Chapter 9

The Structure of General Lineons

In this chapter, it is assumed again that all linear spaces under consideration are finite-dimensional except when a statement to the contrary is made. However, they may be spaces over an *arbitrary* field $\mathbb{F}$. If $\mathbb{F}$ is to be the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers, it will be explicitly stated. We will make frequent use of the definitions and results of Sects.81 and 82, which remain valid for finite-dimensional spaces over $\mathbb{F}$ even if $\mathbb{F}$ is not $\mathbb{R}$.

91 Elementary Decompositions

Let $\mathbf{L}$ be a lineon on a given linear space $\mathcal{V}$.

Definition 1: *We say that a subspace $\mathcal{M}$ of $\mathcal{V}$ is a* **minimal L-space** [**maximal L-space**] *if $\mathcal{M}$ is minimal [maximal] (with respect to inclusion) among all $\mathbf{L}$-subspaces of $\mathcal{V}$ that are different from the zero space $\{\mathbf{0}\}$ [the whole space $\mathcal{V}$].*

If $\mathcal{V}$ is not itself a zero-space, then there always exist minimal and maximal $\mathbf{L}$-spaces. In fact, if $\mathcal{U}$ is an $\mathbf{L}$-space and $\mathcal{U} \neq \{\mathbf{0}\}$ [$\mathcal{U} \neq \mathcal{V}$], then $\mathcal{U}$ includes [is included in] a minimal [maximal] $\mathbf{L}$-space. The following result follows directly from Def.1.

Proposition 1: *Let $\mathcal{U}$ be an $\mathbf{L}$-subspace of $\mathcal{V}$. If $\mathcal{M}$ is a minimal $\mathbf{L}$-space, then either $\mathcal{M} \cap \mathcal{U} = \{\mathbf{0}\}$ or else $\mathcal{M} \subset \mathcal{U}$. If $\mathcal{W}$ is a maximal $\mathbf{L}$-space, then either $\mathcal{W} + \mathcal{U} = \mathcal{V}$ or else $\mathcal{U} \subset \mathcal{W}$.*

The next result is an immediate consequence of Prop.2 of Sect.21 and Prop.1 of Sect.82.

Proposition 2: *A subspace $\mathcal{M}$ of $\mathcal{V}$ is a minimal [maximal] $\mathbf{L}$-space if and only if the annihilator $\mathcal{M}^{\perp}$ of $\mathcal{M}$ is a maximal [minimal] $\mathbf{L}^{\top}$-space.*

Definition 2: *We say that the lineon $\mathbf{L}$ on $\mathcal{V}$ is an* **elementary lineon** *if there is exactly one minimal $\mathbf{L}$-space. If $\mathbf{L}$ is an arbitrary lineon on $\mathcal{V}$, an $\mathbf{L}$-subspace $\mathcal{U}$ of $\mathcal{V}$ is called an* **elementary L-space** *if $\mathbf{L}_{|\mathcal{U}}$ is elementary. We say that a decomposition of $\mathcal{V}$* (see Sect.81) *is an* **elementary L-decomposition** *of $\mathcal{V}$ if all of its terms are elementary $\mathbf{L}$-spaces.*

The following result is an immediate consequence of Def.2.

Proposition 3: *Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ be an elementary lineon and $\mathcal{M}$ its (only) minimal $\mathbf{L}$-space. If $\mathcal{U}$ is a non-zero $\mathbf{L}$-space, then $\mathcal{M} \subset \mathcal{U}$ and $\mathbf{L}_{|\mathcal{U}} \in \mathrm{Lin}\,\mathcal{U}$ is again an elementary lineon and the (only) minimal $\mathbf{L}_{|\mathcal{U}}$-space is again $\mathcal{M}$.*

We say that a lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is **simple** if $\mathcal{V} \neq \{\mathbf{0}\}$ and if there are no $\mathbf{L}$-subspaces other than $\{\mathbf{0}\}$ and $\mathcal{V}$. A simple lineon is elementary; its only minimal $\mathbf{L}$-space is $\mathcal{V}$ and its only maximal $\mathbf{L}$-space is $\{\mathbf{0}\}$. If $\mathbf{L}$ is elementary but not simple and if $\mathcal{W}$ is a maximal $\mathbf{L}$-space, then $\mathcal{W}$ includes the only minimal $\mathbf{L}$-space of $\mathcal{V}$. A lineon of the form $\lambda\mathbf{1}_{\mathcal{V}},\ \lambda \in \mathbb{F}$, is elementary (and then simple) if and only if $\dim\mathcal{V} = 1$.

The significance of Def.2 lies in the following theorem, which reduces the study of the structure of general lineons to that of elementary lineons.

Elementary Decomposition Theorem: *For every lineon $\mathbf{L}$ there exists an elementary $\mathbf{L}$-decomposition.*

The proof is based on three lemmas. The proof of the first of these will be deferred to Sect.93; it is the same as Cor.2 to the Structure Theorem for Elementary Lineons.

Lemma 1: *A lineon $\mathbf{L}$ is elementary if and only if there is exactly one maximal $\mathbf{L}$-space.*

Lemma 2: *Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, let $\mathcal{W}$ be a maximal $\mathbf{L}$-space and let $\mathcal{E}$ be an $\mathbf{L}$-space that is minimal among all subspaces of $\mathcal{V}$ with the property $\mathcal{W}+\mathcal{E} = \mathcal{V}$. Then $\mathcal{E}$ is an elementary $\mathbf{L}$-space.*

Proof: By Prop.1 we have $\mathcal{E} \not\subset \mathcal{W}$ and hence $\mathcal{E}\cap\mathcal{W} subsetneqq \mathcal{E}$. Now let $\mathcal{U}$ be an $\mathbf{L}$-space that is properly included in $\mathcal{E}$, i.e. an $\mathbf{L}_{|\mathcal{E}}$-subspace of $\mathcal{E}$ other than $\mathcal{E}$. In view of the assumed minimality of $\mathcal{E}$, we have $\mathcal{W}+\mathcal{U} subsetneqq \mathcal{V}$. Using Prop.1 again, it follows that $\mathcal{U} \subset \mathcal{W}$ and hence $\mathcal{U} \subset \mathcal{E} \cap \mathcal{W}$. We conclude that $\mathcal{W} \cap \mathcal{E}$ is the only maximal $\mathbf{L}_{|\mathcal{E}}$-space and hence, by Lemma 1, that $\mathcal{E}$ is elementary. ∎

Lemma 3: *Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and let $\mathcal{E}$ be an elementary $\mathbf{L}$-space with greatest possible dimension. Then there is an $\mathbf{L}$-space that is a supplement of $\mathcal{E}$ in $\mathcal{V}$.*

Proof: Let $\mathcal{M}$ be the (only) minimal $\mathbf{L}$-space included in $\mathcal{E}$. By Prop.2, $\mathcal{M}^{\perp}$ is then a maximal $\mathbf{L}^{\top}$-space. We choose a subspace $\tilde{\mathcal{E}}$ of $\mathcal{V}^{*}$ that is minimal among all subspaces of $\mathcal{V}^*$ with the property $\mathcal{M}^\perp + \tilde{\mathcal{E}} = \mathcal{V}^*$. By Lemma 2, $\tilde{\mathcal{E}}$ is an elementary $\mathbf{L}^\top$-space.

In view of (21.11) and (22.5), we have $\{\mathbf{0}\} = \mathcal{V}^{*\perp} = (\mathcal{M}^\perp + \tilde{\mathcal{E}})^\perp = \mathcal{M}^{\perp\perp} \cap \tilde{\mathcal{E}}^\perp = \mathcal{M} \cap \tilde{\mathcal{E}}^\perp$ and hence $\mathcal{M} \not\subset \tilde{\mathcal{E}}^\perp$ and also $\mathcal{M} \not\subset \tilde{\mathcal{E}}^\perp \cap \mathcal{E}$. Since $\tilde{\mathcal{E}}^\perp \cap \mathcal{E}$ is an $\mathbf{L}$-subspace of $\mathcal{V}$, it follows from Prop.1 that

$$\tilde{\mathcal{E}}^\perp \cap \mathcal{E} = \{\mathbf{0}\}. \tag{91.1}$$

Using Prop.4 of Sect.17 and the Formula (21.15) for Dimension of Annihilators, we easily conclude from (91.1) that

$$\dim \tilde{\mathcal{E}} \geq \dim \mathcal{E}. \tag{91.2}$$

Since $\tilde{\mathcal{E}}$ is elementary, we can repeat the argument above with $\mathbf{L}$ replaced by $\mathbf{L}^\top$ and $\mathcal{E}$ replaced by $\tilde{\mathcal{E}}$. Since $\mathbf{L}^{\top\top} = \mathbf{L}$, we thus find an elementary $\mathbf{L}$-subspace $\mathcal{E}'$ of $\mathcal{V}$ such that

$$\dim \mathcal{E}' \geq \dim \tilde{\mathcal{E}}. \tag{91.3}$$

Since $\mathcal{E}$ was assumed to have the greatest possible dimension, we must have $\dim \mathcal{E}' \leq \dim \mathcal{E}$ and hence, by (91.2) and (91.3), $\dim \tilde{\mathcal{E}} = \dim \mathcal{E}$, which gives

$$\dim \mathcal{V} = \dim \tilde{\mathcal{E}}^\perp + \dim \tilde{\mathcal{E}} = \dim \tilde{\mathcal{E}}^\perp + \dim \mathcal{E}.$$

In view of (91.1), it follows from Prop.5 of Sect.17 that $\tilde{\mathcal{E}}^\perp$ is a supplement of $\mathcal{E}$. ■

Proof of the Theorem: We will show that there exists a collection of elementary $\mathbf{L}$-spaces which, when interpreted as a self-indexed family, is a decomposition of $\mathcal{V}$.

We proceed by induction over the dimension of $\operatorname{Dom}\mathbf{L}$. If $\dim \operatorname{Dom}\mathbf{L} = 0$, then the empty collection is an elementary $\mathbf{L}$-decomposition. Assume then, that $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ with $\dim \mathcal{V} > 0$ is given and that the assertion is valid for every lineon whose domain has a dimension strictly less than $\dim \mathcal{V}$.

Since $\mathcal{V}$ is not the zero-space, there exist minimal $\mathbf{L}$-spaces and hence elementary $\mathbf{L}$-spaces. We choose an elementary $\mathbf{L}$-space $\mathcal{E}$ with greatest possible dimension. By Lemma 3, we can choose an $\mathbf{L}$-subspace $\mathcal{U}$ that is a supplement of $\mathcal{E}$ in $\mathcal{V}$. Since $\dim \mathcal{U} = \dim \mathcal{V} - \dim \mathcal{E} < \dim \mathcal{V}$, we can apply the induction hypothesis to $\mathbf{L}_{|\mathcal{U}}$ and determine a collection $\mathfrak{F}$ of elementary $\mathbf{L}_{|\mathcal{U}}$-spaces which, self-indexed, is a decomposition of $\mathcal{U}$. The elements of $\mathfrak{F}$ are elementary $\mathbf{L}$-spaces. Since $\mathcal{E}$ is a supplement of

$\mathcal{U} = \sum(\mathcal{F} \mid \mathcal{F} \in \mathfrak{F})$, it follows from Prop.2,(iii) of Sect.81 that $\mathfrak{F} \cup \{\mathcal{E}\}$ is an elementary **L**-decomposition of $\mathcal{V}$. ■

We say that a lineon $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ is **semi-simple** if there is an **L**-decomposition $(\mathcal{U}_i \mid i \in I)$ of $\mathcal{V}$ such that $\mathbf{L}_{|\mathcal{U}_i}$ is simple for each $i \in I$. Of course, such a decomposition is an elementary decomposition.

If $\mathcal{V}$ is a genuine inner product space, then every normal lineon **N** (and hence every skew, symmetric, or orthogonal lineon) on $\mathcal{V}$ is semi-simple. Indeed, if we choose a basis **e** as in the Corollary to the Structure Theorem for Normal Lineons of Sect.88 and then define

$$\mathcal{U}_k := \left\{ \begin{array}{lll} \operatorname{Lsp}\{\mathbf{e}_{2k-1}, \mathbf{e}_{2k}\} & \text{for all} & k \in m^{]} \\ \operatorname{Lsp}\{\mathbf{e}_{2m+k}\} & \text{for all} & k \in (n-m)^{]} \setminus m^{]} \end{array} \right\}, \tag{91.4}$$

then $(\mathcal{U}_k \mid k \in (n-m)^{]})$ is a decomposition of $\mathcal{V}$ such that $\mathbf{L}_{|\mathcal{U}_k}$ is simple for each $k \in (n-m)^{]}$.

Pitfall: The collection of subspaces that are the terms of an elementary **L**-decomposition is, in general, not uniquely determined by **L**. For example, if $\mathbf{L} = \mathbf{1}_{\mathcal{V}}$, and if $\mathbf{b} := (\mathbf{b}_i \mid i \in I)$ is a basis of $\mathcal{V}$, then $(\operatorname{Lsp}\{\mathbf{b}_i\} \mid i \in I)$ is an elementary $\mathbf{1}_{\mathcal{V}}$-decomposition of $\mathcal{V}$. If $\dim \mathcal{V} > 1$, then the collection $\{\operatorname{Lsp}\{\mathbf{b}_i\} \mid i \in I\}$ depends on the choice of the basis **b** and is not uniquely determined by $\mathcal{V}$. ■

Notes 91

(1) The concept of an elementary lineon and the term "elementary" in the sense of Def.2, were introduced by me in 1970 (see Part E of the Introduction). I chose this term to correspond to the commonly accepted term "elementary divisor". In fact, the elementary divisors of a lineon can be matched, if counted with appropriate multiplicity, with the terms of any elementary decomposition of the lineon (see Sect.95).

(2) The approach to the structure of general lineons presented in this Chapter was developed by me before 1970 and is very different from any that I have seen in the literature. The conventional approaches usually arrive at an elementary decomposition only after an intermediate (often called "primary") decomposition, which I was able to bypass. Also, the conventional treatments make heavy use of the properties of the ring of polynomials over $\mathbb{F}$ (unique factorization, principal ideal ring property, etc.). My approach uses only some of the most basic concepts and facts of linear algebra as presented in Chaps.1 and 2. Not even determinants are needed. The proofs are all based, in essence, on dimensional considerations.

92 Lineonic Polynomial Functions

Let $\mathbb{F}$ be any field. The elements of $\mathbb{F}^{(\mathbb{N})}$, i.e. the sequences in $\mathbb{F}$ indexed on $\mathbb{N}$ and with finite support (see (07.10)), are called **polynomials** over $\mathbb{F}$. As we noted in Sect.14, $\mathbb{F}^{(\mathbb{N})}$ is a linear space over $\mathbb{F}$. It is infinite-dimensional. We use the abbreviation

$$\iota := \delta_1^{\mathbb{N}}, \tag{92.5}$$

where $\delta^{\mathbb{N}}$ is the standard basis of $\mathbb{F}^{(\mathbb{N})}$ (see Sect.16). The space $\mathbb{F}^{(\mathbb{N})}$ acquires the structure of a commutative ring (see Sect.06) if we define its multiplication $(p, q) \mapsto pq$ by

$$(pq)_n := \sum_{k \in n^{[}} p_k q_{n-k} \quad \text{for all} \quad n \in \mathbb{N}. \tag{92.6}$$

This multiplication is characterized by the condition that the n'th power ι^n of ι be given by

$$\iota^n = \delta_n^{\mathbb{N}} \quad \text{for} \quad n \in \mathbb{N}. \tag{92.7}$$

The unity of $\mathbb{F}^{(\mathbb{N})}$ is $\iota^0 = \delta_0^{\mathbb{N}}$. We identify $\mathbb{F}$ with the subring $\mathbb{F}\iota^0$ of $\mathbb{F}^{(\mathbb{N})}$ and hence write

$$\xi = \xi\iota^0 = \xi\delta_0^{\mathbb{N}} \quad \text{for all} \quad \xi \in \mathbb{F}, \tag{92.8}$$

so that $1 = \iota^0$ denotes the unity of both $\mathbb{F}$ and $\mathbb{F}^{(\mathbb{N})}$. We have

$$p = \sum_{k \in \mathbb{N}} p_k \iota^k \quad \text{for all} \quad p \in \mathbb{F}^{(\mathbb{N})}. \tag{92.9}$$

The **degree** of a non-zero polynomial p is defined by

$$\deg p := \max \operatorname{Supt} p = \max\{k \in \mathbb{N} \mid p_k \neq 0\} \tag{92.10}$$

(see (07.9)). If $p \in \mathbb{F}^{(\mathbb{N})}$ is zero or if $\deg p \leq n$, then

$$p = \sum_{k \in (n+1)^{[}} p_k \iota^k. \tag{92.11}$$

We have

$$\deg(pq) = \deg p + \deg q \tag{92.12}$$

for all $p, q \in \left(\mathbb{F}^{(\mathbb{N})}\right)^{\times}$. We say that $p \in \left(\mathbb{F}^{(\mathbb{N})}\right)^{\times}$ is a **monic polynomial** if $p_{\deg p} = 1$. If this is the case, then

$$p = \iota^{\deg p} + \sum_{k \in (\deg p)^{[}} p_k \iota^k. \tag{92.13}$$

The **polynomial function** $(\xi \mapsto p(\xi)) : \mathbb{F} \to \mathbb{F}$ associated with a given polynomial $p \in \mathbb{F}^{(\mathbb{N})}$ is defined by

$$p(\xi) := \sum_{k\in\mathbb{N}} p_k \xi^k \quad \text{for all} \quad \xi \in \mathbb{F}. \tag{92.14}$$

We call $p(\xi)$ the **value** of p at ξ. For every given $\xi \in \mathbb{F}$, the mapping $(p \mapsto p(\xi)) : \mathbb{F}^{(\mathbb{N})} \to \mathbb{F}$ is linear and preserves products, i.e. we have

$$p(\xi)q(\xi) = (pq)(\xi) \quad \text{for all} \quad p, q \in \mathbb{F}^{(\mathbb{N})}. \tag{92.15}$$

Remark: If $\mathbb{F} := \mathbb{R}$, then there is a one-to-one correspondence between polynomials and the associated polynomial functions. In fact, using only basic facts of elementary calculus, one can show that $f : \mathbb{R} \to \mathbb{R}$ is a polynomial function if and only if, for some $n \in \mathbb{N}$, f is n times differentiable and $f^{(n)} = 0$ (see Problem 4 in Chap.1). If this is the case, then there is exactly one $p \in \mathbb{R}^{(\mathbb{N})}$ such that $f = (\xi \mapsto p(\xi))$ and, if $f \neq 0$, then $\deg p < n$. In the case when $\mathbb{F} := \mathbb{R}$, one often identifies the ring $\mathbb{R}^{(\mathbb{N})}$ of polynomials over $\mathbb{R}$ with the ring of associated polynomial functions; then ι as defined by (92.1) becomes identified with the identity mapping of $\mathbb{R}$ as in the abbreviation (08.26).

If $\mathbb{F}$ is any infinite field, one can easily show that there is still a one-to-one correspondence between polynomials and polynomial functions. However, if $\mathbb{F}$ is a finite field, then *every* function from $\mathbb{F}$ to $\mathbb{F}$ is a polynomial function associated with infinitely many different polynomials. ■

Let $\mathcal{V}$ be a linear space over $\mathbb{F}$. The **lineonic polynomial function** $(\mathbf{L} \mapsto p(\mathbf{L})) : \mathrm{Lin}\mathcal{V} \to \mathrm{Lin}\mathcal{V}$ associated with a given $p \in \mathbb{F}^{(\mathbb{N})}$ is defined by

$$p(\mathbf{L}) := \sum_{k\in\mathbb{N}} p_k \mathbf{L}^k \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}\mathcal{V}. \tag{92.16}$$

We call $p(\mathbf{L})$ the **value** of p at $\mathbf{L}$. Again, for every given $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, the mapping $(p \mapsto p(\mathbf{L})) : \mathbb{F}^{(\mathbb{N})} \to \mathrm{Lin}\mathcal{V}$ is linear and preserves products, i.e. we have

$$p(\mathbf{L})q(\mathbf{L}) = (pq)(\mathbf{L}) \quad \text{for all} \quad p, q \in \mathbb{F}^{(\mathbb{N})}. \tag{92.17}$$

The following statements are easily seen to be valid for every lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and every polynomial $p \in \mathbb{F}^{(\mathbb{N})}$.

(I) $\mathbf{L}$ commutes with $p(\mathbf{L})$.

(II) Every $\mathbf{L}$-invariant subspace of $\mathcal{V}$ is also $p(\mathbf{L})$-invariant.

(III) Null $p(\mathbf{L})$ and $\mathrm{Rng}\, p(\mathbf{L})$ are $\mathbf{L}$-invariant.

(IV) If $\mathcal{U}$ is an $\mathbf{L}$-subspace of $\mathcal{V}$, then

$$p(\mathbf{L}_{|\mathcal{U}}) = p(\mathbf{L})_{|\mathcal{U}}. \tag{92.18}$$

(V) We have

$$p(\mathbf{L}^\top) = (p(\mathbf{L}))^\top. \tag{92.19}$$

Pitfall: To call $p(\xi)$, as defined by (92.10), and $p(\mathbf{L})$, as defined by (92.12), the *value* of the polynomial p at ξ and $\mathbf{L}$, respectively, is merely a figure of speech. One must often carefully distinguish between the polynomial p, the polynomial function associated with p, and the lineonic polynomial functions associated with p. For example, if p is a polynomial over $\mathbb{C}$, the polynomial function associated with the termwise complex-conjugate $\overline{p}$ of p is not the same as the value-wise complex-conjugate of the polynomial function associated with p. In some contexts it is useful to introduce explicit notations for the polynomial functions associated with p (see, e.g. Problem 13 in Chapt.6). ■

Let a lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ be given. It is clear that the intersection of any collection of $\mathbf{L}$-invariant subspaces of $\mathcal{V}$ is again $\mathbf{L}$-invariant. We denote the span-mapping corresponding to the collection of all $\mathbf{L}$-subspaces of $\mathcal{V}$ by $\mathrm{Lsp}_{\mathbf{L}}$ (see Sect.03). If $\mathcal{S} \in \mathrm{Sub}\,\mathcal{V}$ we call $\mathrm{Lsp}_{\mathbf{L}}\mathcal{S}$ the **linear L-span** of $\mathcal{S}$; it is the smallest $\mathbf{L}$-space that includes the set $\mathcal{S}$. It is easily seen that

$$\mathrm{Lsp}_{\mathbf{L}}\mathcal{S} = \mathrm{Lsp}\{\mathbf{L}^k\mathbf{v} \mid \mathbf{v} \in \mathcal{S},\ k \in \mathbb{N}\}, \tag{92.20}$$

and, for each $\mathbf{v} \in \mathcal{V}$,

$$\mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\} = \{p(\mathbf{L})\mathbf{v} \mid p \in \mathbb{F}^{(\mathbb{N})}\}. \tag{92.21}$$

Proposition 1: *If $p \in \left(\mathbb{F}^{(\mathbb{N})}\right)^\times$ and $\mathbf{v} \in \mathrm{Null}\ p(\mathbf{L})$, then*

$$\dim \mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\} \leq \deg p. \tag{92.22}$$

Proof: We put $m := \deg p$. Since $\mathbf{v} \in \mathrm{Null}\ p(\mathbf{L})$, we have

$$\mathbf{0} = p(\mathbf{L})\mathbf{v} = p_m(\mathbf{L}^m\mathbf{v}) + \sum_{k \in m^[} p_k(\mathbf{L}^k\mathbf{v}).$$

Since $p_m \neq 0$, it follows that $\mathbf{L}^m\mathbf{v} \in \mathrm{Lsp}\{\mathbf{L}^k\mathbf{v} \mid k \in m^[\}$. Hence, for every $r \in \mathbb{N}$, we have

$$\mathbf{L}^{m+r}\mathbf{v} \in (\mathbf{L}^r)_>(\mathrm{Lsp}\{\mathbf{L}^k\mathbf{v} \mid k \in m^[\}) = \mathrm{Lsp}\{\mathbf{L}^{k+r}\mathbf{v} \mid k \in m^[\}.$$

Using induction over $r \in \mathbb{N}$, we conclude that $\mathbf{L}^{m+r}\mathbf{v} \in \mathrm{Lsp}\{\mathbf{L}^k\mathbf{v} \mid k \in m^{[}\}$ for all $r \in \mathbb{N}$ and hence, in view of (92.16), that

$$\mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\} = \mathrm{Lsp}\{\mathbf{L}^k\mathbf{v} \mid \ k \in m^{[}\}.$$

Since the list $(\mathbf{L}^k\mathbf{v} \mid \ k \in m^{[})$ has $m = \deg p$ terms, it follows from the Characterization of Dimension of Sect.17 that (92.18) holds. ■

Proposition 2: *For every lineon* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$*, there is a unique monic polynomial* p *of least degree whose value at* $\mathbf{L}$ *is zero. This polynomial* p *is called the* **minimal polynomial** *of* $\mathbf{L}$*.*

Proof: Since $\mathrm{Lin}\mathcal{V}$ is finite-dimensional, the sequence $(\mathbf{L}^k \mid k \in \mathbb{N})$ must be linearly dependent. In view of (92.12) this means that there is a $p \in \left(\mathbb{F}^{(\mathbb{N})}\right)^{\times}$ such that $p(\mathbf{L}) = \mathbf{0}$. It follows that there is a monic polynomial q of least degree such that $q(\mathbf{L}) = \mathbf{0}$. If q' were another such polynomial, then $(q - q')(\mathbf{L}) = \mathbf{0}$. If $q \neq q'$, then $\deg (q - q') < \deg q = \deg q'$, and hence a suitable multiple of $q - q'$ would be a monic polynomial with a degree strictly less than $\deg q$ whose value at $\mathbf{L}$ is still zero. ■

If $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is given and if $\mathcal{U}$ is an $\mathbf{L}$-subspace of $\mathcal{V}$ then the minimal polynomial of $\mathbf{L}_{|\mathcal{U}}$ is the monic polynomial of least degree such that $\mathcal{U} \subset \mathrm{Null}\ q(\mathbf{L})$. Using poetic license, we sometimes call this polynomial the *minimal polynomial* of $\mathcal{U}$.

Proposition 3: *Let* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *and* $\mathbf{v} \in \mathcal{V}$ *be given, and let* q *be the minimal polynomial of* $\mathbf{L}$*. If* $\mathcal{V} = \mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\}$*, then* $\dim \mathcal{V} = \deg q$*.*

Proof: Put $n := \dim \mathcal{V}$ and consider the list $(\mathbf{L}^k\mathbf{v} \mid k \in (n+1)^{[})$. Since this list has $n + 1$ terms, it follows from the Characterization of Dimension that it must be a linearly dependent list. This means that

$$0 = \sum_{k \in (n+1)^{[}} h_k(\mathbf{L}^k\mathbf{v}) = h(\mathbf{L})\mathbf{v}$$

for some non-zero polynomial h with $\deg h \leq n$. By (92.13), we have

$$h(\mathbf{L})(p(\mathbf{L})\mathbf{v}) = (hp)(\mathbf{L})\mathbf{v} = ((ph)(\mathbf{L}))\mathbf{v} = p(\mathbf{L})(h(\mathbf{L})\mathbf{v}) = \mathbf{0}$$

for all $p \in \mathbb{F}^{(\mathbb{N})}$. Since $\mathcal{V} = \{p(\mathbf{L})\mathbf{v} \mid p \in \mathbb{F}^{(\mathbb{N})}\}$ by (92.17), it follows that h has the value zero at $\mathbf{L}$. Therefore, since q is the minimal polynomial of $\mathbf{L}$, we have $\deg q \leq \deg h \leq n$. On the other hand, application of Prop.1 to q yields $n = \dim \mathcal{V} \leq \deg q$. ■

We say that a polynomial p is **prime** if (i) p is monic, (ii) $\deg p > 0$, (iii) p is *not* the product of two monic polynomials that are both different from p. We will see in the next section that in order to understand the structure of elementary lineons, one must know the structure of prime polynomials. Unfortunately, for certain fields $\mathbb{F}$ (for example for $\mathbb{F} := \mathbb{Q}$) it is very difficult to describe all possible prime polynomials over $\mathbb{F}$. However in the case when $\mathbb{F}$ is $\mathbb{C}$ or $\mathbb{R}$, the following theorem gives a simple description.

Theorem on Prime Polynomials Over $\mathbb{C}$ and $\mathbb{R}$: *A polynomial p over $\mathbb{C}$ is prime if and only if it is of the form $p = \iota - \zeta$ for some $\zeta \in \mathbb{C}$.*

A polynomial p over $\mathbb{R}$ is prime if and only if it is either of the form $p = \iota - \lambda$ for some $\lambda \in \mathbb{R}$ or else of the form $p = (\iota - \mu)^2 + \kappa^2$ for some $(\mu, \kappa) \in \mathbb{R} \times \mathbb{P}^\times$.

The proof of this theorem depends on the following Lemma.

Lemma 1: *If p is a polynomial over $\mathbb{C}$ with $\deg p \geq 1$, then the equation $?\, z \in \mathbb{C},\ p(z) = 0$ has at least one solution.*

The assertion of this Lemma is included in Part (d) of Problem 13 in Chap.6. The proof is quite difficult, but Problem 13 in Chap.6 gives an outline from which the reader can construct a detailed proof.

Lemma 2: *Let p be a monic polynomial over a field $\mathbb{F}$ and let $\xi \in \mathbb{F}$ be such that $p(\xi) = 0$. Then there is a unique monic polynomial q over $\mathbb{F}$ such that*

$$p = (\iota - \xi)q. \tag{92.23}$$

The proof of this lemma, which is easy, is based on what is usually called "long division" of polynomials. We also leave the details to the reader.

Proof of the Theorem: Let p be a prime polynomial over $\mathbb{C}$. By Lemma 1, we can find $\zeta \in \mathbb{C}$ such that $p(\zeta) = 0$. By Lemma 2, we then have $p = (\iota - \zeta)q$ for some monic polynomial q and hence, since p is prime, $p = \iota - \zeta$ and $q = 1$.

Let now p be a prime polynomial over $\mathbb{R}$. If there exists a $\lambda \in \mathbb{R}$ such that $p(\lambda) = 0$ we have $p = (\iota - \lambda)q$ for some monic polynomial q and hence, since p is prime, $p = (\iota - \lambda)$ and $q = 1$. Assume, then, that $p(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$. Since $\mathbb{R} \subset \mathbb{C}$ and hence $\mathbb{R}^{(\mathbb{N})} \subset \mathbb{C}^{(\mathbb{N})}$, p is also a monic polynomial over $\mathbb{C}$. Hence, by Lemma 1, we can find $\zeta \in \mathbb{C} \setminus \mathbb{R}$ such that $p(\zeta) = 0$ and hence, by Lemma 2, a monic polynomial $q \in \mathbb{C}^{(\mathbb{N})}$ such that $p = (\iota - \zeta)q$. Since $p \in \mathbb{R}^{(\mathbb{N})}$, we have

$$0 = \overline{p(\zeta)} = \overline{p}(\overline{\zeta}) = p(\overline{\zeta}) = 0.$$

where $\overline{p}$ denotes the termwise complex-conjugate of p. It follows that $(\overline{\zeta} - \zeta)q(\overline{\zeta}) = 0$. Since $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we have $\overline{\zeta} - \zeta \neq 0$ and hence $q(\overline{\zeta}) = 0$. Using Lemma 2 again, we can find a monic polynomial $r \in \mathbb{C}^{(\mathbb{N})}$ such that $q = (\iota - \overline{\zeta})r$ and hence

$$p = (\iota - \zeta)q = (\iota - \zeta)(\iota - \overline{\zeta})r.$$

If we put $\mu := \operatorname{Re}\zeta$ and $\kappa := |\operatorname{Im}\zeta|$ then $(\iota - \zeta)(\iota - \overline{\zeta}) = (\iota - \mu)^2 + \kappa^2$ and hence

$$p = ((\iota - \mu)^2 + \kappa^2)r. \tag{92.24}$$

Now, since $p = \overline{p}$, it follows from (92.20) that

$$0 = ((\iota - \mu)^2 + \kappa^2)(r - \overline{r}).$$

Since $r - \overline{r} \in \mathbb{R}^{(\mathbb{N})}$ and since the polynomial function associated with $((\iota - \mu)^2 + \kappa^2)$ has strictly positive values, the polynomial function associated with $r - \overline{r}$ must be the zero function. In view of the Remark above, it follows that $r = \overline{r}$ and hence $r \in \mathbb{R}^{(\mathbb{N})}$. Since p is prime, it follows from (92.20) that $p = (\iota - \mu)^2 + \kappa^2$ and $r = 1$.

It is easy to see that polynomials of the forms described in the Theorem are in fact prime. ■

93 The Structure of Elementary Lineons

s The following result is the basis for understanding the structure of elementary lineons.

Structure Theorem for Elementary Lineons: *Let* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *be an elementary lineon and let* $\mathcal{M}$ *be the (only) minimal* $\mathbf{L}$*-space. Then:*

(a) *There is a unique monic polynomial* q *such that*

$$\mathcal{M} = \operatorname{Null}\ q(\mathbf{L}) \quad \text{and} \quad \deg q = \dim \mathcal{M}. \tag{93.1}$$

This polynomial q *is prime.*

(b) $\dim \mathcal{V}$ *is a multiple of* $\dim \mathcal{M}$*, i.e. there is a (unique)* $d \in \mathbb{N}^\times$ *such that*

$$\dim \mathcal{V} = d \dim \mathcal{M}. \tag{93.2}$$

(c) *There are exactly* $d + 1$ $\mathbf{L}$*-spaces; they are given by*

$$\mathcal{H}_k := \operatorname{Null}\ q^k(\mathbf{L}) = \operatorname{Rng} q^{d-k}(\mathbf{L}), \quad k \in (d+1)^{[}, \tag{93.3}$$

and their dimensions are

$$\dim \mathcal{H}_k = k \dim \mathcal{M}, \quad k \in (d+1)^{[}. \tag{93.4}$$

In particular, we have

$$\mathcal{H}_0 = \{\mathbf{0}\}, \quad \mathcal{H}_1 = \mathcal{M}, \quad \mathcal{H}_d = \mathcal{V}. \tag{93.5}$$

The polynomial q will be called the **prime polynomial** *of* $\mathbf{L}$ *and the number d the* **depth** *of* $\mathbf{L}$.

Proof of Part (a): Let q be a polynomial that satisfies (93.1). We choose $\mathbf{v} \in \mathcal{M}^\times$. Since $\{\mathbf{v}\} \subset \mathcal{M}$ and since $\mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\}$ is the smallest $\mathbf{L}$-space that includes $\{\mathbf{v}\}$, it follows that $\mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\} \subset \mathcal{M}$. On the other hand, since $\mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\}$ is a non-zero $\mathbf{L}$-subspace, it follows from Prop.3 of Sect.91 that $\mathcal{M} \subset \mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\}$. We conclude that $\mathcal{M} = \mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\} = \mathrm{Lsp}_{\mathbf{L}_{|\mathcal{M}}}\{\mathbf{v}\}$. By Prop.3 of Sect.92, it follows that $\dim \mathcal{M}$ is the degree of the minimal polynomial of $\mathbf{L}_{|\mathcal{M}}$. Since q has the value zero at $\mathbf{L}_{|\mathcal{M}}$ by (93.1) and since $\deg q = \dim \mathcal{M}$, it follows that q must *be* the minimal polynomial of $\mathbf{L}_{|\mathcal{M}}$, and hence is uniquely determined by $\mathbf{L}$.

On the other hand, if q is *defined* to be the minimal polynomial of $\mathbf{L}_{|\mathcal{M}}$, it is not hard to verify, using Prop.3 of Sect.91 and Props.1 and 3 of Sect.92, that (93.1) holds.

To prove that q is prime, assume that $q = q_1 q_2$, where q_1 and q_2 are monic polynomials. It cannot happen that both Null $q_1(\mathbf{L}) := \{\mathbf{0}\}$ and Null $q_2(\mathbf{L}) = \{\mathbf{0}\}$, because this would imply $\mathcal{M} = \text{Null } p(\mathbf{L}) = \{\mathbf{0}\}$. Assume, for example, that Null $q_1(\mathbf{L}) \neq \{\mathbf{0}\}$ and choose $\mathbf{v} \in (\text{Null } q_1(\mathbf{L}))^\times$. By Prop.3 of Sect.91 it follows that $\mathcal{M} \subset \mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\}$ and hence, by Prop.1 of Sect.92, that $\deg q = \dim \mathcal{M} \leq \dim \mathrm{Lsp}_\mathbf{L}\{\mathbf{v}\} \leq \deg q_1$. In view of (92.8), this is possible only when $\deg q = \deg q_1$ and $\deg q_2 = 0$ and hence $q_1 = q$ and $q_2 = 1$. ∎

The proof of the remaining assertions will be based on the following

Lemma: *There is a $d \in \mathbb{N}^\times$ such that*

$$q^d(\mathbf{L}) = \mathbf{0}, \quad \dim \mathcal{V} = d \dim \mathcal{M}, \tag{93.6}$$

and

$$\dim \mathrm{Rng}\, q^k(\mathbf{L}) = (d-k) \dim \mathcal{M} \quad \text{for all} \quad k \in d]. \tag{93.7}$$

Proof: Let $j \in \mathbb{N}$ be given and assume that $q^j(\mathbf{L}) \neq \mathbf{0}$, i.e. that $\mathcal{U} := \mathrm{Rng}\, q^j(\mathbf{L})$ is not the zero-space. By Prop.3 of Sect.91 we then have $\mathcal{M} \subset \mathcal{U}$. By Part (a), it follows that $\mathcal{M} = \text{Null } q(\mathbf{L}) = \text{Null } (q(\mathbf{L})_{|\mathcal{U}})$. Noting that $\mathrm{Rng}\,(q(\mathbf{L})_{|\mathcal{U}}) = \mathrm{Rng}\, q^{j+1}(\mathbf{L})$, we can apply the Theorem on Dimensions of Range and Nullspace of Sect.17 to $q(\mathbf{L})_{|\mathcal{U}}$ and obtain

$$\dim \mathrm{Rng}\, q^{j+1}(\mathbf{L}) + \dim \mathcal{M} = \dim \mathrm{Rng}\, q^j(\mathbf{L}) \tag{93.8}$$

for all $j \in \mathbb{N}$ such that $q^j(\mathbf{L}) \neq \mathbf{0}$. For $j = 0$ (93.8) reduces to

$$\dim \operatorname{Rng} q(\mathbf{L}) + \dim \mathcal{M} = \dim \mathcal{V}. \tag{93.9}$$

Using induction, we conclude from (93.8) and (93.9) that

$$\dim \operatorname{Rng} q^{j+1}(\mathbf{L}) = \dim \mathcal{V} - (j+1)\dim \mathcal{M} \tag{93.10}$$

for all $j \in \mathbb{N}$ for which $q^j(\mathbf{L}) \neq \mathbf{0}$. Since the right side of (93.10) becomes negative for large enough j, there must be a $d \in \mathbb{N}^\times$ such that $q^d(\mathbf{L}) = \mathbf{0}$ and $0 = \dim \mathcal{V} - d(\dim \mathcal{M})$, which proves (93.6). We have $q^{k-1}(\mathbf{L}) \neq \mathbf{0}$ for all $k \in d^]$. Hence, if we substitute $j := k - 1$ and $\dim \mathcal{V} = d(\dim \mathcal{M})$ into (93.10), we obtain (93.7). ∎

The part $(93.6)_2$ of the Lemma asserts the validity of Part (b) of the Theorem.

Proof of Part (c): Let $k \in (d+1)^[$ be given and put

$$\mathcal{H}_k := \text{Null } q^k(\mathbf{L}). \tag{93.11}$$

We apply the Theorem on Dimensions of Range and Nullspace to $q^k(\mathbf{L})$ and use the Lemma to obtain

$$\dim \mathcal{H}_k = \dim \mathcal{V} - \dim \operatorname{Rng} q^k(\mathbf{L}) = k \dim \mathcal{M}, \tag{93.12}$$

which proves (93.4). On the other hand, by $(93.6)_1$, we have

$$\{\mathbf{0}\} = \operatorname{Rng} q^d(\mathbf{L}) = q^k(\mathbf{L})_>(\operatorname{Rng} q^{d-k}(\mathbf{L}))$$

and hence $\operatorname{Rng} q^{d-k}(\mathbf{L}) \subset \mathcal{H}_k$. But by (93.12) and (93.7) we have $\dim \mathcal{H}_k = \dim \operatorname{Rng} q^{d-k}(\mathbf{L})$ and hence $\mathcal{H}_k = \operatorname{Rng} q^{d-k}(\mathbf{L})$, which proves (93.3).

Now let $\mathcal{U}$ be an $\mathbf{L}$-subspace of $\mathcal{V}$. If $\mathcal{U} = \{\mathbf{0}\}$, then $\mathcal{U} = \mathcal{H}_0$. If $\mathcal{U} \neq \{\mathbf{0}\}$, then Prop.3 of Sect.81 applies. Hence we can apply the Lemma to $\mathbf{L}_{|\mathcal{U}}$ instead of to $\mathbf{L}$ and conclude that there is a $k \in \mathbb{N}^\times$ such that $\dim \mathcal{U} = k \dim \mathcal{M}$ and

$$0 = q^k(\mathbf{L}_{|\mathcal{U}}) = q^k(\mathbf{L})_{|\mathcal{U}}, \quad \text{i.e.} \quad \mathcal{U} \subset \text{Null } q^k(\mathbf{L}) = \mathcal{H}_k.$$

Since $\dim \mathcal{U} = k \dim \mathcal{M} = \dim \mathcal{H}_k$ by (93.12), it follows that $\mathcal{H}_k = \mathcal{U}$. Since $\mathcal{U}$ was an arbitrary $\mathbf{L}$-subspace of $\mathcal{V}$, Part (c) follows. ∎

We now list several corollaries. The first is an immediate consequence of Part (c) of the Theorem.

Corollary 1: *A lineon $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ is elementary if and only if the collection of $\mathbf{L}$-spaces is totally ordered by inclusion, which means that, given any two $\mathbf{L}$-spaces, one of them must be included in the other.*

The next two corollaries will be proved together.

Corollary 2: *A lineon* $\mathbf{L}$ *is elementary if and only if there is exactly one maximal* $\mathbf{L}$*-space.*

Corollary 3: *If the lineon* $\mathbf{L}$ *is elementary, so is* $\mathbf{L}^\top$*, and* $\mathbf{L}^\top$ *has the same prime polynomial and the same depth as* $\mathbf{L}$*.*

Proof: If $\mathbf{L}$ is elementary, with prime polynomial q and depth d, then, by Part (c) of the Theorem, $\mathcal{H}_{d-1} := \operatorname{Null}\ q^{d-1}(\mathbf{L})$ is the only maximal $\mathbf{L}$-space.

If there is only one maximal $\mathbf{L}$-space $\mathcal{W}$, then $\mathcal{W}^\perp$ is the only minimal $\mathbf{L}^\top$-space and hence $\mathbf{L}^\top$ is elementary. In particular, if $\mathbf{L}$ is elementary, so is $\mathbf{L}^\top$. Hence, if $\mathbf{L}$ has only one maximal $\mathbf{L}$-space, we can apply this observaton to $\mathbf{L}^\top$ and conclude that $(\mathbf{L}^\top)^\top = \mathbf{L}$ must be elementary. Thus, Cor.2 and the first assertion of Cor.3 are proved.

If q is the prime polynomial of $\mathbf{L}$, then $\mathcal{W} := \mathcal{H}_{d-1} = \operatorname{Rng} q(\mathbf{L})$ is the only maximal $\mathbf{L}$-space and hence

$$\mathcal{W}^\perp = (\operatorname{Rng} q(\mathbf{L}))^\perp = \operatorname{Null}\ q(\mathbf{L})^\top = \operatorname{Null}\ q(\mathbf{L}^\top)$$

(see (21.13) and (92.15)) is the only minimal $\mathbf{L}^\top$-space. We have

$$\dim \mathcal{W}^\perp = \dim \mathcal{V} - \dim \mathcal{W} = \dim \mathcal{V} - \dim \operatorname{Rng} q(\mathbf{L})$$

$$= \dim \operatorname{Null}\ q(\mathbf{L}) = \dim \mathcal{M} = \deg\ q$$

by the Formula for Dimension of Annihilators, the Theorem on Dimensions of Range and Nullspace and Part (a) of the Theorem. The uniqueness assertion of Part (a) of the Theorem, applied to $\mathbf{L}^\top$ instead of to $\mathbf{L}$, shows that q is also the prime polynomial of $\mathbf{L}^\top$. Since $\dim \mathcal{V}^* = \dim \mathcal{V}$ and $\dim \mathcal{W}^\perp = \dim \mathcal{M}$, it follows from part (b) of the Theorem that d is also the depth of $\mathbf{L}^\top$. ∎

Corollary 4: *If* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *is elementary, if* $\mathcal{W}$ *is the (only) maximal* $\mathbf{L}$*-space and if* $\mathbf{v} \in \mathcal{V} \setminus \mathcal{W}$*, then* $\mathcal{V} = \operatorname{Lsp}_{\mathbf{L}}\{\mathbf{v}\}$*.*

Proof: We have $\operatorname{Lsp}_{\mathbf{L}}\{\mathbf{v}\} \not\subset \mathcal{W}$. Since all $\mathbf{L}$-spaces other than $\mathcal{V}$ must be included in $\mathcal{W}$, it follows that $\operatorname{Lsp}_{\mathbf{L}}\{\mathbf{v}\} = \mathcal{V}$. ∎

Corollary 5: *Let* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *be an elementary lineon with prime polynomial* q *and depth* d*. Then* $\dim \mathcal{V} = \deg q^d$ *and* q^d *is the minimal polynomial of* $\mathbf{L}$*.*

Proof: Since $\mathcal{V} = \operatorname{Null}\ q^d(\mathbf{L})$ by Part (c) of the Theorem, q^d has the value zero at $\mathbf{L}$. By Cor.4 and Prop.3 of Sect.92, the degree of the minimal polynomial of $\mathbf{L}$ must be $\dim \mathcal{V}$. On the other hand, by (92.8), $(93.1)_2$, and (93.2), we have $\deg q^d = d \deg q = d(\dim \mathcal{M}) = \dim \mathcal{V}$. Hence q^d must be the minimal polynomial of $\mathbf{L}$. ∎

Remark: Another proof of the Structure Theorem can be based on the fact, known to readers familiar with algebra, that the ring $\mathbb{F}^{(\mathbb{N})}$ of polynomials is a principal-ideal ring. The proof goes as follows: First, one shows that if $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is *cyclic* in the sense that $\mathcal{V} = \mathrm{Lsp}_{\mathbf{L}}\{\mathbf{v}\}$ for some $\mathbf{v} \in \mathcal{V}$, then a subspace $\mathcal{U}$ of $\mathcal{V}$ is an $\mathbf{L}$-space if and only if $\mathcal{U} = \mathrm{Rng}\, p(\mathbf{L})$ for some monic divisor p of the minimal polynomial of $\mathbf{L}$. One deduces from this that if there is only one maximal $\mathbf{L}$-space, the minimal polynomial of $\mathbf{L}$ must be of the form q^d, where q is a prime polynomial. Using an argument like the one used in the proofs of Cor.2, and 3, one proves that Cor.2 and 3 are valid. The remainder of the proof is then very easy. ■

As in the case when $\mathbb{F} := \mathbb{R}$ (see Sect.82) the **spectrum** of a lineon $\mathbf{L}$ on a linear space $\mathcal{V}$ over $\mathbb{F}$ is defined by

$$\mathrm{Spec}\,\mathbf{L} := \{\sigma \in \mathbb{F} \mid \ \mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}}) \neq \{\mathbf{0}\}\}. \tag{93.13}$$

If $\mathbf{L}$ is elementary, then $\mathrm{Spec}\,\mathbf{L}$ is non-empty only in the exceptional case described as follows.

Proposition 1: *If $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is elementary and has a non-empty spectrum, then this spectrum is a singleton, the only minimal $\mathbf{L}$-space is one-dimensional, the prime polynomial of $\mathbf{L}$ is $\iota - \sigma$ when $\sigma :\in \mathrm{Spec}\,\mathbf{L}$, and the depth of $\mathbf{L}$ is $n := \dim \mathcal{V}$. Also, there are exactly $n+1$ $\mathbf{L}$-subspaces and they are given by*

$$\mathcal{H}_k := \mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}})^k = \mathrm{Rng}\,(\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}})^{n-k}, \quad k \in (n+1)^{[}. \tag{93.14}$$

Proof: Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ by given. If $\sigma \in \mathrm{Spec}\,\mathbf{L}$, then every one-dimensional subspace of $\mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}})$ is evidently a minimal $\mathbf{L}$-space. Therefore, if $\mathrm{Spec}\,\mathbf{L}$ has more than one element or if $\dim \mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}}) \geq 2$ for some $\sigma \in \mathrm{Spec}\,\mathbf{L}$, there are at least two distinct minimal $\mathbf{L}$-spaces. Hence, if $\mathbf{L}$ is elementary, $\mathrm{Spec}\,\mathbf{L}$ can have only one element. If σ is this element, then $\mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}})$ is one-dimensional and it is the only minimal $\mathbf{L}$-space. The remaining statements follow immediately from the Structure Theorem for Elementary Lineons. ■

A lineon $\mathbf{L}$ is said to be **nilpotent** if $\mathbf{L}^m = 0$ for some $m \in \mathbb{N}^{\times}$. The least such m is then called the **nilpotency** of $\mathbf{L}$.

Proposition 2: *A lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is both elementary and non-invertible if and only if it is nilpotent with a nilpotency equal to* $\dim \mathcal{V}$.

Proof: By Prop.1 of Sect.18 and (93.13), $\mathbf{L}$ is non-invertible if and only if $0 \in \mathrm{Spec}\,\mathbf{L}$. In view of this fact, the assertion follows immediately from Prop.1. ■

Notes 93

(1) The use of the terms "prime polynomial" and "depth" in the sense described in the Structure Theorem for Elementary Lineons was recently proposed by J. J. Schäffer.

(2) What we call simply the "nilpotency" of a nilpotent lineon is sometimes called the "index of nilpotence".

94 Canonical Matrices

We assume that a linear space $\mathcal{V}$ over the field $\mathbb{F}$ is given. In this section, we suggest methods for finding bases of $\mathcal{V}$ relative to which a given lineon on $\mathbf{L}$ has a matrix of a simple and illuminating form. Such a matrix is called a *canonical matrix.* In view of the Elementary Decomposition Theorem of Sect.91, it is sufficient to consider only elementary lineons. Indeed, let $(\mathcal{E}_i \mid i \in I)$ be an elementary $\mathbf{L}$-decomposition for a given $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ and let, for each $i \in I$, a basis $\mathbf{b}^{(i)}$ be determined such that the matrix M_i of $\mathbf{L}_{|\mathcal{E}_i}$ relative to $\mathbf{b}^{(i)}$ is canonical. Then one can "put together" (possibly with the help of reindexing) the bases $\mathbf{b}^{(i)},\ i \in I$, to obtain a basis $\mathbf{b}$ of $\mathcal{V}$ such that the only non-zero terms of the matrix M of $\mathbf{L}$ relative to $\mathbf{b}$ are those in blocks of the form M_i along the diagonal. An illustration will be given at the end of this section.

We first deal with elementary lineons having a non-empty spectrum.

Proposition 1: *If $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ is elementary and if its spectrum is not empty, then there is a list-basis $\mathbf{b} := (\mathbf{b}_i \mid i \in n^]),\ n := \dim\mathcal{V}$, and $\sigma \in \mathbb{F}$ such that*

$$\mathbf{L}\mathbf{b}_i = \left\{ \begin{array}{lll} \sigma\mathbf{b}_i & \text{if} & i = 1 \\ \sigma\mathbf{b}_i + \mathbf{b}_{i-1} & \text{if} & i \in n^] \setminus \{1\} \end{array} \right\}. \qquad (94.1)$$

Proof: By Prop.1 of Sect.93, there is $\sigma \in \mathbb{F}$ such that $\operatorname{Spec}\mathbf{L} = \{\sigma\}$ and the only maximal $\mathbf{L}$-space is $\mathcal{H}_{n-1} = \operatorname{Rng}(\mathbf{L} - \sigma\mathbf{1}_\mathcal{V})$. We choose $\mathbf{v} \in \mathcal{V}\setminus\mathcal{H}_{n-1}$ and define the list $\mathbf{b}$ by $\mathbf{b}_i := (\mathbf{L} - \sigma\mathbf{1}_\mathcal{V})^{n-i}\mathbf{v}$ for all $i \in n^]$. We then have $(\mathbf{L} - \sigma\mathbf{1}_\mathcal{V})\mathbf{b}_1 = (\mathbf{L} - \sigma\mathbf{1}_\mathcal{V})^n\mathbf{v} = \mathbf{0}$ and $(\mathbf{L} - \sigma\mathbf{1}_\mathcal{V})\mathbf{b}_i = \mathbf{b}_{i-1}$ for all $i \in n^] \setminus \{1\}$, which proves (94.1). ∎

The matrix $[\mathbf{L}]_\mathbf{b}$ of $\mathbf{L}$ relative to a basis $\mathbf{b}$ for which (94.1) holds is a canonical matrix. It is given by

$$([\mathbf{L}]_\mathbf{b})_{i,j} = \left\{ \begin{array}{lll} \sigma & \text{if} & j = i \\ 1 & \text{if} & j = i+1 \\ 0 & \text{otherwise} & \end{array} \right\}. \qquad (94.2)$$

If n is small, it can be recorded explicitly in the form

$$[\mathbf{L}]_{\mathbf{b}} = \begin{bmatrix} \sigma & 1 & & & & \\ & \sigma & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \sigma & 1 \\ & & & & & \sigma \end{bmatrix}, \tag{94.3}$$

where zeros are omitted.

From now on we confine ourselves to the case when $\mathbb{F} := \mathbb{C}$ or $\mathbb{F} := \mathbb{R}$.

Let $\mathcal{V}$ be a linear space over $\mathbb{C}$, let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ be an elementary lineon, and let q be its prime polynomial. By the Theorem on Prime Polynomials over $\mathbb{C}$ and $\mathbb{R}$ of Sect.92, q must be of the form $q = \iota - \sigma$ for some $\sigma \in \mathbb{C}$. By (93.1) the minimal $\mathbf{L}$-space is $\mathcal{M} = \mathrm{Null}\ q(\mathbf{L}) = \mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}})$. Since $\mathcal{M} \neq \{\mathbf{0}\}$, it follows that σ belongs to the $\mathbb{C}$-spectrum of $\mathbf{L}$. Hence this spectrum is not empty. Therefore, Prop.1 applies and one can find a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in n^{]})$ such that the matrix $[\mathbf{L}]_{\mathbf{b}}$ is given by (94.2).

In sum, if $\mathbb{F} := \mathbb{C}$, the situation covered by Prop.1 of Sect.93 and Prop.1 above is general rather than exceptional.

We now assume that $\mathcal{V}$ is a linear space over $\mathbb{R}$ and that $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is an elementary lineon. Let q be its prime polynomial. By the Theorem on Prime Polynomials over $\mathbb{C}$ and $\mathbb{R}$, we must have either $q = \iota - \lambda$ for some $\lambda \in \mathbb{R}$ or else $q = (\iota - \mu)^2 + \kappa^2$ for some $(\mu, \kappa) \in \mathbb{R} \times \mathbb{P}^{\times}$. In the former case, Prop.1 applies again. The latter case is covered by the following result.

Proposition 2: *Let* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *be an elementary lineon whose prime polynomial is* $q = (\iota - \mu)^2 + \kappa^2,\ (\mu, \kappa) \in \mathbb{R} \times \mathbb{P}^{\times}$*, and whose depth is* d*. Then there is a list-basis* $\mathbf{b} := (\mathbf{b}_i \mid i \in n^{]}),\ n := \dim \mathcal{V} = 2d$*, such that*

$$\left.\begin{array}{rcl} \mathbf{L}\mathbf{b}_1 &=& \mu \mathbf{b}_1 + \kappa \mathbf{b}_2, \\ \mathbf{L}\mathbf{b}_2 &=& -\kappa \mathbf{b}_1 + \mu \mathbf{b}_2, \end{array}\right\} \tag{94.4}$$

$$\left.\begin{array}{rcl} \mathbf{L}\mathbf{b}_{2k+1} &=& \mu \mathbf{b}_{2k+1} + \kappa \mathbf{b}_{2k+2} + \mathbf{b}_{2k-1} \\ \mathbf{L}\mathbf{b}_{2k+2} &=& -\kappa \mathbf{b}_{2k+1} + \mu \mathbf{b}_{2k+2} + \mathbf{b}_{2k} \end{array}\right\} \quad \text{for all} \quad k \in (d-1)^{]}. \tag{94.5}$$

For each $k \in d^{]}$*, the space* $\mathrm{Lsp}\{\mathbf{b}_{2k-1}, \mathbf{b}_{2k}\}$ *is a supplement of the* $\mathbf{L}$*-space* $\mathcal{H}_k$ *in the* $\mathbf{L}$*-space* $\mathcal{H}_{k-1}$ (see (93.3)).

Proof: By the Structure Theorem for Elementary Lineons of Sect.93, the $\mathbf{L}$-subspaces are given by (93.3), and $\mathcal{H}_{d-1} = \mathrm{Null}\ q^{d-1}(\mathbf{L})$ is the only maximal $\mathbf{L}$-space. We use the abbreviation $\mathbf{D} := \mathbf{L} - \mu \mathbf{1}_{\mathcal{V}}$, so that $q(\mathbf{L}) =$

$\mathbf{D}^2 + \kappa^2 \mathbf{1}_{\mathcal{V}}$. We choose $\mathbf{v} \in \mathcal{V} \setminus \mathcal{H}_{d-1}$ and define the lists $\mathbf{e} := (\mathbf{e}_k \mid k \in d^]})$ and $\mathbf{f} := (\mathbf{f}_k \mid k \in d^{]})$ recursively by

$$\mathbf{e}_1 := \mathbf{v}, \quad \mathbf{f}_1 := \frac{1}{\kappa}\mathbf{D}\mathbf{e}_1, \tag{94.6}$$

$$\left.\begin{array}{rcl} \mathbf{e}_{k+1} & := & \mathbf{D}\mathbf{e}_k + \kappa\mathbf{f}_k \\ \mathbf{f}_{k+1} & := & \mathbf{D}\mathbf{f}_k - \kappa\mathbf{e}_k \end{array}\right\} \quad \text{for all} \quad k \in (d-1)^{]}. \tag{94.7}$$

An easy calculation shows that

$$\begin{array}{rcl} \kappa\mathbf{e}_1 & = & \frac{1}{\kappa}q(\mathbf{L})\mathbf{e}_1 - \mathbf{D}\mathbf{f}_1, \\ \kappa\mathbf{e}_{k+1} & = & q(\mathbf{L})\mathbf{f}_k - \mathbf{D}\mathbf{f}_{k+1} \quad \text{for all} \quad k \in (d-1)^{]}, \end{array}$$

and

$$\kappa\mathbf{e}_{k+1} = q(\mathbf{L})(\mathbf{D}\mathbf{f}_{k-1} - \mathbf{f}_k) + 2\kappa^2\mathbf{f}_k \quad \text{for all} \quad k \in (d-1)^{]}.$$

Since $q(\mathbf{L})^d = \mathbf{0}$, it follows that

$$\left.\begin{array}{rcll} q^{d-1}(\mathbf{L})\mathbf{e}_k & = & -\frac{1}{\kappa}\mathbf{D}q^{d-1}(\mathbf{L})\mathbf{f}_k & \text{for all} \quad k \in d^{]}, \\ q^{d-1}(\mathbf{L})\mathbf{f}_k & = & \frac{1}{2\kappa}q^{d-1}(\mathbf{L})\mathbf{e}_{k+1} & \text{for all} \quad k \in (d-1)^{]} \end{array}\right\}. \tag{94.8}$$

Using the fact that $\mathbf{v} = \mathbf{e}_1 \notin \mathcal{H}_{d-1} = \text{Null}\, q^{d-1}(\mathbf{L})$ and hence $q(\mathbf{L})^{d-1}\mathbf{e}_1 \neq \mathbf{0}$, we conclude from (94.8), by induction, that

$$q(\mathbf{L})^{d-1}\mathbf{e}_k \neq \mathbf{0} \quad \text{and} \quad q(\mathbf{L})^{d-1}\mathbf{f}_k \neq \mathbf{0} \quad \text{for all} \quad k \in d^{]}. \tag{94.9}$$

We now define the list $\mathbf{b} := (\mathbf{b}_i \mid i \in n^{]})$ by

$$\left.\begin{array}{rcl} \mathbf{b}_{2k-1} & := & q(\mathbf{L})^{d-k}\mathbf{e}_k \\ \mathbf{b}_{2k} & := & q(\mathbf{L})^{d-k}\mathbf{f}_k \end{array}\right\} \quad \text{for all} \quad k \in d^{]}. \tag{94.10}$$

It follows easily from (94.6) and (94.7) that (94.4) and (94.5) hold and hence that $\text{Lsp}\{\mathbf{b}_i \mid i \in n^{]}\}$ is $\mathbf{L}$-invariant. On the other hand, we have $\mathbf{b}_{n-1} = \mathbf{b}_{2d-1} = \mathbf{e}_d$ by (94.10), and hence $\mathbf{b}_{n-1} \notin \text{Null}\, q(\mathbf{L})^{d-1} = \mathcal{H}_{d-1}$ by (94.9). Since $\mathcal{H}_{d-1}$ is the only maximal $\mathbf{L}$-space, it follows from Cor.4 of Sect.93 that $\text{Lsp}_{\mathbf{L}}\{\mathbf{b}_{n-1}\} = \mathcal{V}$. We conclude that $\text{Lsp}\{\mathbf{b}_i \mid i \in n^{]}\} = \mathcal{V}$ and hence, by the Theorem on Characterization of Dimension, that $\mathbf{b}$ is indeed a basis of $\mathcal{V}$.

The fact that $\text{Lsp}\{\mathbf{b}_{2k-1}, \mathbf{b}_{2k}\}$ is a supplement of $\mathcal{H}_k$ in $\mathcal{H}_{k-1}$ is an immediate consequence of (94.9), (94.10), and (93.3). ■

The matrix $[\mathbf{L}]_{\mathbf{b}}$ of $\mathbf{L}$ relative to a basis $\mathbf{b}$ for which (94.4) and (94.5) hold is a canonical matrix. It is given by

$$([\mathbf{L}]_{\mathbf{b}})_{i,j} = \left\{ \begin{array}{rll} \mu & \text{if} & j = i \\ \kappa & \text{if} & j = i+1 \text{ and } i \text{ is odd} \\ -\kappa & \text{if} & j = i-1 \text{ and } i \text{ is even} \\ 1 & \text{if} & j = i+2 \\ 0 & \text{otherwise} & \end{array} \right\}. \tag{94.11}$$

If n is small, it can be recorded explicitly in the form

$$[\mathbf{L}]_{\mathbf{b}} = \begin{bmatrix} \mu & \kappa & 1 & 0 & & & & & & \\ -\kappa & \mu & 0 & 1 & & & & & & \\ & & \mu & \kappa & 1 & 0 & & & & \\ & & -\kappa & \mu & 0 & 1 & & & & \\ & & & & \cdot & & \cdot & & & \\ & & & & & \cdot & & \cdot & & \\ & & & & & & \mu & \kappa & 1 & 0 \\ & & & & & & -\kappa & \mu & 0 & 1 \\ & & & & & & & & \mu & \kappa \\ & & & & & & & & -\kappa & \mu \end{bmatrix}, \tag{94.12}$$

where zeros are omitted.

We now illustrate our results by considering a linear space $\mathcal{V}$ over $\mathbb{R}$ with $\dim \mathcal{V} = 4$. If $\mathbf{L}$ is a lineon on $\mathcal{V}$, we can then find a basis $\mathbf{b} := (\mathbf{b}_i \mid i \in 4^]))$ such that the matrix $[\mathbf{L}]_{\mathbf{b}}$ has one of the following 9 forms (zeros are not written):

$$\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}, \begin{bmatrix} \sigma & 1 & & \\ & \sigma & & \\ & & \lambda_1 & \\ & & & \lambda_2 \end{bmatrix}, \begin{bmatrix} \mu & \kappa & & \\ -\kappa & \mu & & \\ & & \lambda_1 & \\ & & & \lambda_2 \end{bmatrix},$$

$$\begin{bmatrix} \sigma & 1 & & \\ & \sigma & 1 & \\ & & \sigma & \\ & & & \lambda \end{bmatrix}, \begin{bmatrix} \sigma_1 & 1 & & \\ & \sigma_1 & & \\ & & \sigma_2 & 1 \\ & & & \sigma_2 \end{bmatrix}, \begin{bmatrix} \mu & \kappa & & \\ -\kappa & \mu & & \\ & & \sigma & 1 \\ & & & \sigma \end{bmatrix},$$

$$\begin{bmatrix} \mu_1 & \kappa_1 & & \\ -\kappa_1 & \mu_1 & & \\ & & \mu_2 & \kappa_2 \\ & & -\kappa_2 & \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma & 1 & & \\ & \sigma & 1 & \\ & & \sigma & 1 \\ & & & \sigma \end{bmatrix}, \begin{bmatrix} \mu & \kappa & 1 & \\ -\kappa & \mu & & 1 \\ & & \mu & \kappa \\ & & -\kappa & \mu \end{bmatrix}.$$

The number of terms in an elementary decomposition of $\mathbf{L}$ is 4 in the first form, 3 in the second and third, 2 in the fourth up to the seventh, and 1 in the last two. The first, third, and seventh forms apply when $\mathbf{L}$ is semi-simple.

Pitfall: If $\mathcal{V}$ is not the zero space, the basis relative to which the matrix of an elementary lineon has the form (94.2) or (94.11) is never uniquely determined by $\mathbf{L}$. Indeed, the construction of these bases involved the choice of an element outside the maximal $\mathbf{L}$-space, and different choices give rise to different bases. ■

Remark: Props.1 and 2 can be used to prove the following result: If $\mathcal{V}$ is a field over $\mathbb{C}$ or $\mathbb{R}$, then every $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ has an additive decomposition $(\mathbf{N}, \mathbf{S})$, $\mathbf{N}, \mathbf{S} \in \operatorname{Lin}\mathcal{V}$, such that $\mathbf{L} = \mathbf{N} + \mathbf{S}$, $\mathbf{N}$ is nilpotent, $\mathbf{S}$ is semi-simple, and $\mathbf{N}$ and $\mathbf{S}$ commute. Indeed, if $\mathbf{L}$ is elementary, one can define $\mathbf{S}$ to be the lineon whose matrix, relative to a basis for which (94.2) or (94.11) hold, is obtained by replacing 1 on the right side of (94.2) or (94.11) by 0. Then one can define $\mathbf{N} := \mathbf{L} - \mathbf{S}$ and prove that $(\mathbf{N}, \mathbf{S})$ is a decomposition with the desired properties. The case when $\mathbf{L}$ is not elementary can be reduced to the case when it is by applying the Elementary Decomposition Theorem.

Actually the result just described remains valid for a large class of fields (often called *perfect fields*), and the decomposition can be proved to be unique. ■

Notes 94

(1) A matrix of the type described by (94.2) or (94.3) is often called an "elementary Jordan matrix". A matrix whose only non-zero terms are in blocks of the form described by (94.2) or (94.3) along the diagonal is often called a "Jordan matrix", "Jordan form", or "Jordan canonical form". Using this terminology, we may say, in consequence of the results of Sects.91-93 and of Prop.1, that for every lineon on a linear space over $\mathbb{C}$, we can find a basis such that the matrix of the lineon relative to that basis is a Jordan matrix.

95 Similarity, Elementary Divisors

Definitions *We say that the lineon* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *is* **similar** *to the lineon* $\mathbf{L}' \in \operatorname{Lin}\mathcal{V}'$ *if there is a linear isomorphism* $\mathbf{A} : \mathcal{V} \to \mathcal{V}'$ *such that*

$$\mathbf{L}' = \mathbf{A}\mathbf{L}\mathbf{A}^{-1}, \quad \text{i.e.} \quad \mathbf{L}'\mathbf{A} = \mathbf{A}\mathbf{L}. \tag{95.1}$$

Assume that $\mathbf{L}' \in \operatorname{Lin}\mathcal{V}'$ is similar to $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$. By Cor. 2 to the Characterization of Dimension (Sect.17) we then must have $\dim \mathcal{V} = \dim \mathcal{V}'$. Also, the following results, all easily proved, are valid.

(I) A subspace $\mathcal{U}$ of $\mathcal{V}$ is $\mathbf{L}$-invariant if and only if $\mathbf{A}_{>}(\mathcal{U})$ is $\mathbf{L}'$-invariant. (See Prop.3 of Sect.82).

(II) $\mathbf{L}$ is elementary if and only if $\mathbf{L}'$ is elementary. If this is the case, then $\mathbf{L}$ and $\mathbf{L}'$ have the same prime polynomial and the same depth.

(III) A decomposition $(\mathcal{E}_i \mid i \in I)$ of $\mathcal{V}$ is an elementary $\mathbf{L}$-decomposition if and only if $(\mathbf{A}_{>}(\mathcal{E}_i) \mid i \in I)$ is an elementary $\mathbf{L}'$-decomposition of $\mathcal{V}'$.

Roughly, similar lineons have the same intrinsic structure.

The following result shows, roughly, that elementary decompositions of similar lineons can be made to correspond.

Similarity Theorem for Lineons: *Let $(\mathcal{E}_i \mid i \in I)$ and $(\mathcal{E}'_i \mid i \in I')$ be elementary decompositions of the given lineons $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and $\mathbf{L}' \in \mathrm{Lin}\mathcal{V}'$, respectively. If $\mathbf{L}'$ is similar to $\mathbf{L}$, then there is an invertible mapping $\varphi : I \to I'$ such that $\mathbf{L}_{|\mathcal{E}_i}$ is similar to $\mathbf{L}_{|\mathcal{E}'_{\varphi(i)}}$ for all $i \in I$.*

The proof will be based on the following:

Lemma: *Assume that* (95.1) *holds for given $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, $\mathbf{L}' \in \mathrm{Lin}\mathcal{V}'$, and $\mathbf{A} \in \mathrm{Lis}(\mathcal{V}, \mathcal{V}')$. Let $\mathcal{E}$,$\mathcal{U}$ be supplementary $\mathbf{L}$-subspaces of $\mathcal{V}$ and let $\mathcal{E}'$,$\mathcal{U}'$ be supplementary $\mathbf{L}'$-subspaces of $\mathcal{V}'$. Let $\mathbf{E} \in \mathrm{Lin}\mathcal{V}$ be the idempotent for which $\mathrm{Null}\ \mathbf{E} = \mathcal{E}$ and $\mathrm{Rng}\,\mathbf{E} = \mathcal{U}$ and let $\mathbf{F} \in \mathrm{Lin}\mathcal{V}'$ be the idempotent for which $\mathrm{Null}\ \mathbf{F} = \mathcal{U}'$ and $\mathrm{Rng}\,\mathbf{F} = \mathcal{E}'$* (see Prop.4 of Sect.19). *Then:*

(a) *$\mathrm{Null}\ (\mathbf{FA})$ is an $\mathbf{L}$-subspace of $\mathcal{V}$ and $\mathrm{Null}\ (\mathbf{EA}^{-1})$ is an $\mathbf{L}'$-subspace of $\mathcal{V}'$.*

(b) *$\mathbf{A}_{>}(\mathrm{Null}\ (\mathbf{FA}|_{\mathcal{E}})) = \mathrm{Null}\ (\mathbf{EA}^{-1}|_{\mathcal{U}'})$.*

(c) *If $\dim \mathcal{E} \geq \dim \mathcal{E}'$ and $\mathrm{Null}\ (\mathbf{FA}|_{\mathcal{E}}) = \{\mathbf{0}\}$, then $\mathbf{L}_{|\mathcal{E}}$ is similar to $\mathbf{L}'_{|\mathcal{E}'}$, and $\mathbf{L}_{|\mathcal{U}}$ is similar to $\mathbf{L}'_{|\mathcal{U}'}$.*

Proof: It follows immediately from the definitions of $\mathbf{E}$ and $\mathbf{F}$ that $\mathbf{E}$ commutes with $\mathbf{L}$ and that $\mathbf{F}$ commutes with $\mathbf{L}'$.

Let $\mathbf{v} \in \mathrm{Null}\ (\mathbf{FA})$ be given. Using (95.1) we obtain $\mathbf{0} = \mathbf{L}'(\mathbf{FA})\mathbf{v} = \mathbf{F}(\mathbf{L}'\mathbf{A})\mathbf{v} = \mathbf{F}(\mathbf{AL})\mathbf{v} = \mathbf{FA}(\mathbf{Lv})$ and hence $\mathbf{Lv} \in \mathrm{Null}\ (\mathbf{FA})$. Since $\mathbf{v} \in \mathrm{Null}\ (\mathbf{FA})$ was arbitrary, it follows that $\mathrm{Null}\ (\mathbf{FA})$ is $\mathbf{L}$-invariant. Interchanging the roles of $\mathbf{L}$ and $\mathbf{L}'$ we find that $\mathrm{Null}\ (\mathbf{EA}^{-1})$ is $\mathbf{L}'$ invariant. Hence Part (a) is proved.

Let $\mathbf{v} \in \mathcal{V}$ be given. Then

$$\begin{aligned} & \mathbf{v} \in \mathrm{Null}\ (\mathbf{FA}|_{\mathcal{E}}) = \mathrm{Null}\ (\mathbf{FA}) \cap \mathcal{E} \\ \Longleftrightarrow\ & \mathbf{v} \in \mathcal{E} = \mathrm{Null}\ \mathbf{E} \text{ and } \mathbf{F}(\mathbf{Av}) = \mathbf{0} \\ \Longleftrightarrow\ & \mathbf{0} = \mathbf{Ev} = (\mathbf{EA}^{-1})(\mathbf{Av}) \text{ and } \mathbf{Av} \in \mathrm{Null}\ \mathbf{F} = \mathcal{U}' \\ \Longleftrightarrow\ & \mathbf{Av} \in \mathrm{Null}\ (\mathbf{EA}^{-1}) \cap \mathcal{U}' = \mathrm{Null}\ (\mathbf{EA}^{-1}|_{\mathcal{U}'}). \end{aligned}$$

Since $\mathbf{v} \in \mathcal{V}$ was arbitrary, Part (b) follows.

If $\dim \mathcal{E} \geq \dim \mathcal{E}'$ and Null $(\mathbf{FA}|_{\mathcal{E}}) = \{\mathbf{0}\}$ we can apply the Pigeonhole Principle for Linear Mappings of Sect.17 to $(\mathbf{FA})|_{\mathcal{E}}^{\mathcal{E}'}$ and conclude that $\dim \mathcal{E} = \dim \mathcal{E}'$ and that $(\mathbf{FA})|_{\mathcal{E}}^{\mathcal{E}'}$ is invertible. Using (95.1) we obtain

$$(\mathbf{FA})|_{\mathcal{E}}^{\mathcal{E}'}\mathbf{L}_{|\mathcal{E}} = (\mathbf{FAL})|_{\mathcal{E}}^{\mathcal{E}'} = (\mathbf{FL}'\mathbf{A})|_{\mathcal{E}}^{\mathcal{E}'} = (\mathbf{L}'\mathbf{FA})|_{\mathcal{E}}^{\mathcal{E}'} = \mathbf{L}'_{|\mathcal{E}'}(\mathbf{FA})|_{\mathcal{E}}^{\mathcal{E}'}$$

and hence that $\mathbf{L}'_{|\mathcal{E}'}$ is similar to $\mathbf{L}_{|\mathcal{E}}$. By part (b) we also have
Null $(\mathbf{EA}^{-1}|_{\mathcal{U}'}) = \{\mathbf{0}\}$. Since

$$\dim \mathcal{U}' = \dim \mathcal{V}' - \dim \mathcal{E}' = \dim \mathcal{V} - \dim \mathcal{E} = \dim \mathcal{U}$$

we can apply the same argument as just given to the case when $\mathbf{L}$ and $\mathbf{L}'$ are interchanged and conclude that $\mathbf{L}_{|\mathcal{U}}$ is similar to $\mathbf{L}'_{|\mathcal{U}'}$. ∎

Proof of the Theorem: We choose a space of greatest dimension from the collection $\{\mathcal{E}_i \mid i \in I\} \cup \{\mathcal{E}'_i \mid i \in I'\}$. Without loss of generality we may assume that we have chosen $\mathcal{E}_j,\ j \in I$. Let $\mathcal{M}$ be the (only) minimal $\mathbf{L}$-space included in $\mathcal{E}_j$. Let $(\mathbf{E}'_i \mid i \in I)$ be the family of idempotents associated with the decomposition $(\mathcal{E}'_i \mid i \in I')$ (see Prop.5 of Sect.81). By (81.5), we have

$$\sum_{i \in I'}(\mathbf{E}'_i\mathbf{A})|_{\mathcal{M}} = (\sum_{i \in I'}\mathbf{E}'_i)\mathbf{A}|_{\mathcal{M}} = \mathbf{A}|_{\mathcal{M}}.$$

Since $\mathbf{A}$ is invertible and $\mathcal{M} \neq \{\mathbf{0}\}$, we may choose $j' \in I'$ such that $(\mathbf{E}_{j'}\mathbf{A})|_{\mathcal{M}} \neq \{\mathbf{0}\}$. We now abbreviate $\mathcal{E} := \mathcal{E}_j,\ \mathcal{E}' := \mathcal{E}'_{j'}$,
$\mathcal{U} := \sum(\mathcal{E}_i \mid i \in I \setminus \{j\})$, $\mathcal{U}' := \sum(\mathcal{E}'_i \mid i \in I' \setminus \{j'\})$ and $\mathbf{F} := \mathbf{E}'_{j'}$. The Lemma then applies. By part (a), Null $(\mathbf{FA})$ is an $\mathbf{L}$-subspace of $\mathcal{V}$ and so is $\mathcal{E} \cap \text{Null } (\mathbf{FA}) = \text{Null } (\mathbf{FA}|_{\mathcal{E}})$. Since $\mathbf{FA}|_{\mathcal{M}} \neq \{\mathbf{0}\}$ and $\mathcal{M} \subset \mathcal{E}$, we cannot have $\mathcal{M} \subset \text{Null } (\mathbf{FA}|_{\mathcal{E}})$. Hence, since $\mathcal{E}$ is elementary, it follows from Prop.3 of Sect.91 that Null $(\mathbf{FA}|_{\mathcal{E}}) = \{\mathbf{0}\}$. In view of the maximality assumption on the dimension of $\mathcal{E} := \mathcal{E}_j$, we have
$\dim \mathcal{E} \geq \dim \mathcal{E}'$. Hence we can apply Part (c) of the Lemma to conclude that $\dim \mathcal{E}_j = \dim \mathcal{E}'_{j'}$, that $\mathbf{L}_{|\mathcal{E}_j}$ is similar to $\mathbf{L}'_{|\mathcal{E}'_{j'}}$, and that $\mathbf{L}_{|\mathcal{U}_j}$ is similar to $\mathbf{L}'_{|\mathcal{U}'_{j'}}$. The desired result now follows immediately by induction, using Prop.6 of Sect.81. ∎

If we apply the Similarity Theorem to the case when $\mathbf{L}' = \mathbf{L}$ we obtain

Corollary 1: *All elementary* $\mathbf{L}$*-decompositions for a given lineon* $\mathbf{L}$ *have the same number of terms. Moreover, if two such decompositions are given, then the terms of one decomposition can be matched with the terms of the other such that the adjustments of* $\mathbf{L}$ *to the matched* $\mathbf{L}$*-spaces are similar.*

In view of the statement (II) above, this Corollary implies that the prime polynomials and the depths of the adjustments of a lineon $\mathbf{L}$ to the terms in an elementary $\mathbf{L}$-decomposition, if each is counted with an appropriate "multiplicity", do not depend on the decomposition but only on $\mathbf{L}$. More precisely, denoting the set of all powers of prime polynomials over $\mathbb{F}$ by $\mathfrak{P}$, one can associate with each lineon $\mathbf{L}$ a unique **elementary multiplicity** function

$$\mathrm{emult}_{\mathbf{L}} : \mathfrak{P} \to \mathbb{N}$$

with the following property: In *every* elementary decomposition of $\mathbf{L}$, there are exactly $\mathrm{emult}_{\mathbf{L}}(q^d)$ terms whose minimal polynomial is q^d. The support of $\mathrm{emult}_{\mathbf{L}}$ is finite and the members of this support are called the **elementary divisors** of $\mathbf{L}$. They are all divisors of the **characteristic polynomial** chp of $\mathbf{L}$, which is defined by

$$\mathrm{chp}_{\mathbf{L}} := \prod_{q^d \in \mathfrak{P}} (q^d)^{\mathrm{emult}_{\mathbf{L}}(q^d)}. \tag{95.2}$$

It follows from Cor.5 of Sect.93 and Prop.4 of Sect.81 that the degree of the characteristic polynomial (95.2) is $\dim \mathcal{V}$. It is evident that $\mathrm{chp}_{\mathbf{L}}(\mathbf{L}) = \mathbf{0}$. Therefore the degree of the minimal polynomial of $\mathbf{L}$ cannot exceed the dimension of the domain $\mathcal{V}$ of $\mathbf{L}$.

Remark 1: In Vol.II we will give another definition of the characteristic polynomial, a definition in terms of *determinants*. Then (95.2) and $\mathrm{chp}_{\mathbf{L}}(\mathbf{L}) = \mathbf{0}$ will become theorems. The determinant of $\mathbf{L}$ will turn out to be given by $\det \mathbf{L} = (-1)^{\dim \mathcal{V}} (\mathrm{chp}_{\mathbf{L}})_0$. ■

The following consequence of the Similarity Theorem is now evident.

Corollary 2: *If the lineon* $\mathbf{L}'$ *is similar to the lineon* $\mathbf{L}$, *then* $\mathbf{L}'$ *and* $\mathbf{L}$ *have the same elementary multiplicity functions, the same elementary divisors, and the same characteristic polynomial, i.e. we have* $\mathrm{emult}_{\mathbf{L}} = \mathrm{emult}_{\mathbf{L}'}$ *and* $\mathrm{chp}_{\mathbf{L}} = \mathrm{chp}_{\mathbf{L}'}$.

Remark 2: The converse of Cor.2 is also true: if $\mathrm{emult}_{\mathbf{L}} = \mathrm{emult}_{\mathbf{L}'}$ then $\mathbf{L}'$ is similar to $\mathbf{L}$ (see Problem 2). However, if $\mathbf{L}$ and $\mathbf{L}'$ merely have the same *set* of elementary divisors (without counting multiplicity), or if $\mathbf{L}$ and $\mathbf{L}'$ merely have the same characteristic polynomial, then they need not be equivalent. ■

96 Problems for Chapter 9

(1) Consider

$$L := \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{bmatrix} \in \operatorname{Lin}\mathbb{R}^3.$$

(a) Find an elementary L-decomposition of $\mathbb{R}^3$.

(b) Determine the elementary divisors, the characteristic polynomial, and the minimal polynomial of L.

(c) Find a basis $b := (b_1, b_2, b_3)$ of $\mathbb{R}^3$ such that the matrix $[L]_b$ is canonical in the sense described in Sect.94.

(2) Let $\mathcal{V}$ and $\mathcal{V}'$ be linear spaces (over any field $\mathbb{F}$). Let $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$, $\mathbf{L}' \in \operatorname{Lin}\mathcal{V}'$ be given and let q and q' be their respective minimal polynomials.

(a) Assume that $q = q'$ and $\deg q = \dim \mathcal{V} = \dim \mathcal{V}'$. Show that $\mathbf{L}'$ is then similar to $\mathbf{L}$.

(d) Assume that $\mathbf{L}$ and $\mathbf{L}'$ are both elementary and have the same prime polynomial and depth. Show that $\mathbf{L}'$ is then similar to $\mathbf{L}$. (Hint: Use Cor.5 to the Structure Theorem for Elementary Lineons.)

(c) Assume that $\mathbf{L}$ and $\mathbf{L}'$ have the same elementary multiplicity function, i.e. that $\operatorname{emult}_{\mathbf{L}} = \operatorname{emult}_{\mathbf{L}'}$ (see Sect.95). Show that $\mathbf{L}'$ is then similar to $\mathbf{L}$. (Hint: Choose elementary decompositions for $\mathbf{L}$ and $\mathbf{L}'$, use Part (b) above, and then Prop.6 of Sect.81.)

(3) Let $(\mathcal{E}_i \mid i \in I)$ be a decomposition of a given linear space $\mathcal{V}$ and let $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$.

(a) Put

$$\mathcal{E}'_j := \textstyle\bigcap_{i \in I \setminus \{j\}} \mathcal{E}_i^{\perp} \quad \text{for all} \quad j \in I. \tag{P9.1}$$

Show that $(\mathcal{E}'_j \mid j \in I)$ is a decomposition of $\mathcal{V}^*$.

(b) Prove: If $(\mathcal{E}_i \mid i \in I)$ is an elementary $\mathbf{L}$-decomposition, then $(\mathcal{E}'_j \mid j \in I)$ as defined by $(P9.1)$, is an elementary $\mathbf{L}^{\top}$-decomposition and $(\mathbf{L}_{|\mathcal{E}_i})^{\top}$ is similar to $(\mathbf{L}^{\top})_{|\mathcal{E}'_i}$ for each $i \in I$.

(c) Prove that $\mathbf{L}^\top$ is similar to $\mathbf{L}$. (Hint: Use Cor.3 to the Structure Theorem for Elementary Lineons and the results of Problem 2.)

(4) Let $\mathcal{V}$ be a linear space over $\mathbb{R}$ and let $\mathbf{L}$ be a lineon on $\mathcal{V}$.

(a) Show: If $\mathbf{L}$ is elementary, then either $\operatorname{Spec}\mathbf{L}$ is a singleton, in which case $\operatorname{Pspec}\mathbf{L}$ (see Def.2 of Sect.88) is empty, or else $\operatorname{Spec}\mathbf{L}$ is empty, in which case $\operatorname{Pspec}\mathbf{L}$ is a singleton.

(b) Assume that $\mathbf{L}$ is elementary and that $\operatorname{Spec}\mathbf{L}$ is a singleton. Put $\sigma :\in \operatorname{Spec}\mathbf{L}$. Show that

$$\mathbf{L} = \sigma \mathbf{1}_{\mathcal{V}} + \mathbf{N} \tag{P9.2}$$

for some nilpotent lineon $\mathbf{N}$ with nilpotency $n := \dim\mathcal{V}$ (see Sect.93).

(c) Assume that $\mathbf{L}$ is elementary and that $\operatorname{Spec}\mathbf{L}$ is empty. Put $(\mu, \kappa) :\in \operatorname{Pspec}\mathbf{L}$. Show that $\dim\mathcal{V}$ must be even and that

$$\mathbf{L} = \mu \mathbf{1}_{\mathcal{V}} + \kappa\mathbf{J} + \mathbf{N} \tag{P9.3}$$

for some $\mathbf{J} \in \operatorname{Lin}\mathcal{V}$ satisfying $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$ and some nilpotent lineon $\mathbf{N}$ that commutes with $\mathbf{L}$ and has nilpotency $d := \frac{1}{2}\dim\mathcal{V}$.

(d) Prove: $\operatorname{Spec}\mathbf{L}$ cannot be empty when $\dim\mathcal{V}$ is odd and $\operatorname{Pspec}\mathbf{L}$ cannot be empty when $\operatorname{Spec}\mathbf{L}$ is empty.

(5) Let $\mathcal{V}$ be a linear space over $\mathbb{R}$ and let $\mathbf{L}$ be a lineon on $\mathcal{V}$.

(a) Let $(\mathcal{E}_i \mid i \in I)$ be a decomposition of $\mathcal{V}$ all of whose terms are $\mathbf{L}$-spaces. Prove that the terms are then also $(\exp_{\mathcal{V}}(\mathbf{L}))$-spaces, that

$$(\exp_{\mathcal{V}}(\mathbf{L}))_{|\mathcal{E}_i} = \exp_{\mathcal{E}_i}(\mathbf{L}_{|\mathcal{E}_i}) \quad \text{for all} \quad i \in I, \tag{P9.4}$$

and that

$$\exp_{\mathcal{V}}(\mathbf{L}) = \textstyle\sum_{i\in I} \exp_{\mathcal{E}_i}(\mathbf{L}_{|\mathcal{E}_i})|^{\mathcal{V}}\mathbf{P}_i, \tag{P9.5}$$

where $(\mathbf{P}_i \mid i \in I)$ is the family of projections associated with the decomposition (see Prop.5 of Sect.81). (Hint: Use a proof similar to the one of Prop.3 of Sect.85.)

Remark: Since this result applies, in particular, to elementary decompositions, we see that the problem of evaluating the exponential of an arbitrary lineon is reduced to the problem of evaluating the exponential of *elementary* lineons. ■

(b) Assume that $\mathbf{L}$ is elementary. Show that

$$\exp_{\mathcal{V}}(\mathbf{L}) = \mathrm{e}^{\sigma} \sum_{k \in n^{[}} \tfrac{1}{k!} \mathbf{N}^k \tag{P9.6}$$

if $\mathbf{L}$ is of the form $(P9.2)$ of Problem 4 and that

$$\exp_{\mathcal{V}}(\mathbf{L}) = \mathrm{e}^{\mu} (\cos \kappa \mathbf{1}_{\mathcal{V}} + \sin \kappa\, \mathbf{J}) \sum_{k \in d^{[}} \tfrac{1}{k!} \mathbf{N}^k \tag{P9.7}$$

if $\mathbf{L}$ is of the form $(P9.3)$. (Hint: Use the results of Problem 9 in Chap.6.)