The Conceptual Infrastructure of Mathematics

0. Introduction

By conceptual *infrastructure* of mathematics I mean the concepts, the terminology, the symbols, and the notations that mathematicians and people who apply mathematics use in their daily professional activities. This is very different from what is often called the conceptual *foundation* of mathematics, which deals critically with the deeper meaning of these concepts, an activity which only some mathematicians, and these only occasionally, worry about. I believe that the infrastructure of mathematics was and still is being developed in four stages (all quotations in Sects 1 - 2 are from *Mathematical Thought from Ancient to Modern Times*, 1972, by Morris Kline):

1. Axioms, Theorems, and Proofs

"Mathematics as an organized, independent, and reasoned discipline did not exist before the classical Greeks of the period from 600 to 300 B.C.". The Greeks introduced mathematics as a deductive discipline, with axioms, theorems, and proofs . The Greeks had some knowledge of number theory (e.g. they had the proof that there is no largest prime number), but most of their contributions dealt with geometry. Euclids Elements (about 300 B.C.) gave a comprehensive account of Greek mathematics, and it has been part of the infrastructure of mathematics ever since.

2. Efficient Notations, Variables

Fundamental additions to the infrastructure were introduced in the 16th and 17th century "The advance in algebra that proved far more significant than the technical progress of the 16th century was the introduction of better symbolism." Familiar notations such as $+, -, <, >, =, \sqrt{}$ and the use of parentheses, square brackets, and braces were introduced during that time (For example, $=$ was introduced by Robert Recorde in 1557.) "The most significant change in the character of algebra was introduced in connection with symbolism by Francois Vieta (1540 - 1603)... He was the first to use letters purposefully and systematically, not just to represent an unknown or powers of an unknown but as general coefficients. Improvements in Vieta's use of letters are due to Descartes (1596 - 1650). He used first letters of the alphabet for known quantities and last letters for unknowns,... Leibniz (1646 - 1716) must be mentioned in the history of symbolism,... He made prolonged studies of the various notations, asked the opinions of his contemporaries, and then chose the best... Thus, by the end of the 17th century, the deliberate use of symbolism - as opposed to the incidental and accidental use - and the awareness of the power of generality it confers entered mathematics."

The idea of a function was developed in the 17th century and made the the invention of differential calculus possible. It was Leibniz, in the late 1600's, who introduced the concepts as well as the terms *variable, constant*, and *parameter*.

A function was defined to be the relation between an independent variable and a dependent variable. Newton used the term *fluent* for such a relation. It was Leibniz who introduced also the concepts as well as the notations of calculus, such as $\frac{\mathrm{d}y}{\mathrm{d}x}$ and the integral sign $\int$.

Without the infrastructure introduced in the 16th and 17th century the rapid advances of mathematics in the 18th and 19th century would have been impossible.

3. Sets, Mapping, and Mathematical Structures

The end of the 19th century saw the beginning of a fundamental shift in the infrastructure of mathematics, starting with the work on the concept of a set by Georg Cantor. More and more, mathematics is seen to be dealing with sets and mappings, rather than with variables. The concepts of domain and codomain of a mapping are made explicit. Functions are special kinds of mappings and are considered as objects in themselves and are given names, generic or specific. Instead of $y = f(x)$ to denote a function, in the language of sets and mappings one uses just the single letter f by itself and considers the domain $\mathrm{Dom}\, f$ and codomain $\mathrm{Cod}\, f$ of f to be parts of the specification of f. So it becomes possible to consider sets (spaces) of functions rather than only specific functions considered one by one. A *functional* is a mapping whose domain is a set of functions. The now standard notations of the language of sets, such as $\in$ for membership, $\subset$ for inclusion, $\cup$ for union, and $\cap$ for intersection are fairly recent. They were not yet used when I first learned about set theory while in high school (from the little book by E.Kamke, published first in the early 1930's). Even more recent are notations such as $f : D \longrightarrow C$ and $U := \{x \in S \mid ...x...\}$.

Using the concept of a set as a background, the past 100 years or so saw the introduction of the idea of an abstract mathematical structure. Examples of such structures are groups, rings, fields, topological spaces, linear spaces. The multi-volume work of Bourbaki, *Éléments de Mathématique*, starting in 1939, explicitly introduced the concept of a mathematical structure and gave a systematic description of the important structures, using the language of sets and mappings.

4. Categories and Functors

In 1942 , Eilenberg and Mac Lane introduced the concepts as well as the terms *category, functor* and *natural transformation*, and thereby added very importantly to the infrastructure of mathematics.

Eilenberg died in January 1998, and the November 1998 issue of the *Notices of the American Mathematical Society* contains the following descriptions of his role and his views, which bear upon the conceptual infrastructure of mathematics.

Alex Heller: "As I perceived it then, Sammy considered that the highest value in mathematics was to be found, not in specious depth nor in overcoming of overwhelming difficulty, but rather in providing the definitive clarity that would illuminate its underlying order. This was to be accomplished by elucidating the true structure of the objects of mathematics. Let me hasten to say that this was in no sense an ontological quest: the true structure was intrinsic to mathematics and was to be discerned only by doing more mathematics. Sammy had no patience for metaphysical argument.. He was not a Platonist: equally, he was not a non-Platonist. It might be more to the point to make a different distinction: Sammys mathematical aesthetic was classical rather than romantic.

Category theory was one of Sammys principal tools in his search for mathematical reality. Category theory also developed into a mathematical subject with its own honorable history and practitioners, beginning with MacLane and including, notably, F.W.Lawvere, Sammys most remarkable student, who saw it as a foundation for all of mathematics and justified this intuition with such innovations as categorical semantics and topos theory. Sammy did not, I think want to be reckoned a member of this school. I believe, in fact, that he would have rejected the idea that mathematics needed a foundation. Category theory was for him only a tool - in fact, a powerful one - for expanding our understanding.. It was his willingness to search for this understanding at an ever higher level that really set him apart and that made him, in my estimation, the author of a revolution in mathematics as notable as that initiated by Cantors invention of set theory, Like Cantor, Sammy has changed the way we think about mathematics."

Hyman Bass: "He fit squarely into the tradition of Hilbert, E.Artin, E.Noether, and Bourbaki: He was a champion of the axiomatic unification that so dominated the early postwar mathematics. His philosophy was that the aims of mathematics are to find and articulate with clarity and economy the underlying principles that govern mathematical phenomena. Complexity and opaqueness were, for him, signs of insufficient understanding. He sought not just theorems, but ways to make the truth transparent, natural, inevitable for the 'right thinking' person. It was this 'right thinking', not just facts, that Sammy tried to teach and that, in many domains, he succeeded in teaching to a whole generation of mathematicians."

Eilenberg was the only non-French member of the Bourbaki-group.

5. The present situation

At the present time, all four stages of the conceptual infrastructure described above are used to some extent. High school geometry is still done more or less as described in Euclids Elements. (I remember being forced to memorize by rote some of Euclids axioms in an 8th grade geometry course; it almost turned me away from mathematics.) High school algebra and elementary calculus are still done, in most cases, using the language of variables and constants as described in stage 2 above. On this level, a function is still denoted by $y = f(x)$ rather than just f. People are reluctant to use sin or cos by themselves rather than $\sin x$ or $\cos x$. The identity function of the real numbers is described by $y = x$.

The language of sets and mappings as described in stage 3 above provides for much more precision, clarity, and economy than the language of variables and constants. In 1972 (?), I taught an elementary calculus course here at CMU in which I systematically used the language of sets and mappings. (I introduced the notation ι for the identity function of the reals.) There is still much resistance against the language of sets and mappings. An attempt to introduce this language into high school mathematics, known as the "New Math", turned out to be an utter failure, because teachers cannot teach something that they do not understand themselves. During my Ph.D. Final in 1954, the examiners circulated a note with "too much Bourbaki" on it. A review of a volume of selected papers by me and collaborators downgrades much of my work as merely "presenting well-known and well-understood concepts in continuum mechanics or thermodynamics in the idiom of elementary set theory".

I am convinced that the concepts of sets, mappings, and mathematical structures will eventually displace the older modes of conceptual mathematics. But this change requires more and better concepts, terminology, and notation than the illogical and ambiguous mishmash that is now considered to be standard. During the past 25 years or so, Juan Schäffer and I have developed such an infrastructure for the basic parts of mathematics (in connection with our undergraduate honors program entitled *Mathematical Studies*). Although we had no interaction with Eilenberg, our philosophy has been the same as the one ascribed to him above: "His philosophy was that the aims of mathematics are to find and articulate with clarity and economy the underlying principles that govern mathematical phenomena. Complexity and opaqueness were, for him, signs of insufficient understanding. He sought not just theorems, but ways to make the truth transparent, natural, inevitable for the 'right thinking' person."

The language of categories and functors is used extensively in algebraic topology but not yet very much in other fields of mathematics.

One should carefully distinguish between the *language* of sets and mappings and the *language* of categories and functors on the one hand and *set theory* and *category theory* on the other. The latter have become specialties and have been pursued by specialists in their fields, but as such have had very little influence on the rest of mathematics.

6. Isocategories

A *mathematical structure* is obtained by prescribing a list of *ingredients* and postulating a list of *axioms*. For example, to obtain a *topological space*, one must specify a set and a collection of subsets of it (the *open sets*) and then require that certain axioms be satisfied. Perhaps the simplest example of an algebraic structure is that of a *pre-monoid* : the ingredients are a set M and a mapping $cmb : M \times M \longrightarrow M$, called *combination* in the abstract or *multiplication, addition*, or *composition* in special cases. The only axiom is the associative law.*Monoids* and *groups* are obtained from pre-monoids by adding ingredients and axioms.

A *species of structure* consists of all mathematical structures obtained by a particular recipe. For example, the species of topological spaces consists of all topological spaces. For each species, there is a natural definition of isomorphism between two members of the species. Thus, every species gives rise immediately to an *isocategory*: its objects are the members of the species and its morphisms are the isomorphisms. Most branches of mathematics deal extensively with procedures that start with a given structure and construct from it a new structure that may or may not belong to the same species as the given one. For example, from each set S one can construct the set Sub$S$ of all subsets of S and also the group Perm$S$ of all permutatioms of S. Both Sub and Perm may be viewed as *isofunctors*. From the point of view of category *theory*, isocategories may not be particularly interesting, but I believe that they can provide a powerful conceptual tool for articulating with clarity and economy the underlying principles that govern many mathematical phenomena. Most other examples of categories can be obtained from isocategories by extending the class of isomorphism to a larger class of morphisms. For example, one can use the isocategory of topological spaces to construct an enlarged category of topological spaces by taking all continuous mappings or all open mappings to be the morphisms.

In a paper entitled **Isocategories and Tensor Functors**, written in 1992, I have shown how one can gain better insight into many aspects of linear algebra and differential geometry by using a special type of isofunctor which I call *tensor functor*.