in \mathrm{Iso}(k,n)} \lambda_s \sigma_s(\mathbf{b} \circ t) = \lambda_t - \lambda_t = 0$$

for all $t \in \mathrm{Iso}(k,n)$. It follows from the Lemma that $\boldsymbol{\rho} = 0$ and hence that λ, as defined by (13.3), is indeed a solution of (13.2). ∎

In view of (11.15) we have the following

Corollary 1. *We have*

$$\dim \mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R}) = \binom{\dim\mathcal{V}}{k}. \tag{13.4}$$

In particular, $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ *is a zero-space when* $k > \dim\mathcal{V}$*.*

Since $\mathrm{Iso}(n,n) = \{n^]\}$, the Theorem for $k := n$ reduces to the following

Corollary 2. *We have*

$$\mathrm{Skew}_n(\mathcal{V}^n, \mathbb{R}) = \mathbb{R}(\bigwedge \mathbf{b}^*) \tag{13.5}$$

and

$$(\bigwedge \mathbf{b}^*)(\mathbf{b}) = 1. \tag{13.6}$$

Proposition 1. *Let* $\boldsymbol{\phi} \in \mathcal{V}^{*k}$ *be given. Then* $\bigwedge \boldsymbol{\phi} = 0$ *if and only if* $\boldsymbol{\phi}$ *is linearly dependent.*

Proof: Put $\mathcal{A} := \mathrm{Lsp}\,\mathrm{Rng}\boldsymbol{\phi}$, which is a subspace of $\mathcal{V}^*$. We then have $\boldsymbol{\phi} \in \mathcal{A}^k$. Using the identification $\mathcal{A} \cong \mathcal{A}^{**}$ (see Sect.22 of Vol.I), we may consider $\bigwedge \boldsymbol{\phi}$ not only as an element of $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$ but also as an element of $\mathrm{Skew}_k(\mathcal{A}^{*k}, \mathbb{R})$. The latter interpretation is related to the former by

$$(\bigwedge \boldsymbol{\phi})(\hat{\mathbf{f}}) = (\bigwedge \boldsymbol{\phi})(\mathbf{f}) \quad \text{for all} \quad \mathbf{f} \in \mathcal{V}^k \tag{13.7}$$

when $\hat{\mathbf{f}} \in \mathcal{A}^{*k}$ is defined by

$$\hat{\mathbf{f}}_j := \mathbf{f}_j|_{\mathcal{A}} \quad \text{for all} \quad j \in k^],$$

where $\mathbf{f}_j$ on the right must be interpreted as an element of $\mathcal{V}^{**}$. Since the mapping $(\mathbf{u} \mapsto \mathbf{u}|_{\mathcal{A}}) : \mathcal{V}^{**} \longrightarrow \mathcal{A}^*$ is surjective (see Prop.6 of Sect.21 of Vol.I), it follows from (13.7) that $\bigwedge \boldsymbol{\phi}$, when regarded as an elemant of $\mathrm{Skew}_k(\mathcal{A}^{*k}, \mathbb{R})$, is zero if and only if it is zero when regarded as an element of $\mathrm{Skew}_k(\mathcal{V}^k, \mathbb{R})$.

Assume that $\boldsymbol{\phi}$ is linearly dependent. Then $\dim\mathcal{A}^* = \dim\mathcal{A} = \dim \mathrm{Lsp}\,\mathrm{Rng}\boldsymbol{\phi} < k$. It follows from Cor.1 that $\mathrm{Skew}_k(\mathcal{A}^{*k}, \mathbb{R})$ is a zero-space and hence that $\bigwedge \boldsymbol{\phi} = 0$.

Assume that $\boldsymbol{\phi}$ is linearly independent. Then $\boldsymbol{\phi}$ is a basis of $\mathcal{A} \cong \mathcal{A}^{**}$ and hence the dual of a basis $\boldsymbol{\phi}^*$ of $\mathcal{A}^*$. It follows from Cor.2, applied to the case when $\mathcal{V}$ is replaced by $\mathcal{A}^*$, n by k, and $\mathbf{b}$ by $\boldsymbol{\phi}^*$, that $(\bigwedge \boldsymbol{\phi})(\boldsymbol{\phi}^*) = 1$ and hence that $\bigwedge \boldsymbol{\phi} \neq \mathbf{0}$. ∎

Proposition 2. *We have*

$$(\bigwedge \boldsymbol{\phi})(\mathbf{f}) = (\bigwedge \mathbf{f})(\boldsymbol{\phi}) \tag{13.8}$$

for all $\mathbf{f} \in \mathcal{V}^n$ *and all* $\boldsymbol{\phi} \in \mathcal{V}^{*n}$.

Proof: Let $\mathbf{f} \in \mathcal{V}^n$ be given. If $\mathbf{f}$ is linearly dependent, then both sides of (13.8) are zero by Prop.1 above and Prop.7 of Sect.11.

Assume, then, that $\mathbf{f}$ is linearly independent and hence a basis of $\mathcal{V}$. It is clear from Defs.1 and 2 of Sect.11 , from the linearity of the mapping Λ defined be (11.4), and the bilinearity of the tensor-product that the mapping

$$(\boldsymbol{\phi} \mapsto (\bigwedge \boldsymbol{\phi})(\mathbf{f})) : \mathcal{V}^{*n} \longrightarrow \mathbb{R} \tag{13.9}$$

is multilinear. Since a given $\boldsymbol{\phi} \in \mathcal{V}^{*n}$ is linearly independent if it fails to be injective, it follows from Prop.1 above and Prop.8 of Sect.11 that the mapping (13.9) is skew and hence belongs to $\mathrm{Skew}_k(\mathcal{V}^{*k}, \mathbb{R})$. By Cor.2 above, applied to the case when $\mathcal{V}$ is replaced by $\mathcal{V}^*$ and $\mathbf{b}$ by $\mathbf{f}^*$, it follows that the mapping (13.9) is a scalar multiple of $\bigwedge \mathbf{f}$. But, in view of (13.6), both the mapping (13.9) and $\bigwedge \mathbf{f}$ have the value 1 at $\mathbf{f}^*$. Hence $\bigwedge \mathbf{f}$ is the same as the mapping (13.9), which shows that (13.8) holds for all $\boldsymbol{\phi} \in \mathcal{V}^{*n}$. ∎

Proposition 3. *Let* $\mathcal{V}$ *and* $\mathcal{W}$ *be linear spaces and let* $\mathbf{L} \in \mathrm{Lin}(\mathcal{W}, \mathcal{V})$ *be given. For every* $k \in \mathbb{N}$ *we then have*

$$(\bigwedge \boldsymbol{\phi})(\mathbf{L}^{\times k}\mathbf{g}) = \bigwedge((\mathbf{L}^\top)^{\times k}\boldsymbol{\phi})(\mathbf{g}) \tag{13.10}$$

for all $\boldsymbol{\phi} \in \mathcal{V}^{*k}$ *and all* $\mathbf{g} \in \mathcal{W}^k$.

Proof: We proceed by induction over $k \in \mathbb{N}$. For $k = 0$ (13.10) reduces to $1 = 1$. Assume, then, that $k \in \mathbb{N}$ with $k \geq 1$ is given and that the assertion becomes valid after k has been replaced by $k - 1$.

Given $\boldsymbol{\phi} \in \mathcal{V}^{*k}$ and $\mathbf{g} \in \mathcal{W}^k$, we infer from (11.10) and (11.8) that

$$\begin{aligned}(\bigwedge \boldsymbol{\phi})(\mathbf{L}^{\times k}\mathbf{g}) &= ((\bigwedge \boldsymbol{\phi}|_{(k-1)^]}) \wedge \boldsymbol{\phi}_k)(\mathbf{L}^{\times k}\mathbf{g}) = \\ &= \sum_{j \in k^]} (-1)^{k-j} (\boldsymbol{\phi}_k(\mathbf{L}\mathbf{g}_j))(\bigwedge \boldsymbol{\phi}|_{(k-1)^]}(\mathrm{del}_j(\mathbf{L}^{\times k}\mathbf{g})).\end{aligned} \tag{13.11}$$

Let $j \in k^]$ be given. Since $\mathrm{del}_j(\mathbf{L}^{\times k}\mathbf{g}) = \mathbf{L}^{\times(k-1)}\mathrm{del}_j(\mathbf{g})$, the induction hypothesis yields

$$\bigwedge \boldsymbol{\phi}|_{(k-1)^]}(\mathrm{del}_j(\mathbf{L}^{\times k}\mathbf{g})) = \bigwedge((\mathbf{L}^\top)^{\times(k-1)}\boldsymbol{\phi}|_{(k-1)^]})(\mathrm{del}_j\mathbf{g}).$$

Using this result and the fact that $\boldsymbol{\phi}_k(\mathbf{L}\mathbf{g}_j) = (\mathbf{L}^\top\boldsymbol{\phi}_k)\mathbf{g}_j$, we conclude from (13.11) that

$$(\bigwedge \boldsymbol{\phi})(\mathbf{L}^{\times k}\mathbf{g}) = \sum_{j \in k^]} (-1)^{(k-j)}((\mathbf{L}^\top\boldsymbol{\phi}_k)\mathbf{g}_j)((\mathbf{L}^\top)^{\times k}\boldsymbol{\phi})|_{(k-1)^]}(\mathrm{del}_j\mathbf{g}).$$

Using (11.8) and (11.10) again, we obtain the desired result (13.10). ∎

14. Determinants

We assume again that a linear space $\mathcal{V}$ is given and we put $n := \dim\mathcal{V}$.

Theorem on Characterization of the Determinant. *There is exactly one function* $\det : \operatorname{Lin}\mathcal{V} \longrightarrow \mathbb{R}$ *that satisfies*

$$\boldsymbol{\omega} \circ \mathbf{L}^{\times n} = \det(\mathbf{L})\boldsymbol{\omega} \tag{14.1}$$

for all $\boldsymbol{\omega} \in \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R})$ *and all* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$. *We call* det *the* **determinant-function** *for* $\mathcal{V}$ *and its value* $\det(\mathbf{L})$ *at a given* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *the* **determinant** *of* $\mathbf{L}$.

Proof: Let $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ and $\boldsymbol{\omega} \in \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R})$ be given.In view of Prop.3 of Sect.11, we have $\boldsymbol{\omega} \circ \mathbf{L}^{\times n} \in \operatorname{Lin}_n(\mathcal{V}^n, \mathbb{R})$ and, since

$$(\boldsymbol{\omega} \circ \mathbf{L}^{\times n})^{\sim(p,q)} = \boldsymbol{\omega}^{\sim(p,q)} \circ \mathbf{L}^{\times n} = -\boldsymbol{\omega} \circ \mathbf{L}^{\times n}$$

for all $p, q \in n^{]}$ with $p \neq q$, we have $\boldsymbol{\omega} \circ \mathbf{L}^{\times n} \in \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R})$ (see Def.2 of Sect.11). Therefore, since $\boldsymbol{\omega} \circ \mathbf{L}^{\times n} \in \operatorname{Lin}_n(\mathcal{V}^n, \mathbb{R})$ was arbitrary, we may consider the mapping

$$(\boldsymbol{\omega} \mapsto \boldsymbol{\omega} \circ \mathbf{L}^{\times n}) : \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R}) \longrightarrow \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R}). \tag{14.2}$$

In view of Prop.1 of Sect.14 of Vol.I, this mapping is linear. Now, by Cor.1 of Sect.13, we have $\dim \operatorname{Skew}_n(\mathcal{V}^n, \mathbb{R}) = 1$. It follows that the mapping (14.2) must be scalar multiplication with a number in $\mathbb{R}$, which we denote by $\det(\mathbf{L})$. ∎

Let I be any finite index set with $\#I = n$. By enumerating I, it follws from the Theorem just proved that

$$\boldsymbol{\omega} \circ \mathbf{L}^{\times I} = \det(\mathbf{L})\boldsymbol{\omega} \tag{14.3}$$

holds for all $\boldsymbol{\omega} \in \operatorname{Skew}_I(\mathcal{V}^I, \mathbb{R})$ and all $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$.

Of course, the determinant function depends on $\mathcal{V}$. We write $\det^{\mathcal{V}}$ instead of det when we wish to emphasize this fact.

Pitfall: the determinant function det is *not* linear except when $n = 1$. ∎

It is evident from (14.1) that

$$\det(\mathbf{1}_{\mathcal{V}}) = 1. \tag{14.4}$$

Basic Rules for the Determinant. *Let* $s \in \mathbb{R}$ *and* $\mathbf{L}, \mathbf{M} \in \operatorname{Lin}\mathcal{V}$ *be given. We then have*

$$\det(s\mathbf{L}) = s^n \det(\mathbf{L}), \tag{14.5}$$

$$\det(\mathbf{LM}) = \det(\mathbf{L})\det(\mathbf{M}), \tag{14.6}$$

$$\mathbf{L} \ \text{ is invertible if and only if } \det(\mathbf{L}) \neq 0, \tag{14.7}$$

$$\det(\mathbf{L}^{-1}) = \det(\mathbf{L})^{-1} \ \text{ if } \ \mathbf{L} \in \operatorname{Lis}\mathcal{V}, \tag{14.8}$$

$$\det^{\mathcal{V}^*}(\mathbf{L}^\top) = \det^{\mathcal{V}}(\mathbf{L}). \tag{14.9}$$

Proof: The assertion (14.5) follows from (14.1) because $\boldsymbol{\omega}\circ(s\mathbf{L})^{\times n} = s^n(\boldsymbol{\omega}\circ\mathbf{L}^{\times n})$ for all $\boldsymbol{\omega} \in \mathrm{Lin}_n(\mathcal{V}^n, \mathbb{R})$

The assertion (14.6) follows from (14.1) because $(\mathbf{LM})^{\times n} = \mathbf{L}^{\times n}\mathbf{M}^{\times n}$.

If $\mathbf{L}$ is invertible, then (14.6) with $\mathbf{M} := \mathbf{L}^{-1}$ and (14.4) yield

$$\det(\mathbf{L})\det(\mathbf{L}^{-1}) = \det(\mathbf{L}\mathbf{L}^{-\mathbf{1}}) = \det(\mathbf{1}_{\mathcal{V}}) = 1 \tag{14.10}$$

and hence $\det(\mathbf{L}) \neq 0$.

We now choose a list-basis $\mathbf{b}$ of $\mathcal{V}$. Using (14.1) with $\omega := \bigwedge \mathbf{b}^*$, it follows from (12.6) that

$$\det(\mathbf{L}) = \det(\mathbf{L})(\bigwedge \mathbf{b}^*)(\mathbf{b}) = (\bigwedge \mathbf{b}^*)(\mathbf{L}^{\times n}\mathbf{b}). \tag{14.11}$$

Now, if $\mathbf{L}$ fails to be invertible, then $\mathbf{L}^{\times n}\mathbf{b}$ is linearly dependent (see Prop.2 of Sect.16 of Vol.I). Hence, by Prop.2 of Sect.11 , the right side of (14.11) is zero and so we have $\det(\mathbf{L}) = 0$, which completes the proof of (14.7).

The assertion (14.8) is an immediate consequence of (14.10).

To prove (14.9), we apply Props.3 and 2 of Sect.13 to the right side of (14.11) and obtain

$$\det(\mathbf{L}) = (\bigwedge \mathbf{b}^*)(\mathbf{L}^{\times n}\mathbf{b}) = (\bigwedge(\mathbf{L}^\top)^{\times n}\mathbf{b}^*))(\mathbf{b}) = (\bigwedge \mathbf{b})((\mathbf{L}^\top)^{\times n}\mathbf{b}^*).$$

Using (14.11) with $\mathcal{V}$ replaced by $\mathcal{V}^*$ and $\mathbf{L}$ replaced by $\mathbf{L}^\top$ and with $\mathbf{b}$ and $\mathbf{b}^*$ interchanged, we arrive at the desired result (14.9). ▮

Proposition 1. *Let a linear space $\mathcal{V}'$ and a linear isomorphism $\mathbf{A} : \mathcal{V} \longrightarrow \mathcal{V}'$ be given. Then*

$$\det^{\mathcal{V}'}(\mathbf{A}\mathbf{L}\mathbf{A}^{-\mathbf{1}}) = \det^{\mathcal{V}}(\mathbf{L}) \quad \text{for all } \ \mathbf{L} \in \mathrm{Lin}\mathcal{V}. \tag{14.11}$$

Proof: By Cor.2 of the Theorem on Characterisation of Dimension of Vol.I we have $\dim\mathcal{V}' = \dim\mathcal{V} = n$. Let $\boldsymbol{\omega} \in \mathrm{Skew}_n(\mathcal{V}^n, \mathbb{R})$ be given. Noting that $\boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times n} \in \mathrm{Skew}_n(\mathcal{V}'^n, \mathbb{R})$, we apply (14.1) and then (14.1) again with $\mathcal{V}$, $\mathbf{L}$, and $\boldsymbol{\omega}$ replaced by $\mathcal{V}'$, $\mathbf{A}\mathbf{L}\mathbf{A}^{-1}$, and $\boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times n}$ and obtain

$$\det^{\mathcal{V}}(\mathbf{L})\boldsymbol{\omega} = \boldsymbol{\omega} \circ \mathbf{L}^{\times n} = (\boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times n}) \circ (\mathbf{A}\mathbf{L}\mathbf{A}^{-1})^{\times n} \circ \mathbf{A}^{\times n} =$$

$$= \det^{\mathcal{V}'}(\mathbf{A}\mathbf{L}\mathbf{A}^{-\mathbf{1}})(\boldsymbol{\omega} \circ (\mathbf{A}^{-1})^{\times n}) \circ \mathbf{A}^{\times n} = \det^{\mathcal{V}'}(\mathbf{A}\mathbf{L}\mathbf{A}^{-\mathbf{1}})(\boldsymbol{\omega}).$$

Since $\boldsymbol{\omega} \in \mathrm{Skew}_n(\mathcal{V}^n, \mathbb{R})$ was arbitrary, (14.11) follows. ▮

Now let a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ be given. If we apply Prop.1 to the case when $\mathbf{A} := \mathrm{lnc}_{\mathbf{b}}{}^{-1}$, we see that

$$\det(\mathbf{L}) = \det([\mathbf{L}]_{\mathbf{b}}) \ \text{ for all } \ \mathbf{L} \in \mathrm{Lin}\mathcal{V}, \tag{14.13),}$$

where $[\mathbf{L}]_{\mathbf{b}}$ is the matrix of $\mathbf{L}$ relative to $\mathbf{b}$ (see (18.6) of Vol.I).

We assume now that a finite index set I, a subset J of I, and $\mathbf{h} \in \mathcal{V}^{I \setminus J}$ are given. Note that for every $\boldsymbol{\omega} \in \mathrm{Skew}_I(\mathcal{V}^I, \mathbb{R})$ we have $\boldsymbol{\omega}(\mathbf{h}.J) \in \mathrm{Skew}_J(\mathcal{V}^{I \setminus J}, \mathbb{R})$ (see Sect.01).

Lemma. *Let a subspace $\mathcal{U}$ of $\mathcal{V}$, a projection $\mathbf{P} : \mathcal{V} \longrightarrow \mathcal{W}$ to a subspace $\mathcal{W}$ of $\mathcal{V}$ be given such that* $\text{Null}\,\mathbf{P} = \mathcal{U}$ *(see Sect.19 of Vol.I) be given. Also, let* $\boldsymbol{\omega} \in \text{Skew}_I(\mathcal{V}^I, \mathbb{R})$ *and* $\mathbf{g} \in \mathcal{V}^J$ *be given. Assume that* $\mathcal{U} \subset \text{Lsp}(\text{Rng}\mathbf{h})$. *Then*

$$\boldsymbol{\omega}(\mathbf{h}.J)(\mathbf{g}) = \omega(\mathbf{h}.J)(\mathbf{P}^{\times J}\mathbf{g}) \tag{14.14}$$

Proof: We proceed by induction over $\#J$. The assertion is trivial when $\#J = 0$. Assume, then, that $\#J > 0$ and that the assertion becomes valid after J has been replaced by a subset J' having one element less.

Put $K := I \backslash J$, choose $j \in J$ and put $J' := J \backslash \{j\}$ and $K' := I \backslash J' = K \cup \{j\}$. Define $\mathbf{h}' \in \mathcal{V}^{K'}$ by

$$\mathbf{h}'_i := \begin{cases} \mathbf{h}_i & \text{if } \; i \in K \\ \mathbf{g}_j & \text{if } \; i = j \end{cases} \quad \text{for all } \; i \in K'. \tag{14.15}$$

Then $\text{Rng}\mathbf{h} \subset \text{Rng}\mathbf{h}'$ and hence $\mathcal{U} \subset \text{Lsp}(\text{Rng}\mathbf{h}')$. By the induction hypothesis, we have

$$\boldsymbol{\omega}(\mathbf{h}.J)(\mathbf{g}) = \boldsymbol{\omega}(\mathbf{h}'.J')(\mathbf{g}|_{J'}) = \boldsymbol{\omega}(\mathbf{h}'.J')(\mathbf{P}^{\times J'}\mathbf{g}|_{J'}). \tag{14.16}$$

We now define $\mathbf{f} \in \mathcal{V}^I$ by $\mathbf{f}|_{K'} := \mathbf{h}'$ and $\mathbf{f}|_{J'} := \mathbf{P}^{\times J'}\mathbf{g}|_{J'}$. In view of (14.15) we then have $\mathbf{f}_j = \mathbf{h}'_j = \mathbf{g}_j$ and hence

$$\boldsymbol{\omega}(\mathbf{h}'.J')(\mathbf{P}^{\times J'}\mathbf{g}|_{J'}) = \boldsymbol{\omega}(f) = \boldsymbol{\omega}(f.j)(\mathbf{g}_j). \tag{14.17}$$

Since $\boldsymbol{\omega}(f.j) : \mathcal{V} \longrightarrow \mathbb{R}$ is linear we have

$$\boldsymbol{\omega}(f.j)(\mathbf{g}_j) = \boldsymbol{\omega}(f.j)(\mathbf{P}\mathbf{g}_j) + \boldsymbol{\omega}(\mathbf{f}.j)(\mathbf{g}_j - \mathbf{P}\mathbf{g}_j). \tag{14.18}$$

Since $\mathbf{g}_j - \mathbf{P}\mathbf{g}_j \in \text{Null}\mathbf{P} = \mathcal{U}$ and since $\mathbf{h} = (\mathbf{f}|_{I\backslash\{j\}})|_K$, we have $\mathcal{U} \subset \text{Lsp}(\text{Rng}\mathbf{h}) \subset \text{Lsp}(\text{Rng}(\mathbf{f}|_{I\backslash\{j\}})$ and hence $\mathbf{g}_j - \mathbf{P}\mathbf{g}_j \in \text{Lsp}(\text{Rng}(\mathbf{f}|_{I\backslash\{j\}})$. It follows from Prop.8 of Sect.15 of Vol.I that $(\mathbf{f}.j)(\mathbf{g}_j - \mathbf{P}\mathbf{g}_j) \in \mathcal{V}^I$ is a linearly dependent family. By Prop.7 of Sect.11 we conlude that the second term on the right side of (14.18) is zero and hence, by (14.17) and (14.16), that $\boldsymbol{\omega}(\mathbf{f}\,.j)(\mathbf{P}\mathbf{g}_j) = \boldsymbol{\omega}(\mathbf{h}\,.J)(\mathbf{g})$. Since $\mathbf{f}|_K = \mathbf{h}$ and $(\mathbf{f}|_J.j)(\mathbf{P}\mathbf{g}_j) = (\mathbf{P}^{\times J'}\mathbf{g}|_{J'}.j)(\mathbf{P}\mathbf{g}_j) = \mathbf{P}^{\times J}\mathbf{g}$, we have $(\mathbf{f}.j)(\mathbf{P}\mathbf{g}_j) = (\mathbf{h}.J)(\mathbf{P}^{\times J}\mathbf{g})$ and thus the desired conclusion (14.14). ∎

Proposition 2. *Let* $\mathbf{L} \in \text{Lin}\mathcal{V}$, *an* $\mathbf{L}$*-subspace* $\mathcal{U}$ *of* $\mathcal{V}$, *and a projection* $\mathbf{P} : \mathcal{V} \longrightarrow \mathcal{W}$ *to a subspace* $\mathcal{W}$ *of* $\mathcal{V}$ *be given such that* $\text{Null}\,\mathbf{P} = \mathcal{U}$. *(See Def.1 of Sect.18 and Def.1 of Sect 19 of Vol.I.) Then*

$$\det{}^{\mathcal{V}}(\mathbf{L}) = \det{}^{\mathcal{U}}(\mathbf{L}_{|\mathcal{U}})\det{}^{\mathcal{W}}(\mathbf{P}\mathbf{L}|_{\mathcal{W}}), \tag{14.19}$$

Proof: We choose a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ of $\mathcal{V}$ such that $\mathbf{b}|_K$ is a basis of $\mathcal{U}$ and $\mathbf{b}|_J$ is a basis of $\mathcal{W}$ for suitable subsets J and K of I such that $K = I \backslash J$.

Let $\boldsymbol{\omega} \in \text{Skew}_I(\mathcal{V}^I, \mathbb{R})$ be given. By (14.3) we then have

$$\det{}^{\mathcal{V}}(\mathbf{L})\boldsymbol{\omega}(\mathbf{b}) = \boldsymbol{\omega}(\mathbf{L}^{\times I}\mathbf{b}) = \boldsymbol{\omega}(\mathbf{L}^{\times J}\mathbf{b}|_J\,.K)(\mathbf{L}^{\times K}\mathbf{b}|_K). \tag{14.20}$$

Since $\mathcal{U}$ is $\mathbf{L}$-invariant and since $\mathbf{b}|_K \in \mathcal{U}^K$, we have $\mathbf{L}^{\times K}\mathbf{b}|_K = (\mathbf{L}_{|\mathcal{U}})^{\times K}\mathbf{b}|_K \in \mathcal{U}^K$. Since $\boldsymbol{\omega}(\mathbf{L}^{\times J}\mathbf{b}|_J .K)|_{\mathcal{U}^K} \in \mathrm{Skew}_K(\mathcal{U}^K, \mathbb{R})$, it follows from (14.3) and (14.20) that

$$\det{}^{\mathcal{V}}(\mathbf{L})\boldsymbol{\omega}(\mathbf{b}) = \det{}^{\mathcal{U}}(\mathbf{L}_{|\mathcal{U}})\boldsymbol{\omega}(\mathbf{L}^{\times J}\mathbf{b}|_J .K)(\mathbf{b}|_K). \tag{14.21}$$

It is clear that $(\mathbf{L}^{\times J}\mathbf{b}|_J .K)(\mathbf{b}|_K) = (\mathbf{b}|_K .J)(\mathbf{L}^{\times J}\mathbf{b}|_J)$. Therefore, since $\mathrm{Null}\mathbf{P} = \mathcal{U} = \mathrm{Lsp}\,\mathrm{Rng}\mathbf{b}|_K$, we can apply the Lemma with $\mathbf{h} := \mathbf{b}|_K$ and $\mathbf{g} := (\mathbf{L}^{\times J}\mathbf{b}|_J)$ and infer from (14.21) that

$$\det{}^{\mathcal{V}}(\mathbf{L})\boldsymbol{\omega}(\mathbf{b}) = \det{}^{\mathcal{U}}(\mathbf{L}_{|\mathcal{U}})\boldsymbol{\omega}(\mathbf{b}|_K .J)(\mathbf{P}^{\times J}\mathbf{L}^{\times J}\mathbf{b}|_J). \tag{14.22}$$

Since $\mathbf{PL}|_{\mathcal{W}} \in \mathrm{Lin}\mathcal{W}$, since $\mathbf{b}|_J \in \mathcal{W}^J$, since $\mathbf{P}^{\times J}\mathbf{L}^{\times J}\mathbf{b}|_J = (\mathbf{PL}|_{\mathcal{W}})^{\times J}\mathbf{b}|_J$, and since $\boldsymbol{\omega}(\mathbf{b}|_K .J)|_{\mathcal{W}^J} \in \mathrm{Skew}_J(\mathcal{W}^J, \mathbb{R})$, we can apply (14.3) again and conclude from (14.22) that

$$\det{}^{\mathcal{V}}(\mathbf{L})\boldsymbol{\omega}(\mathbf{b}) = \det{}^{\mathcal{U}}(\mathbf{L}_{|\mathcal{U}})\det{}^{\mathcal{W}}(\mathbf{PL}_{|\mathcal{W}})\boldsymbol{\omega}(\mathbf{b}|_K .J)(\mathbf{b}|_J). \tag{14.23}$$

Since $\boldsymbol{\omega}(\mathbf{b}) = \boldsymbol{\omega}(\mathbf{b}|_K.J)(\mathbf{b}|_J)$, the desired result follows from (14.23) when we put $\boldsymbol{\omega} := \bigwedge \mathbf{b}^*$ and observe (12.6). ∎

Proposition 3. *Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and an $\mathbf{L}$-decomposition $(\mathcal{U}_i \mid i \in I)$ of $\mathcal{V}$, i. e., a decomposition of $\mathcal{V}$ whose terms are $\mathbf{L}$-invariant, be given (see Def.1 of Sect.81 of Vol.I). Then*

$$\det{}^{\mathcal{V}}(\mathbf{L}) = \prod_{i\in I} \det{}^{\mathcal{U}_i}(\mathbf{L}_{|\mathcal{U}_i}). \tag{14.24}$$

Proof: We proceed by induction over $\#I$. The assertion is trivial when $\#I = 1$. Assume that $\#I > 1$ and that the assertion becomes valid after I has been replaced by an index set having one less element.

We choose $j \in I$ and put $I' := I\backslash\{j\}$. By Prop.2 of Sect.81 of Vol.I, $\mathcal{W} := \sum(\mathcal{U}_i \mid i \in I')$ is a supplement of $\mathcal{U}_j$. It is clear that $\mathcal{W}$ is $\mathbf{L}$-invariant and hence that $\mathbf{PL}|_{\mathcal{W}} = \mathbf{L}_{|\mathcal{W}}$ when $\mathbf{P}$ is the projection of $\mathcal{V}$ on $\mathcal{W}$ with $\mathrm{Null}\mathbf{P} = \mathcal{U}_j$. Using Prop.2, we conclude that

$$\det{}^{\mathcal{V}}(\mathbf{L}) = \det{}^{\mathcal{U}_j}(\mathbf{L}_{|\mathcal{U}_j})\det{}^{\mathcal{W}}(\mathbf{L}|_{\mathcal{W}}).$$

Applying the induction hypothesis to $\mathbf{L}_{|\mathcal{W}}$, we see that (14.24) follows. ∎

Proposition 4. *Given $\mathbf{v} \in \mathcal{V}$ and $\boldsymbol{\lambda} \in \mathcal{V}^*$, we have*

$$\det(\mathbf{1}_{\mathcal{V}} + \mathbf{v} \otimes \boldsymbol{\lambda}) = 1 + \boldsymbol{\lambda}\mathbf{v}. \tag{14.25}$$

Proof: The assertion is trivial if $\boldsymbol{\lambda} = \mathbf{0}$. Assume that $\boldsymbol{\lambda} \neq \mathbf{0}$. Then we may choose $\mathbf{w} \in \mathcal{V}$ such that $\boldsymbol{\lambda} w = 1$. It is clear that $\mathbf{P} := (\mathbf{w} \otimes \boldsymbol{\omega})|^{\mathcal{W}}$ is a projection from $\mathcal{V}$ on $\mathcal{W} := \mathbb{R}\mathbf{w}$ with $\mathcal{U} := \mathrm{Null}\mathbf{P} = \{\boldsymbol{\lambda}\}^{\perp}$. Putting $\mathbf{L} := \mathbf{1}_{\mathcal{V}} + \mathbf{v} \otimes \boldsymbol{\lambda}$, we see that $\mathcal{U}$ is $\mathbf{L}$-invariant and that $\mathbf{L}_{|\mathcal{U}} = \mathbf{1}_{\mathcal{U}}$. It follows that

$$\mathbf{PLw} = \mathbf{P}(\mathbf{w} + (\boldsymbol{\lambda}\mathbf{w})\mathbf{v}) = \mathbf{P}(\mathbf{w} + \mathbf{v}) = \mathbf{w} + \mathbf{Pw} = \mathbf{w} + (\boldsymbol{\lambda}\mathbf{v})\mathbf{w} = (1 + \boldsymbol{\lambda}\mathbf{v})\mathbf{w}.$$

Since $\mathcal{W} := \mathbb{R}\mathbf{w}$ we conclude that $\mathbf{PL}|_{\mathcal{W}} = (1+\boldsymbol{\lambda}\mathbf{v})\mathbf{1}_{\mathcal{W}}$. Therefore, using Prop.3 and (14.4) we obtain

$$\det{}^{\mathcal{V}}(\mathbf{L}) = \det{}^{\mathcal{U}}(\mathbf{1}_{\mathcal{U}})\det{}^{\mathcal{W}}((1+\boldsymbol{\lambda}\mathbf{v})\mathbf{1}_{\mathcal{W}}) = 1+\boldsymbol{\lambda}\mathbf{v},$$

which is the desised result. ∎

15. Invariants, characteristic polynomials.

As before, we assume that a linear space $\mathcal{V}$ is given and we put $n := \dim\mathcal{V}$. We assume that $n \geq 1$.

Definition 1. *A function* $f : \mathrm{Lin}\mathcal{V} \longrightarrow \mathbb{R}$ *is called an* **invariant** *if*

$$f(\mathbf{ALA}^{-1}) = f(\mathbf{L}) \quad \text{for all} \quad \mathbf{A} \in \mathrm{Lis}\mathcal{V},\ \mathbf{L} \in \mathrm{Lin}\mathcal{V}. \tag{15.1}$$

Theorem on Characterization of Principal Invariants. *For each* $k \in n^{]}$ *there is exactly one function* $\mathrm{inv}_k : \mathrm{Lin}\mathcal{V} \longrightarrow \mathbb{R}$ *such that*

$$\det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = s^n + \sum_{j \in n^{[}} (-1)^{n-j} s^j \mathrm{inv}_{n-j}(\mathbf{L}) \tag{15.2}$$

for all $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *and all* $s \in \mathbb{R}$ *. The function* inv_k *is an invariant; it is called the* k-**th principal invariant function**. *Given* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *the value* $\mathrm{inv}_k(\mathbf{L})$ *is called the k-th principal invariant of* $\mathbf{L}$.

For every list-basis $\mathbf{b}$ *of* $\mathcal{V}$ *we have*

$$\mathrm{inv}_k(\mathbf{L}) = \sum_{J \in \mathrm{Fin}_k(n^{]})} (\bigwedge \mathbf{b}^*)(\mathbf{b}|_{n^{]}\setminus J} \,.J)(\mathbf{L}^{\times J}\mathbf{b}|_J). \tag{15.3}$$

Proof: Let a list-basis $\mathbf{b}$ of $\mathcal{V}$ be given. Since $(\bigwedge \mathbf{b}^*)(\mathbf{b}) = 1$ (see (12.6)), it follows from (14.1) that

$$\det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = (\bigwedge \mathbf{b}^*)((s\mathbf{1}_{\mathcal{V}} - \mathbf{L})^{\times n}\mathbf{b}) = (\bigwedge \mathbf{b}^*)(s\mathbf{b} - \mathbf{L}^{\times n}\mathbf{b}).$$

Using Prop.10 of Sect.11 with $\mathbf{M} := \bigwedge \mathbf{b}^*$ and using $(\bigwedge \mathbf{b}^*)(\mathbf{b}) = 1$ again, we conclude that (15.2) holds when $\mathrm{inv}_k(\mathbf{L})$, for each $k \in n^{]}$, is given by (15.3). This shows that $s \mapsto \det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})) : \mathbb{R} \longrightarrow \mathbb{R}$ is a polynomial function. Since this function determines a unique polynomial (see the Remark in Sect.92 of Vol.I), it follows that $\mathrm{inv}_k(\mathbf{L})$, for each $k \in n^{]}$, is uniquely determined by $\mathbf{L}$ and does not depend on the choice of the basis $\mathbf{b}$.

The fact that inv_k is an invariant according to Def.1 is an immediate consequence of Prop.1 of Sect.14. ∎

Of course, the principal invariant functions depend on the space $\mathcal{V}$. We write $\mathrm{inv}_k^{\mathcal{V}}$ instead of inv_k when we wish to emphasize this fact.

Proposition 1. *For each $k \in n^]$ we have*

$$\mathrm{inv}_k(s\mathbf{L}) = s^k \mathrm{inv}_k(\mathbf{L}) \ \text{ for all } \ s \in \mathbb{R}\ , \mathbf{L} \in \mathrm{Lin}\mathcal{V}. \tag{15.4}$$

Proof: Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and $s, t \in \mathbb{R}$ be given. It follws from (14.5) and (15.2) with s replaced by t that

$$\det(s(t\mathbf{1}_{\mathcal{V}} - \mathbf{L})) = s^n \det(t\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = s^n (t^n + \sum_{j \in n^[} (-1)^{n-j} t^j \mathrm{inv}_{n-j}(\mathbf{L})). \tag{15.5}$$

On the other hand, it follows from (15.2) with s replaced by ts and $\mathbf{L}$ by $s\mathbf{L}$ that

$$\det(s(t\mathbf{1}_{\mathcal{V}} - \mathbf{L})) = \det((ts)\mathbf{1}_{\mathcal{V}} - s\mathbf{L}) = (ts)^n + \sum_{j \in n^[} (-1)^{n-j} (ts)^j \mathrm{inv}_{n-j}(s\mathbf{L}) \tag{15.6}$$

Since $t \in \mathbb{R}$ was arbitrary, the desired result follows from (15.5) and (15.6) by comparing the coefficients of like powers of t. ∎

Proposition 2. *We have*

$$\det = \mathrm{inv}_n \quad \text{and} \quad \mathrm{tr} = \mathrm{inv}_1. \tag{15.7}$$

For all $\mathbf{v} \in \mathcal{V}$ *and* $\boldsymbol{\lambda} \in \mathcal{V}^*$ *we have*

$$\mathrm{inv}_k(\mathbf{v} \otimes \boldsymbol{\lambda}) = 0 \quad \text{when} \ \ k \in n^] \setminus \{1\} \tag{15.8}$$

Proof: *Let* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *be given. Putting* $s = 0$ *in (15.2) gives* $\det(-\mathbf{L}) = (-1)^n \mathrm{inv}_n(\mathbf{L})$. *Since (14.5) with* $s := -1$ *gives* $\det(-\mathbf{L}) = (-1)^n \det(\mathbf{L})$, *the first of (15.7) follows.*

Now let $\mathbf{v} \in \mathcal{V}$ and $\boldsymbol{\lambda} \in \mathcal{V}^*$ be given. It follows from (14.5) and Prop.4 of Sect.14 that

$$\det(s\mathbf{1}_{\mathcal{V}} - \mathbf{v} \otimes \boldsymbol{\lambda}) = \det(s(\mathbf{1}_{\mathcal{V}} - \frac{1}{s}\boldsymbol{\lambda}\mathbf{v})) = s^n \det(\mathbf{1}_{\mathcal{V}} - \frac{1}{s}\mathbf{v} \otimes \boldsymbol{\lambda}) =$$

$$= s^n (1 - \frac{1}{s}\boldsymbol{\lambda}\mathbf{v}) = s^n - s^{n-1}\boldsymbol{\lambda}\mathbf{v} \tag{15.9}$$

for all $s \in \mathbb{R}^\times$. In order that (15.9) be compatible with (15.2) when $\mathbf{L} := \mathbf{v} \otimes \boldsymbol{\lambda}$ (15.9) must be valid and we must have $\mathrm{inv}_1(\mathbf{v} \otimes \boldsymbol{\lambda}) = \boldsymbol{\lambda}\mathbf{v}$. Since $\mathrm{inv}_1 : \mathrm{Lin}\mathcal{V} \longrightarrow \mathbb{R}$ is linear by (15.3) and since $\mathbf{v} \in \mathcal{V}$ and $\boldsymbol{\lambda} \in \mathcal{V}^*$ were arbitrary, the second of (15.7) follows from the Characterization of the Trace in Sect.26 of Vol.I. ∎

We now assume that a lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is given. For the considerations below, the distinction between polynomial functions and polynomials, which are members of $\mathbb{R}^{(\mathbb{N})}$, is important. (See Sect.92 of Vol.I.)

Definition 2. *The polynomial determined by the polynomial function* $(s \mapsto \det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})) : \mathbb{R} \longrightarrow \mathbb{R}$ *is called the* **characteristic polynomial** *of* $\mathbf{L}$ *and is denoted by* $\mathrm{chp}_{\mathbf{L}} \in \mathbb{R}^{(\mathbb{N})}$, *so that*

$$\mathrm{chp}_{\mathbf{L}}(s) = \det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) \quad \text{for all} \quad s \in \mathbb{R}. \tag{15.10}$$

This definition and (15.2) give immediately the following result.

Propsition 3. *The characteristic polynomial* $\mathrm{chp}_{\mathbf{L}}$ *is a monic polynomial of degree* n *(see Sect.92 of Vol.I), so that* $(\mathrm{chp}_{\mathbf{L}})_n = 1$ *and* $(\mathrm{chp}_{\mathbf{L}})_j = 0$ *for all* $j \in \mathbb{N}$ *with* $j > n$. *The first* n *terms of* $\mathrm{chp}_{\mathbf{L}}$ *are given by*

$$(\mathrm{chp}_{\mathbf{L}})_j = (-1)^{n-j}\mathrm{inv}_{n-j}(\mathbf{L}) \quad \text{for all} \quad j \in n^{[}. \tag{15.11}$$

Proposition 4. *The spectrum* $\mathrm{Spec}\mathbf{L}$ *(as defined in Def.1 of Sect.82 of Vol.I) consists of the roots of the characteristic polynomial of* $\mathbf{L}$, *i.e.,*

$$\mathrm{Spec}\mathbf{L} = \{\sigma \in \mathbb{R} \mid \mathrm{chp}_{\mathbf{L}}(\sigma) = 0\}. \tag{15.11}$$

Proof: Since $\mathrm{Null}(\sigma\mathbf{1}_{\mathcal{V}} - \mathbf{L}) \neq \{\mathbf{0}\}$ if and only if $\det(\sigma\mathbf{1}_{\mathcal{V}} - \mathbf{L})$ is invertible by Prop.1 of Sect.18 of Vol.I, (15.11) follows immediately from (14.7) and Def.2 above. ∎

The purpose of the remainder of this Section is to show that Def.2 is consistent with the definition of characteristic polynomial given by (95.2) in Vol.I.

We say that $\mathbf{L}$ is a **cyclic lineon** if there is a $\mathbf{v} \in \mathcal{V}$ such that $\mathcal{V} = \mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\}$, where $\mathrm{Lsp}_{\mathbf{L}}$ denotes the $\mathbf{L}$-span as defined by (92.16) of Vol.I. (see also the Remark in Sect.93 of Vol.I.)

Proposition 5. *Assume that* $\mathbf{L}$ *is cyclic and denote the minimal polynomial of* $\mathbf{L}$ *(as defined in Prop.2 of Sect.92 of Vol.I) by* q. *Then*

$$\det(\mathbf{L}) = (-1)^n q_0. \tag{15.13}$$

Proof: We choose $\mathbf{v} \in \mathcal{V}$ such that $\mathcal{V} = \mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\}$. It is easily seen that

$$\mathbf{b} := (\mathbf{L}^{i-1)}\mathbf{v} \mid i \in n^{]}) \tag{15.14}$$

is a list basis of $\mathcal{V}$. Since the degree of q is n by Prop.3 of Sect.92 of Vol.I and since q is monic, we have $q = \iota^n + \sum(q_k\iota^k \mid k \in n^{[})$. Since $q(\mathbf{L}) = \mathbf{0}$, we conclude that

$$\mathbf{L}^n\mathbf{v} = -\sum_{k\in n^{[}} q_k\mathbf{L}^k\mathbf{v} = \sum_{k\in n^{[}} q_k\mathbf{b}_{k+1}. \tag{15.15}$$

By (15.14) we have

$$\mathbf{L}^{\times n}\mathbf{b} = (\mathbf{L}^i\mathbf{v} \mid i \in n^{]}) = ((\mathbf{L}^i\mathbf{v} \mid i \in (n-1)^{]})\,.n)(\mathbf{L}\mathbf{b}_n) =$$

$$= ((\mathbf{b}_{i+1} \mid i \in (n-1)^{]})\,.n)(\mathbf{L}\mathbf{b}_n). \tag{15.16}$$

Now put $\boldsymbol{\omega} := \bigwedge \mathbf{b}^* \in \mathrm{Skew}_n(\mathcal{V}^n, \mathbb{R})$. Since $\boldsymbol{\omega}((\mathbf{b}^{i+1} \mid i \in (n-1)^{]})\,.n) : \mathcal{V} \longrightarrow \mathbb{R}$ is linear, it follows from (15.15) and (15.16) that

$$\boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}) = -\sum_{k \in n^{]}} q_k \boldsymbol{\omega}((\mathbf{b}_{i+1} \mid i \in (n-1)^{]})\,.n)\mathbf{b}_{k+1}). \tag{15.17}$$

Since the term $\mathbf{b}_k$ occurs twice in the list $(\mathbf{b}_{i+1} \mid i \in (n-1)^{]})\,.n)\mathbf{b}_{k+1}$ except when $k = 0$, it follows from Prop.8 of Sect.11 that the only non-zero term in the sum on the right of (15.17) is the one for which $k = 0$, so that

$$(\mathbf{L}^{\times n}\mathbf{b}) = -q_0 \boldsymbol{\omega}((\mathbf{b}_{i+1} \mid i \in (n-1)^{]})\,.n)\mathbf{b}_1). \tag{15.18}$$

Now, it is easily seen that the list $(\mathbf{b}_{i+1} \mid i \in (n-1)^{]})\,.n)\mathbf{b}_1$, which informally can be written as $(\mathbf{b}_2, \mathbf{b}_3, \cdots, \mathbf{b}_{n-1}, \mathbf{b}_1)$, can be obtained from the list $\mathbf{b}$ by $(n-1)$ switches. Hence, since $\boldsymbol{\omega}$ is skew, it follows from (15.18) that

$$\boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}) = (-1)^n q_0 \boldsymbol{\omega}(\mathbf{b}).$$

Since $\boldsymbol{\omega}(\mathbf{b}) = 1$ by (12.6), the desired result (15.13) is a direct consequence of (14.1). ∎

Proposition 6. *Assume that* $\mathbf{L}$ *is cyclic. Then the minimal polynomial of* $\mathbf{L}$ *coincides with its characteristic polynomial.*

Proof: Denote the minimal polynomial of $\mathbf{L}$ and let $s \in \mathbb{R}$ be given. Since $p(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = (p \circ (s - \iota))(\mathbf{L})$ holds for every polynomial p, it is easily seen that $(-1)^n(q \circ (s - \iota))$ is the minimal polynomial of $s\mathbf{1}_{\mathcal{V}} - \mathbf{L}$. Also, it is easily seen that $s\mathbf{1}_{\mathcal{V}} - \mathbf{L}$ is cyclic. Hence we can apply Prop.5 to the case when $\mathbf{L}$ is replaced by $s\mathbf{1}_{\mathcal{V}} - \mathbf{L}$ and conclude that

$$\det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = (-1)^n(-1)^n(q \circ (s - \iota))_0 = q(s)$$

. Since $s \in \mathbb{R}$ was arbitrary, th desired result follows from(14.9). ∎

Proposition 7. *Let* $(\mathcal{U}_i \mid i \in I)$ *be an* $\mathbf{L}$*-decomposition of* $\mathcal{V}$ *(see Prop.2 of Sect.14) such that* $\mathbf{L}_{|\mathcal{U}_i}$ *is cyclic for every* $i \in I$*. We then have*

$$\mathrm{chp}_{\mathbf{L}} = \prod_{i \in I} q_i\ , \tag{15.19}$$

where q_i *is the minimal polynomial of* $\mathbf{L}_{|\mathcal{U}_i}$ *for every* $i \in I$*.*

Proof: Let $s \in \mathbb{R}$ be given. It is clear that $(\mathcal{U}_i \mid i \in I)$ is also an $(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})$-decomposition of $\mathcal{V}$ and that $(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})_{|\mathcal{U}_i} = s\mathbf{1}_{\mathcal{U}_i} - \mathbf{L}_{|\mathcal{U}_i}$ for each $i \in I$. Therefore, Prop.2 of Sect.14 yields

$$\det{}^{\mathcal{V}}(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = \prod_{i \in I} \det{}^{\mathcal{U}_i}(s\mathbf{1}_{\mathcal{U}_i} - \mathbf{L}_{|\mathcal{U}_i}). \tag{15.20}$$

Since $\mathbf{L}_{|\mathcal{U}_i}$ is cyclic for each $i \in I$, it follows from Prop.6 that $q_i(s) = \mathrm{chp}_{\mathbf{L}_{|\mathcal{U}_i}}(s)$ for each $i \in I$. Using (15.10) with $\mathbf{L}$ replaced by $\mathbf{L}_{|\mathcal{U}_i}$ and observing that $s \in \mathbb{R}$ was arbitrary, we see that (15.20) gives the desired result (15.19). ▮

By the Elementary Decomposition Theorem of Sect.91, Vol.I, the lineon $\mathbf{L}$ has an elementary decomposition $(\mathcal{E}_i \mid i \in I)$. By Cor.4 of Sect.93, Vol.I, $\mathbf{L}_{|\mathcal{E}_i}$ is cyclic for each $i \in I$. Therefore Prop.7 can be applied and (15.19) is valid when, for each $i \in I$, q_i is the minimal polynomial of $\mathbf{L}_{|\mathcal{E}_i}$ and hence an elementary divisor of $\mathbf{L}$. (See Sect.95 od Vol.I.) The multiplicity of each elementary divisor in the family $(q_i \mid i \in I)$ is $\mathrm{emult}_{\mathbf{L}}$. Thus, (15.19) agrees with (95.2) of Vol.I, showing that Def.2 of the present section is indeed consistent with (95.2) of Vol.I. As was pointed out in Sect.95 of Vol.I, we have $\mathrm{chp}_{\mathbf{L}}(\mathbf{L}) = \mathbf{0}$. In the next section, we shall give a new proof of this fact, a proof that does not make use of the Elementary Decomposition Theorem of Vol.I.

16. Adjugates, covariants

As in the previous section, we assume that a linear space $\mathcal{V}$ is given, we put $n := \dim\mathcal{V}$ and assume that $n \geq 1$.

Theorem on Characterization of Adjugates. *There is exactly one mapping* $\mathrm{adj} : \mathrm{Lin}\mathcal{V} \longrightarrow \mathrm{Lin}\mathcal{V}$ *such that*

$$\det(\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}) - \det(\mathbf{L}) = \boldsymbol{\lambda}\mathrm{adj}(\mathbf{L})\mathbf{v} \tag{16.1}$$

for all $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, $\mathbf{v} \in \mathcal{V}$ *, and* $\boldsymbol{\lambda} \in \mathcal{V}^*$. *We call* adj *the* **adjugate-mapping** *of* $\mathcal{V}$ *and its value* $\mathrm{adj}(\mathbf{L})$ *at a given* $\mathbf{L}$ *the adjugate of* $\mathbf{L}$.

For every list-basis $\mathbf{b}$ *of* $\mathcal{V}$ *we have*

$$\mathrm{adj}(\mathbf{L}) = \sum_{j \in n^]} \mathbf{b}_j \otimes (\bigwedge \mathbf{b}^*)(\mathbf{L}^{\times *}\mathbf{b}\,.j)\ . \tag{16.2}$$

Proof: Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, $\mathbf{v} \in \mathcal{V}$, $\boldsymbol{\lambda} \in \mathcal{V}^*$ and a list-basis $\mathbf{b}$ of $\mathcal{V}$ be given and put $\boldsymbol{\omega} := \bigwedge \mathbf{b}^*$. Observing (12.6), it follows from the Theorem on Characterisation of Determinants of Sect.14 that

$$\det(\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}) - \det(\mathbf{L}) = \omega(\mathbf{L}^{\times n}\mathbf{b} + (\mathbf{v} \otimes \boldsymbol{\lambda})^{\times n}\mathbf{b}) - \boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}). \tag{16.3}$$

We have $((\mathbf{v} \otimes \boldsymbol{\lambda})^{\times n}\mathbf{b})_i = (\mathbf{v} \otimes \boldsymbol{\lambda})\mathbf{b}_i = (\boldsymbol{\lambda}\mathbf{b}_i)\mathbf{v}$ for all $i \in I$, i.e., all terms of the list $(\mathbf{v} \otimes \boldsymbol{\lambda})^{\times n}\mathbf{b}$ are multiples of $\mathbf{v}$. Therefore, every restriction of this list to a subset of $n^]$ that has two or more members is linearly dependent. Hence, By Prop.7 of Sect.11, if we apply (11.6) to the case when $\mathbf{M}$, $\mathbf{f}$, $\mathbf{g}$ are replaced by $\boldsymbol{\omega}$, $\mathbf{L}^{\times n}\mathbf{b}$, $(\mathbf{v} \otimes \boldsymbol{\lambda})^{\times n}\mathbf{b}$, respectively, we see that only the terms on the right side of (11.6) corresponding to a singleton J can be non-zero. Thus (11.6) yields

$$\boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b} + (\mathbf{v} \otimes \boldsymbol{\lambda})^{\times n}\mathbf{b}) = \boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}) + \sum_{j \in n^]} \boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}\,.j)((\boldsymbol{\lambda}\mathbf{b}_i)\mathbf{v}).$$

Therefore, by (16.3), we have

$$\det(\mathbf{L}+\mathbf{v}\otimes\boldsymbol{\lambda})-\det(\mathbf{L})=\sum_{j\in n^]}(\boldsymbol{\lambda}\mathbf{b}_i)\boldsymbol{\omega}(\mathbf{L}^{\times n}\mathbf{b}\,.j)(\mathbf{v}),\tag{16.4}$$

which shows that (16.1) holds when $\mathrm{adj}(\mathbf{L})$ is given by (16.2). We also infer from (16.4) that the mapping

$$((\mathbf{v},\boldsymbol{\lambda})\mapsto\det(\mathbf{L}+\mathbf{v}\otimes\boldsymbol{\lambda})-\det(\mathbf{L})):\mathcal{V}\times\mathcal{V}^*\longrightarrow\mathbb{R}$$

is bilinear. This mapping is identified with $\mathrm{adj}(\mathbf{L})$ by the identifications $\mathrm{Lin}_2(\mathcal{V}\times\mathcal{V}^*,\mathbb{R})\cong\mathrm{Lin}(\mathcal{V},\mathrm{Lin}(\mathcal{V}^*,\mathbb{R}))=\mathrm{Lin}(\mathcal{V},\mathcal{V}^{**})\cong\mathrm{Lin}\mathcal{V}$ and hence $\mathrm{adj}(\mathbf{L})$ does not depend on the choice of the basis $\mathbf{b}$. ▮

Of course, the adjugate mapping depends on $\mathcal{V}$ and we write $\mathrm{adj}^{\mathcal{V}}$ instead of just adj when we wish to emphasize this fact.

It follows immediately from (16.1) and Prop.4 of Sect.14 that

$$\mathrm{adj}(\mathbf{0})=\mathbf{0},\quad\mathrm{adj}(\mathbf{1}_{\mathcal{V}})=\mathbf{1}_{\mathcal{V}}.\tag{16.5}$$

Basic Rules for the Adjugate. *Let $\mathbf{L}\in\mathrm{Lin}\mathcal{V}$ and $s\in\mathbb{R}$ be given. We then have*

$$\mathrm{adj}(s\mathbf{L})=s^{n-1}\mathrm{adj}(\mathbf{L}),\tag{16.6}$$

$$\mathrm{adj}(\mathbf{L})\mathbf{L}=\det(\mathbf{L})\mathbf{1}_{\mathcal{V}},\tag{16.7}$$

$$\mathrm{adj}(\mathbf{L})\text{ is invertible if and only if }\mathbf{L}\text{ is invertible,}\tag{16.8}$$

$$\mathrm{adj}(\mathbf{L}^{-1})=(\mathrm{adj}(\mathbf{L}))^{-1}=\frac{1}{\det(\mathbf{L})}\mathbf{L}\quad\text{if}\quad\mathbf{L}\in\mathrm{Lis}\mathcal{V},\tag{16.9}$$

$$(\mathrm{adj}^{\mathcal{V}}(\mathbf{L}))^{\top}=\mathrm{adj}^{\mathcal{V}^*}(\mathbf{L}^{\top}).\tag{16.10}$$

Proof: Using (16.1) and (14.5) each twice, we obtain

$$s^n\boldsymbol{\lambda}\mathrm{adj}(\mathbf{L})\mathbf{v}=\det(s\mathbf{L}+s\mathbf{v}\otimes\boldsymbol{\lambda})-\det(s\mathbf{L})=\boldsymbol{\lambda}(\mathrm{adj}(s\mathbf{L}))(s\mathbf{v})=s\boldsymbol{\lambda}\mathrm{adj}(s\mathbf{L})\mathbf{v}$$

for all $\mathbf{v}\in\mathcal{V}$ and $\boldsymbol{\lambda}\in\mathcal{V}^*$ and hence that $s^n\mathrm{adj}(\mathbf{L})=s\mathrm{adj}(s\mathbf{L})$, which implies (16.6) when $s\neq 0$. For $s:=0$ (16.6) reduces to $(16.5)_1$.

Using (14.25) twice and also using (14.6) and (16.1) we obtain

$$\det(\mathbf{L})(1+\boldsymbol{\lambda}\mathbf{v})=\det(\mathbf{L}(\mathbf{1}_{\mathcal{V}}+\mathbf{v}\otimes\boldsymbol{\lambda}))=\det(\mathbf{L}+(\mathbf{L}\mathbf{v})\otimes\boldsymbol{\lambda})=\det(\mathbf{L})+\lambda\mathrm{adj}(\mathbf{L})\mathbf{L}\mathbf{v}$$

and hence $\det(\mathbf{L})\boldsymbol{\lambda}\mathbf{v}=\boldsymbol{\lambda}(\mathrm{adj}(\mathbf{L})\mathbf{L})\mathbf{v}$ for all $\mathbf{v}\in\mathcal{V}$ and $\boldsymbol{\lambda}\in\mathcal{V}^*$, which is equivalent to (16.7).

The rule (16.8) is an immediate consequence of (14.7) and (16.7).

The rule (16.9) follows from (16.8) by using (16.7) and then (16.7) again with $\mathbf{L}$ replaced by $\mathbf{L}^{-1}$.

The rule (16.10) is an immediate consequence of the rule (14.9), the characterization (16.1), and (25.6) of Vol.I. ▮

Definition. *A mapping* $\mathbf{F} : \text{Lin}\mathcal{V} \longrightarrow \text{Lin}\mathcal{V}$ *is called a* **covariant** *if*

$$\mathbf{F}(\mathbf{A}\mathbf{L}\mathbf{A}^{-1}) = \mathbf{A}\mathbf{F}(\mathbf{L})\mathbf{A}^{-1} \quad \text{for all} \quad \mathbf{A} \in \text{Lis}\mathcal{V}\ , \mathbf{L} \in \text{Lin}\mathcal{V}. \tag{16.11}$$

Theorem on Characterization of Principal Covariants. *For each* $k \in n^{]}$ *there is exactly one mapping* $\text{cov}_k : \text{Lin}\mathcal{V} \longrightarrow \text{Lin}\mathcal{V}$ *such that*

$$\text{adj}(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = -\sum_{j \in n^{[}} (-1)^{n-j} s^j \text{cov}_{n-j}(\mathbf{L}) \tag{16.11}$$

for all $\mathbf{L} \in \text{Lin}\mathcal{V}$ *and all* $s \in \mathbb{R}$. *The mapping* cov_k *is a covariant; it is called the* k**-th principal covariant mapping**. *Given* $\mathbf{L} \in \text{Lin}\mathcal{V}$ *the value* $\text{cov}_k(\mathbf{L})$ *is called the* k*-th principal cavariant of* $\mathbf{L}$.

We have

$$\text{inv}_k(\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}) - \text{inv}_k(\mathbf{L}) = \boldsymbol{\lambda}\text{cov}_k(\mathbf{L})\mathbf{v} \tag{16.13}$$

for all $\mathbf{L} \in \text{Lin}\mathcal{V},\ \mathbf{v} \in \mathcal{V},\ \boldsymbol{\lambda} \in \mathcal{V}^*$ *and* $k \in n^{]}$.

Proof: Let $\mathbf{L} \in \text{Lin}\mathcal{V},\ \mathbf{v} \in \mathcal{V},\ \boldsymbol{\lambda} \in \mathcal{V}^*$ and $s \in \mathbb{R}$ be given. It follows from (16.1) with $\mathbf{L}$ replaced by $s\mathbf{1}_{\mathcal{V}} - \mathbf{L}$ and $\mathbf{v}$ by $-\mathbf{v}$ that

$$\det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L} + (-\mathbf{v}) \otimes \boldsymbol{\lambda}) - \det(s\mathbf{1}_{\mathcal{V}} - \mathbf{L}) = -\boldsymbol{\lambda}\text{adj}(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})\mathbf{v}.$$

Using (15.2) with $\mathbf{L}$ replaced by $\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}$ and then again with $\mathbf{L}$ itself we obtain

$$-\sum_{j \in n^{[}} (-1)^{n-j} s^j (\text{inv}_{n-j}(\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}) - \text{inv}_{n-j}(\mathbf{L})) = \boldsymbol{\lambda}\text{adj}(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})\mathbf{v}. \tag{16.14}$$

Since $s \in \mathbb{R}$ was arbitrary, we conclude that $(s \mapsto \boldsymbol{\lambda}\text{adj}(s\mathbf{1}_{\mathcal{V}} - \mathbf{L})\mathbf{v})$ is a polynomial function. Since this function determines a unique polynomial (see the Remark in Sect.92 of Vol.I), one can easily deduce from (16.14) that, for each $k \in n^{]}$, the mapping

$$((\mathbf{v}, \boldsymbol{\lambda}) \mapsto (\text{inv}_k(\mathbf{L} + \mathbf{v} \otimes \boldsymbol{\lambda}) - \text{inv}_k(\mathbf{L})) : \mathcal{V} \times \mathcal{V}^* \longrightarrow \mathbb{R}$$

is bilinear. Hence, in view of the identification $\text{Lin}_2(\mathcal{V} \times \mathcal{V}^*, \mathbb{R}) \cong \text{Lin}\mathcal{V}$, there is exactly one $\text{cov}_k(\mathbf{L}) \in \text{Lin}\mathcal{V}$ such that (16.13) holds for all $\mathbf{v} \in \mathcal{V},\ \boldsymbol{\lambda} \in \mathcal{V}^*$. The formula (16.11) is an immediate consequence of the fact that (16.14) and (16.13) hold for all $\mathbf{v} \in \mathcal{V},\ \boldsymbol{\lambda} \in \mathcal{V}^*$.

The fact that cov_k is a covariant is an immediate consequence of Prop.1 of Sect.14 and the characterizations (16.1) and (16.11). It can also be deduced from (16.13) and the fact that the principal invariants are indeed invariants in the sense of Def.1 of Sect.15. ∎

Of course, the principal covariant mappings depend on the space $\mathcal{V}$. We write $\text{cov}_k^{\mathcal{V}}$ instead of cov_k when we wish to emphasize this fact.

Formulas for Principal Covariants. *Let a lineon* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *be given. We then have*

$$\mathrm{cov}_1(\mathbf{L}) = \mathbf{1}, \qquad \mathrm{cov}_n(\mathbf{L}) = \mathrm{adj}(\mathbf{L}), \tag{16.15}$$

and the principal covariants of $\mathbf{L}$ *can be expressed explicitly in terms of its principal invariants by*

$$\mathrm{cov}_k(\mathbf{L}) = (-1)^{k-1}\mathbf{L}^{k-1} - \sum_{j\in(k-1)^]} (-1)^j \mathrm{inv}_{k-j}(\mathbf{L})\mathbf{L}^{j-1} \quad \text{for all } k \in n^]. \tag{16.16}$$

Proof: To obtain $(16.15)_2$ it suffices to evaluate (16.11) at $s := 0$ and to use (16.6) with $s := -1$.

Let $s \in \mathbb{R}$ be given. If we write (16.7) with $\mathbf{L}$ replaced by $s\mathbf{1}_\mathcal{V} - \mathbf{L}$ and then use (16.11) and (15.1) we find

$$-\sum_{j\in n^[} (-1)^{n-j} s^j \mathrm{cov}_{n-j}(\mathbf{L})(s\mathbf{1}_\mathcal{V} - \mathbf{L}) = s^n \mathbf{1}_\mathcal{V} + \sum_{j\in n^]} (-1)^{n-j} s^j \mathrm{inv}_{n-j}(\mathbf{L})\mathbf{1}_\mathcal{V}.$$

A short and easy calculation yields

$$\sum_{j\in n^]} (-1)^{n-j} s^j \mathrm{cov}_{n-j+1}(\mathbf{L}) =$$

$$= s^n \mathbf{1}_\mathcal{V} + \sum_{j\in n^[} (-1)^{n-j} s^j (\mathrm{inv}_{n-j}(\mathbf{L})\mathbf{1}_\mathcal{V} - \mathrm{cov}_{n-j}(\mathbf{L})\mathbf{L}). \tag{16.17}$$

Since $s \in \mathbb{R}$ was arbitrary und since polynomials are determined by the corresponding polynomial functions, the terms corresponding to the same powers of s on the two sides of (16.17) must agree. We conclude that $\mathrm{cov}_1(\mathbf{L}) = \mathbf{1}_\mathcal{V}$,

$$\mathrm{cov}_{k+1}(\mathbf{L}) = \mathrm{inv}_k(\mathbf{L})\mathbf{1}_\mathcal{V} - \mathrm{cov}_k(\mathbf{L})\mathbf{L} \qquad \text{for all } k \in (n-1)^], \tag{16.18}$$

and $\mathrm{inv}_n(\mathbf{L})\mathbf{1}_\mathcal{V} - \mathrm{cov}_n(\mathbf{L})\mathbf{L} = \mathbf{0}$. The first of these three conclusions is $(16.15)_1$, the second, namely (16.18), implies (16.16) by induction, and the third is merely a restatement of (16.7) when $(16.15)_2$ and $(15.7)_1$ are observed. ∎

Proposition 1. *For any given* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *we have*

$$\mathrm{chp}_\mathbf{L}(\mathbf{L}) = \mathbf{0}. \tag{16.19}$$

Proof: For $k := n$, (16.16) and $(16.15)_1$ give

$$\mathbf{L}^{n-1} - \sum_{j\in(n-1)^]} (-1)^{n-j} \mathrm{inv}_{n-j}(\mathbf{L})\mathbf{L}^{j-1} + (-1)^n \mathrm{adj}(\mathbf{L}) = \mathbf{0}.$$

After multiplying this equation from the right by $\mathbf{L}$ and using (16.7), $(15.7)_1$, and (15.11), we conclude that (16.19) holds. ∎