Finite Dimesional Spaces Vol.II
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0. Preliminaries

01 Insertions, switches, symmetric and skew mappings.

Let a family $(A_i \mid i \in I)$ of sets be given. Recall (Sect.04 of Vol.I) that for each family $a \in \times(A_i \mid i \in I)$ and each $j \in I$. the mapping

$$(a.j) : A_j \longrightarrow \underset{i\in I}{\times} A_i$$

is defined by the rule

$$((a.j)(x))_i := \left\{ \begin{array}{lll} a_i & \text{if} & i \in I\backslash\{j\} \\ x & \text{if} & i = j \end{array} \right\} \quad \text{for all } x \in A_j. \tag{01.1}$$

Moreover, when ϕ is a mapping with $\mathrm{Dom}\phi = \times(A_i \mid i \in I)$, we use the notation

$$\phi(a.j) := \phi \circ (a.j) \ : \ A_j \longrightarrow \mathrm{Cod}\phi. \tag{01.2}$$

Analogously, for each $a \in \times(A_i \mid i \in I)$ and each $(p,q) \in I^2$ with $p \neq q$, we define the mapping

$$a.(p,q) : A_p \times A_q \longrightarrow \underset{i\in I}{\times} A_i$$

by

$$((a.(p,q)(x,y))_i := \begin{cases} a_i & \text{if} \quad i \in I\backslash\{p,q\}, \\ x & \text{if} \quad i = p, \\ y & \text{if} \quad i = q \end{cases} \tag{01.3}$$

for all $(x,y) \in A_p \times A_q$. Moreover,when ϕ is a mapping with $\mathrm{Dom}\phi = \times(A_i \mid i \in I)$, we use the notation

$$\phi(a.(p,q)) := \phi \circ (a.(p,q)) : A_p \times A_q \longrightarrow \mathrm{Cod}\phi, \tag{01.4}$$

so that

$$\phi(a.(p,q))(a_p, a_q) = \phi(a) \quad \text{for all } a \in \underset{i\in I}{\times} A_i. \tag{01.5}$$

Assume now that a subset J of I and $a \in \times(A_i \mid i \in I)$ or $a \in \times(A_i \mid i \in I\backslash J)$ are given. We define the mapping

$$(a.J) : \times(A_i \mid i \in J) \longrightarrow \times(A_i \mid i \in I)$$

by

$$((a.J)(b))_i := \begin{cases} a_i & \text{if} \quad i \in I\backslash J \\ b_i & \text{if} \quad i \in J \end{cases} \quad \text{for all } b \in \times(A_i \mid i \in J). \tag{01.6}$$

If ϕ is a mapping with $\text{Dom}\phi = \times(A_i \mid i \in I)$ we write $\phi(a.J) := \phi \circ (a.J)$, in analogy to (01.2) and (01.4).

The mappings defined by (01.2), (01.3), and (01.6) are called **insertion mappings**. They are related by $(a.j) = (a.\{j\}) = (a|_{I\setminus\{j\}}.\{j\})$ when $j \in I$ and $(a.(p,q)) = (a.\{p,q\}) = (a|_{I\setminus\{p,q\}}.\{p,q\})$ when $p, q \in I$ with $p \neq q$.

From now on we assume that a set S, an index set I, and a linear space $\mathcal{W}$ are given. Given any family $a \in S^I$ and given $(p,q) \in I^2$ with $p \neq q$, we use the notation

$$a^{\sim(p,q)} := (a.(p,q))(a_q, a_p) \ \in S^I. \tag{01.7}$$

Roughly, $a^{\sim(p,q)}$ is obtained from a by switching the p-term with the q-term and hence the operation $\sim(p,q)$ is called a **switch**. We assume now that a mapping $\mathbf{M} : S^I \longrightarrow \mathcal{W}$ is given. We define $\mathbf{M}^{\sim(p,q)} : S^I \longrightarrow \mathcal{W}$ by

$$\mathbf{M}^{\sim(p,q)}(a) := \mathbf{M}(a^{\sim(p,q)}) \ \text{ for all } a \in S^I. \tag{01.8}$$

Definition 1. *The mapping* $\mathbf{M} : S^I \longrightarrow \mathcal{W}$ *is said to be* **symmetric** *if*

$$\mathbf{M}^{\sim(p,q)} = \mathbf{M} \ \text{ for all } (p,q) \in I \text{ with } p \neq q; \tag{01.9}$$

it is said to be **skew** *if*

$$\mathbf{M}^{\sim(p,q)} = -\mathbf{M} \ \text{ for all } (p,q) \in I \text{ with } p \neq q. \tag{01.10}$$

If the given mapping $\mathbf{M} : S^I \longrightarrow \mathcal{W}$ is symmetric [skew], so is the mapping $(\mathbf{M}.J) : S^J \longrightarrow \mathcal{W}$ for every subset J of I.

We now consider the case when $I := k^]$ for some $k \in \mathbb{N}$. Recall the abbreviation $S^k := S^{k^]}$ (see Sect.02 of Vol.I). The following criterion is clear from Def.1.

Proposition 1. *A given mapping* $\mathbf{M} : S^k \longrightarrow \mathcal{W}$ *is symmetric [skew] if and only if*

$$\mathbf{M}^{\sim(p,p+j)} = \mathbf{M} \quad [\ \mathbf{M}^{\sim(p,p+j)} = -\mathbf{M}\ ] \tag{01.11}$$

for all $p, j \in (k-1)^]$ *with* $p + j \leq k$.

The following gives a much weaker sufficient condition.

Proposition 2. *If a given mapping* $\mathbf{M} : S^k \longrightarrow \mathcal{W}$ *satisfies*

$$\mathbf{M}^{\sim(p,p+1)} = \mathbf{M} \quad [\ \mathbf{M}^{\sim(p,p+1)} = -\mathbf{M}\ ] \ \text{ for all } p \in (k-1)^] \tag{01.12}$$

then it is symmetric [skew].

Proof: First we note that

$$a^{\sim(p,q+1)} = (((a^{\sim(q,q+1)})^{\sim(p,q)})^{\sim(q,q+1)}) \tag{01.13}$$

holds for all $a \in S^k$ and $p, q \in (k-1)^]$ with $p < q$, as is easily verified from the definition (01.7).

We now assume that (01.12) holds. Let $p \in (k-1)^]$ be given. We will show, by induction over j, that (01.11) is valid for this given p. For $j := 1$ (01.11) reduces to (01.12). Let $j \in (k-p-1)^]$ be given and assume that (01.11) is valid for this given j. Let $a \in S^k$ be given. We then have, by (01.8) and (01.13),

$$\mathbf{M}^{\sim(p,p+j+1)}(a) = \mathbf{M}(a^{\sim(p,p+j+1)}) = \mathbf{M}(((a^{\sim(p+j,p+j+1)})^{\sim(p,p+j)})^{\sim(p+j,p+j+1)})$$
$$= \mathbf{M}^{\sim(p+j,p+j+1)}((a^{\sim(p+j,p+j+1)})^{\sim(p,p+j)})$$

Using (01.12) with p replaced by $p+j$ and then using (01.8) again, we find that

$$\mathbf{M}^{\sim(p,p+j+1)}(a) = \pm\mathbf{M}((a^{\sim(p+j,p+j+1)})^{\sim(p,p+j)}) = \pm\mathbf{M}^{\sim(p,p+j)}(a^{\sim(p+j,p+j+1)}).$$

Using the induction hypothesis we obtain

$$\mathbf{M}^{\sim(p,p+j+1)}(a) = \mathbf{M}(a^{\sim(p+j,p+j+1)}) = \mathbf{M}^{\sim(p+j,p+j+1)}(a).$$

(The minus sign of $\pm$ applies to the condition (01.12) in brackets.)

Using (01.12) again with p replaced by $p+j$ we conclude that

$$\mathbf{M}^{\sim(p,p+j+1)}(a) = \pm\mathbf{M}(a),$$

which shows that (01.11) is valid with j replaced by $j+1$, completing the induction. ▮

Let a mapping $f := D \longrightarrow C$ and an index set I be given. Recall (see Sect.04 of Vol.I) that the cross-power $f^{\times I} : D^I \longrightarrow C^I$ is defined by

$$f^{\times I}(x) := (f(x_i) \mid i \in I) \quad \text{for all} \;\; x \in D^I. \tag{01.14}$$

For each $k \in \mathbb{N}$ we use the abbreviation $f^{\times k} := f^{\times k^]}$.

Now let, in addition, a mapping $\mathbf{M} : C^I \longrightarrow \mathcal{W}$ be given. If $\mathbf{M}$ is symmetric [skew], so is $\mathbf{M} \circ f^{\times I}$.

02 Multilinear mappings.

Definition 1. *Let a family of linear spaces $(\mathcal{V}_i \mid i \in I)$ and a linear space $\mathcal{W}$ be given. We say that the mapping*

$$\mathbf{M} : \mathop{\times}_{i\in I} \mathcal{V}_i \longrightarrow \mathcal{W} \tag{02.1}$$

is **multilinear** *if, for every $\mathbf{f} \in \times(\mathcal{V}_i \mid i \in I)$ and every $j \in I$, the mapping $\mathbf{M}(\mathbf{f}.j) : \mathcal{V}_j \longrightarrow \mathcal{W}$ is linear, so that $\mathbf{M}(\mathbf{f}.j) \in \mathrm{Lin}(\mathcal{V}_j, \mathcal{W})$. The set of all multilinear mappings from $\times(\mathcal{V}_i \mid i \in I)$ to $\mathcal{W}$ will be denoted by $\mathrm{Lin}_I(\times(\mathcal{V}_i \mid i \in I), \mathcal{W})$.*

If the mapping (02.1) is multilinear and if $(p,q) \in I^2$ with $p \neq q$ is given, then $\mathbf{M}(\mathbf{f}.(p,q)) : \mathcal{V}_p \times \mathcal{V}_q \longrightarrow \mathcal{W}$ is bilinear for each $\mathbf{f} \in \times(\mathcal{V}_i \mid i \in I)$.

The following three results generalize Props.1, 3, and 4 of Sect.24 of Vol.I.

Proposition 1. $\mathrm{Lin}_I(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})$ *is a subspace of* $\mathrm{Map}(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})$.

If $I:=\emptyset$, then $\times(\mathcal{V}_i \mid i\in I)=\{\emptyset\}$ and $\mathrm{Lin}_\emptyset(\{\emptyset\},\mathcal{W})=\mathrm{Map}(\{\emptyset\},\mathcal{W})$, which we identify with $\mathcal{W}$. If $\#I=1$, then $\mathrm{Lin}_I(\mathcal{V}^I,\mathcal{W})=\mathrm{Lin}(\mathcal{V}^I,\mathcal{W})$, which we usually identify with $\mathrm{Lin}(\mathcal{V},\mathcal{W})$, especially when $I:=1^]$. If $\#I>1$, we have

$$\mathrm{Lin}_I(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})\ \cap\ \mathrm{Lin}(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})=\{\emptyset\}.$$

Proposition 2. *The composite of a multilinear mapping with a linear mapping is again multilinear. More precisely, given a family of linear spaces* $(\mathcal{V}_i \mid i\in I)$*, given linear spaces* $\mathcal{W}$ *and* $\mathcal{W}'$*, and given a multilinear mapping* $\mathbf{M}\in\mathrm{Lin}_I(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})$ *and a linear mapping* $\mathbf{L}\in\mathrm{Lin}(\mathcal{W},\mathcal{W}')$*, we have* $\mathbf{LM}\in\mathrm{Lin}_I(\times(\mathcal{V}_i \mid i\in I),\mathcal{W}')$.

Proposition 3. *The composite of the cross-product of a family of linear mappings with a multilinear mapping is again multilinear. More precisely, given families of linear spaces* $(\mathcal{V}_i \mid i\in I)$ *and* $(\mathcal{V}'_i \mid i\in I)$*, given a linear space* $\mathcal{W}$ *and, for each* $i\in I$*, a linear mapping* $\mathbf{L}_i\in\mathrm{Lin}(\mathcal{V}_i,\mathcal{V}'_i)$*, and given a multilinear mapping* $\mathbf{M}\in\mathrm{Lin}_I(\times(\mathcal{V}'_i \mid i\in I),\mathcal{W})$*, we have* $\mathbf{M}\circ\times(\mathbf{L}_i \mid i\in I)\in\ \mathrm{Lin}_I(\times(\mathcal{V}_i \mid i\in I),\mathcal{W})$.

From now on we assume that an index set I and linear spaces $\mathcal{V}$ and $\mathcal{W}$ are given.The set of all multilinear mappings from $\mathcal{V}^I$ to $\mathcal{W}$ that are also symmetric [skew] in the sense of Def.1 of Sect.01 will be denoted by $\mathrm{Sym}_I(\mathcal{V}^I,\mathcal{W})$ [$\mathrm{Skew}_I(\mathcal{V}^I,\mathcal{W})$]. If $I:=k^]$ for some $k\in\mathbb{N}$, we use the abbreviations

$$\mathrm{Lin}_k(\mathcal{V}^k,\mathcal{W}):=\mathrm{Lin}_{k^]}(\mathcal{V}^{k^]},\mathcal{W}),$$

$$\mathrm{Sym}_k(\mathcal{V}^k,\mathcal{W}):=\mathrm{Sym}_{k^]}(\mathcal{V}^{k^]},\mathcal{W}),\quad \mathrm{Skew}_k(\mathcal{V}^k,\mathcal{W}):=\mathrm{Skew}_{k^]}(\mathcal{V}^{k^]},\mathcal{W}),\qquad (02.2)$$

which are consistent with Def.5 of Sect.24 of Vol.I.

The following result is an immediate consequence of Def.1 of Sect.01.

Proposition 4. $\mathrm{Sym}_I(\mathcal{V}^I,\mathcal{W})$ *and* $\mathrm{Skew}_I(\mathcal{V}^I,\mathcal{W})$ *are subspaces of* $\mathrm{Lin}_I(\mathcal{V}^I,\mathcal{W})$. *If* $\#I=0$ *or* 1 *then* $\mathrm{Sym}_I(\mathcal{V}^I,\mathcal{W})=\mathrm{Skew}_I(\mathcal{V}^I,\mathcal{W})=\mathrm{Lin}_I(\mathcal{V}^I,\mathcal{W})$. *If* $\#I\geq 2$*, then* $\mathrm{Sym}_I(\mathcal{V}^I,\mathcal{W})$ *is disjunct from* $\mathrm{Skew}_I(\mathcal{V}^I,\mathcal{W})$.

Pitfall: Prop.7 of Sect.24 of Vol.I states, in effect, that $\mathrm{Sym}_I(\mathcal{V}^I,\mathcal{W})$ and $\mathrm{Skew}_I(\mathcal{V}^I,\mathcal{W})$ are actually supplementary in $\mathrm{Lin}_I(\mathcal{V}^I,\mathcal{W})$ when $\#I=2$. They are not supplementary when $\#I>2$, $\dim\mathcal{V}>1$, and $\dim\mathcal{W}>2$. ▮

Given $(p,q)\in I$ with $p\neq q,\mathbf{f}\in\mathcal{V}^I$, and $\mathbf{M}\in\mathrm{Lin}_I(\mathcal{V}^I,\mathcal{W})$ it is clear that $\mathbf{M}(\mathbf{f}.(p,q))\in\mathrm{Lin}_2(\mathcal{V}^2,\mathcal{W})$ and that

$$(\mathbf{M}(\mathbf{f}.(p,q)))^{\sim}(\mathbf{f}_p,\mathbf{f}_q)=\mathbf{M}^{\sim(p,q)}(\mathbf{f}),\qquad (02.3)$$

where the switch on the left is defined according to (24.4) of Vol.I and the switch on the right according to (01.8) and (01.7). Using this fact, we obtain

Proposition 5. *A given* $\mathbf{M} \in \mathrm{Lin}_I(\mathcal{V}^I, \mathcal{W})$ *is symmetric [skew] if and only if*

$$\mathbf{M}(\mathbf{f}.(p,q)) \in \mathrm{Sym}_2(\mathcal{V}^2, \mathcal{W}), \quad [\ \mathbf{M}(\mathbf{f}.(p,q)) \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{W})\ ]$$

for all $\mathbf{f} \in \mathcal{V}^I$ *and all* $(p,q) \in I$ *with* $p \neq q$.

In the case when $I = k^{]}$ for a given $k \in \mathbb{N}$, one can modify Prop.5 in the following manner.

Proposition 6. *A given* $\mathbf{M} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$ *is symmetric [skew] if and only if*

$$\mathbf{M}(\mathbf{f}.(p,p+1)) \in \mathrm{Sym}_2(\mathcal{V}^2, \mathcal{W}), \quad [\ \mathbf{M}(\mathbf{f}.(p,p+1)) \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{W})\ ]$$

for all $\mathbf{f} \in \mathcal{V}^k$ *and all* $p \in (k-1)^{]}$.

The proof of Prop.6 is immediate from Prop.2 of Sect.01.

Proposition 7. *For any given* $\mathbf{M} \in \mathrm{Skew}_I(\mathcal{V}^I, \mathcal{W})$ *and any given linearly dependent* $\mathbf{f} \in \mathcal{V}^I$ *we have* $\mathbf{M}(\mathbf{f}) = \mathbf{0}$.

Proof: It follows from, Prop.1 of Sect.15 of Vol.I that we may choose $\lambda \in (\mathbf{R}^{(I)})^{\times}$ such that $\mathrm{lnc}_{\mathbf{f}}\lambda = \sum(\lambda_i \mathbf{f}_i \mid i \in \ I) = \mathbf{0}$. Since $\lambda \neq 0$, we may choose $j \in I$ such that $\lambda_j \neq 0$. It follows that

$$\mathbf{f}_j = -\frac{1}{\lambda_j} \sum_{i \in I \setminus \{j\}} \lambda_i \mathbf{f}_i. \tag{02.4}$$

Since $\mathbf{M}$ is multilinear and hence $\mathbf{M}(\mathbf{f}.j) : \mathcal{V} \longrightarrow \mathcal{W}$ linear, we conclude that

$$\mathbf{M}(\mathbf{f}) = \mathbf{M}(\mathbf{f}.j)(\mathbf{f}_j) = -\frac{1}{\lambda_j} \sum_{i \in I \setminus \{j\}} \lambda_i \mathbf{M}(\mathbf{f}.j)(\mathbf{f}_i). \tag{02.5}$$

Since $\mathbf{M}(\mathbf{f}.j)(\mathbf{f}_i) = \mathbf{M}(\mathbf{f}.(j,i))(\mathbf{f}_i, \mathbf{f}_i)$ and since $\mathbf{M}(\mathbf{f}.(j,i)) \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{W})$ for all $i \in I \setminus \{j\}$ by Prop.5 above, we conclude from Prop.5 of Sect.24 of Vol.I that $\mathbf{M}(\mathbf{f}.j)(\mathbf{f}_i) = 0$ for all $i \in I \setminus \{j\}$ and hence from (02.5) that $\mathbf{M}(\mathbf{f}) = 0$. ∎

Proposition 8. *Let* $\mathbf{M} \in \mathrm{Lin}_I(\mathcal{V}^I, \mathcal{W})$ *be given. Then* $\mathbf{M}$ *is skew if and only if* $\mathbf{M}(\mathbf{f}) = 0$ *for all* $\mathbf{f} \in \mathcal{V}^I$ *that fail to be injective, i.e., have repeated terms.*

Proof: The "only if" part is a direct consequence of Prop.7 above and Prop.3 of Sect.15 of Vol.I.

To prove the "if" part, assume that $\mathbf{M}(\mathbf{f}) = 0$ for all non-injective $\mathbf{f} \in \mathcal{V}^I$. Let $\mathbf{g} \in \mathcal{V}^I$ and $(p,q) \in I^2$ with $p \neq q$ be given. For every $\mathbf{u} \in \mathcal{V}$, $\mathbf{f} := (\mathbf{g}.(p,q))(\mathbf{u},\mathbf{u})$ is then non-injective because $\mathbf{f}_p = \mathbf{f}_q = \mathbf{u}$. Hence we have $\mathbf{M}((\mathbf{g}.(p,q))(\mathbf{u},\mathbf{u})) = 0$ for every $\mathbf{u} \in \mathcal{V}$. By Prop.5 of Sect.24 of Vol.I it follows that $\mathbf{M}(\mathbf{g}.(p,q)) \in \mathrm{Skew}_2(\mathcal{V}^2, \mathcal{W})$. Since $\mathbf{g} \in \mathcal{V}^I$ and $(p,q) \in I^2$ with $p \neq q$ were arbitrary, it follows from Prop.5 above that $\mathbf{M} \in \mathrm{Skew}_I(\mathcal{V}^I, \mathcal{W})$. ∎

In the case when $I = k^{]}$ for a given $k \in \mathbb{N}$, one can modify Prop.7 in the following manner.

Proposition 9. *Let* $\mathbf{M} \in \mathrm{Lin}_k(\mathcal{V}^k, \mathcal{W})$ *be given. Then* $\mathbf{M}$ *is skew if and only if* $\mathbf{M}(\mathbf{f}) = 0$ *for all* $\mathbf{f} \in \mathcal{V}^k$ *that have adjacent repeated terms in the sense that* $\mathbf{f}_j = \mathbf{f}_{j+1}$ *for some* $j \in (k-1)^]$.

The proof of Prop.9 is analogous to that of Prop.8 except that Prop.6 rather than Prop.5 is used in it.

For later use we record here a generalization to multilinear mappings of the general distributive law (07.19) of Vol.I:

Proposition 10. *Let* $\mathbf{M} \in \mathrm{Lin}_I(\mathcal{V}^I, \mathcal{W})$ *be given. Then*

$$\mathbf{M}(\mathbf{f}+\mathbf{g}) = \sum_{J \in \mathrm{Sub} I} \mathbf{M}(\mathbf{f}|_{I \setminus J} .J)(\mathbf{g}|_J) \quad \text{for all} \quad \mathbf{f}, \mathbf{g} \in \mathcal{V}^I. \tag{02.6}$$

Given $j, k \in \mathbb{P}^\times$ it is often useful to use the identification

$$\mathcal{V}^j \times \mathcal{V}^k \cong \mathcal{V}^{j+k} \tag{02.6}$$

by identifying a given pair $(\mathbf{x}, \mathbf{y})$ of list $\mathbf{x} \in \mathcal{V}^j$, $\mathbf{y} \in \mathcal{V}^k$ with a single list in $\mathcal{V}^{j+k}$ defined by concatenation in the sense that

$$(\mathbf{x}, \mathbf{y})_i := \left\{ \begin{array}{lcr} x_i & \text{if} & i \in j^] \\ y_{i-j} & \text{if} & i \in (j+1)..(j+k) \end{array} \right\} \quad \text{for all } i \in (j+k)^] \ . \tag{02.7}$$

Also it is often useful to use the identification

$$\mathrm{Lin}_{\,j}(\mathcal{V}^j, \mathrm{Lin}_{\,k}(\mathcal{V}^k, \mathcal{W}) \cong \mathrm{Lin}_{\,k+j}(\mathcal{V}^{j+k}, \mathcal{W}) \tag{02.8}$$

with the following precaution: If $\mathbf{M}$ belongs to the right side of (02.8), then the corresponding element on the left side of (02.8) will be denoted by $\mathbf{M}_{<j>}$. The two are related by

$$\mathbf{M}(\mathbf{x}, \mathbf{y}) = \mathbf{M}_{<j>}(\mathbf{x})(\mathbf{y}) \qquad \text{for all} \quad \mathbf{x} \in \mathcal{V}^j \ , \mathbf{y} \in \mathcal{V}^k \ . \tag{02.9}$$

If $\mathbf{M}$ belongs to the left side of (02.8), then the corresponding element on the right side of (02.8) will be denoted by $\mathbf{M}_{<\rightarrow>}$.

Proposition 11. *Let* $m \in \mathbb{P}^\times$ *and* $\mathbf{S} \in \mathrm{Lin}_{\,m+1}(\mathcal{V}^{m+1}, \mathcal{W})$ *be given. Using identifications and notations of the type (02.8) and (02.9), we have*

$$\mathbf{S}_{<1>} \in \mathrm{Lin}\,(\mathcal{V}, \mathrm{Lin}_{\,m}(\mathcal{V}^m, \mathcal{W})) \ \ , \quad \mathbf{S}_{<2>} \in \mathrm{Lin}\,(\mathcal{V}^2, \mathrm{Lin}_{\,(}\mathcal{V}^{m-1}, \mathcal{W}) \ . \tag{02.8}$$

If $\mathbf{S}_{<2>}$ *is symmetric and if the values of* $\mathbf{S}_{<2>}$ *are symmetric, then* $\mathcal{S}$ *itself is symmetric.*

03 Convex functions

We assume that a finite-dimensional flat space $\mathcal{E}$ with translation space $\mathcal{V}$ and a subset $\mathcal{D}$ of $\mathcal{E}$ are given. The **epigraph** of a function $f : \mathcal{D} \longrightarrow \mathbf{R}$ is defined by

$$\mathrm{Epi}(f) := \{(x, s) \mid x \in \mathcal{D}, s \in f(x) + \mathbb{P}\} \ \subset \ \mathcal{E} \times \mathbf{R} \tag{X.1}$$

We now assume that a function $f : \mathcal{D} \longrightarrow \mathbf{R}$ is given.

Definition 1. *We say that the function f is* **convex** *if its epigraph is a convex subset of $\mathcal{E} \times \boldsymbol{R}$.*

Proposition 1. *If f is convex, then its domain $\mathcal{D}$ is a convex subset of $\mathcal{E}$.*

From now on, we assume that $\mathcal{D}$ is convex.

Proposition 2. *The function $f : \mathcal{D} \longrightarrow \boldsymbol{R}$ is convex if and only if*

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y) \qquad \text{for all } \; x, y \in \mathcal{D} \; , \; \lambda \in [0,1] \; . \quad (X.2)$$

Proposition 3. *Assume that $f : \mathcal{D} \longrightarrow \boldsymbol{R}$ is convex. Let a non-empty family $p := (p_i \mid i \in I)$ of points in $\mathcal{D}$ and $\lambda \in (\mathbb{P}^{(I)})_1$ be given. (See Def.3 of Sect 37 of Vol.I .) Then*

$$f(\mathrm{cxc}_p(\lambda)) \leq \mathrm{cxc}_{f \circ p}(\lambda) \; . \qquad (X.3)$$

Proposition 4. *Assume that $f : \mathcal{D} \longrightarrow \boldsymbol{R}$ is convex. Let a flat space $\mathcal{F}$ and a flat mapping $\alpha : \mathcal{F} \longrightarrow \mathcal{E}$ be given. Then*

$$f \circ \alpha|_{\alpha^{<}(\mathcal{D})}^{\mathcal{D}} : \alpha^{<}(\mathcal{D}) \longrightarrow \boldsymbol{R} \qquad (X.4)$$

is convex.

Proposition 5. *Let Φ be a collection of convex functions with domain $\mathcal{D}$. Assume that, for every $x \in \mathcal{D}$ the set $\{h(x) \mid h \in \Phi\}$ is bounded above and define $g : \mathcal{D} \longrightarrow \boldsymbol{R}$ by*

$$g(x) := \sup\{h(x) \mid h \in \Phi\} \qquad \text{for all} \quad x \in \mathcal{D} \; . \qquad (X.5)$$

Then g is convex.

Proof: *It easily follows from (X.5) that*

$$\mathrm{Epi}(g) = \bigcap_{h \in \Phi} \mathrm{Epi}(h) \; . \qquad (X.6)$$

Hence, by Def.1, $\mathrm{Epi}(g)$ *is the intersection of a collection of convex sets and hence itself convex by Prop.1 of Sect.37 of Vol.I.* ▮

Proposition 6. *Let $(h_i \mid i \in I)$ be a family of convex functions in* $\mathrm{Map}(\mathcal{D}, \boldsymbol{R})$*. Then $\sum_{i \in I} \lambda_i h_i$ is convex for all $\lambda \in \mathbb{P}^{(I)}$.*

Theorem. *If the domain $\mathcal{D}$ of the function f is open (and convex) and if f is convex,then the restriction $f|_{\mathcal{C}}$ of f to every compact subset $\mathcal{C}$ of $\mathcal{D}$ is constricted.*

Proof .... (Use (ii) of Prop.1 of Sect 64 of Vol.I in the case when ν is a diamond norm.)

Corollary. *If the domain $\mathcal{D}$ of the function f is open (and convex) and if f is convex, then it is continuous.*

Proposition 6. *Assume that the domain $\mathcal{D}$ of f is open (and convex) and that f is differentiable. Then f is convex if and only if*

$$(\nabla_x f - \nabla_y f)(x - y) \geq 0 \qquad \text{for all} \quad x, y \in \mathcal{D}\ . \tag{X.7}$$

Proposition 7. *Assume that the domain $\mathcal{D}$ of f is open (and convex) and that f is twice differentiable. Then f is convex if and only if the values of the second gradient $\nabla^{(2)} f : \mathcal{D} \longrightarrow \mathrm{Qu}(\mathcal{V})$ are positive quadratic forms. (See Def.1 of Sect.27 of Vol.I.)*