Chapter 8

Spectral Theory

In this chapter, it is assumed that all linear spaces under consideration are over the real field $\mathbb{R}$ and that all linear and inner-product spaces are finite-dimensional. However, in Sects.89 and 810 we deal with linear spaces over $\mathbb{R}$ that are at the same time linear spaces over the complex field $\mathbb{C}$. Most of this chapter actually deals with a given finite-dimensional genuine inner-product space. Some of the definitions remain meaningful and some of the results remain valid if the space is infinite-dimensional or if $\mathbb{R}$ is replaced by an arbitrary field.

81 Disjunct Families, Decompositions

We assume that a linear space $\mathcal{V}$ and a finite family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$ are given. There is a natural summing mapping from the product space $\times(\mathcal{U}_i \mid i \in I)$ (see Sect.14) to $\mathcal{V}$, defined by

$$(\mathbf{u} \mapsto \Sigma_I \mathbf{u}) \ : \ \mathop{\times}_{i \in I} \mathcal{U}_i \to \mathcal{V}. \tag{81.1}$$

This summing mapping is evidently linear.

Definition 1: *We say that the finite family* $(\mathcal{U}_i \mid i \in I)$ *of subspaces of* $\mathcal{V}$ *is* **disjunct** *if the summing mapping* (81.1) *of the family is injective; we say that the family is a* **decomposition** *of* $\mathcal{V}$ *if the summing mapping is invertible.*

It is clear that every restriction of a disjunct family is again disjunct. In other words, if $(\mathcal{U}_i \mid i \in I)$ is disjunct and $J \in \operatorname{Sub} I$, then $(\mathcal{U}_j \mid j \in J)$ is again disjunct. The *union* of a disjunct family cannot be the same as its sum unless all but at most one of the terms are zero-spaces. A disjunct family

$(\mathcal{U}_i \mid i \in I)$ is a decomposition of $\mathcal{V}$ if and only if its member-wise sum is $\mathcal{V}$, i.e. $\sum(\mathcal{U}_i \mid i \in I) = \mathcal{V}$ (see (07.14)).

Since the summing mapping is injective if and only if its nullspace is zero, we have the following result:

Proposition 1: *A finite family $(\mathcal{U}_i \mid i \in I)$ of subspaces is disjunct if and only if, for every family $\mathbf{u} \in \times(\mathcal{U}_i \mid i \in I)$, $\sum(\mathbf{u}_i \mid i \in I) = \mathbf{0}$ is possible only when $\mathbf{u}_i = \mathbf{0}$ for all $i \in I$.*

The following result gives various characterizations of disjunct families and decompositions.

Proposition 2: *Let $(\mathcal{U}_i \mid i \in I)$ be a finite family of subspaces of $\mathcal{V}$. Then the following are equivalent:*

(i) *The family $(\mathcal{U}_i \mid i \in I)$ is disjunct [a decomposition of $\mathcal{V}$].*

(ii) *For every $j \in I$, $\mathcal{U}_j$ and $\sum(\mathcal{U}_i \mid i \in I\setminus\{j\})$ are disjunct [supplementary in $\mathcal{V}$].*

(iii) *If $I \neq \emptyset$ then for some $j \in I$, $\mathcal{U}_j$ and $\sum(\mathcal{U}_i \mid i \in I \setminus \{j\})$ are disjunct [supplementary in $\mathcal{V}$] and the family $(\mathcal{U}_i \mid i \in I \setminus \{j\})$ is disjunct.*

Proof: We prove only the assertions concerning disjunctness. The rest then follows from

$$\sum(\mathcal{U}_i \mid i \in I) = \mathcal{U}_j + \sum(\mathcal{U}_i \mid i \in I \setminus \{j\}),$$

valid for all $j \in I$.

(i) $\Rightarrow$ (ii): Assume that $(\mathcal{U}_i \mid i \in I)$ is disjunct and that $j \in I$ and $\mathbf{w} \in \mathcal{U}_j \cap \sum(\mathcal{U}_i \mid i \in I\setminus\{j\})$ are given. Then we may choose $(\mathbf{u}_i \mid i \in I\setminus\{j\})$ such that $\mathbf{w} = \sum(\mathbf{u}_i \mid i \in I\setminus\{j\})$ and hence $(-\mathbf{w}) + \sum(\mathbf{u}_i \mid i \in I\setminus\{j\}) = \mathbf{0}$. By Prop.1, it follows that $\mathbf{w} = \mathbf{0}$. We conclude that $\mathcal{U}_j \cap \sum(\mathcal{U}_i \mid i \in I\setminus\{j\}) = \{\mathbf{0}\}$.

(ii) $\Rightarrow$ (i): Assume that $(\mathcal{U}_i \mid i \in I)$ fails to be disjunct. By Prop.1 we can then find $j \in I$ and $\mathbf{w} \in \mathcal{U}_j^{\times}$ such that $\mathbf{w} + \sum(\mathbf{u}_i \mid i \in I \setminus \{j\}) = \mathbf{0}$ for a suitable choice of $\mathbf{u} \in \times(\mathcal{U}_i \mid i \in I \setminus \{j\})$. Then $\mathbf{w} \in \mathcal{U}_j \cap \sum(\mathcal{U}_i \mid i \in I \setminus \{j\})$ and hence, since $\mathbf{w} \neq \mathbf{0}$, $\mathcal{U}_j$ and $\sum(\mathcal{U}_i \mid i \in I \setminus \{j\})$ are not disjunct.

(i) $\Rightarrow$ (iii): This follows from (i) $\Rightarrow$ (ii) and the fact that a restriction of a disjunct family is disjunct.

(iii) $\Rightarrow$ (i): Choose $j \in I$ according to (iii) and let $\mathbf{u} \in \times(\mathcal{U}_i \mid i \in I)$ such that $\sum(\mathbf{u}_i \mid i \in I) = \mathbf{0}$ be given. Then $\mathbf{u}_j = -\sum(\mathbf{u}_i \mid i \in I \setminus \{j\})$ and hence $\mathbf{u}_j = \mathbf{0}$ because $\mathcal{U}_j$ and $\sum(\mathcal{U}_i \mid i \in I \setminus \{j\})$ are disjunct. It follows that $\sum(\mathbf{u}_i \mid i \in I \setminus \{j\}) = \mathbf{0}$ and hence $\mathbf{u}|_{I\setminus\{j\}} = \mathbf{0}$ because $(\mathcal{U}_i \mid i \in I \setminus \{j\})$ is disjunct. We conclude that $\mathbf{u} = \mathbf{0}$ and hence, by Prop.1, that $(\mathcal{U}_i \mid i \in I)$ is disjunct. ∎

It follows from Prop.2 that a pair $(\mathcal{U}_1, \mathcal{U}_2)$ of subspaces of $\mathcal{V}$ is disjunct [a decomposition of $\mathcal{V}$] if and only if $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjunct [supplementary in $\mathcal{V}$] in the sense of Def.2 of Sect.12.

The following result is an easy consequence of Prop.2.

Proposition 3: *Let $(\mathcal{U}_i \mid i \in I)$ be a disjunct family of subspaces of $\mathcal{V}$, let J be a subset of I, and let $\mathcal{W}_j$ be a subspace of $\mathcal{U}_j$ for each $j \in J$. Then $(\mathcal{W}_j \mid j \in J)$ is again a disjunct family of subspaces of $\mathcal{V}$. Moreover, if $(\mathcal{W}_j \mid j \in J)$ is a decomposition of $\mathcal{V}$, so is $(\mathcal{U}_i \mid i \in I)$ and we have $\mathcal{W}_j = \mathcal{U}_j$ for all $j \in J$ and $\mathcal{U}_i = \{\mathbf{0}\}$ for all $i \in I \setminus J$.*

Using Prop.2 above and Prop.4 of Sect.17 one easily obtains the following result by induction.

Proposition 4: *A finite family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$ is disjunct if and only if*

$$\dim\Big(\sum_{i\in I}\mathcal{U}_i\Big) = \sum_{i\in I}\dim\mathcal{U}_i. \tag{81.2}$$

In particular, if $(\mathcal{U}_i \mid i \in I)$ is a decomposition of $\mathcal{V}$, then $\dim\mathcal{V} = \sum(\dim\mathcal{U}_i \mid i \in I)$.

Let $(\mathbf{f}_i \mid i \in I)$ be a finite family of non-zero elements in $\mathcal{V}$. This family is linearly independent, or a basis of $\mathcal{V}$, depending on whether the family $(\mathrm{Lsp}\{\mathbf{f}_i\} \mid i \in I)$ of one-dimensional subspaces of $\mathcal{V}$ is disjunct, or a decomposition of $\mathcal{V}$, respectively.

The following result generalizes Prop.4 of Sect.19 and is easily derived from that proposition and Prop.2.

Proposition 5: *Let $(\mathcal{U}_i \mid i \in I)$ be a decomposition of $\mathcal{V}$. Then there is a unique family $(\mathbf{P}_i \mid i \in I)$ of projections $\mathbf{P}_i : \mathcal{V} \to \mathcal{U}_i$ such that*

$$\mathbf{v} = \sum_{i\in I}\mathbf{P}_i\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}, \tag{81.3}$$

and we have

$$\mathcal{U}_i \subset \mathrm{Null}\ \mathbf{P}_j \quad \text{for all} \quad i, j \in I \quad \text{with} \quad i \neq j. \tag{81.4}$$

Also, there is a unique family $(\mathbf{E}_i \mid i \in I)$ of idempotent lineons on $\mathcal{V}$ such that $\mathcal{U}_i = \mathrm{Rng}\,\mathbf{E}_i$,

$$\sum_{i\in I}\mathbf{E}_i = \mathbf{1}_{\mathcal{V}}, \tag{81.5}$$

and

$$\mathbf{E}_i\mathbf{E}_j = \mathbf{0} \quad \text{for all} \quad i, j \in I \quad \text{with} \quad i \neq j. \tag{81.6}$$

We have

$$\text{Null } \mathbf{E}_j = \sum_{i \in I \setminus \{j\}} \mathcal{U}_i \quad \text{for all} \quad j \in I \tag{81.7}$$

and

$$\mathbf{E}_i = \mathbf{P}_i|^{\mathcal{V}} \quad \text{and} \quad \mathbf{P}_i = \mathbf{E}_i|^{\mathcal{U}_i} \quad \text{for all} \quad i \in I. \tag{81.8}$$

The family $(\mathbf{P}_i \mid i \in I)$ *is called the* **family of projections** *and the family* $(\mathbf{E}_i \mid i \in I)$ *the* **family of idempotents** *associated with the given decomposition.*

The following result, a generalization of Prop.5 of Sect.19 and Prop.2 of Sect.16, shows how linear mappings with domain $\mathcal{V}$ are determined by their restrictions to each term of a decomposition of $\mathcal{V}$.

Proposition 6: *Let* $(\mathcal{U}_i \mid i \in I)$ *be a decomposition of* $\mathcal{V}$*. For every linear space* $\mathcal{V}'$ *and every family* $(\mathbf{L}_i \mid i \in I)$ *with* $\mathbf{L}_i \in \text{Lin}(\mathcal{U}_i, \mathcal{V}')$ *for all* $i \in I$*, there is exactly one* $\mathbf{L} \in \text{Lin}(\mathcal{V}, \mathcal{V}')$ *such that*

$$\mathbf{L}_i = \mathbf{L}|_{\mathcal{U}_i} \quad \text{for all} \quad i \in I. \tag{81.9}$$

It is given by

$$\mathbf{L} := \sum_{i \in I} \mathbf{L}_i \mathbf{P}_i, \tag{81.10}$$

where $(\mathbf{P}_i \mid i \in I)$ *is the family of projections associated with the decomposition.*

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(1) I introduced the term "disjunct" in the sense of Def.1 for pairs of subspaces (see Note (2) to Sect.12); J. J. Schäffer suggested that the term be used also for arbitrary families of subspaces.

(2) In most of the literature, the phrase "$\mathcal{V}$ is the direct sum of the family $(\mathcal{U}_i \mid i \in I)$" is used instead of "$(\mathcal{U}_i \mid i \in I)$ is a decomposition of $\mathcal{V}$". The former phrase is actually absurd because it confuses a property of the family $(\mathcal{U}_i \mid i \in I)$ with a property of the sum of the family (see also Note (3) to Sect.12). Even Bourbaki falls into this trap.

82 Spectral Values and Spectral Spaces

We assume that a linear space $\mathcal{V}$ and a lineon $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ are given. Recall that a subspace $\mathcal{U}$ of $\mathcal{V}$ is called an $\mathbf{L}$-space if it is $\mathbf{L}$-invariant, i.e. if $\mathbf{L}_>(\mathcal{U}) \subset \mathcal{U}$ (Def.1 of Sect.18). Assume that $\mathcal{U}$ is an $\mathbf{L}$-space. Then we can consider the adjustment $\mathbf{L}_{|\mathcal{U}} \in \mathrm{Lin}\mathcal{U}$. It is clear that a subspace $\mathcal{W}$ of $\mathcal{U}$ is an $\mathbf{L}$-space if and only if it is a $\mathbf{L}_{|\mathcal{U}}$-space.

Proposition 1: *A subspace $\mathcal{U}$ of $\mathcal{V}$ is $\mathbf{L}$-invariant if and only if its annihilator $\mathcal{U}^\perp$ is $\mathbf{L}^\top$-invariant.*

Proof: Assume that $\mathcal{U}$ is $\mathbf{L}$-invariant, i.e. that $\mathbf{L}_>(\mathcal{U}) \subset \mathcal{U}$. By the Theorem on Annihilators and Transposes of Sect.21 we then have

$$\mathcal{U}^\perp \subset (\mathbf{L}_>(\mathcal{U}))^\perp = (\mathbf{L}^\top)^<(\mathcal{U}^\perp),$$

which means that $\mathcal{U}^\perp$ is $\mathbf{L}^\top$-invariant. Interchanging the roles of $\mathbf{L} = (\mathbf{L}^\top)^\top$ and $\mathbf{L}^\top$ and of $\mathcal{U}^\perp$ and $\mathcal{U} = (\mathcal{U}^\perp)^\perp$ we see that the $\mathbf{L}^\top$-invariance of $\mathcal{U}^\perp$ implies the $\mathbf{L}$-invariance of $\mathcal{U}$. ∎

Definition 1: *For every $\sigma \in \mathbb{R}$, we write*

$$\mathrm{Sps}_{\mathbf{L}}(\sigma) := \mathrm{Null}\ (\mathbf{L} - \sigma \mathbf{1}_{\mathcal{V}}). \tag{82.1}$$

This is an $\mathbf{L}$-space; if it is non-zero, we call it the **spectral space** *of $\mathbf{L}$ for σ. The* **spectrum** *of $\mathbf{L}$ is defined to be*

$$\mathrm{Spec}\,\mathbf{L} := \{\sigma \in \mathbb{R} \mid \mathrm{Sps}_{\mathbf{L}}(\sigma) \neq \{\mathbf{0}\}\}, \tag{82.2}$$

and its elements are called the **spectral values** *of $\mathbf{L}$. If $\sigma \in \mathrm{Spec}\,\mathbf{L}$, then the non-zero members of $\mathrm{Sps}_{\mathbf{L}}(\sigma)$ are called* **spectral vectors** *of $\mathbf{L}$ for σ. The family $(\mathrm{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{L})$ of subspaces of $\mathbf{L}$ is called the* **family of spectral spaces** *of $\mathbf{L}$. The* **multiplicity function** $\mathrm{mult}_{\mathbf{L}} : \mathbb{R} \to \mathbb{N}$ *of $\mathbf{L}$ is defined by*

$$\mathrm{mult}_{\mathbf{L}}(\sigma) := \dim(\mathrm{Sps}_{\mathbf{L}}(\sigma)); \tag{82.3}$$

its value $\mathrm{mult}_{\mathbf{L}}(\sigma) \in \mathbb{N}^\times$ at $\sigma \in \mathrm{Spec}\,\mathbf{L}$ is called the **multiplicity** *of the spectral value σ.*

We note that

$$\mathrm{Spec}\,\mathbf{L} = \{\sigma \in \mathbb{R} \mid \mathrm{mult}_{\mathbf{L}}(\sigma) \neq 0\}, \tag{82.4}$$

i.e. that $\mathrm{Spec}\,\mathbf{L}$ is the support of $\mathrm{mult}_{\mathbf{L}}$ (see Sect.07).

It is evident that $\sigma \in \mathbb{R}$ is a spectral value of $\mathbf{L}$ if and only if

$$\mathbf{Lu} = \sigma \mathbf{u} \tag{82.5}$$

for some $\mathbf{u} \in \mathcal{V}^{\times}$. Every $\mathbf{u} \in \mathcal{V}^{\times}$ for which (82.5) holds is a spectral vector of $\mathbf{L}$ for σ. In fact, $\mathrm{Sps}_{\mathbf{L}}(\sigma)$ is the largest among the subspaces $\mathcal{U}$ of $\mathcal{V}$ for which

$$\mathbf{L}|_{\mathcal{U}} = \sigma \mathbf{1}_{\mathcal{U}\subset\mathcal{V}}. \tag{82.6}$$

Proposition 2: *If the lineons* $\mathbf{L}$ *and* $\mathbf{M}$ *commute, then every spectral space of* $\mathbf{M}$ *is* $\mathbf{L}$*-invariant.*

Proof: Note that $\mathbf{L}$ and $\mathbf{M}$ commute if and only if $\mathbf{L}$ and $\mathbf{M} - \sigma\mathbf{1}_{\mathcal{V}}$ commute, no matter what $\sigma \in \mathbb{R}$ is. Hence, by (82.1), it is sufficient to show that $\mathrm{Null}\ \mathbf{M}$ is $\mathbf{L}$-invariant. But for every $\mathbf{u} \in \mathrm{Null}\ \mathbf{M}$, we have

$$\mathbf{0} = \mathbf{L}(\mathbf{M}\mathbf{u}) = (\mathbf{L}\mathbf{M})\mathbf{u} = (\mathbf{M}\mathbf{L})\mathbf{u} = \mathbf{M}(\mathbf{L}\mathbf{u}),$$

i.e. $\mathbf{L}\mathbf{u} \in \mathrm{Null}\ \mathbf{M}$, which proves what was needed. ∎

Let $\mathbf{E} \in \mathrm{Lin}\mathcal{V}$ be an idempotent. By Prop.3, (iv) of Sect.19, we then have

$$\mathrm{Rng}\,\mathbf{E} = \mathrm{Sps}_{\mathbf{E}}(1). \tag{82.7}$$

It is easily seen that $\mathrm{Sps}_{\mathbf{E}}(1)$ and $\mathrm{Sps}_{\mathbf{E}}(0) = \mathrm{Null}\ \mathbf{E}$ are the only spaces of the form $\mathrm{Sps}_{\mathbf{E}}(\sigma),\ \ \sigma \in \mathbb{R}$, that can be different from zero and hence that $\mathrm{Spec}\,\mathbf{E} \subset \{0, 1\}$. In fact, we have $\mathrm{Spec}\,\mathbf{E} = \{0, 1\}$ unless $\mathbf{E} = \mathbf{0}$ or $\mathbf{E} = \mathbf{1}_{\mathcal{V}}$. The multiplicities of the spectral values 0 and 1 of $\mathbf{E}$ are given, in view of (26.11), by

$$\mathrm{mult}_{\mathbf{E}}(1) = \mathrm{tr}\,\mathbf{E}, \quad \mathrm{mult}_{\mathbf{E}}(0) = \dim\mathcal{V} - \mathrm{tr}\,\mathbf{E}. \tag{82.8}$$

The following result, which is easily proved, shows how linear isomorphisms affect invariance, spectra, spectral spaces, and multiplicity:

Proposition 3: *Let* $\mathcal{V}, \mathcal{V}'$ *be linear spaces, let* $\mathbf{A} : \mathcal{V} \to \mathcal{V}'$ *be a linear isomorphism, and let* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ *be a lineon, so that* $\mathbf{A}\mathbf{L}\mathbf{A}^{-1} \in \mathrm{Lin}\mathcal{V}'$.

(a) *If* $\mathcal{U}$ *is an* $\mathbf{L}$*-space then* $\mathbf{A}_{>}(\mathcal{U})$ *is an* $(\mathbf{A}\mathbf{L}\mathbf{A}^{-1})$*-space.*

(b) *For every* $\sigma \in \mathbb{R}$*, we have*

$$\mathbf{A}_{>}(\mathrm{Sps}_{\mathbf{L}}(\sigma)) = \mathrm{Sps}_{\mathbf{A}\mathbf{L}\mathbf{A}^{-1}}(\sigma).$$

(c) $\mathrm{Spec}\,\mathbf{L} = \mathrm{Spec}\,(\mathbf{A}\mathbf{L}\mathbf{A}^{-1})$ and $\mathrm{mult}_{\mathbf{L}} = \mathrm{mult}_{\mathbf{A}\mathbf{L}\mathbf{A}^{-1}}$.

Theorem on Spectral Spaces: *The spectrum of a lineon on* $\mathcal{V}$ *has at most* $\dim\mathcal{V}$ *members and the family of its spectral spaces is disjunct.*

The proof will be based on the following Lemma:

Lemma: *Let* $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$*, let* S *be a finite subset of* $\mathrm{Spec}\,\mathbf{L}$ *and let* $\mathbf{f} := (\mathbf{f}_\sigma \mid \sigma \in S)$ *be such that* $\mathbf{f}_\sigma \in (\mathrm{Sps}_{\mathbf{L}}(\sigma))^{\times}$ *for all* $\sigma \in S$*. Then* $\mathbf{f}$ *is linearly independent.*

Proof: We proceed by induction over $\sharp S$. The assertion is trivial when $\sharp S = 0$, because $\mathbf{f}$ must then be the empty family. Assume, then, that $\sharp S \geq 1$ and that the assertion is valid when S is replaced by S' with $\sharp S' = \sharp S - 1$. Let $\lambda \in \text{Null}\ (\text{lnc}_{\mathbf{f}}) \subset \mathbb{R}^S$, so that $\sum(\lambda_\sigma \mathbf{f}_\sigma \mid \sigma \in S) = \mathbf{0}$. Since $\mathbf{L}\mathbf{f}_\sigma = \sigma\mathbf{f}_\sigma$, it follows that

$$\mathbf{0} = \mathbf{L}(\sum_{\sigma \in S} \lambda_\sigma \mathbf{f}_\sigma) = \sum_{\sigma \in S} \lambda_\sigma \sigma \mathbf{f}_\sigma.$$

Now choose $\tau \in S$ and put $S' := S \setminus \{\tau\}$. We then obtain

$$\mathbf{0} = \tau \sum_{\sigma \in S} \lambda_\sigma \mathbf{f}_\sigma - \sum_{\sigma \in S} \lambda_\sigma \sigma \mathbf{f}_\sigma = \sum_{\sigma \in S'} \lambda_\sigma (\tau - \sigma) \mathbf{f}_\sigma.$$

Since $\tau - \sigma \neq 0$ for all $\sigma \in S'$ and since $\sharp S' = \sharp S - 1$, we can apply the induction hypothesis to $((\tau - \sigma)\mathbf{f}_\sigma \mid \sigma \in S')$ and conclude that $\lambda_\sigma = 0$ for all $\sigma \in S'$, so that $\mathbf{0} = \sum_{\sigma \in S} \lambda_\sigma \mathbf{f}_\sigma = \lambda_\tau \mathbf{f}_\tau$. Since $\mathbf{f}_\tau \neq \mathbf{0}$ it also follows that $\lambda_\tau = 0$ and hence that $\lambda = 0$. Since $\lambda \in \text{Null}\ (\text{lnc}_{\mathbf{f}})$ was arbitrary, we conclude that $\text{Null}\ (\text{lnc}_{\mathbf{f}}) = \{\mathbf{0}\}$. ■

Proof of the Theorem: The fact that $\sharp \text{Spec}\, \mathbf{L} \leq \dim \mathcal{V}$ follows from the Lemma and the Characterization of Dimension of Sect.17. To show that $(\text{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \text{Spec}\, \mathbf{L})$ is disjunct, we need only observe that $\sum(\mathbf{u}_\sigma \mid \sigma \in \text{Spec}\, \mathbf{L}) = \mathbf{0}$ with $\mathbf{u} \in \times(\text{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \text{Spec}\, \mathbf{L})$ implies that $\sum(\mathbf{u}_\sigma \mid \sigma \in S) = \mathbf{0}$, where $S := \text{Supt}\, \mathbf{u}$. The Lemma states that this is possible only when $S = \emptyset$, i.e. when $\mathbf{u}_\sigma = \mathbf{0}$ for all $\sigma \in \text{Spec}\, \mathbf{L}$. By Prop.1, this shows that the family $(\text{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \text{Spec}\, \mathbf{L})$ is disjunct. ■

Since a disjunct family with more than one non-zero term cannot have $\mathcal{V}$ as its union, the following is an immediate consequence of the Theorem just proved.

Proposition 4: *If every* $\mathbf{v} \in \mathcal{V}^\times$ *is a spectral vector of* $\mathbf{L} \in \text{Lin}\mathcal{V}$ *then* $\mathbf{L} = \lambda \mathbf{1}_\mathcal{V}$ *for some* $\lambda \in \mathbb{R}$.

Remark: If $\dim \mathcal{V} = 0$, i.e. if $\mathcal{V}$ is a zero space, then the spectrum of the only member $\mathbf{1}_\mathcal{V} = \mathbf{0}$ of $\text{Lin}\mathcal{V}$ is empty. ■

Proposition 5: *If the sum of the family* $(\text{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \text{Spec}\, \mathbf{L})$ *of spectral spaces of a lineon* $\mathbf{L} \in \text{Lin}\mathcal{V}$ *is* $\mathcal{V}$*, and hence the family is a decomposition of* $\mathcal{V}$*, and if* $(\mathbf{E}_\sigma \mid \sigma \in \text{Spec}\, \mathbf{L})$ *is the associated family of idempotents, then*

$$\mathbf{L} = \sum_{\sigma \in \text{Spec}\, \mathbf{L}} \sigma \mathbf{E}_\sigma \tag{82.9}$$

and

$$\text{tr}\mathbf{L} = \sum_{\sigma \in \text{Spec}\, \mathbf{L}} (\text{mult}_{\mathbf{L}}(\sigma))\sigma. \tag{82.10}$$

Proof: In view of (82.6), we have $\mathbf{L}|_{\mathrm{Sps}_{\mathbf{L}}(\sigma)} = \sigma\mathbf{1}_{\mathrm{Sps}_{\mathbf{L}}(\sigma)\subset\mathcal{V}}$ for all $\sigma \in \mathrm{Spec}\,\mathbf{L}$. It follows from Prop.6 of Sect.81 and (81.8) that (82.9) holds. The equation (82.10) follows from (82.9) and (82.3) using Prop.5 of Sect.26. ∎

Proposition 6: *Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ be given. Assume that Z is a finite subset of $\mathbb{R}$ and that $(\mathcal{W}_\sigma \mid \sigma \in Z)$ is a decomposition of $\mathcal{V}$ whose terms $\mathcal{W}_\sigma$ are non-zero $\mathbf{L}$-spaces and satisfy*

$$\mathbf{L}_{|\mathcal{W}_\sigma} = \sigma\mathbf{1}_{\mathcal{W}_\sigma} \quad \text{for all} \quad \sigma \in Z. \tag{82.11}$$

Then $Z = \mathrm{Spec}\,\mathbf{L}$ and $\mathcal{W}_\sigma = \mathrm{Sps}_{\mathbf{L}}(\sigma)$ for all $\sigma \in Z$.

Proof: Since $\mathcal{W}_\sigma$ is non-zero for all $\sigma \in Z$, it follows from (82.11) that $\sigma \in \mathrm{Spec}\,\mathbf{L}$ and $\mathcal{W}_\sigma \subset \mathrm{Sps}_{\mathbf{L}}(\sigma)$ for all $\sigma \in Z$. The assertion is now an immediate consequence of Prop.3 of Sect.81. ∎

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(1) A large number of terms for our "spectral value" can be found in the literature. The most common is "eigenvalue". Others combine the adjectives "proper", "characteristic", "latent", or "secular" in various combinations with the nouns "value", "number", or "root". I believe it is economical to use the adjective "spectral", which fits with the commonly accepted term "spectrum".

(2) In infinite-dimensional situations, one must make a distinction between the sets $\{\sigma \in \mathbb{R} \mid \ (\mathbf{L} - \sigma\mathbf{1}_{\mathcal{V}})$ is not invertible$\}$ and $\{\sigma \in \mathbb{R} \mid \mathrm{Null}\ (\mathbf{L} - \sigma\mathbf{1}_{\mathcal{V}}) \neq \{\mathbf{0}\}\}$. It is the former that is usually called the spectrum. The latter is usually called the "point-spectrum".

(3) When considering spectral values or spectral spaces, most people replace "spectral" with "eigen", or another of the adjectives mentioned in Note (1).

(4) The notation $\mathrm{Sps}_{\mathbf{L}}(\sigma)$ for a spectral space is introduced here for the first time. I could find only ad hoc notations in the literature.

83 Orthogonal Families of Subspaces

We assume that an inner-product space $\mathcal{V}$ is given. We say that two subsets $\mathcal{S}$ and $\mathcal{T}$ of $\mathcal{V}$ are **orthogonal**, and we write $\mathcal{S}\perp\mathcal{T}$, if every element of $\mathcal{S}$ is orthogonal to every element of $\mathcal{T}$, i.e. if $\mathbf{u}\cdot\mathbf{v} = 0$ for all $\mathbf{u} \in \mathcal{S},\ \mathbf{v} \in \mathcal{T}$. This is the case if and only if $\mathcal{S} \subset \mathcal{T}^\perp$. We say that a family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$ is **orthogonal** if its terms are pairwise orthogonal, i.e. if $\mathcal{U}_i\perp\mathcal{U}_j$ for all $i, j \in I$ with $i \neq j$.

Proposition 1: *A family $(\mathcal{U}_i \mid i \in I)$ of subspaces of $\mathcal{V}$ is orthogonal if and only if for all $j \in I$*

$$\sum_{i \in I \setminus \{j\}} \mathcal{U}_i \subset \mathcal{U}_j^{\perp}. \tag{83.1}$$

Proof: If (83.1) holds for all $j \in I$, then

$$\mathcal{U}_k \subset \sum_{i \in I \setminus \{j\}} \mathcal{U}_i \subset \mathcal{U}_j^{\perp}$$

and hence $\mathcal{U}_k \perp \mathcal{U}_j$ for all $j, k \in I$ with $j \neq k$. If the family is orthogonal, then $\mathcal{U}_i \perp \mathcal{U}_j$ and hence $\mathcal{U}_i \subset \mathcal{U}_j^{\perp}$ for all $j \in I$ and $i \in I \setminus \{j\}$. Since $\mathcal{U}_j^{\perp}$ is a subspace of $\mathcal{V}$, (83.1) follows. ■

Using Prop.2 of Sect.1 and the Characterization of Regular Subspaces of Sect.41 one immediately obtains the next two results.

Proposition 2: *If the terms of an orthogonal family are regular subspaces then it is a disjunct family.*

Proposition 3: *A family $(\mathcal{U}_i \mid i \in I)$ of non-zero subspaces of $\mathcal{V}$ is an orthogonal decomposition of $\mathcal{V}$ if and only if all the terms of the family are regular subspaces of $\mathcal{V}$ and*

$$\sum_{i \in I \setminus \{j\}} \mathcal{U}_i = \mathcal{U}_j^{\perp} \quad \text{for all} \quad j \in I. \tag{83.2}$$

Proposition 4: *A decomposition of $\mathcal{V}$ is orthogonal if and only if all the terms of the family of idempotents associated with the decomposition are symmetric.*

Proof: Let $(\mathcal{U}_i \mid i \in I)$ be a decomposition of $\mathcal{V}$ and let $(\mathbf{E}_i \mid i \in I)$ be the family of idempotents associated with it (see Prop.5 of Sect.81). From Prop.3 and (81.7) it is clear that the decomposition is orthogonal if and only if $\text{Null}\ \mathbf{E}_j = \mathcal{U}_j^{\perp} = (\text{Rng}\,\mathbf{E}_j)^{\perp}$ for all $j \in I$; and this is equivalent, by Prop.3 of Sect.41, to the statement that all the terms of $(\mathbf{E}_i \mid i \in I)$ are symmetric. ■

Proposition 5: *The family of spectral spaces of a symmetric lineon is orthogonal.*

Proof: Let $\mathbf{S} \in \text{Sym}\mathcal{V}$ and $\sigma, \tau \in \text{Spec}\,\mathbf{S}$ with $\sigma \neq \tau$ be given. For all $\mathbf{u} \in \text{Sps}_{\mathbf{S}}(\sigma)$ and $\mathbf{w} \in \text{Sps}_{\mathbf{S}}(\tau)$ we then have $\mathbf{S}\mathbf{u} = \sigma\mathbf{u}$ and $\mathbf{S}\mathbf{w} = \tau\mathbf{w}$. It follows that

$$\sigma(\mathbf{w} \cdot \mathbf{u}) = \mathbf{w} \cdot (\sigma\mathbf{u}) = \mathbf{w} \cdot \mathbf{S}\mathbf{u} = (\mathbf{S}\mathbf{w}) \cdot \mathbf{u} = (\tau\mathbf{w}) \cdot \mathbf{u} = \tau(\mathbf{w} \cdot \mathbf{u}),$$

and hence $(\sigma - \tau)(\mathbf{w} \cdot \mathbf{u}) = 0$ for all $\mathbf{u} \in \text{Sps}_{\mathbf{S}}(\sigma)$ and $\mathbf{w} \in \text{Sps}_{\mathbf{S}}(\tau)$. Since $\sigma - \tau \neq 0$, we conclude that $\text{Sps}_{\mathbf{S}}(\sigma) \perp \text{Sps}_{\mathbf{S}}(\tau)$. ■

84 The Structure of Symmetric Lineons

There are lineons with an empty spectrum, i.e. with no spectral values at all. The concepts of spectral value and spectral space are insufficient for the description of the structure of general lineons (see Chap.9). However, a *symmetric* lineon on a *genuine* inner-product space is completely determined by its family of spectral spaces. The following theorem, whose proof depends on the Theorem on Attainment of Extrema of Sect.58 and on the Constrained Extremum Theorem of Sect.69, is the key to the derivation of this result.

Theorem on the Extreme Spectral Values of a Symmetric Lineon: *If $\mathcal{V}$ is a non-zero genuine inner-product space, then every symmetric lineon $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ has a non-empty spectrum. In fact, the least and greatest members of* $\mathrm{Spec}\,\mathbf{S}$ *are given by*

$$\min \mathrm{Spec}\,\mathbf{S} = \min \overset{\sqcap}{\mathbf{S}}\,|_{\mathrm{Usph}\mathcal{V}}, \tag{84.1}$$

$$\max \mathrm{Spec}\,\mathbf{S} = \max \overset{\sqcap}{\mathbf{S}}\,|_{\mathrm{Usph}\mathcal{V}}, \tag{84.2}$$

where $\overset{\sqcap}{\mathbf{S}} : \mathcal{V} \to \mathbb{R}$ *is the quadratic form associated with* $\mathbf{S}$ (see (27.13)) *and where* $\mathrm{Usph}\mathcal{V}$ *is the unit sphere of* $\mathcal{V}$ *defined by* (42.9).

Proof: $\mathrm{Usph}\mathcal{V}$ is not empty because $\mathcal{V}$ is non-zero, and $\mathrm{Usph}\mathcal{V}$ is closed and bounded because it is the boundary of the unit ball $\mathrm{Ubl}\mathcal{V}$ (see Prop.3 of Sect.52 and Prop.12 of Sect.53). By Prop.3 of Sect.66, $\overset{\sqcap}{\mathbf{S}}$ is of class C^1 and hence continuous. By the Theorem on Attainment of Extrema, it follows that $\overset{\sqcap}{\mathbf{S}}\,|_{\mathrm{Usph}\mathcal{V}}$ attains a maximum and a minimum. Assume that one of these extrema is attained at $\mathbf{u} \in \mathrm{Usph}\mathcal{V}$. By Cor.1 to the Constrained Extremum Theorem, we must have $\nabla_{\mathbf{u}} \overset{\sqcap}{\mathbf{S}} = \sigma \nabla_{\mathbf{u}} \mathrm{sq}$ for some $\sigma \in \mathbb{R}$. Since $(\nabla_{\mathbf{u}}\mathrm{sq})\mathbf{v} = 2\mathbf{u} \cdot \mathbf{v}$ and $\nabla_{\mathbf{u}} \overset{\sqcap}{\mathbf{S}}\, \mathbf{v} = \sigma(\mathbf{S}\mathbf{u}) \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$ by (66.17), we conclude that $2(\mathbf{S}\mathbf{u}) \cdot \mathbf{v} = \sigma(2\mathbf{u} \cdot \mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$, i.e. $\mathbf{S}\mathbf{u} = \sigma\mathbf{u}$. Since $|\mathbf{u}| = 1$, it follows that $\sigma \in \mathrm{Spec}\,\mathbf{S}$ and that $\overset{\sqcap}{\mathbf{S}}(\mathbf{u}) = \mathbf{u} \cdot \mathbf{S}\mathbf{u} = \sigma|\mathbf{u}|^2 = \sigma$. We conclude that the right sides of both (84.1) and (84.2) belong to $\mathrm{Spec}\,\mathbf{S}$.

Now let $\tau \in \mathrm{Spec}\,\mathbf{S}$ be given. We may then choose $\mathbf{u} \in \mathrm{Usph}\mathcal{V}$ such that $\mathbf{S}\mathbf{u} = \tau\mathbf{u}$. Since $\mathbf{u} \cdot \mathbf{u} = 1$, we have $\tau = (\tau\mathbf{u}) \cdot \mathbf{u} = (\mathbf{S}\mathbf{u}) \cdot \mathbf{u} = \overset{\sqcap}{\mathbf{S}}(\mathbf{u}) \in \mathrm{Rng}\, \overset{\sqcap}{\mathbf{S}}\,|_{\mathrm{Usph}\mathcal{V}}$. Since $\tau \in \mathrm{Spec}\,\mathbf{S}$ was arbitrary, the assertions (84.1) and (84.2) follow. ∎

The next theorem, which completely describes the structure of symmetric lineons on genuine inner product spaces, is one of the most important theorems of all of mathematics. It has numerous applications not only in many branches of mathematics, but also in almost all branches of theoretical physics, both classical and modern. The reason is that symmetric lineons appear in many contexts, often unexpectedly.

Spectral Theorem: *Let $\mathcal{V}$ be a genuine inner-product space. A lineon on $\mathcal{V}$ is symmetric if and only if the family of its spectral spaces is an orthogonal decomposition of $\mathcal{V}$.*

Proof: Let $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ be given. Since all spectral spaces of $\mathbf{S}$ are $\mathbf{S}$-invariant, so is their sum $\mathcal{U} := \sum(\mathrm{Sps}_{\mathbf{S}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$. By Prop.1 of Sect.82, the orthogonal supplement $\mathcal{U}^\perp$ of $\mathcal{U}$ is also $\mathbf{S}$-invariant, and $\mathbf{S}_{|\mathcal{U}^\perp} \in \mathrm{Sym}\mathcal{U}^\perp$ is meaningful. Now, by the preceding theorem, if $\mathcal{U}^\perp$ is non-zero, $\mathbf{S}_{|\mathcal{U}^\perp}$ must have a spectral vector $\mathbf{w} \in (\mathcal{U}^\perp)^\times$. Of course, $\mathbf{w}$ is then also a spectral vector of $\mathbf{S}$ and hence $\mathbf{w} \in \mathcal{U}$. Therefore, $\mathcal{U}^\perp \neq \{\mathbf{0}\}$ implies $\mathcal{U} \cap \mathcal{U}^\perp \neq \{\mathbf{0}\}$, which contradicts the fact that, in a genuine inner-product space, all subspaces are regular. It follows that $\mathcal{U}^\perp = \{\mathbf{0}\}$ and hence $\mathcal{V} = \mathcal{U} = \sum(\mathrm{Sps}_{\mathbf{S}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$. By Props.2 and 5 of Sect.83 it follows that $(\mathrm{Sps}_{\mathbf{S}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ is an orthogonal decomposition of $\mathcal{V}$.

Assume, now, that $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ is such that $(\mathrm{Sps}_{\mathbf{L}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{L})$ is an orthogonal decomposition of $\mathcal{V}$. We can then apply Prop.5 of Sect.82 and conclude that (82.9) holds. Since the decomposition was assumed to be orthogonal, each of the idempotents $\mathbf{E}_\sigma$ in (82.9) is symmetric by Prop.4 of Sect.83. Therefore, by (82.9), $\mathbf{L}$ is symmetric. ∎

We assume now that a genuine inner-product space $\mathcal{V}$ is given. We list two corollaries to the Spectral Theorem. The first follows by applying Prop.5 of Sect.81, Prop.5 of Sect.82, and Prop.4 of Sect.83.

Corollary 1: *For every $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ there is exactly one family $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ of non-zero symmetric idempotents such that*

$$\mathbf{E}_\sigma \mathbf{E}_\tau = \mathbf{0} \quad \text{for all} \quad \sigma, \tau \in \mathrm{Spec}\,\mathbf{S} \quad \text{with} \quad \sigma \neq \tau, \tag{84.3}$$

$$\sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} \mathbf{E}_\sigma = \mathbf{1}_{\mathcal{V}}, \tag{84.4}$$

and

$$\sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} \sigma \mathbf{E}_\sigma = \mathbf{S}. \tag{84.5}$$

The family $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ is called the **spectral resolution** *of $\mathbf{S}$. Its terms are called the* **spectral idempotents** *of $\mathbf{S}$.*

The second corollary is obtained by choosing orthonormal bases in each of the spectral spaces of $\mathbf{S}$.

Corollary 2: *A lineon $\mathbf{S}$ on $\mathcal{V}$ is symmetric if and only if there exists an orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ of $\mathcal{V}$ whose terms are spectral vectors of $\mathbf{S}$. If this is the case we have*

$$\mathbf{S}\mathbf{e}_i = \lambda_i \mathbf{e}_i \quad \text{for all} \quad i \in I, \tag{84.6}$$

where $\lambda := (\lambda_i \mid i \in I)$ *is a family of real numbers whose range is* $\operatorname{Spec}\mathbf{S}$*. In fact each* $\sigma \in \operatorname{Spec}\mathbf{S}$ *occurs exactly* $\operatorname{mult}_{\mathbf{S}}(\sigma)$ *times in the family* λ*.*

The matrix of $\mathbf{S}$ *relative to the basis* $\mathbf{e}$ *is diagonal.*

Of course, the orthonormal basis $\mathbf{e}$ in Cor.2 is *not* uniquely determined by $\mathbf{S}$ if $\dim\mathcal{V} > 0$. For example, if we replace any or all the terms of $\mathbf{e}$ by their opposites, we get another orthonormal basis whose terms are also spectral vectors of $\mathbf{S}$.

Proposition 1: *Let* $\mathcal{V}$ *and* $\mathcal{V}'$ *be genuine inner product spaces. Then* $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$ *and* $\mathbf{S}' \in \operatorname{Sym}\mathcal{V}'$ *have the same multiplicity function,* i.e. $\operatorname{mult}_{\mathbf{S}} = \operatorname{mult}_{\mathbf{S}'}$ *if and only if there is an orthogonal isomorphism* $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ *such that*

$$\mathbf{S}' = \mathbf{R}\mathbf{S}\mathbf{R}^{-1} = \mathbf{R}\mathbf{S}\mathbf{R}^{\top}. \tag{84.7}$$

If (84.7) *holds, then*

$$\mathbf{R}_{>}(\operatorname{Sps}_{\mathbf{S}}(\sigma)) = \operatorname{Sps}_{\mathbf{S}'}(\sigma) \quad \text{for all} \quad \sigma \in \operatorname{Spec}\mathbf{S}. \tag{84.8}$$

Proof: Assume that $\operatorname{mult}_{\mathbf{S}} = \operatorname{mult}_{\mathbf{S}'}$. By Cor.2 above, we may then choose orthonormal bases $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ and $\mathbf{e}' := (\mathbf{e}'_i \mid i \in I)$ of $\mathcal{V}$ and $\mathcal{V}'$, respectively, such that (84.6) and

$$\mathbf{S}'\mathbf{e}'_i = \lambda_i \mathbf{e}'_i \quad \text{for all} \quad i \in I \tag{84.9}$$

hold with one and the same family $\lambda := (\lambda_i \mid i \in I)$. Let $\mathbf{R} : \mathcal{V} \to \mathcal{V}'$ be the linear isomorphism for which $\mathbf{R}\mathbf{e}_i = \mathbf{e}'_i$ for all $i \in I$. By Prop.4 of Sect.43, $\mathbf{R}$ is an orthogonal isomorphism. By (84.9) and (84.6) we have

$$\mathbf{S}'\mathbf{R}\mathbf{e}_i = \lambda_i(\mathbf{R}\mathbf{e}_i) = \mathbf{R}(\lambda_i\mathbf{e}_i) = \mathbf{R}\mathbf{S}\mathbf{e}_i$$

for all $i \in I$. Since $\mathbf{e}$ is a basis, it follows that $\mathbf{S}'\mathbf{R} = \mathbf{R}\mathbf{S}$, which is equivalent to (84.7).

If (84.7) holds, then (84.8) and the equality $\operatorname{mult}_{\mathbf{S}} = \operatorname{mult}_{\mathbf{S}'}$ follow from Prop.3 of Sect.82. ∎

Using Prop.1 above and Prop.2 of Sect.82, we immediately obtain

Proposition 2: $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$ *commutes with* $\mathbf{R} \in \operatorname{Orth}\mathcal{V}$ *if and only if the spectral spaces of* $\mathbf{S}$ *are* $\mathbf{R}$*-invariant.*

In the special case when all spectral values of $\mathbf{S}$ have multiplicity 1, the condition of Prop.2 is equivalent to the following one: If $\mathbf{e}$ is an orthonormal basis such that the matrix of $\mathbf{S}$ relative to $\mathbf{e}$ is diagonal (see Cor.2 above), then the matrix of $\mathbf{R}$ relative to $\mathbf{e}$ is also diagonal and its diagonal terms are either 1 or -1.

We are now able to prove the result (52.20) already announced in Sect.52.

Proposition 3: *If $\mathcal{V}$ and $\mathcal{V}'$ are genuine inner-product spaces and if $n := \dim \mathcal{V} > 0$, then*

$$|\mathbf{L}| \geq ||\mathbf{L}|| \geq \frac{1}{\sqrt{n}}|\mathbf{L}| \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}'). \tag{84.10}$$

Proof: Let $\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}')$ be given. The inequality $|\mathbf{L}| \geq ||\mathbf{L}||$ is an immediate consequence of (52.4), with both ν and ν' replaced by $|\cdot|$, and Prop.3 of Sect.44.

Since $\mathbf{S} := \mathbf{L}^\top\mathbf{L}$ belongs to $\mathrm{Sym}\mathcal{V}$, we can consider the spectral resolution $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ of $\mathbf{S}$ mentioned in Cor.1 above. By (84.5) and (26.11), we obtain

$$\mathrm{tr}\,\mathbf{S} = \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} \sigma\,\mathrm{tr}\,\mathbf{E}_\sigma = \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} \sigma \dim \mathrm{Rng}\,\mathbf{E}_\sigma. \tag{84.11}$$

Since $(\mathrm{Rng}\,\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ is a decomposition of $\mathcal{V}$, it follows from (84.11) and Prop.4 of Sect.81 that

$$\mathrm{tr}\,\mathbf{S} \leq n \max(\mathrm{Spec}\,\mathbf{S}).$$

Using the Theorem on the Extreme Spectral Values of a Symmetric Lineon, we conclude that

$$\mathrm{tr}\,\mathbf{S} \leq n \max \overline{\overline{\mathbf{S}}}\,|_{\mathrm{Usph}\mathcal{V}}.$$

Since $\overline{\overline{\mathbf{S}}}\,(\mathbf{v}) = \mathbf{v} \cdot \mathbf{S}\mathbf{v} = \mathbf{v} \cdot \mathbf{L}^\top\mathbf{L}\mathbf{v} = |\mathbf{L}\mathbf{v}|^2$ for all $\mathbf{v} \in \mathcal{V}$, it follows that

$$\mathrm{tr}\,\mathbf{S} \leq n \max\{|\mathbf{L}\mathbf{v}|^2 \mid \mathbf{v} \in \mathrm{Usph}\mathcal{V}\}\ .$$

Since $\mathrm{tr}\,\mathbf{S} = \mathrm{tr}(\mathbf{L}^\top\mathbf{L}) = |\mathbf{L}|^2$ by (44.13), it follows from (52.3), with both ν and ν' replaced by $|\cdot|$, that $|\mathbf{L}|^2 \leq n||\mathbf{L}||^2$. ■

Remark: If $\mathbf{L}$ is a tensor product, i.e. if $\mathbf{L} = \mathbf{w} \otimes \mathbf{v}$ for some $\mathbf{v} \in \mathcal{V} \cong \mathcal{V}^*$ and $\mathbf{w} \in \mathcal{V}'$, then $|\mathbf{L}| = ||\mathbf{L}||$, as easily seen from (44.14) and Part (a) of Problem 7 in Chap.5. If $\mathbf{L}$ is orthogonal, we have $||\mathbf{L}|| = \frac{1}{\sqrt{n}}|\mathbf{L}|$ by Problem 5 in Chap.5. Hence either of the inequalities in (84.10) can become an equality. ■

85 Lineonic Extensions
Lineonic Square Roots and Logarithms

We assume that a genuine inner-product space $\mathcal{V}$ is given. In Sect.41, we noted the identification

$$\mathrm{Sym}\mathcal{V} \cong \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R})$$

and in Sect. 27, we discussed the isomorphism $(\mathbf{S} \mapsto \overset{\frown}{\mathbf{S}}) : \mathrm{Sym}_2(\mathcal{V}^2, \mathbb{R}) \to \mathrm{Qu}\mathcal{V}$. We thus see that each symmetric lineon $\mathbf{S}$ on $\mathcal{V}$ corresonds to a quadratic form $\overset{\frown}{\mathbf{S}}$ on $\mathcal{V}$ given by

$$\overset{\frown}{\mathbf{S}}(\mathbf{u}) := \mathbf{S}(\mathbf{u}, \mathbf{u}) = (\mathbf{S}\mathbf{u}) \cdot \mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}. \tag{85.1}$$

The identity lineon $\mathbf{1}_{\mathcal{V}} \in \mathrm{Sym}\mathcal{V}$ is identified with the inner product ip on $\mathcal{V}$ and hence $\overset{\frown}{\mathbf{1}}_{\mathcal{V}} = \overset{\frown}{ip} = \mathrm{sq}$. In view of Def.1 of Sect.27, it is meaningful to speak of **positive [strictly positive] symmetric lineons**. We use the notations

$$\mathrm{Pos}\mathcal{V} := \{\mathbf{S} \in \mathrm{Sym}\mathcal{V} \mid \overset{\frown}{\mathbf{S}}(\mathbf{u}) \geq 0 \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}\} \tag{85.2}$$

and

$$\mathrm{Pos}^+\mathcal{V} := \{\mathbf{S} \in \mathrm{Sym}\mathcal{V} \mid \overset{\frown}{\mathbf{S}}(\mathbf{u}) > 0 \quad \text{for all} \quad \mathbf{u} \in \mathcal{V}^\times\} \tag{85.3}$$

for the sets of all positive and strictly positive symmetric lineons, respectively. Since $\mathcal{V}$ is genuine, we have $\mathbf{1}_{\mathcal{V}} \in \mathrm{Pos}^+\mathcal{V}$. By Prop.3 of Sect.27 we have

$$\overset{\frown}{\mathbf{L}^\top\mathbf{S}\mathbf{L}} = \overset{\frown}{\mathbf{S}} \circ \mathbf{L} \tag{85.4}$$

for all $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ and all $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$.

The following result is an immediate consequence of (85.4):

Proposition 1: *If $\mathbf{S} \in \mathrm{Pos}\mathcal{V}$ and $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$, then $\mathbf{L}^\top\mathbf{S}\mathbf{L} \in \mathrm{Pos}\mathcal{V}$. Moreover, $\mathbf{L}^\top\mathbf{S}\mathbf{L}$ is strictly positive if and only if $\mathbf{S}$ is strictly positive and $\mathbf{L}$ is invertible. In particular, we have $\mathbf{L}^\top\mathbf{L} \in \mathrm{Pos}\mathcal{V}$, and $\mathbf{L}^\top\mathbf{L} \in \mathrm{Pos}^+\mathcal{V}$ if and only if $\mathbf{L}$ is invertible.*

The following result is a corollary to the Theorem on the Extreme Spectral Values of a Symmetric Lineon (see Sect.84).

Proposition 2: *A symmetric lineon is positive [strictly positive] if and only if all its spectral values are positive [strictly positive]. In other words,*

$$\mathrm{Pos}\mathcal{V} = \{\mathbf{S} \in \mathrm{Sym}\mathcal{V} \mid \mathrm{Spec}\,\mathbf{S} \subset \mathbb{P}\}, \tag{85.5}$$

$$\mathrm{Pos}^+\mathcal{V} = \{\mathbf{S} \in \mathrm{Sym}\mathcal{V} \mid \mathrm{Spec}\,\mathbf{S} \subset \mathbb{P}^\times\}. \tag{85.6}$$

Let a function $f : \mathbb{R} \to \mathbb{R}$ be given. We define the **lineonic extension** $f_{(\mathcal{V})} : \mathrm{Sym}\mathcal{V} \to \mathrm{Sym}\mathcal{V}$ of f by

$$f_{(\mathcal{V})}(\mathbf{S}) := \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} f(\sigma)\mathbf{E}_\sigma \tag{85.7}$$

when $(\mathbf{E}_\sigma \mid \sigma \in \operatorname{Spec}\mathbf{S})$ is the spectral resolution of $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$. If the domain of f is $\mathbb{P}$ or $\mathbb{P}^\times$ instead of $\mathbb{R}$, we still use (85.7), but, in view of Prop.2, the domain of $f_{(\mathcal{V})}$ must be taken to be $\operatorname{Pos}\mathcal{V}$ or $\operatorname{Pos}^+\mathcal{V}$, respectively. A similar observation applies to the codomains of f and $f_{(\mathcal{V})}$.

Let f and g be functions whose domains and codomains are one of $\mathbb{R}$, $\mathbb{P}$, $\mathbb{P}^\times$ each. The following rules are evident from the definition (85.7):

(I) For all $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$, $\mathbf{S} \in \operatorname{Pos}\mathcal{V}$, or $\mathbf{S} \in \operatorname{Pos}^+\mathcal{V}$, as appropriate, the spectrum of $f_{(\mathcal{V})}(\mathbf{S})$ is given by

$$\operatorname{Spec}(f_{(\mathcal{V})}(\mathbf{S})) = f_>(\operatorname{Spec}\mathbf{S}). \tag{85.8}$$

Moreover, if $f|_{\operatorname{Spec}\mathbf{S}}$ is injective, then $f_{(\mathcal{V})}(\mathbf{S})$ and $\mathbf{S}$ have the same spectral spaces and the same spectral idempotents.

(II) If $\operatorname{Dom} f = \operatorname{Cod} g$, then $\operatorname{Dom} f_{(\mathcal{V})} = \operatorname{Cod} g_{(\mathcal{V})}$ and

$$(f \circ g)_{(\mathcal{V})} = f_{(\mathcal{V})} \circ g_{(\mathcal{V})}. \tag{85.9}$$

(III) If f is invertible, so is $f_{(\mathcal{V})}$ and

$$(f_{(\mathcal{V})})^{\leftarrow} = (f^{\leftarrow})_{(\mathcal{V})}. \tag{85.10}$$

As an example, we consider the lineonic extension $\iota^n{}_{(\mathcal{V})} : \operatorname{Sym}\mathcal{V} \to \operatorname{Sym}\mathcal{V}$ of the real nth power function $\iota^n : \mathbb{R} \to \mathbb{R}$.

By (85.7), we have

$$\iota^n{}_{(\mathcal{V})}(\mathbf{S}) = \sum_{\sigma \in \operatorname{Spec}\mathbf{S}} \sigma^n \mathbf{E}_\sigma \quad \text{for all} \quad n \in \mathbb{N}. \tag{85.11}$$

when $(\mathbf{E}_\sigma \mid \sigma \in \operatorname{Spec}\mathbf{S})$ is the spectral resolution of $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$. Using the fact that $\mathbf{E}_\sigma^2 = \mathbf{E}_\sigma$ for all $\sigma \in \operatorname{Spec}\mathbf{S}$ and that $\mathbf{E}_\sigma\mathbf{E}_\tau = 0$ for all $\sigma, \tau \in \operatorname{Spec}\mathbf{S}$ with $\sigma \neq \tau$ (see (81.6)), one easily infers from (85.11) by induction that

$$\iota^n{}_{(\mathcal{V})}(\mathbf{S}) = \mathbf{S}^n \quad \text{for all} \quad \mathbf{S} \in \operatorname{Sym}\mathcal{V}. \tag{85.12}$$

Hence, the lineonic extension of the real nth power is nothing but the adjustment $\operatorname{pow}_{n|\operatorname{Sym}\mathcal{V}}$ of the lineonic nth power defined by (66.18).

Of particular importance is the lineonic extention of the real square function $\iota^2{}_{|\mathbb{P}}$, adjusted to $\mathbb{P}$. This function is invertible and the inverse is the positive real square root function $\sqrt{\ } := (\iota^2{}_{|\mathbb{P}})^{\leftarrow} : \mathbb{P} \to \mathbb{P}$. We call the lineonic extension of $\sqrt{\ }$ the **lineonic square root** and denote it by

$$\operatorname{sqrt} := \sqrt{\ }_{(\mathcal{V})} : \operatorname{Pos}\mathcal{V} \to \operatorname{Pos}\mathcal{V}.$$

We also write $\sqrt{\mathbf{S}} := \mathrm{sqrt}(\mathbf{S})$ when $\mathbf{S} \in \mathrm{Pos}\mathcal{V}$. By (85.7) we have

$$\mathrm{sqrt}(\mathbf{S}) = \sqrt{\mathbf{S}} = \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} \sqrt{\sigma}\mathbf{E}_\sigma \tag{85.13}$$

when $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ is the spectral resolution of $\mathbf{S} \in \mathrm{Pos}\mathcal{V}$.

The following result follows directly from the rules (I) and (III) above and from (85.12).

Lineonic Square Root Theorem: *For every* $\mathbf{S} \in \mathrm{Pos}\mathcal{V}$*, the lineonic square root* $\sqrt{\mathbf{S}}$ *given by* (85.13) *is the only solution of the equation*

$$?\ \mathbf{T} \in \mathrm{Pos}\mathcal{V},\ \ \mathbf{T}^2 = \mathbf{S}. \tag{85.14}$$

The spectrum of $\sqrt{\mathbf{S}}$ *is given by*

$$\mathrm{Spec}\,\sqrt{\mathbf{S}} = \{\sqrt{\sigma} \mid \sigma \in \mathrm{Spec}\,\mathbf{S}\}, \tag{85.15}$$

and we have

$$\mathrm{Sps}_{\sqrt{\mathbf{S}}}(\sqrt{\sigma}) = \mathrm{Sps}_{\mathbf{S}}(\sigma) \quad \text{for all} \quad \sigma \in \mathrm{Spec}\,\mathbf{S}. \tag{85.16}$$

It turns out that the domain $\mathrm{Pos}\mathcal{V}$ of the lineonic square root sqrt is *not* an open subset of the linear space $\mathrm{Sym}\mathcal{V}$ and hence it makes no sense to ask whether sqrt is differentiable. However, as we will see, $\mathrm{Pos}^+\mathcal{V}$ *is* an open subset of $\mathrm{Sym}\mathcal{V}$ and the restriction of sqrt to $\mathrm{Pos}^+\mathcal{V}$ is of class C^1. Actually, $\mathrm{Pos}^+\mathcal{V}$ is sqrt-invariant because the lineonic extension of the *strictly* positive real square function $(\iota^2{}_{|\mathbb{P}^\times})^\leftarrow$ must be the adjustment

$$\mathrm{sqrt}^+ := \mathrm{sqrt}_{|\mathrm{Pos}^+\mathcal{V}} \tag{85.17}$$

of the square root. By (85.12) and the rule (III) above, sqrt^+ is the inverse of the adjustment to $\mathrm{Pos}^+\mathcal{V}$ of the lineonic square function pow_2:

$$\mathrm{sqrt}^+ = (\mathrm{pow}_2{}^+)^\leftarrow, \quad \text{where} \quad \mathrm{pow}_2{}^+ := \mathrm{pow}_{2|\mathrm{Pos}^+\mathcal{V}}. \tag{85.18}$$

We call sqrt^+ the **strict lineonic square root**.

Theorem on the Smoothness of the Strict Lineonic Square Root: *The set* $\mathrm{Pos}^+\mathcal{V}$ *of strictly positive lineons is an open subset of* $\mathrm{Sym}\mathcal{V}$ *and the strict lineonic square-root* $\mathrm{sqrt}^+ : \mathrm{Pos}^+\mathcal{V} \to \mathrm{Pos}^+\mathcal{V}$ *is of class* C^1.

Proof: Let $\mathbf{S} \in \mathrm{Pos}^+\mathcal{V}$ be given and let σ be the least spectral value of $\mathbf{S}$. By Prop.2, σ is strictly positive and by (84.1), we have

$$(\mathbf{S}\mathbf{v}) \cdot \mathbf{v} \geq \sigma|\mathbf{v}|^2 \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{85.19}$$

In view of (52.19), we have $|\mathbf{Tv}| \leq ||\mathbf{T}||\,|\mathbf{v}|$ for every $\mathbf{T} \in \mathrm{Sym}\mathcal{V}$ and all $\mathbf{v} \in \mathcal{V}$, where $||\mathbf{T}||$ is the operator-norm of $\mathbf{T}$. Therefore, the Inner-Product Inequality of Sect.42 yields

$$(\mathbf{Tv}) \cdot \mathbf{v} \geq -|\mathbf{Tv}|\,|\mathbf{v}| \geq -||\mathbf{T}||\,|\mathbf{v}|^2 \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{85.20}$$

Combining (85.19) and (85.20), we see that

$$((\mathbf{S}+\mathbf{T})\mathbf{v}) \cdot \mathbf{v} \geq (\sigma - ||\mathbf{T}||)|\mathbf{v}|^2 \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

It follows that $\mathbf{S}+\mathbf{T}$ is strictly positive if $||\mathbf{T}|| < \sigma$ i.e. if $\mathbf{T} \in \sigma\mathrm{Ce}(||\cdot||)$. Hence $\mathrm{Pos}^+\mathcal{V}$ includes the neighborhood $\mathbf{S} + \sigma(\mathrm{Ce}(||\cdot||) \cap \mathrm{Sym}\mathcal{V})$ of $\mathbf{S}$ in $\mathrm{Sym}\mathcal{V}$. Since $\mathbf{S} \in \mathrm{Pos}^+\mathcal{V}$ was arbitrary, it follows that $\mathrm{Pos}^+\mathcal{V}$ is open in $\mathrm{Sym}\mathcal{V}$.

Since $\mathrm{Pos}^+\mathcal{V}$ is open in $\mathrm{Sym}\mathcal{V}$, it makes sense to say, and it is true by Prop.4 of Sect.66, that ${\mathrm{pow}_2}^+$ is of class C^1. By (66.19), the gradient of ${\mathrm{pow}_2}^+$ is given by

$$(\nabla_{\mathbf{S}}{\mathrm{pow}_2}^+)\mathbf{U} = \mathbf{US} + \mathbf{SU} \quad \text{for all} \quad \mathbf{U} \in \mathrm{Sym}\mathcal{V}. \tag{85.21}$$

Let $\mathbf{U} \in \mathrm{Null}\ (\nabla_{\mathbf{S}}{\mathrm{pow}_2}^+)$ be given, so that $\mathbf{US} + \mathbf{SU} = \mathbf{0}$. We then have

$$\begin{aligned}(\mathbf{S}+\gamma\mathbf{U})^2 &= \mathbf{S}^2 + \gamma(\mathbf{SU}+\mathbf{US}) + \gamma^2\mathbf{U}^2 = \mathbf{S}^2 + \gamma^2\mathbf{U}^2 \\ &= \mathbf{S}^2 - \gamma(\mathbf{SU}+\mathbf{US}) + \gamma^2\mathbf{U}^2 = (\mathbf{S}-\gamma\mathbf{U})^2\end{aligned}$$

for all $\gamma \in \mathbb{P}^\times$. Since $\mathrm{Pos}^+\mathcal{V}$ is open in $\mathrm{Sym}\mathcal{V}$, we may choose $\gamma \in \mathbb{P}^\times$ such that both $\mathbf{S}+\gamma\mathbf{U}$ and $\mathbf{S}-\gamma\mathbf{U}$ belong to $\mathrm{Pos}^+\mathcal{V}$. We then have

$${\mathrm{pow}_2}^+(\mathbf{S}+\gamma\mathbf{U}) = {\mathrm{pow}_2}^+(\mathbf{S}-\gamma\mathbf{U}).$$

Since ${\mathrm{pow}_2}^+$ is injective, it follows that $\mathbf{S}+\gamma\mathbf{U} = \mathbf{S}-\gamma\mathbf{U}$ and hence that $\mathbf{U} = \mathbf{0}$. It follows that $\mathrm{Null}\ (\nabla_{\mathbf{S}}{\mathrm{pow}_2}^+) = \{\mathbf{0}\}$. By the Pigeonhole Principle for Linear Mappings, it follows that $\nabla_{\mathbf{S}}{\mathrm{pow}_2}^+$ is invertible. The Local Inversion Theorem yields that $\mathrm{sqrt}^+ = ({\mathrm{pow}_2}^+)^{\leftarrow}$ is of class C^1. ∎

Since, by (68.4), we have

$$\nabla_{\mathbf{S}}\mathrm{sqrt} = (\nabla_{\sqrt{\mathbf{S}}}{\mathrm{pow}_2}^+)^{-1} \quad \text{for all} \quad \mathbf{S} \in \mathrm{Pos}^+\mathcal{V},$$

we can conclude from (85.21) that for every $\mathbf{S} \in \mathrm{Pos}^+\mathcal{V}$ and every $\mathbf{V} \in \mathrm{Sym}\mathcal{V}$, the value $(\nabla_{\mathbf{S}}\mathrm{sqrt}^+)\mathbf{V}$ is the unique solution of the equation

$$?\ \mathbf{U} \in \mathrm{Sym}\mathcal{V}, \quad \mathbf{V} = \mathbf{U}\sqrt{\mathbf{S}} + \sqrt{\mathbf{S}}\mathbf{U}. \tag{85.22}$$

If $\mathbf{V}$ commutes with $\mathbf{S}$ then the solution of (85.22) is given by

$$(\nabla_{\mathbf{S}}\mathrm{sqrt}^{+})\mathbf{V} = \frac{1}{2}\sqrt{\mathbf{S}}^{-1}\mathbf{V}, \tag{85.23}$$

which is consistent with the formula for the derivative of the square-root function of elementary calculus. If $\mathbf{V}$ does not commute with $\mathbf{S}$, then there is no simple explicit formula for $(\nabla_{\mathbf{S}}\mathrm{sqrt}^{+})\mathbf{V}$.

As another example of a lineonic extension we consider $(\exp|^{\mathbb{P}^{\times}})_{(\mathcal{V})}$, where exp is the real exponential function (see Sect.08).

Proposition 3: *The lineonic extension of* $\exp|^{\mathbb{P}^{\times}}$ *coincides with an adjustment of the lineonic exponential defined in Prop.2 of Sect.612. Specifically, we have*

$$(\exp|^{\mathbb{P}^{\times}})_{(\mathcal{V})} = \exp_{\mathcal{V}}|^{\mathrm{Pos}^{+}\mathcal{V}}_{\mathrm{Sym}\mathcal{V}}. \tag{85.24}$$

Proof: Let $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ be given and let $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ be its spectral resolution. We define $\mathbf{F} : \mathbb{R} \to \mathrm{Lin}\mathcal{V}$ by

$$\mathbf{F} := \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} (\exp \circ (\iota\sigma))\mathbf{E}_\sigma. \tag{85.25}$$

Differentiation gives

$$\mathbf{F}^{\cdot} = \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} (\exp \circ (\iota\sigma))\sigma\mathbf{E}_\sigma. \tag{85.26}$$

Using the fact that $\mathbf{E}_\sigma^2 = \mathbf{E}_\sigma$ for all $\sigma \in \mathrm{Spec}\,\mathbf{S}$ and that $\mathbf{E}_\sigma\mathbf{E}_\tau = \mathbf{0}$ for all $\sigma, \tau \in \mathrm{Spec}\,\mathbf{S}$ with $\sigma \neq \tau$, we infer from (84.5), (85.25), and (85.26) that $\mathbf{F}^{\cdot} = \mathbf{S}\mathbf{F}$. Since, by (85.25) and (84.4), $\mathbf{F}(0) = \sum(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S}) = \mathbf{1}_{\mathcal{V}}$ we can use Prop.4 of Sect.612 to conclude that $\mathbf{F} = \exp_{\mathcal{V}} \circ (\iota\mathbf{S})$. Evaluation of (85.25) at $1 \in \mathbb{R}$ hence gives

$$(\exp|^{\mathbb{P}^{\times}})_{(\mathcal{V})}(\mathbf{S}) = \mathbf{F}(1) = \exp_{\mathcal{V}}(\mathbf{S}).$$

Since $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$ was arbitrary, (85.24) follows. ∎

Since $\exp|^{\mathbb{P}^{\times}}$ is invertible, and since its inverse is the logarithm $\log := (\exp|^{\mathbb{P}^{\times}})^{\leftarrow} : \mathbb{P}^{\times} \to \mathbb{R}$, it follows from Prop.3 and rule (III) above that $\exp_{\mathcal{V}}|^{\mathrm{Pos}^{+}\mathcal{V}}_{\mathrm{Sym}\mathcal{V}}$ is invertible and that $\log_{(\mathcal{V})}$ is its inverse. We call $\log_{(\mathcal{V})}$ the **lineonic logarithm** and also write $\log \mathbf{S} := \log_{(\mathcal{V})}(\mathbf{S})$ when $\mathbf{S} \in \mathrm{Pos}^{+}\mathcal{V}$. In sum, we have the following result.

Lineonic Logarithm Theorem: *For every* $\mathbf{S} \in \mathrm{Pos}^{+}\mathcal{V}$*, the lineonic logarithm* $\log \mathbf{S}$*, given by*

$$\log \mathbf{S} = \sum_{\sigma \in \mathrm{Spec}\,\mathbf{S}} (\log \sigma)\mathbf{E}_\sigma, \tag{85.27}$$

where $(\mathbf{E}_\sigma \,|\, \sigma \in \operatorname{Spec}\mathbf{S})$ *is the spectral resolution of* $\mathbf{S}$, *is the only solution of the equation*

$$? \ \mathbf{T} \in \operatorname{Sym}\mathcal{V}, \ \exp_{\mathcal{V}}(\mathbf{T}) = \mathbf{S}. \tag{85.28}$$

The spectrum of $\log \mathbf{S}$ *is given by*

$$\operatorname{Spec}(\log \mathbf{S}) = \log_{>}(\operatorname{Spec}\mathbf{S}) \tag{85.29}$$

and we have

$$\operatorname{Sps}_{\log \mathbf{S}}(\log \sigma) = \operatorname{Sps}_{\mathbf{S}}(\sigma) \quad \text{for all} \quad \sigma \in \operatorname{Spec}\mathbf{S}. \tag{85.30}$$

Remark: Since Dom $\log_{(\mathcal{V})} = \operatorname{Pos}^+\mathcal{V}$ is an open subset of Sym$\mathcal{V}$ by the Theorem on Smoothness of the Strict Lineonic Square Root, it is meaningful to ask whether $\log_{(\mathcal{V})}$ is of class C^1. In fact it is, but the proof is far from trivial. (See Problem 6 at the end of this chapter.) ■

Notes 85

(1) The terms "positive semidefinite" instead of "positive" and "positive definite" instead of "strictly positive" are often used in the literature in connection with symmetric lineons. (See also Note (1) to Sect.27.)

(2) The Theorem on the Smoothness of the Lineonic Square Root, with a proof somewhat different from the one above, was contained in notes that led to this book. These notes were the basis of the corresponding theorem and proof in "An Introduction to Continuum Mechanics" by M.E. Gurtin (Academic Press, 1981). I am not aware of any other place in the literature where the Theorem is proved or even mentioned.

86 Polar Decomposition

Let $\mathcal{V}$ be a genuine inner-product space. In view of the identifications $\operatorname{Lin}\mathcal{V} \cong \operatorname{Lin}_2(\mathcal{V}^2, \mathbb{R})$, $\operatorname{Sym}\mathcal{V} \cong \operatorname{Sym}_2(\mathcal{V}^2, \mathbb{R})$, and $\operatorname{Skew}\mathcal{V} \cong \operatorname{Skew}_2(\mathcal{V}^2, \mathbb{R})$ (see Sect.41), we can phrase Prop.6 of Sect.24 in the following way:

Additive Decomposition Theorem: *To every lineon* $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ *corresponds a unique pair* $(\mathbf{S}, \mathbf{A})$ *of lineons such that*

$$\mathbf{L} = \mathbf{S} + \mathbf{A}, \quad \mathbf{S} \in \operatorname{Sym}\mathcal{V}, \ \mathbf{A} \in \operatorname{Skew}\mathcal{V}. \tag{86.1}$$

In fact, $\mathbf{S}$ *and* $\mathbf{A}$ *are given by*

$$\mathbf{S} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^\top), \quad \mathbf{A} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^\top). \tag{86.2}$$

The pair $(\mathbf{S}, \mathbf{A})$ *is called the* **additive decomposition** *of* $\mathbf{L}$.

The following theorem asserts the existence and uniqueness of certain *multiplicative* decompositions for invertible lineons.

Polar Decomposition Theorem *To every invertible lineon* $\mathbf{L} \in \mathrm{Lis}\mathcal{V}$ *corresponds a unique pair* $(\mathbf{R}, \mathbf{S})$ *such that*

$$\mathbf{L} = \mathbf{RS}, \quad \mathbf{R} \in \mathrm{Orth}\mathcal{V}, \ \ \mathbf{S} \in \mathrm{Pos}^{+}\mathcal{V}. \tag{86.3}$$

In fact, $\mathbf{S}$ *and* $\mathbf{R}$ *are given by*

$$\mathbf{S} = \sqrt{\mathbf{L}^{\top}\mathbf{L}}, \quad \mathbf{R} = \mathbf{L}\mathbf{S}^{-1}. \tag{86.4}$$

Also, there is a unique pair $(\mathbf{R}', \mathbf{S}')$ *such that*

$$\mathbf{L} = \mathbf{S}'\mathbf{R}', \ \ \mathbf{R}' \in \mathrm{Orth}\mathcal{V}, \ \ \mathbf{S}' \in \mathrm{Pos}^{+}\mathcal{V}. \tag{86.5}$$

In fact, $\mathbf{R}'$ *coincides with* $\mathbf{R}$*, and* $\mathbf{S}'$ *is given by*

$$\mathbf{S}' = \mathbf{R}\mathbf{S}\mathbf{R}^{\top}. \tag{86.6}$$

The pair $(\mathbf{R}, \mathbf{S})$ $[(\mathbf{R}', \mathbf{S}')]$ *for which* (86.3) [(86.5)] *holds is called the* **right [left] polar decomposition** *of* $\mathbf{L}$.

Proof: Assume that (86.3) holds for a given pair $(\mathbf{R}, \mathbf{S})$. Since $\mathbf{S}^{\top} = \mathbf{S}$ and $\mathbf{R}^{\top}\mathbf{R} = \mathbf{1}_{\mathcal{V}}$ we then have

$$\mathbf{L}^{\top}\mathbf{L} = (\mathbf{RS})^{\top}(\mathbf{RS}) = \mathbf{S}^{\top}\mathbf{R}^{\top}\mathbf{R}\mathbf{S} = \mathbf{S}^{2}.$$

Since $\mathbf{L}^{\top}\mathbf{L}$ as well as $\mathbf{S}$ belongs to $\mathrm{Pos}^{+}\mathcal{V}$ (see Prop.1 of Sect.85), it follows from the Lineonic Square Root Theorem that (86.4) must be valid and hence that $\mathbf{S}$ and $\mathbf{R}$, if they exist, are uniquely determined by $\mathbf{L}$. To prove existence, we *define* $\mathbf{S}$ and $\mathbf{R}$ by (86.4). We then have $\mathbf{L} = \mathbf{RS}$ and $\mathbf{S} \in \mathrm{Pos}^{+}\mathcal{V}$. To prove that $\mathbf{R} \in \mathrm{Orth}\mathcal{V}$ we need only observe that

$$\mathbf{R}^{\top}\mathbf{R} = (\mathbf{L}\mathbf{S}^{-1})^{\top}(\mathbf{L}\mathbf{S}^{-1}) = \mathbf{S}^{-1}\mathbf{L}^{\top}\mathbf{L}\mathbf{S}^{-1} = \mathbf{S}^{-1}\mathbf{S}^{2}\mathbf{S}^{-1} = \mathbf{1}_{\mathcal{V}}.$$

Assume now that (86.5) holds for a given pair $(\mathbf{R}', \mathbf{S}')$. We then have

$$\mathbf{L} = \mathbf{S}'\mathbf{R}' = (\mathbf{R}'\mathbf{R}'^{\top})\mathbf{S}'\mathbf{R}' = \mathbf{R}'(\mathbf{R}'^{\top}\mathbf{S}'\mathbf{R}').$$

In view of Prop.1 of Sect.85, $\mathbf{R}'^{\top}\mathbf{S}'\mathbf{R}'$ belongs to $\mathrm{Pos}^{+}\mathcal{V}$. It follows from the already proved uniqueness of the pair $(\mathbf{R}, \mathbf{S})$ that $\mathbf{R}' = \mathbf{R}$ and that $\mathbf{S} = \mathbf{R}'^{\top}\mathbf{S}'\mathbf{R}'$, and hence (86.6) holds. Therefore, $\mathbf{R}'$ and $\mathbf{S}'$, if they exist, are uniquely determined by $\mathbf{S}$ and $\mathbf{R}$, and hence by $\mathbf{L}$. To prove existence, we need only *define* $\mathbf{R}'$ and $\mathbf{S}'$ by $\mathbf{R}' := \mathbf{R}$ and $\mathbf{S}' := \mathbf{R}\mathbf{S}\mathbf{R}^\top$. It is then evident that (86.5) holds. ■

If we apply the Theorem to the case when $\mathbf{L}$ is replaced by $\mathbf{L}^\top$, we obtain

$$\mathbf{S}' = \sqrt{\mathbf{L}\mathbf{L}^\top}, \quad \mathbf{R}' = \mathbf{R} = \mathbf{S}'^{-1}\mathbf{L}^\top. \tag{86.7}$$

The polar decompositions can be illustrated geometrically. Assume that (86.3) holds, choose an orthonormal basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ of spectral vectors of $\mathbf{S}$ (see Cor.2 to the Spectral Theorem), and consider the "unit cube"

$$\mathcal{C} := \mathrm{Box}(\frac{1}{2}\mathbf{e}) := \{\mathbf{v} \in \mathcal{V} \mid \ |\mathbf{v} \cdot \mathbf{e}_i| \leq \frac{1}{2} \quad \text{for all} \quad i \in I\}.$$

Since $\mathbf{S}$ is strictly positive, we have $\mathbf{S}\mathbf{e}_i = \lambda_i \mathbf{e}_i$ for all $i \in I$ where $\lambda := (\lambda_i \mid i \in I)$ is a family of strictly positive numbers. The lineon $\mathbf{S}$ will "stretch and/or compress" the cube $\mathcal{C} = \mathrm{Box}(\frac{1}{2}\mathbf{e})$ into the "rectangular box"

$$\mathbf{S}_>(\mathcal{C}) = \mathrm{Box}(\frac{\lambda_i}{2}\mathbf{e}_i \mid i \in I),$$

whose sides have the lengths $\lambda_i, \ i \in I$. Finally, the orthogonal lineon $\mathbf{R}$ will "rotate" or "rotate and reflect" (see end of Sect.88) the rectangular box $\mathbf{S}_>(\mathcal{C})$, the result being the rectangular box

$$\mathbf{L}_>(\mathcal{C}) = \mathbf{R}_>(\mathbf{S}_>(\mathcal{C})) = \mathrm{Box}(\frac{\lambda_i}{2}\mathbf{R}\mathbf{e}_i \mid i \in I)$$

congruent to $\mathbf{S}_>(\mathcal{C})$. Thus, the right polar decomposition (86.3) shows that the transformation of the cube $\mathcal{C}$ into the rectangular box $\mathbf{L}_>(\mathcal{C})$ can be obtained by first "stretching and/or compressing" and then "rotating" or "rotating and reflecting". The left polar decomposition (86.5) can be used to show that $\mathbf{L}_>(\mathcal{C})$ can be obtained from $\mathcal{C}$ also by first "rotating" or "rotating and reflecting" and then "stretching and/or compressing". In the case when $\dim \mathcal{V} = 2$ these processes are illustrated in Figure 1.

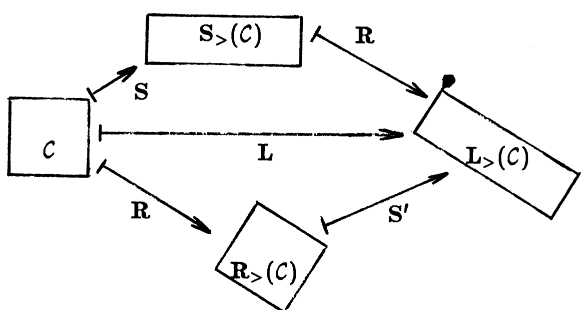

Figure 1.

The Polar Decomposition Theorem shows that there are mappings

$$\begin{array}{lcl} \mathrm{or} & : & \mathrm{Lis}\mathcal{V} \rightarrow \mathrm{Lin}\mathcal{V}, \\ \mathrm{rp} & : & \mathrm{Lis}\mathcal{V} \rightarrow \mathrm{Pos}^{+}\mathcal{V}, \\ \mathrm{lp} & : & \mathrm{Lis}\mathcal{V} \rightarrow \mathrm{Pos}^{+}\mathcal{V}, \end{array}$$

such that $\mathrm{Rng}\,\mathrm{or} \subset \mathrm{Orth}\mathcal{V}$ and

$$\mathbf{L} = \mathrm{or}(\mathbf{L})\mathrm{rp}(\mathbf{L}) = \mathrm{lp}(\mathbf{L})\mathrm{or}(\mathbf{L}) \tag{86.8}$$

for all $\mathbf{L} \in \mathrm{Lis}\mathcal{V}$. For $\mathbf{L} := \mathbf{1}_{\mathcal{V}}$, we get

$$\mathrm{or}(\mathbf{1}_{\mathcal{V}}) = \mathrm{rp}(\mathbf{1}_{\mathcal{V}}) = \mathrm{lp}(\mathbf{1}_{\mathcal{V}}) = \mathbf{1}_{\mathcal{V}}. \tag{86.9}$$

In view of (86.4), we have

$$\mathrm{rp}(\mathbf{L}) = \mathrm{sqrt}^{+}(\mathbf{L}^{\top}\mathbf{L}), \quad \mathrm{or}(\mathbf{L}) = \mathbf{L}(\mathrm{rp}(\mathbf{L}))^{-1} \tag{86.10}$$

for all $\mathbf{L} \in \mathrm{Lis}\mathcal{V}$, where sqrt^{+} is the strict lineonic square-root function defined in the previous section. It is easily seen, also, that

$$\mathrm{lp}(\mathbf{L}) = \mathrm{rp}(\mathbf{L}^{\top}) = \mathrm{sqrt}^{+}(\mathbf{L}\mathbf{L}^{\top}) \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lis}\mathcal{V}. \tag{86.11}$$

Proposition 1: *The mappings* or, rp*, and* lp *characterized by the Polar Decomposition Theorem are of class* C^1*. Their gradients at* $\mathbf{1}_{\mathcal{V}}$ *are given by*

$$(\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{or})\mathbf{M} = \frac{1}{2}(\mathbf{M} - \mathbf{M}^{\top}), \tag{86.12}$$

$$(\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{rp})\mathbf{M} = (\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{lp})\mathbf{M} = \frac{1}{2}(\mathbf{M} + \mathbf{M}^{\top}) \tag{86.13}$$

for all $\mathbf{M} \in \mathrm{Lin}\mathcal{V}$*.*

Proof: Since $(\mathbf{L} \mapsto \mathbf{L}^{\top}\mathbf{L}) : \mathrm{Lis}\mathcal{V} \rightarrow \mathrm{Pos}^{+}\mathcal{V}$ is of class C^1 by the General Product Rule and since $\mathrm{sqrt}^{+} : \mathrm{Pos}^{+}\mathcal{V} \rightarrow \mathrm{Pos}^{+}\mathcal{V}$ is of class C^1 by the Theorem on the Smoothness of the Strict Lineonic Square Root, it follows from $(86.10)_1$ and the Chain Rule that rp is of class C^1. It is an immediate consequence of $(86.11)_1$ that lp is also of class C^1 and that

$$(\nabla_{\mathbf{L}}\mathrm{lp})\mathbf{M} = (\nabla_{\mathbf{L}^{\top}}\mathrm{rp})\mathbf{M}^{\top} \tag{86.14}$$

for all $\mathbf{L} \in \mathrm{Lis}\mathcal{V}, \ \mathbf{M} \in \mathrm{Lin}\mathcal{V}$.

Using the Differentiation Theorem for Inversion Mappings of Sect.68 we conclude from $(86.10)_2$ that or is also of class C^1.

If we differentiate $(86.8)_1$ with respect to $\mathbf{L}$ using the General Product Rule, we obtain

$$\mathbf{M} = ((\nabla_{\mathbf{L}}\mathrm{or})\mathbf{M})\mathrm{rp}(\mathbf{L}) + \mathrm{or}(\mathbf{L})((\nabla_{\mathbf{L}}\mathrm{rp})\mathbf{M})$$

for all $\mathbf{L} \in \mathrm{Lis}\mathcal{V}$, $\mathbf{M} \in \mathrm{Lin}\mathcal{V}$. In view of (86.9), for $\mathbf{L} := \mathbf{1}_{\mathcal{V}}$ this yields

$$\mathbf{M} = (\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{or})\mathbf{M} + (\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{rp})\mathbf{M} \quad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}\mathcal{V}. \tag{86.15}$$

Since the codomain $\mathrm{Pos}^{+}\mathcal{V}$ of rp is an open subset of $\mathrm{Sym}\mathcal{V}$, the codomain of $\nabla_{\mathbf{L}}\mathrm{rp}$ is $\mathrm{Sym}\mathcal{V}$ for all $\mathbf{L} \in \mathrm{Lis}\mathcal{V}$. In particular,

$$(\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{rp})\mathbf{M} \in \mathrm{Sym}\mathcal{V} \quad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}\mathcal{V}. \tag{86.16}$$

Since $\mathrm{Rng}\,\mathrm{or} \subset \mathrm{Orth}\mathcal{V}$, it follows from Prop.2 of Sect.66 that

$$(\nabla_{\mathbf{1}_{\mathcal{V}}}\mathrm{or})\mathbf{M} \in \mathrm{Skew}\mathcal{V} \quad \text{for all} \quad \mathbf{M} \in \mathrm{Lin}\mathcal{V}. \tag{86.17}$$

Comparing (86.15)–(86.17) with (86.1), we see that for every $\mathbf{M} \in \mathrm{Lin}\mathcal{V}$,

$$\mathbf{U} := (\nabla_{1_{\mathcal{V}}}\mathrm{rp})\mathbf{M}, \quad \mathbf{A} := (\nabla_{1_{\mathcal{V}}}\mathrm{or})\mathbf{M}$$

gives the additive decomposition $(\mathbf{U}, \mathbf{A})$ of $\mathbf{M}$. Hence (86.12) and (86.13) follow from the Additive Decomposition Theorem. ■

There are no simple explicit formulas for $\nabla_{\mathbf{L}}\mathrm{rp}, \nabla_{\mathbf{L}}\mathrm{or}$, and $\nabla_{\mathbf{L}}\mathrm{lp}$ when $\mathbf{L} \neq \mathbf{1}_{\mathcal{V}}$.

The results (86.12) and (86.13) show, roughly, that for "infinitesimal" $\mathbf{M} \in \mathrm{Lin}\mathcal{V}$, the right and left polar decompositions of $\mathbf{1}_{\mathcal{V}} + \mathbf{M}$ coincide and are given by $(\mathbf{1}_{\mathcal{V}} + \mathbf{A}, \mathbf{1}_{\mathcal{V}} + \mathbf{U})$ when $(\mathbf{U}, \mathbf{A})$ is the additive decomposition of $\mathbf{M}$.

Notes 86

(1) In the literature on continuum mechanics (including some of my own papers, I must admit) Prop.1 is taken tacitly for granted. I know of no other place in the literature where it is proved.

87 The Structure of Skew Lineons

A genuine inner-product space $\mathcal{V}$ is assumed to be given. First, we deal with an important special type of skew lineons:

Definition 1: *We say that a lineon on* $\mathcal{V}$ *is a* **perpendicular turn** *if it is both skew and orthogonal.*

If $\mathbf{U}$ is a perpendicular turn and $\mathbf{u} \in \mathcal{V}$, then $|\mathbf{Uu}| = |\mathbf{u}|$ because $\mathbf{U}$ is orthogonal, and $(\mathbf{Uu}) \cdot \mathbf{u} = 0$, i.e. $\mathbf{Uu} \perp \mathbf{u}$, because $\mathbf{U}$ is skew. Hence $\mathbf{U}$ "turns" every vector into one that is perpendicular and has the same magnitude.

The following two results are immediate consequences of the definition.

Proposition 1: *A lineon* $\mathbf{U}$ *on* $\mathcal{V}$ *is a perpendicular turn if and only if it satisfies any two of the following three conditions:* (a) $\mathbf{U} \in \mathrm{Skew}\mathcal{V}$, (b) $\mathbf{U} \in \mathrm{Orth}\mathcal{V}$, (c) $\mathbf{U}^2 = -\mathbf{1}_{\mathcal{V}}$.

Proposition 2: *Let* $\mathbf{U}$ *be a perpendicular turn on* $\mathcal{V}$ *and let* $\mathbf{u} \in \mathcal{V}$ *be a unit vector. Then* $(\mathbf{u}, \mathbf{Uu})$ *is an orthonormal pair and its span* $\mathrm{Lsp}\{\mathbf{u}, \mathbf{Uu}\}$ *is a two-dimensional* $\mathbf{U}$*-space.*

The structure of perpendicular turns is described by the following result.

Structure Theorem for Perpendicular Turns: *If* $\mathbf{U}$ *is a perpendicular turn, then there exists an orthonormal subset* $\mathfrak{c}$ *of* $\mathcal{V}$ *such that* $(\mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\} \,|\, \mathbf{e} \in \mathfrak{c})$ *is an orthogonal decomposition of* $\mathcal{V}$ *whose terms are two-dimensional* $\mathbf{U}$*-spaces. The set* $\mathfrak{c}$ *can be chosen so as to contain any single prescribed unit vector. We have* $2\sharp\mathfrak{c} = \dim \mathcal{V}$.

Proof: Consider orthonormal subsets $\mathfrak{c}$ of $\mathcal{V}$ such that $(\mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\} \mid \mathbf{e} \in \mathfrak{c})$ is an orthogonal family of subspaces of $\mathcal{V}$. It is a trivial consequence of Prop.2 that the empty subset of $\mathcal{V}$ and also singletons with unit vectors have this property. Choose a *maximal* set $\mathfrak{c}$ with this property. It is clear that $\mathfrak{c}$ can be chosen so as to contain any prescribed unit vector. Since all the spaces $\mathrm{Lsp}\{\mathbf{u}, \mathbf{Uu}\}$ are $\mathbf{U}$-spaces by Prop.2, it follows that the sum

$$\mathcal{U} := \sum_{\mathbf{e} \in \mathfrak{c}} \mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\}$$

of the family $(\mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\} \mid \mathbf{e} \in \mathfrak{c})$ is also a $\mathbf{U}$-space. Since $\mathbf{U}$ is skew, it follows from Prop.1 of Sect.82 that the orthogonal supplement $\mathcal{U}^\perp$ of $\mathcal{U}$ is also a $\mathbf{U}$-space. Now if $\mathcal{U}^\perp \neq \{\mathbf{0}\}$, we may choose a unit vector $\mathbf{f} \in \mathcal{U}^\perp$. Since $\mathcal{U}^\perp$ is a $\mathbf{U}$-space, we have $\mathrm{Lsp}\{\mathbf{f}, \mathbf{Uf}\} \subset \mathcal{U}^\perp$. It follows that $\mathfrak{c} \cup \{\mathbf{f}\}$ is a subset of $\mathcal{V}$ such that $(\mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\} \mid \mathbf{e} \in \mathfrak{c} \cup \{\mathbf{f}\})$ is an orthogonal family, which contradicts the maximality of $\mathfrak{c}$. We conclude that $\mathcal{U}^\perp = \{\mathbf{0}\}$ and hence $\mathcal{U} = \mathcal{V}$, which means that $(\mathrm{Lsp}\{\mathbf{e}, \mathbf{Ue}\} \mid \mathbf{e} \in \mathfrak{c})$ is an orthogonal decomposition of $\mathcal{V}$. By Prop.2, the terms of the decomposition are two-dimensional and by Prop.4 of Sect.81, we have $2(\sharp\mathfrak{c}) = \dim \mathcal{V}$. ■

Corollary: *Perpendicular turns on* $\mathcal{V}$ *can exist only when* $\dim \mathcal{V}$ *is even. Let* $\mathbf{U}$ *be such a turn and let* $m := \frac{1}{2} \dim \mathcal{V}$*. Then there exists an orthonormal* *basis* $\mathbf{e} := (\mathbf{e}_i \mid i \in (2m)^{]})$ *such that*

$$\mathbf{U}\mathbf{e}_i = \left\{ \begin{array}{rl} \mathbf{e}_{i+1} & \text{when} \quad i \text{ is odd} \\ -\mathbf{e}_{i-1} & \text{when} \quad i \text{ is even} \end{array} \right\} \tag{87.1}$$

for all $i \in (2m)^{]}$.

Proof: The basis $\mathbf{e}$ is obtained by choosing an orthonormal set $\mathfrak{c}$ as in the Theorem, so that $\sharp\mathfrak{c} = m$, and enumerating it by odd numbers, so that $\mathfrak{c} = \{\mathbf{e}_i \mid i \in 2(m^{]}) - 1\}$. The $\mathbf{e}_i$ for $i \in 2(m^{]})$ are then defined by $\mathbf{e}_i := \mathbf{U}\mathbf{e}_{i-1}$. Since $\mathbf{U}^2 = -\mathbf{1}_{\mathcal{V}}$ by (c) of Prop.1, we have $\mathbf{U}\mathbf{e}_i = \mathbf{U}^2\mathbf{e}_{i-1} = -\mathbf{e}_{i-1}$ when $i \in 2(m^{]})$. ∎

If m is small, then the $(2m)$-by-$(2m)$ matrix $[\mathbf{U}]_{\mathbf{e}}$ of $\mathbf{U}$ relative to any basis for which (87.1) holds can be recorded explicitly in the form

$$[\mathbf{U}]_{\mathbf{e}} = \begin{bmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & -1 \\ & & & & & & 1 & 0 \end{bmatrix}, \tag{87.2}$$

where zeros are omitted.

In order to deal with skew lineons, we need the following analogue of Def.1 of Sect.82.

Definition 2: *Let* $\mathbf{L}$ *be a lineon on* $\mathcal{V}$. *For every* $\kappa \in \mathbb{P}^{\times}$, *we write*

$$\mathrm{Qsps}_{\mathbf{L}}(\kappa) := \mathrm{Null}\ (\mathbf{L}^2 + \kappa^2 \mathbf{1}_{\mathcal{V}}). \tag{87.3}$$

This is an $\mathbf{L}$*-space; if it is non-zero, we call it the* **quasi-spectral space** *of* $\mathbf{L}$ *for* κ. *The* **quasi-spectrum** *of* $\mathbf{L}$ *is defined to be*

$$\mathrm{Qspec}\ \mathbf{L} := \{\kappa \in \mathbb{P}^{\times} \mid \mathrm{Qsps}_{\mathbf{L}}(\kappa) \neq \{\mathbf{0}\}\}. \tag{87.4}$$

In view of Def.1 of Sect.82 we have

$$\mathrm{Qspec}\ \mathbf{L} = \{\sqrt{\sigma} \mid \sigma \in \mathrm{Spec}\,(-\mathbf{L}^2) \cap \mathbb{P}^{\times}\} \tag{87.5}$$

and

$$\mathrm{Qsps}_{\mathbf{L}}(\kappa) = \mathrm{Sps}_{-\mathbf{L}^2}(\kappa^2) \quad \text{for all} \quad \kappa \in \mathbb{P}^{\times} \tag{87.6}$$

for all $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$.

The proof of the following result is based on the Spectral Theorem.

Structure Theorem for Skew Lineons: *A lineon* $\mathbf{A}$ *on* $\mathcal{V}$ *is skew if and only if* (i) $\mathrm{Rng}\,\mathbf{A} = (\mathrm{Null}\ \mathbf{A})^{\perp}$, (ii) *the family of quasi-spectral spaces of* $\mathbf{A}$ *is an orthogonal decomposition of* $\mathrm{Rng}\,\mathbf{A}$*, and* (iii) *for each* $\kappa \in \mathrm{Qspec}\ \mathbf{A}$ *we have*

$$\mathbf{A}_{|\mathcal{W}_\kappa} = \kappa \mathbf{U}_\kappa, \tag{87.7}$$

where $\mathcal{W}_\kappa := \mathrm{Qsps}_{\mathbf{A}}(\kappa)$ *and where* $\mathbf{U}_\kappa \in \mathrm{Lin}\mathcal{W}_\kappa$ *is a perpendicular turn.*

Proof: Assume that $\mathbf{A}$ is skew. It follows from (22.9) that $\mathrm{Rng}\,\mathbf{A} = \mathrm{Rng}\,(-\mathbf{A}) = \mathrm{Rng}\,(\mathbf{A}^{\top}) = (\mathrm{Null}\ \mathbf{A})^{\perp}$, i.e. that (i) is valid. Since $\mathbf{A}^{\top}\mathbf{A} = (-\mathbf{A})\mathbf{A} = -\mathbf{A}^2$ is positive by Prop.1 of Sect.85, it follows from Prop.2 of Sect.85 that $\mathrm{Spec}\,(-\mathbf{A}^2) \subset \mathbb{P}$. Hence it follows from (87.5) that

$$\mathrm{Qspec}\ \mathbf{A} = \{\sqrt{\sigma} \mid \sigma \in \mathrm{Spec}\,(-\mathbf{A}^2) \setminus \{0\}\}.$$

In view of (87.6) we conclude that the spectral spaces of $-\mathbf{A}^2$ are $\mathrm{Qsps}_{\mathbf{A}}(\kappa)$ with $\kappa \in \mathrm{Qspec}\ \mathbf{A}$, and also $\mathrm{Sps}_{-\mathbf{A}^2}(0)$ if this subspace is not zero. Since it is easily seen that $\mathrm{Sps}_{-\mathbf{A}^2}(0) = \mathrm{Null}\ \mathbf{A}^2 = \mathrm{Null}\ \mathbf{A}$, it follows from the Spectral Theorem, applied to $-\mathbf{A}^2$, that $(\mathrm{Qsps}_{\mathbf{A}}(\kappa) \mid \kappa \in \mathrm{Qspec}\ \mathbf{A})$ is an orthogonal decomposition of $(\mathrm{Null}\ \mathbf{A})^{\perp} = \mathrm{Rng}\,\mathbf{A}$, i.e. that (ii) is valid. Now let $\kappa \in \mathrm{Qspec}\ \mathbf{A}$ be given. By (87.3) we have $(\mathbf{A}^2 + \kappa^2 \mathbf{1}_{\mathcal{V}})_{|\mathcal{W}_\kappa} = \mathbf{0}$ when $\mathcal{W}_\kappa := \mathrm{Qsps}_{\mathbf{A}}(\kappa)$ and hence, since $\mathcal{W}_\kappa$ is an $\mathbf{A}$-space, $(\frac{1}{\kappa}\mathbf{A}_{|\mathcal{W}_\kappa})^2 = -\mathbf{1}_{\mathcal{W}_\kappa}$. Since $\frac{1}{\kappa}\mathbf{A}_{|\mathcal{W}_\kappa} \in \mathrm{Skew}\mathcal{W}_\kappa$, it follows from Prop.1 that (iii) is valid.

Assume now that the conditions (i), (ii), and (iii) hold and put $\mathcal{W}_\circ := \mathrm{Null}\ \mathbf{A}$ and $K := (\mathrm{Qspec}\ \mathbf{A}) \cup \{0\}$. Then $(\mathcal{W}_\kappa \mid \kappa \in K)$ is an orthogonal decomposition of $\mathcal{V}$. Let $(\mathbf{E}_\kappa \mid \kappa \in K)$ be the associated family of idempotents (see Prop.5 of Sect.81). Using Prop.6 of Sect.81, (81.8), and (87.7), we find that

$$\mathbf{A} = \sum_{\kappa \in \mathrm{Qspec}\ \mathbf{A}} \kappa \mathbf{U}_\kappa|^{\mathcal{V}} \mathbf{E}_\kappa|^{\mathcal{W}_\kappa}. \tag{87.8}$$

Since $\mathbf{U}_\kappa \in \mathrm{Skew}\mathcal{W}_\kappa$ and, by Prop.4 of Sect.83, $\mathbf{E}_\kappa \in \mathrm{Sym}\mathcal{V}$ for each $\kappa \in \mathrm{Qspec}\ \mathbf{A}$, it is easily seen that each term in the sum on the right side of (87.8) is skew and hence that $\mathbf{A}$ is skew. ∎

Assume that $\mathbf{A} \in \mathrm{Skew}\mathcal{V}$. It follows from (iii) of the Theorem just proved and the Corollary to the Structure Theorem for Perpendicular Turns that $\dim \mathcal{W}_\kappa$ is even for all $\kappa \in \mathrm{Qspec}\mathbf{A}$. Hence, by (ii) and Prop.3 of Sect.81, we obtain the following

Proposition 3: *If* $\mathbf{A}$ *is a skew lineon on* $\mathcal{V}$*, then* $\dim(\mathrm{Rng}\,\mathbf{A})$ *is even. In particular, if the dimension of* $\mathcal{V}$ *is odd, then there exist no invertible skew lineons on* $\mathcal{V}$*.*

The following corollary to the Structure Theorem above is obtained by choosing, in each of the quasi-spectral spaces, an orthonormal basis as described in the Corollary to the Structure Theorem for Perpendicular Turns.

Corollary: *Let* $\mathbf{A}$ *be a skew lineon on* $\mathcal{V}$ *and put* $n := \dim \mathcal{V}$, $m := \frac{1}{2}\dim(\operatorname{Rng}\mathbf{A})$. *Then there is an orthonormal basis* $\mathbf{e} := (\mathbf{e}_i \mid i \in n^]) $ *of* $\mathcal{V}$ *and a list* $(\kappa_k \mid k \in m^])$ *of strictly positive real numbers such that*

$$\left.\begin{array}{lcl} \mathbf{A}\mathbf{e}_{2k-1} & = & \kappa_k \mathbf{e}_{2k} \\ \mathbf{A}\mathbf{e}_{2k} & = & -\kappa_k \mathbf{e}_{2k-1} \end{array}\right\} \quad \text{for all} \quad k \in m^], \tag{87.9}$$

$$\mathbf{A}\mathbf{e}_i = \mathbf{0} \quad \text{for all} \quad i \in n^] \setminus (2m)^].$$

The only non-zero terms in the matrix $[\mathbf{A}]_{\mathbf{e}}$ of $\mathbf{A}$ relative to a basis $\mathbf{e}$ for which (87.9) holds are those in blocks of the form $\begin{bmatrix} 0 & -\kappa_k \\ \kappa_k & 0 \end{bmatrix}$ along the diagonal. For example, if $\dim \mathcal{V} = 3$ and $\mathbf{A} \neq \mathbf{0}$, then $[\mathbf{A}]_{\mathbf{e}}$ has the form

$$[\mathbf{A}]_{\mathbf{e}} = \begin{bmatrix} 0 & -\kappa & 0 \\ \kappa & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{for some} \quad \kappa \in \mathbb{P}^\times. \tag{87.10}$$

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(1) I am introducing the term "perpendicular turn" here for the first time; I believe it is very descriptive.

(2) The concepts of a quasi-spectrum, quasi-spectral space, and the notations of Def.2 are introduced here for the first time.

88 Structure of Normal and Orthogonal Lineons

Again, a genuine inner-product space $\mathcal{V}$ is assumed to be given.

Definition 1: *We say that a lineon on* $\mathcal{V}$ *is* **normal** *if it commutes with its transpose.*

It is clear that symmetric, skew, and orthogonal lineons are all normal.

Proposition 1: *A lineon* $\mathbf{N} \in \operatorname{Lin}\mathcal{V}$ *is normal if and only if the two terms* $\mathbf{S} := \frac{1}{2}(\mathbf{N} + \mathbf{N}^\top)$ *and* $\mathbf{A} := \frac{1}{2}(\mathbf{N} - \mathbf{N}^\top)$ *of the additive decomposition* $(\mathbf{S}, \mathbf{A})$ *of* $\mathbf{N}$ *commute. If this is the case, then any two of* $\mathbf{N}, \mathbf{N}^\top, \mathbf{S}$ *and* $\mathbf{A}$ *commute.*

Proof: We have $\mathbf{N} = \mathbf{S} + \mathbf{A}$ and, since $\mathbf{S} \in \mathrm{Sym}\mathcal{V}$, $\mathbf{A} \in \mathrm{Skew}\mathcal{V}$, $\mathbf{N}^\top = \mathbf{S} - \mathbf{A}$. It follows that $\mathbf{N}\mathbf{N}^\top - \mathbf{N}^\top\mathbf{N} = 2(\mathbf{A}\mathbf{S} - \mathbf{S}\mathbf{A})$. Therefore, $\mathbf{N}$ and $\mathbf{N}^\top$ commute if and only if $\mathbf{S}$ and $\mathbf{A}$ commute. ∎

Proposition 2: *A lineon* $\mathbf{N} \in \mathrm{Lin}\mathcal{V}$ *is normal if and only if*

$$|\mathbf{N}\mathbf{v}| = |\mathbf{N}^\top\mathbf{v}| \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}. \tag{88.1}$$

If this is the case, then Null $\mathbf{L}$ = Null $\mathbf{L}^\top$.

Proof: We have $|\mathbf{N}\mathbf{v}|^2 = \mathbf{N}\mathbf{v} \cdot \mathbf{N}\mathbf{v} = \mathbf{v} \cdot \mathbf{N}^\top\mathbf{N}\mathbf{v}$ and $|\mathbf{N}^\top\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{N}\mathbf{N}^\top\mathbf{v}$ for all $\mathbf{v} \in \mathcal{V}$. Hence (88.1) holds if and only if the quadratic forms corresponding to $\mathbf{N}^\top\mathbf{N}$ and $\mathbf{N}\mathbf{N}^\top$ are the same (see (85.1)). This is the case if and only if $\mathbf{N}^\top\mathbf{N}$ and $\mathbf{N}\mathbf{N}^\top$ are the same (see Sect.27). ∎

Proposition 3: *An invertible lineon* $\mathbf{N} \in \mathrm{Lis}\mathcal{V}$ *is normal if and only if the two terms of its right [left] polar decomposition commute* (see Sect.86).

Proof: It is clear from the Polar Decomposition Theorem of Sect.86 that the terms of the right or of the left polar decomposition commute if and only if the right and the left polar decompositions coincide, i.e. if and only if $\mathbf{S} = \mathbf{S}'$ in the notation of the statement of this theorem. But by $(86.4)_1$ and $(86.7)_1$, we have $\mathbf{S} = \mathbf{S}'$ if and only if $\mathbf{N}\mathbf{N}^\top = \mathbf{N}^\top\mathbf{N}$. ∎

In order to deal with the structure of normal lineons, we need the following analogue of Def.2 of Sect.87.

Definition 2: *Let* $\mathbf{L}$ *be a lineon on* $\mathcal{V}$. *For every* $\nu \in \mathbb{R} \times \mathbb{P}^\times$ *we write*

$$\mathrm{Psps}_{\mathbf{L}}(\nu) := \mathrm{Null}\ ((\mathbf{L} - \nu_1\mathbf{1}_\mathcal{V})^2 + \nu_2^2\mathbf{1}_\mathcal{V}). \tag{88.2}$$

This is an $\mathbf{L}$*-space; if it is not zero, we call it the* **pair-spectral space** *of* $\mathbf{L}$ *for* ν. *The* **pair-spectrum** *of* $\mathbf{L}$ *is defined to be*

$$\mathrm{Pspec}\,\mathbf{L} := \{\nu \in \mathbb{R} \times \mathbb{P}^\times \mid \mathrm{Psps}_{\mathbf{L}}(\nu) \neq \{\mathbf{0}\}\}. \tag{88.3}$$

The proof of the following theorem is based on the Structure Theorem for Skew Lineons and the Spectral Theorem.

Structure Theorem for Normal Lineons: *A lineon* $\mathbf{N}$ *on* $\mathcal{V}$ *is normal if and only if the family of its pair-spectral spaces is an orthogonal decomposition of* $\mathcal{V}$ *and, for each* $\nu \in \mathrm{Pspec}\,\mathbf{N}$, *we have*

$$\mathbf{N}_{|\mathcal{W}_\nu} = \left\{ \begin{array}{ll} \nu_1\mathbf{1}_{\mathcal{W}_\nu} & \text{if} \quad \nu_2 = 0 \\ \nu_1\mathbf{1}_{\mathcal{W}_\nu} + \nu_2\mathbf{U}_\nu & \text{if} \quad \nu_2 \neq 0 \end{array} \right\}, \tag{88.4}$$

where $\mathcal{W}_\nu := \mathrm{Psps}_{\mathbf{N}}(\nu)$, *and where* $\mathbf{U}_\nu \in \mathrm{Lin}\mathcal{W}_\nu$ *is a perpendicular turn.*

First we prove an auxiliary result:

Lemma: *If* $\mathbf{N} \in \mathrm{Lin}\mathcal{V}$ *is normal and if* $(\mathbf{S}, \mathbf{A})$ *is the additive decomposition of* $\mathbf{N}$*, then*

$$\mathrm{Psps}_{\mathbf{N}}((\sigma,\kappa)) = \left\{ \begin{array}{lll} \mathrm{Sps}_{\mathbf{S}}(\sigma) \cap \mathrm{Qsps}_{\mathbf{A}}(\kappa) & \text{if} & \kappa \neq 0 \\ \mathrm{Sps}_{\mathbf{S}}(\sigma) \cap \mathrm{Null}\ \mathbf{A} & \text{if} & \kappa = 0 \end{array} \right\} \tag{88.5}$$

for all $\sigma \in \mathbb{R}$ *and all* $\kappa \in \mathbb{P}$*.*

Proof: Let $\sigma \in \mathbb{R}$ and $\kappa \in \mathbb{P}$ be given and put $\mathbf{M} := \mathbf{N} - \sigma \mathbf{1}_{\mathcal{V}}$ and $\mathbf{T} := \mathbf{S} - \sigma\mathbf{1}_{\mathcal{V}}$. Then $\mathbf{M}$ is normal and $(\mathbf{T}, \mathbf{A})$ is the additive decomposition of $\mathbf{M}$. Using the fact that any two of $\mathbf{M}, \mathbf{M}^{\top}, \mathbf{T}$ and $\mathbf{A}$ commute (see Prop.1), we obtain

$$\begin{aligned} (\mathbf{M}^2 + \kappa^2 \mathbf{1}_{\mathcal{V}})^{\top}(\mathbf{M}^2 + \kappa^2\mathbf{1}_{\mathcal{V}}) &= (\mathbf{M}^{\top}\mathbf{M})^2 + \kappa^2(\mathbf{M}^{\top 2} + \mathbf{M}^2) + \kappa^4 \mathbf{1}_{\mathcal{V}} \\ &= ((\mathbf{T}-\mathbf{A})(\mathbf{T}+\mathbf{A}))^2 + \kappa^2((\mathbf{T}-\mathbf{A})^2 + (\mathbf{T}+\mathbf{A})^2) + \kappa^4\mathbf{1}_{\mathcal{V}} \\ &= \mathbf{T}^4 + 2(\mathbf{AT})^{\top}(\mathbf{AT}) + 2\kappa^2\mathbf{T}^2 + (\mathbf{A}^2 + \kappa^2\mathbf{1}_{\mathcal{V}})^2. \end{aligned}$$

Since $\mathbf{T}, \mathbf{T}^2$ and $(\mathbf{A}^2 + \kappa^2\mathbf{1}_{\mathcal{V}})$ are symmetric, it follows that

$$|(\mathbf{M}^2 + \kappa^2\mathbf{1}_{\mathcal{V}})\mathbf{v}|^2 = |\mathbf{T}^2\mathbf{v}|^2 + 2|\mathbf{ATv}|^2 + 2\kappa^2|\mathbf{Tv}|^2 + |(\mathbf{A}^2 + \kappa^2\mathbf{1}_{\mathcal{V}})\mathbf{v}|^2$$

for all $\mathbf{v} \in \mathcal{V}$, from which we infer that $\mathbf{v} \in \mathrm{Null}\ (\mathbf{M}^2 + \kappa^2\mathbf{1}_{\mathcal{V}}) = \mathrm{Psps}_{\mathbf{N}}(\sigma,\kappa)$ if and only if $\mathbf{v} \in \mathrm{Null}\ \mathbf{T}\ \cap\ \mathrm{Null}\ (\mathbf{A}^2 + \kappa^2\mathbf{1}_{\mathcal{V}})$. Since $\mathrm{Null}\ \mathbf{T} = \mathrm{Sps}_{\mathbf{S}}(\sigma)$, since $\mathrm{Null}\ (\mathbf{A}^2 + \kappa^2\mathbf{1}_{\mathcal{V}}) = \mathrm{Qsps}_{\mathbf{A}}(\kappa)$ when $\kappa \neq 0$ by Def.2 of Sect.82, and since $\mathrm{Null}\ \mathbf{A}^2 = \mathrm{Null}\ \mathbf{A}$ because $\mathbf{A}$ is skew, (88.5) follows. ∎

Proof of Theorem: Assume that $\mathbf{N}$ is normal and that $(\mathbf{S}, \mathbf{A})$ is its additive decomposition. Let $\sigma \in \mathrm{Spec}\,\mathbf{S}$ be given and put $\mathcal{U} := \mathrm{Sps}_{\mathbf{S}}(\sigma)$. By Prop.2 of Sect.82, $\mathcal{U}$ is $\mathbf{A}$-invariant and we may hence consider the adjustment $\mathbf{A}_{|\mathcal{U}} \in \mathrm{Skew}\mathcal{U}$. It is clear that $\mathrm{Null}\ \mathbf{A}_{|\mathcal{U}} = \mathcal{U} \cap \mathrm{Null}\ \mathbf{A}$ and

$$\mathrm{Qsps}_{\mathbf{A}_{|\mathcal{U}}}(\kappa) = \mathcal{U} \cap \mathrm{Qsps}_{\mathbf{A}}(\kappa) \quad \text{for all} \quad \kappa \in \mathbb{P}^{\times}.$$

Hence, by the Lemma, we conclude that

$$\mathrm{Psps}_{\mathbf{N}}(\sigma,\kappa) = \left\{ \begin{array}{lll} \mathrm{Qsps}_{\mathbf{A}_{|\mathcal{U}}}(\kappa) & \text{for all} & \kappa \in \mathbb{P}^{\times} \\ \mathrm{Null}\ \mathbf{A}_{|\mathcal{U}} & \text{if} & \kappa = 0 \end{array} \right\}. \tag{88.6}$$

Since $\mathbf{S}_{|\mathcal{U}} = \sigma\mathbf{1}_{\mathcal{U}}$, $\mathcal{U}$ is $\mathbf{N}$-invariant and

$$\mathbf{N}_{|\mathcal{U}} = \sigma\mathbf{1}_{\mathcal{U}} + \mathbf{A}_{|\mathcal{U}}. \tag{88.7}$$

We now apply the Structure Theorem for Skew Lineons to $\mathbf{A}_{|\mathcal{U}} \in \mathrm{Skew}\ \mathcal{U}$. In view of (88.6), and since $\mathrm{Qspec}\ \mathbf{A}_{|\mathcal{U}} \subset \mathrm{Qspec}\ \mathbf{A}$, we have

$(\mathrm{Psps}_{\mathbf{N}}(\nu) \mid \nu \in \mathrm{Pspec}\,\mathbf{N},\ \nu_1 = \sigma)$ is an orthogonal decomposition of $\mathcal{U}$, and for all $\nu \in \mathrm{Pspec}\,\mathbf{N}$ with $\nu_1 = \sigma$ we have

$$\mathbf{A}_{|\mathcal{W}_\nu} = \mathbf{0} \quad \text{if} \quad \nu_2 = 0 \quad \text{and} \quad \mathbf{A}_{|\mathcal{W}_\nu} = \nu_2 \mathbf{U}_\nu \quad \text{if} \quad \nu_2 \neq 0,$$

where $\mathcal{W}_\nu := \mathrm{Psps}_{\mathbf{N}}(\nu)$ and where $\mathbf{U}_\nu \in \mathrm{Lin}\mathcal{W}_\nu$ is a perpendicular turn. Using this result and (88.7), adjusted to the subspace $\mathcal{W}_\nu = \mathcal{W}_{(\sigma,\nu_2)}$ of $\mathcal{U}$, we infer that (88.4) is valid.

By the Spectral Theorem $(\mathrm{Sps}_{\mathbf{S}}(\sigma) \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ is an orthogonal decomposition of $\mathcal{V}$. Since, as we just have seen, $(\mathrm{Psps}_{\mathbf{N}}(\nu) \mid \nu \in \mathrm{Pspec}\,\mathbf{N},\ \nu_1 = \sigma)$ is an orthogonal decomposition of $\mathrm{Sps}_{\mathbf{S}}(\sigma)$ for each $\sigma \in \mathrm{Spec}\,\mathbf{S}$, it follows that $(\mathrm{Psps}_{\mathbf{N}}(\nu) \mid \nu \in \mathrm{Pspec}\,\mathbf{N})$ is an orthogonal decomposition of $\mathcal{V}$. This completes the proof of the "only-if" -part of the Theorem. The proof of the "if"-part goes the same way as that of the "if"-part of the Structure Theorem for Skew Lineons. We leave the details to the reader. ■

For a symmetric lineon we have $\mathbf{S}$, $\mathrm{Pspec}\,\mathbf{S} = \mathrm{Spec}\,\mathbf{S} \times \{0\}$ and $\mathrm{Psps}_{\mathbf{S}}((\sigma,0)) = \mathrm{Sps}_{\mathbf{S}}(\sigma)$ for all $\sigma \in \mathrm{Spec}\,\mathbf{S}$. For a skew lineon $\mathbf{A}$ we have $\mathrm{Pspec}\,\mathbf{A} = \{0\} \times (\mathrm{Spec}\,\mathbf{A} \cup \mathrm{Qspec}\ \mathbf{A})$ and $\mathrm{Psps}_{\mathbf{A}}(0,\kappa) = \mathrm{Qsps}_{\mathbf{A}}(\kappa)$ for all $\kappa \in \mathrm{Qspec}\ \mathbf{A}$, while $\mathrm{Psps}_{\mathbf{A}}((0,0)) = \mathrm{Null}\ \mathbf{A}$ when $\mathbf{A}$ is not invertible.

As in the previous section, we obtain a corollary to the Structure Theorem just given as follows.

Corollary: *Let* $\mathbf{N}$ *be a normal lineon on* $\mathcal{V}$*. Then there exists a number* $m \in \mathbb{N}$ *with* $2m \leq n := \dim \mathcal{V}$*, an orthonormal basis* $\mathbf{e} := (\mathbf{e}_i \mid i \in n^{]})$ *of* $\mathcal{V}$*, a list* $(\mu_k \mid k \in m^{]})$ *in* $\mathbb{R}$*, a list* $(\kappa_k \mid k \in m^{]})$ *in* $\mathbb{P}^\times$*, and a family* $(\lambda_i \mid i \in n^{]} \setminus (2m)^{]})$ *in* $\mathbb{R}$ *such that*

$$\left.\begin{array}{lcl} \mathbf{N}\mathbf{e}_{2k-1} & = & \mu_k \mathbf{e}_{2k-1} + \kappa_k \mathbf{e}_{2k} \\ \mathbf{N}\mathbf{e}_{2k} & = & \mu_k \mathbf{e}_{2k} - \kappa_k \mathbf{e}_{2k-1} \end{array}\right\} \quad \text{for all} \quad k \in m^{]}, \tag{88.8}$$

$$\mathbf{N}\mathbf{e}_i = \lambda_i \mathbf{e}_i \quad \text{for all} \quad i \in n^{]} \setminus (2m)^{]}.$$

The only non-zero terms in the matrix $[\mathbf{N}]_{\mathbf{e}}$ of $\mathbf{N}$ relative to a basis $\mathbf{e}$ for which (88.8) holds are either on the diagonal or in blocks of the form $\begin{bmatrix} \mu_k & -\kappa_k \\ \kappa_k & \mu_k \end{bmatrix}$ along the diagonal. For example, if $\dim \mathcal{V} = 3$ and if $\mathbf{N}$ is not symmetric, then $[\mathbf{N}]_{\mathbf{e}}$ has the form

$$[\mathbf{N}]_{\mathbf{e}} = \begin{bmatrix} \mu & -\kappa & 0 \\ \kappa & \mu & 0 \\ 0 & 0 & \lambda \end{bmatrix}, \quad \kappa \in \mathbb{P}^\times,\ \lambda,\ \mu \in \mathbb{R}. \tag{88.9}$$

Orthogonal lineons are special normal lineons. To describe their structure, the following definition is useful.

Definition 3: *Let* $\mathbf{L}$ *be a lineon on* $\mathcal{V}$. *For every* $\theta \in \,]0,\pi[$ *we write*

$$\operatorname{Aspec}_{\mathbf{L}}(\theta) := \operatorname{Psps}_{\mathbf{L}}((\cos\theta, \sin\theta)) \tag{88.10}$$

(see (88.2)). *This is an* $\mathbf{L}$*-space; if it is not zero, we call it the* **angle-spectral space** *of* $\mathbf{L}$ *for* θ. *The* **angle-spectrum** *of* $\mathbf{L}$ *is defined to be*

$$\operatorname{Aspec}\mathbf{L} := \{\theta \in \,]0,\pi[\;|\; \operatorname{Asps}_{\mathbf{L}}(\theta) \neq \{\mathbf{0}\}\}. \tag{88.11}$$

The following result is a corollary to the Structure Theorem for Normal Lineons.

Structure Theorem for Orthogonal Lineons: *A lineon* $\mathbf{R}$ *on* $\mathcal{V}$ *is orthogonal if and only if* (i) Null $(\mathbf{R}+\mathbf{1}_{\mathcal{V}}) \perp$ Null $(\mathbf{R}-\mathbf{1}_{\mathcal{V}})$, (ii) *the family of angle-spectral spaces is an orthogonal decomposition of* (Null $(\mathbf{R}+\mathbf{1}_{\mathcal{V}})+$ Null $(\mathbf{R}-\mathbf{1}_{\mathcal{V}}))^{\perp}$, *and* (iii) *for each* $\theta \in \operatorname{Aspec}\mathbf{R}$ *we have*

$$\mathbf{R}_{|\mathcal{U}_\theta} = \cos\theta\mathbf{1}_{\mathcal{U}_\theta} + \sin\theta\mathbf{V}_\theta, \tag{88.12}$$

where $\mathcal{U}_\theta := \operatorname{Asps}_{\mathbf{R}}(\theta)$ *and where* $\mathbf{V}_\theta \in \operatorname{Lin}\mathcal{U}_\theta$ *is a perpendicular turn.*

Proof: The conditions (i), (ii), (iii) are satisfied if and only if the conditions of the Structure Theorem for Normal Lineons are satisfied with $\mathbf{N} := \mathbf{R}$ and

$$\operatorname{Pspec}\mathbf{R} \subset \{(1,0),(-1,0)\} \cup \{(\cos\theta,\sin\theta) \;|\; \theta \in \operatorname{Aspec}\mathbf{R}\}. \tag{88.13}$$

Thus we may assume that $\mathbf{R}$ is normal. Since $(\operatorname{Psps}_{\mathbf{R}}(\nu) \,|\, \nu \in \operatorname{Pspec}\mathbf{R})$ is an orthogonal decomposition of $\mathcal{V}$ whose terms $\mathcal{W}_\nu := \operatorname{Psps}_{\mathbf{R}}(\nu)$ are $\mathbf{R}$-invariant, it is easily seen that $\mathbf{R}$ is orthogonal if and only if $\mathbf{R}_{|\mathcal{W}_\nu}$ is orthogonal for each $\nu \in \operatorname{Pspec}\mathbf{R}$. It follows from (88.4), with $\mathbf{N} := \mathbf{R}$, that this is the case if and only if $\nu_1^2 + \nu_2^2 = 1$ for all $\nu \in \operatorname{Pspec}\mathbf{R}$, which is equivalent to (88.13). If $\nu_2 = 0$ we have $\nu = (1,0)$ or $\nu = (-1,0)$ and $\operatorname{Psps}_{\mathbf{R}}((1,0)) =$ Null $(\mathbf{R}-\mathbf{1}_{\mathcal{V}})$ and $\operatorname{Psps}_{\mathbf{R}}((-1,0)) =$ Null $(\mathbf{R}+\mathbf{1}_{\mathcal{V}})$. If $\nu_2 \neq 0$ we have $\nu = (\cos\theta,\sin\theta)$ for some $\theta \in]0,\pi[$ and (88.4) reduces to (88.12) with $\mathbf{V}_\theta := \mathbf{U}_{(\cos\theta,\sin\theta)}$. ■

As before we obtain a corollary as follows:

Corollary: *Let* $\mathbf{R}$ *be an orthogonal lineon on* $\mathcal{V}$. *Then there exist numbers* $m \in \mathbb{N}$ *and* $p \in \mathbb{N}$ *with* $2m \le p \le n := \dim\mathcal{V}$, *an orthonormal basis* $\mathbf{e} := (\mathbf{e}_i \,|\, i \in n^{]})$ *of* $\mathcal{V}$, *and a list* $(\theta_k \,|\, k \in m^{]})$ *in* $]0,\pi[$ *such that*

$$\left.\begin{array}{lcl} \mathbf{R}\mathbf{e}_{2k-1} & = & \cos\theta_k\mathbf{e}_{2k-1} + \sin\theta_k\mathbf{e}_{2k} \\ \mathbf{R}\mathbf{e}_{2k} & = & \cos\theta_k\mathbf{e}_{2k} - \sin\theta_k\mathbf{e}_{2k-1} \end{array}\right\} \quad \text{for all} \quad k \in m^{]}$$

$$\begin{array}{rcl} \mathbf{R}\mathbf{e}_i &=& -\mathbf{e}_i \quad \text{for all} \quad i \in p^{]} \setminus (2m)^{]}, \\ \mathbf{R}\mathbf{e}_i &=& \mathbf{e}_i \quad \text{for all} \quad i \in n^{]} \setminus p^{]}. \end{array} \tag{88.14}$$

The only non-zero terms in the matrix $[\mathbf{R}]_{\mathbf{e}}$ of $\mathbf{R}$ relative to a basis $\mathbf{e}$ for which (88.14) holds are either 1 or -1 on the diagonal or in blocks of the form $\begin{bmatrix} \cos\theta_k & -\sin\theta_k \\ \sin\theta_k & \cos\theta_k \end{bmatrix}$ along the diagonal. For example, if $\dim \mathcal{V} = 3$ and if $\mathbf{R}$ is not symmetric, then $[\mathbf{R}]_{\mathbf{e}}$ has the form

$$[\mathbf{R}]_{\mathbf{e}} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & \pm 1 \end{bmatrix}, \quad \theta \in]0, \pi[. \tag{88.15}$$

If the $+$ sign is appropriate, then $\mathbf{R}$ is a *rotation* by an angle θ about an axis in the direction of $\mathbf{e}_3$. If the $-$sign is appropriate, then $\mathbf{R}$ is obtained from such a rotation by composing it with a *reflection* in the plane $\{\mathbf{e}_3\}^{\perp}$.

Notes 88

(1) The concepts of a pair-spectrum, pair-spectral space, angle-spectrum, and angle-spectral space, and the notations of Defs.2 and 3 are introduced here for the first time.

89 Complex Spaces, Unitary Spaces

Definition 1: *A* **complex space** *is a linear space* $\mathcal{V}$ *(over* $\mathbb{R}$*) endowed with additional structure by the prescription of a lineon* $\mathbf{J}$ *on* $\mathcal{V}$ *that satisfies* $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$. *We call* $\mathbf{J}$ *the* **complexor** *of* $\mathcal{V}$.

A complex space $\mathcal{V}$ acquires the natural structure of a linear space over the field $\mathbb{C}$ of complex numbers if we stipulate that the scalar multiplication $\mathrm{sm}^{\mathbb{C}} : \mathbb{C} \times \mathcal{V} \to \mathcal{V}$ of $\mathcal{V}$ as a space over $\mathbb{C}$ be given by

$$\mathrm{sm}^{\mathbb{C}}(\zeta, \mathbf{u}) = (\mathrm{Re}\,\zeta)\mathbf{u} + (\mathrm{Im}\,\zeta)\mathbf{J}\mathbf{u} \quad \text{for all} \quad \zeta \in \mathbb{C}, \tag{89.1}$$

where $\mathrm{Re}\,\zeta$ and $\mathrm{Im}\,\zeta$ denote the real and imaginary parts of ζ. Indeed, it is easily verified that $\mathrm{sm}^{\mathbb{C}}$ satisfies the axioms (S1)–(S4) of Sect.11 if we take $\mathbb{F} := \mathbb{C}$. The scalar multiplication $\mathrm{sm} : \mathbb{R} \times \mathcal{V} \to \mathcal{V}$ of $\mathcal{V}$ as a space over $\mathbb{R}$ is simply the restriction of $\mathrm{sm}^{\mathbb{C}}$ to $\mathbb{R} \times \mathcal{V}$:

$$\mathrm{sm} = \mathrm{sm}^{\mathbb{C}}|_{\mathbb{R} \times \mathcal{V}}. \tag{89.2}$$

We use the simplified notation

$$\zeta \mathbf{u} := \mathrm{sm}^{\mathbb{C}}(\zeta, \mathbf{u}) \quad \text{for all} \quad \zeta \in \mathbb{C}, \;\; \mathbf{u} \in \mathcal{V} \tag{89.3}$$

as described in Sect.11.

Conversely, every linear space $\mathcal{V}$ over the field $\mathbb{C}$ of complex numbers has the natural structure of a complex space in the sense of Def.1. The structure of $\mathcal{V}$ as a linear space over $\mathbb{R}$ is obtained simply by restricting the scalar multiplication of $\mathcal{V}$ to $\mathbb{R} \times \mathcal{V}$, and the complexor of $\mathcal{V}$ is the mapping $(\mathbf{u} \mapsto \mathrm{i}\,\mathbf{u}) : \mathcal{V} \to \mathcal{V}$.

Let $\mathcal{V}$ be a complex space. Most of the properties and constructions involving $\mathcal{V}$ discussed in Chaps.1 and 2 depend on whether $\mathcal{V}$ is regarded as a linear space over $\mathbb{R}$ or over $\mathbb{C}$. To remove the resulting ambiguities, we use the prefix "$\mathbb{C}$" or a superscript $\mathbb{C}$ if we refer to the structure of $\mathcal{V}$ as a linear space over $\mathbb{C}$. For example, a non-empty subset $\mathfrak{c}$ of $\mathcal{V}$ that is a $\mathbb{C}$-basis is not a basis relative to the structure of $\mathcal{V}$ as a space over $\mathbb{R}$. However, it is easily seen that $\mathfrak{c} \in \mathrm{Sub}\,\mathcal{V}$ is a $\mathbb{C}$-basis set if and only if $\mathfrak{c} \cup \mathbf{J}_{>}(\mathfrak{c})$ is a basis set of $\mathcal{V}$. If this is the case, then $\mathfrak{c} \cap \mathbf{J}_{>}(\mathfrak{c}) = \emptyset$ and hence, since $\mathbf{J}$ is injective, $\sharp(\mathfrak{c} \cup \mathbf{J}_{>}(\mathfrak{c})) = 2(\sharp\mathfrak{c})$. Therefore, the dimension of $\mathcal{V}$ is a linear space over $\mathbb{R}$ is twice the dimension of $\mathcal{V}$ as a linear space over $\mathbb{C}$:

$$\dim \mathcal{V} = 2 \dim^{\mathbb{C}} \mathcal{V}. \tag{89.4}$$

It follows that $\dim \mathcal{V}$ must be an even number.

Let $\mathcal{V}$ and $\mathcal{V}'$ be complex spaces, with complexors $\mathbf{J}$ and $\mathbf{J}'$, respectively. The set of all $\mathbb{C}$-linear mappings from $\mathcal{V}$ to $\mathcal{V}'$ is easily seen to be given by

$$\mathrm{Lin}^{\mathbb{C}}(\mathcal{V}, \mathcal{V}') = \{\mathbf{L} \in \mathrm{Lin}(\mathcal{V}, \mathcal{V}') \mid \mathbf{L}\mathbf{J} = \mathbf{J}'\mathbf{L}\}. \tag{89.5}$$

In particular, we have $\mathrm{Lin}^{\mathbb{C}}(\mathcal{V}, \mathcal{V}) =: \mathrm{Lin}^{\mathbb{C}}\mathcal{V} = \mathrm{Comm}\,\mathbf{J}$, the commutant-algebra of the complexor $\mathbf{J}$ (see 18.2). The set $\mathbb{C}\mathbf{1}_{\mathcal{V}} := \{\zeta \mathbf{1}_{\mathcal{V}} \mid \zeta \in \mathbb{C}\}$ is a subalgebra of $\mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ and we have

$$\mathbf{J} = \mathrm{i}\,\mathbf{1}_{\mathcal{V}}. \tag{89.6}$$

Let $\mathbf{J}$ be the complexor of a complex space $\mathcal{V}$. Since $(-\mathbf{J})^2 = \mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$, we can also consider $\mathcal{V}$ as a complex space with $-\mathbf{J}$ rather than $\mathbf{J}$ as the designated complexor. We call the resulting structure of $\mathcal{V}$ the **conjugate-complex** structure of $\mathcal{V}$. In view of (89.1), the corresponding scalar multiplication, denoted by $\overline{\mathrm{sm}}^{\mathbb{C}} : \mathbb{C} \times \mathcal{V} \to \mathcal{V}$, is given by

$$\overline{\mathrm{sm}}^{\mathbb{C}}(\zeta, \mathbf{u}) = (\mathrm{Re}\,\zeta)\mathbf{u} - (\mathrm{Im}\,\zeta)\mathbf{J}\mathbf{u} \quad \text{for all} \quad \zeta \in \mathbb{C} \tag{89.7}$$

and we have

$$\overline{\mathrm{sm}}^{\mathbb{C}}(\zeta,\mathbf{u}) = \overline{\zeta}\mathbf{u} \quad \text{for all} \quad \zeta \in \mathbb{C},\ \mathbf{u} \in \mathcal{V}. \tag{89.8}$$

where $\overline{\zeta}$ denotes the complex-conjugate of ζ.

Let $\mathcal{V}$ and $\mathcal{V}'$ be complex spaces, with complexors $\mathbf{J}$ and $\mathbf{J}'$, respectively. We say that a mapping from $\mathcal{V}$ to $\mathcal{V}'$ is **conjugate-linear** if it is $\mathbb{C}$-linear when one of $\mathcal{V}$ and $\mathcal{V}'$ is considered as a linear space over $\mathbb{C}$ relative to its conjugate-complex structure. The set of all conjugate-linear mappings from $\mathcal{V}$ to $\mathcal{V}'$ will be denoted by $\overline{\mathrm{Lin}}^{\mathbb{C}}(\mathcal{V},\mathcal{V}')$, and we have

$$\overline{\mathrm{Lin}}^{\mathbb{C}}(\mathcal{V},\mathcal{V}') = \{\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{V}') \mid \mathbf{LJ} = -\mathbf{J}'\mathbf{L}\}. \tag{89.9}$$

A mapping $\mathbf{L} \in \mathrm{Lin}(\mathcal{V},\mathcal{V}')$ is conjugate-linear if and only if

$$\mathbf{L}(\zeta\mathbf{u}) = \overline{\zeta}\mathbf{Lu} \quad \text{for all} \quad \zeta \in \mathbb{C},\ \mathbf{u} \in \mathcal{V}. \tag{89.10}$$

A mapping $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ is $\mathbb{C}$-linear when *both* $\mathcal{V}$ and $\mathcal{V}'$ are considered as linear spaces over $\mathbb{C}$ relative to their conjugate-complex structures if and only if it is $\mathbb{C}$-linear in the ordinary sense.

Let $\mathcal{V}$ be a complex space with complexor $\mathbf{J}$. Its dual $\mathcal{V}^*$ then acquires the natural structure of a complex space if we stipulate that the complexor of $\mathcal{V}^*$ be $\mathbf{J}^\top \in \mathrm{Lin}\mathcal{V}^*$. The dual $\mathcal{V}^* := \mathrm{Lin}(\mathcal{V},\mathbb{R})$ must be carefully distinguished from the "complex dual" $\mathrm{Lin}^{\mathbb{C}}(\mathcal{V},\mathbb{C})$ of $\mathcal{V}$. However, the following result shows that the two are naturally $\mathbb{C}$-linearly isomorphic.

Proposition 1: *The mapping*

$$(\boldsymbol{\omega} \mapsto \mathrm{Re}\,\boldsymbol{\omega}) : \mathrm{Lin}^{\mathbb{C}}(\mathcal{V},\mathbb{C}) \to \mathcal{V}^*. \tag{89.11}$$

is a $\mathbb{C}$-linear isomorphism. Its inverse $\Gamma \in \mathrm{Lis}^{\mathbb{C}}(\mathcal{V}^*, \mathrm{Lin}^{\mathbb{C}}(\mathcal{V},\mathbb{C}))$ is given by

$$(\Gamma\boldsymbol{\lambda})\mathbf{u} = \boldsymbol{\lambda}\mathbf{u} - \mathrm{i}\,\boldsymbol{\lambda}(\mathbf{Ju}) \quad \text{for all} \quad \mathbf{u} \in \mathcal{V},\ \boldsymbol{\lambda} \in \mathcal{V}^*. \tag{89.12}$$

Proof: We put $\mathcal{V}^\dagger := \mathrm{Lin}^{\mathbb{C}}(\mathcal{V},\mathbb{C})$ and denote the mapping (89.11) by $\Phi : \mathcal{V}^\dagger \to \mathcal{V}^*$, so that

$$(\Phi\boldsymbol{\omega})\mathbf{u} = \mathrm{Re}\,(\boldsymbol{\omega}\mathbf{u}) \quad \text{for all} \quad \boldsymbol{\omega} \in \mathcal{V}^\dagger,\ \mathbf{u} \in \mathcal{V}. \tag{89.13}$$

It is clear that Φ is linear. By (22.3) and (89.6) we have

$$\begin{aligned}((\mathbf{J}^\top\Phi)\boldsymbol{\omega})\mathbf{u} &= (\Phi\boldsymbol{\omega})(\mathbf{Ju}) = \Phi\boldsymbol{\omega}(\mathrm{i}\,\mathbf{u}) \\ &= \Phi(\mathrm{i}\,(\boldsymbol{\omega}\mathbf{u})) = ((\Phi(\mathrm{i}\,\mathbf{1}_{\mathcal{V}^\dagger}))\boldsymbol{\omega})\mathbf{u}\end{aligned}$$

for all $\boldsymbol{\omega} \in \mathcal{V}^\dagger$ and all $\mathbf{u} \in \mathcal{V}$, and hence

$$\mathbf{J}^\top \Phi = \Phi(\mathrm{i}\,\mathbf{1}_{\mathcal{V}^\dagger}).$$

Since $\mathbf{J}^\top$ is the complexor of $\mathcal{V}^*$ and $\mathrm{i}\,\mathbf{1}_{\mathcal{V}^\dagger}$ the complexor of $\mathcal{V}^\dagger$, it follows from (89.5) that Φ is $\mathbb{C}$-linear.

We now *define* $\Gamma : \mathcal{V}^* \to \mathcal{V}^\dagger$ by (89.12). Using (89.13), we then have

$$\begin{aligned} ((\Gamma\Phi)\boldsymbol{\omega})\mathbf{u} &= (\Gamma(\Phi\boldsymbol{\omega}))\mathbf{u} = (\Phi\boldsymbol{\omega})\mathbf{u} - \mathrm{i}\,(\Phi\boldsymbol{\omega})(\mathbf{J}\mathbf{u}) \\ &= \mathrm{Re}\,(\boldsymbol{\omega}\mathbf{u}) - \mathrm{i}\,\mathrm{Re}\,(\boldsymbol{\omega}(\mathbf{J}\mathbf{u})) = \mathrm{Re}\,(\boldsymbol{\omega}\mathbf{u}) - \mathrm{i}\,\mathrm{Re}\,(\mathrm{i}\,(\boldsymbol{\omega}\mathbf{u})) \\ &= \mathrm{Re}\,(\boldsymbol{\omega}\mathbf{u}) + \mathrm{i}\,\mathrm{Im}\,(\boldsymbol{\omega}\mathbf{u}) = \boldsymbol{\omega}\mathbf{u} \end{aligned}$$

for all $\boldsymbol{\omega} \in \mathcal{V}^\dagger$ and all $\mathbf{u} \in \mathcal{V}$. It follows that $\Gamma\Phi = 1_{\mathcal{V}^\dagger}$. In a similar way, one proves that $\Phi\Gamma = 1_{\mathcal{V}^*}$ and hence that $\Gamma = \Phi^{-1}$. ∎

Definition 2: *A* **unitary space** *is an inner-product space $\mathcal{V}$ endowed with additional structure by the prescription of a perpendicular turn* $\mathbf{J}$ *on* $\mathcal{V}$ (see Def.1 of Sect.87).

We say that $\mathcal{V}$ is a **genuine** *unitary space if its structure as an inner-product space is genuine.*

Since $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}}$ by Prop.1 of Sect.87, a unitary space has the structure of a complex space with complexor $\mathbf{J}$ in the sense of Def.1. The identification $\mathcal{V}^* \cong \mathcal{V}$ has to be treated with some care. The natural complex-space structure of $\mathcal{V}^*$ is determined by the complexor $\mathbf{J}^\top$ but, since $\mathbf{J}^\top = -\mathbf{J}$ for a perpendicular turn, the complex structure of $\mathcal{V}^*$, when identified with $\mathcal{V}$, is the *conjugate-complex* structure of $\mathcal{V}$, not the original complex structure. The identification mapping $(\mathbf{v} \mapsto \mathbf{v}\cdot) : \mathcal{V} \to \mathcal{V}^*$ (see Sect.41) is not $\mathbb{C}$-linear but conjugate-linear.

We define the **unitary product** $\mathrm{up} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ of a unitary space $\mathcal{V}$ by

$$\mathrm{up}(\mathbf{u}, \mathbf{v}) := \Gamma(\mathbf{v}\cdot)\mathbf{u} \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}, \tag{89.14}$$

where $\Gamma \in \mathrm{Lis}^{\mathbb{C}}(\mathcal{V}^*, \mathrm{Lin}^{\mathbb{C}}(\mathcal{V}, \mathbb{C}))$ is the natural isomorphism given by (89.12). In view of the remarks above, $\mathrm{up}(\cdot, \mathbf{v}) : \mathcal{V} \to \mathbb{C}$ is $\mathbb{C}$-linear for all $\mathbf{v} \in \mathcal{V}$ and $\mathrm{up}(\mathbf{u}, \cdot) : \mathcal{V} \to \mathbb{C}$ is conjugate-linear for all $\mathbf{u} \in \mathcal{V}$. We use the simplified notation

$$\langle \mathbf{u} | \mathbf{v} \rangle := \mathrm{up}(\mathbf{u}, \mathbf{v}) \quad \text{when} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{89.15}$$

It follows from (89.15), (89.14), and (89.12) that

$$\begin{aligned} \langle \mathbf{u} | \mathbf{v} \rangle &= \mathbf{v} \cdot \mathbf{u} - \mathrm{i}\,(\mathbf{v} \cdot \mathbf{J}\mathbf{u}) = \mathbf{u} \cdot \mathbf{v} + \mathrm{i}\,(\mathbf{u} \cdot \mathbf{J}\mathbf{v}), \\ \mathrm{Re}\,\langle \mathbf{u} | \mathbf{v} \rangle &= \mathbf{u} \cdot \mathbf{v}, \qquad \langle \mathbf{u} | \mathbf{u} \rangle = \mathbf{u}^{\cdot 2} \end{aligned} \tag{89.16}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. The properties of up are reflected in the following rules, valid for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}$ and all $\zeta \in \mathbb{C}$:

$$\begin{aligned} \langle \mathbf{v}|\mathbf{u}\rangle &= \overline{\langle \mathbf{u}|\mathbf{v}\rangle}, & (89.17)\\ \langle \mathbf{u}+\mathbf{v}|\mathbf{w}\rangle &= \langle \mathbf{u}|\mathbf{w}\rangle + \langle \mathbf{v}|\mathbf{w}\rangle, & (89.18)\\ \langle \zeta\mathbf{u}|\mathbf{v}\rangle &= \zeta\langle \mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{u}|\overline{\zeta}\mathbf{v}\rangle. & (89.19)\end{aligned}$$

Let $\mathcal{V}$ be a unitary space. The following result, which is easily proved, describes the properties of transposition of $\mathbb{C}$-lineons.

Proposition 2: *If* $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ *then* $\mathbf{L}^{\top} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ *and* $\mathbf{L}^{\top}$ *is characterized by*

$$\langle \mathbf{L}\mathbf{u}|\mathbf{v}\rangle = \langle \mathbf{u}|\mathbf{L}^{\top}\mathbf{v}\rangle \quad \text{for all} \quad \mathbf{u}, \mathbf{v} \in \mathcal{V}. \tag{89.20}$$

The mapping $(\mathbf{L} \mapsto \mathbf{L}^{\top}) : \mathrm{Lin}^{\mathbb{C}}\mathcal{V} \to \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ *is conjugate-linear, so that*

$$(\zeta\mathbf{L})^{\top} = \overline{\zeta}\mathbf{L}^{\top} \quad \text{for all} \quad \zeta \in \mathbb{C},\ \mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}. \tag{89.21}$$

We use the notation

$$\mathrm{Sym}^{\mathbb{C}}\mathcal{V} = \mathrm{Sym}\mathcal{V} \cap \mathrm{Lin}^{\mathbb{C}}\mathcal{V} \tag{89.22}$$

for the set of all symmetric $\mathbb{C}$-lineons. This set is a subspace, but *not* a $\mathbb{C}$-subspace, of $\mathrm{Lin}^{\mathbb{C}}\mathcal{V}$. In fact, the following result shows how $\mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ can be recovered from $\mathrm{Sym}^{\mathbb{C}}\mathcal{V}$:

Theorem on Real and Imaginary Parts: *Let* $\mathcal{V}$ *be a unitary space. We have*

$$\mathrm{i}\,\mathrm{Sym}^{\mathbb{C}}\mathcal{V} = \mathrm{Skew}\mathcal{V} \cap \mathrm{Lin}^{\mathbb{C}}\mathcal{V} \tag{89.23}$$

and $\mathrm{i}\,\mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ *is a supplement of* $\mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ *in* $\mathrm{Lin}^{\mathbb{C}}\mathcal{V}$*, i.e. to every* $\mathbb{C}$*-lineon* $\mathbf{L}$ *corresponds a unique pair* $(\mathbf{H}, \mathbf{G})$ *such that*

$$\mathbf{L} = \mathbf{H} + \mathrm{i}\,\mathbf{G}, \quad \text{and} \quad \mathbf{H}, \mathbf{G} \in \mathrm{Sym}^{\mathbb{C}}\mathcal{V}. \tag{89.24}$$

$\mathbf{H}$ *is called the* **real part** *and* $\mathbf{G}$ *the* **imaginary part** *of* $\mathbf{L}$.

Proof: Let $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ be given. By (89.21) we have $(\mathrm{i}\,\mathbf{L})^{\top} = -\mathrm{i}\,\mathbf{L}^{\top}$ and hence $\mathbf{L} = -\mathbf{L}^{\top}$ if and only if $(\mathrm{i}\,\mathbf{L}) = (\mathrm{i}\,\mathbf{L})^{\top}$. Since $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ was arbitrary, (89.23) follows.

Again let $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ be given and assume that (89.24) holds. Since $\mathrm{i}\,\mathbf{G} \in \mathrm{Skew}\mathcal{V}$ by (89.23), it follows that $(\mathbf{H}, \mathrm{i}\,\mathbf{G})$ must be the additive decomposition of $\mathbf{L}$ described in the Additive Decomposition Theorem of Sect.86 and hence that

$$\mathbf{H} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^{\top}),\ \ \mathbf{G} = \frac{\mathrm{i}}{2}(\mathbf{L}^{\top} - \mathbf{L}), \tag{89.25}$$

which proves the uniqueness of $(\mathbf{H}, \mathbf{G})$. To prove existence, we *define* $\mathbf{H}$ and $\mathbf{G}$ by (89.25) and observe, using (89.21), that they have the required properties. ∎

Since $\mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ and $\mathrm{i}\,\mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ are linearly isomorphic subspaces of $\mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ we can use Props.5 and 7 of Sect.17 to obtain the following consequence of the Theorem.

Corollary: *Let* $n := \dim^{\mathbb{C}}\mathcal{V} = \frac{1}{2}\dim\mathcal{V}$. *Then* $\dim \mathrm{Lin}^{\mathbb{C}}\mathcal{V} = 2\dim^{\mathbb{C}} \mathrm{Lin}^{\mathbb{C}}\mathcal{V} = 2n^2$ *and* $\dim \mathrm{Sym}^{\mathbb{C}}\mathcal{V} = \dim(\mathrm{Skew}\mathcal{V} \cap \mathrm{Lin}^{\mathbb{C}}\mathcal{V}) = n^2$.

Let $\mathcal{V}, \mathcal{V}'$ be complex spaces. We say that a mapping $\mathbf{U} : \mathcal{V} \to \mathcal{V}'$ is **unitary** if it is orthogonal and $\mathbb{C}$-linear. It is easily seen that $\mathbf{U} : \mathcal{V} \to \mathcal{V}'$ is unitary if and only if it is linear and preserves the unitary product defined by (89.14). The set of all unitary mappings from $\mathcal{V}$ to $\mathcal{V}'$ will be denoted by $\mathrm{Unit}(\mathcal{V}, \mathcal{V}')$, so that

$$\mathrm{Unit}(\mathcal{V}, \mathcal{V}') := \mathrm{Orth}(\mathcal{V}, \mathcal{V}') \cap \mathrm{Lin}^{\mathbb{C}}(\mathcal{V}, \mathcal{V}') \tag{89.26}$$

If $\mathbf{U}$ is unitary and invertible, then $\mathbf{U}^{-1}$ is again unitary (see Prop.1 of Sect.43). In this case, $\mathbf{U}$ is called a **unitary isomorphism**. We write $\mathrm{Unit}\mathcal{V} := \mathrm{Unit}(\mathcal{V}, \mathcal{V})$. It is clear that $\mathrm{Unit}\mathcal{V}$ is a subgroup of $\mathrm{Orth}\mathcal{V}$ and hence of $\mathrm{Lis}\mathcal{V}$. The group $\mathrm{Unit}\mathcal{V}$ is called the **unitary group** of the unitary space $\mathcal{V}$.

Remark: Let I be a finite index set. The linear space $\mathbb{C}^I$ over $\mathbb{C}$ (see Sect.14) has the natural structure of a unitary space whose inner square is given by

$$\lambda^{\cdot 2} := \sum_{i \in I} |\lambda_i|^2 \quad \text{for all} \quad \lambda \in \mathbb{C}^i. \tag{89.27}$$

Of course, the complexor is termwise multiplication by i. The unitary product is given by

$$\langle \lambda | \mu \rangle = \sum_{i \in I} \lambda_i \overline{\mu}_i \quad \text{for all} \quad \lambda, \mu \in \mathbb{C}^I \tag{89.28}$$

∎

Notes 89

(1) As far as I know, the treatment of complex spaces as real linear spaces with additional structure appears nowhere else in the literature. The textbooks just deal with linear spaces over $\mathbb{C}$. I believe that the approach taken here provides better insight, particularly into conjugate-linear structures and mappings.

(2) The term "complexor" is introduced here for the first time.

(3) Some people use the term "anti-linear" or the term "semi-linear" instead of "conjugate-linear".

(4) Most textbooks deal only with what we call "genuine" unitary spaces and use the term "unitary space" only in this restricted sense. The term "complex inner-product space" is also often used in this sense.

(5) What we call "unitary product" is very often called "complex inner-product" or "Hermitian".

(6) The notations $\langle \mathbf{u}, \mathbf{v} \rangle$ and $(\mathbf{u}, \mathbf{v})$ are often used for the value $\langle \mathbf{u} \,|\, \mathbf{v} \rangle$ of the unitary product. (See also Note (3) to Sect.41.)

(7) In most of the literature on theoretical physics, the order of $\mathbf{u}, \mathbf{v}$ in the notation $\langle \mathbf{u} \,|\, \mathbf{v} \rangle$ for the unitary product is reversed. Therefore, in textbooks on theoretical physics, (89.19) is replaced by

$$\langle \mathbf{u} \,|\, \zeta \mathbf{v} \rangle = \zeta \langle \mathbf{u} \,|\, \mathbf{v} \rangle = \langle \overline{\zeta} \mathbf{u} \,|\, \mathbf{v} \rangle.$$

(8) Symmetric $\mathbb{C}$-lineons are very often called "self-adjoint" or "Hermitian". I do not think a special term is needed.

(9) Skew $\mathbb{C}$-lineons are often called "skew-Hermitian" or "anti-Hermitian".

(10) The notations U_n or $\mathrm{U}(n)$ are often used for the unitary group $\mathrm{Unit}\mathbb{C}^n$.

810 Complex Spectra

Let $\mathcal{V}$ be a complex space. As we have seen in Sect.89, $\mathcal{V}$ can be regarded both as a linear space over $\mathbb{R}$ and as a linear space over $\mathbb{C}$. We modify Def.2 of Sect.82 as follows: Let $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ be given. For every $\zeta \in \mathbb{C}$, we write

$$\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta) := \mathrm{Null}\,(\mathbf{L} - \zeta \mathbf{1}_{\mathcal{V}}), \tag{810.1}$$

and, if it is non-zero, call it the **spectral $\mathbb{C}$-space** of $\mathbf{L}$ for ζ. Of course, if ζ is real, we have $\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta) = \mathrm{Sps}_{\mathbf{L}}(\zeta)$.

The spectral $\mathbb{C}$-spaces are $\mathbb{C}$-subspaces of $\mathcal{V}$. The **$\mathbb{C}$-spectrum** of $\mathbf{L}$ is defined by

$$\mathrm{Spec}^{\mathbb{C}}\mathbf{L} := \{\zeta \in \mathbb{C} \mid \mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta) \neq \{\mathbf{0}\}\}. \tag{810.2}$$

It is clear that

$$\mathrm{Spec}\,\mathbf{L} = (\mathrm{Spec}^{\mathbb{C}}\mathbf{L}) \cap \mathbb{R}. \tag{810.3}$$

The **$\mathbb{C}$-multiplicity** function $\mathrm{mult}^{\mathbb{C}}_{\mathbf{L}} : \mathbb{C} \to \mathbb{N}$ of $\mathbf{L}$ is defined by

$$\mathrm{mult}^{\mathbb{C}}_{\mathbf{L}}(\zeta) := \dim^{\mathbb{C}}(\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta)) \quad \text{for all} \quad \zeta \in \mathbb{C}. \tag{810.4}$$

In view of (89.4) we have

$$\mathrm{mult}_{\mathbf{L}}(\sigma) = 2\mathrm{mult}^{\mathbb{C}}_{\mathbf{L}}(\sigma) \quad \text{for all} \quad \sigma \in \mathbb{R}. \tag{810.5}$$

$\mathrm{Spec}^{\mathbb{C}}\mathbf{L}$ is the support of $\mathrm{mult}^{\mathbb{C}}_{\mathbf{L}}$.

The Theorem on Spectral Spaces of Sect.82 has the following analogue, which is proved in the same way.

Proposition 1: *The $\mathbb{C}$-spectrum of a $\mathbb{C}$-lineon $\mathbf{L}$ on a complex space $\mathcal{V}$ has at most $\dim^{\mathbb{C}}\mathcal{V}$ members and the family $(\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta) \mid \zeta \in \mathrm{Spec}^{\mathbb{C}}\mathbf{L})$ of the spectral $\mathbb{C}$-spaces of $\mathbf{L}$ is disjunct.*

The following are analogues of Prop.5 and Prop.6 of Sect.82 and are proved in a similar manner.

Proposition 2: *Let $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ be given and assume that the family $(\mathrm{Sps}_{\mathbf{L}}(\zeta) \mid \zeta \in \mathrm{Spec}^{\mathbb{C}}\mathbf{L})$ of spectral $\mathbb{C}$-spaces is a decomposition of $\mathcal{V}$. If $(\mathbf{E}_\zeta \mid \zeta \in \mathrm{Spec}^{\mathbb{C}}\mathbf{L})$ is the associated family of idempotents then*

$$\mathbf{L} = \sum_{\zeta \in \mathrm{Spec}^{\mathbb{C}}\mathbf{L}} \zeta \mathbf{E}_\zeta. \tag{810.6}$$

Proposition 3: *Let $\mathbf{L} \in \mathrm{Lin}^{\mathbb{C}}\mathcal{V}$ be given. Assume that Z is a finite subset of $\mathbb{C}$ and that $(\mathcal{W}_\zeta \mid \zeta \in Z)$ is a decomposition of $\mathcal{V}$ whose terms are non-zero $\mathbf{L}$-invariant $\mathbb{C}$-subspaces of $\mathcal{V}$ and satisfy*

$$\mathbf{L}_{|\mathcal{W}_\zeta} = \zeta \mathbf{1}_{\mathcal{W}_\zeta} \quad \text{for all} \quad \zeta \in Z. \tag{810.7}$$

Then $Z = \mathrm{Spec}^{\mathbb{C}}\mathbf{L}$ and $\mathcal{W}_\zeta = \mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}}(\zeta)$ for all $\zeta \in Z$.

From now on we assume that $\mathcal{V}$ is a *genuine* unitary space.

Proposition 4: *The $\mathbb{C}$-spectrum of a symmetric $\mathbb{C}$-lineon $\mathbf{S} \in \mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ contains only real numbers, i.e.* $\mathrm{Spec}^{\mathbb{C}}\mathbf{S} = \mathrm{Spec}\,\mathbf{S}$.

Proof: Let $\zeta \in \mathrm{Spec}^{\mathbb{C}}\mathbf{S}$, so that $\mathbf{S}\mathbf{u} = \zeta\mathbf{u}$ for some $\mathbf{u} \in \mathcal{V}^\times$. Using $\mathbf{S} = \mathbf{S}^\top$, (89.20), (89.19), and $(89.16)_4$ we find

$$\begin{aligned} 0 &= \langle \mathbf{S}\mathbf{u} \mid \mathbf{u}\rangle - \langle \mathbf{u} \mid \mathbf{S}\mathbf{u}\rangle = \langle \zeta\mathbf{u} \mid \mathbf{u}\rangle - \langle \mathbf{u} \mid \zeta\mathbf{u}\rangle \\ &= (\zeta - \overline{\zeta})\langle \mathbf{u} \mid \mathbf{u}\rangle = (\zeta - \overline{\zeta})\mathbf{u}^{\cdot 2}. \end{aligned}$$

Since $\mathcal{V}$ is genuine, we must have $\mathbf{u}^{\cdot 2} \neq 0$ and hence $\zeta = \overline{\zeta}$, i.e. $\zeta \in \mathbb{R}$. ∎

Remark: A still shorter proof of Prop.4 can be obtained by applying the Spectral Theorem and Prop.2, but the proof above is more elementary in that it makes no use of the Spectral Theorem. ∎

Let $\mathbf{S} \in \mathrm{Sym}^{\mathbb{C}}\mathcal{V}$ be given. By Prop.4, the family of spectral $\mathbb{C}$-spaces of $\mathbf{S}$ coincides with the family of spectral spaces in the sense of Def.1 of Sect.82.

This family consists of $\mathbb{C}$-subspaces of $\mathcal{V}$ and, by the Spectral Theorem, is an orthogonal decomposition of $\mathcal{V}$. All the terms $\mathbf{E}_\sigma$ in the spectral resolution of $\mathbf{S}$ (see Cor.1 to the Spectral Theorem) are $\mathbb{C}$-linear. There exists a $\mathbb{C}$-basis $\mathbf{e} := (\mathbf{e}_i \mid i \in I)$ of $\mathcal{V}$ and a family $\lambda = (\lambda_i \mid i \in I)$ in $\mathbb{R}$ such that (84.6) holds. Each $\sigma \in \operatorname{Spec}\mathbf{S}$ occurs exactly $\operatorname{mult}_{\mathbf{S}}^{\mathbb{C}}(\sigma)$ times in the family λ.

For $\mathbb{C}$-linear normal lineons, the Structure Theorem of Sect.88 can be replaced by the following simpler version.

Spectral Theorem for Normal $\mathbb{C}$-Lineons: *Let $\mathcal{V}$ be a genuine unitary space. A $\mathbb{C}$-lineon on $\mathcal{V}$ is normal if and only if the family of its spectral $\mathbb{C}$-spaces is an orthogonal decomposition of $\mathcal{V}$.*

Proof: Let $\mathbf{N} \in \operatorname{Lin}^{\mathbb{C}}\mathcal{V}$ be normal. Let $\mathbf{H}, \mathbf{G} \in \operatorname{Sym}^{\mathbb{C}}\mathcal{V}$ be the real and imaginary parts of $\mathbf{N}$ as described in the Theorem on Real and Imaginary Parts of Sect.89, so that $\mathbf{N} = \mathbf{H} + \mathrm{i}\,\mathbf{G}$. From this Theorem and Prop.1 of Sect.88 it follows that $\mathbf{H}$ and $\mathbf{G}$ commute. Let $Z := \operatorname{Spec}\mathbf{H} = \operatorname{Spec}^{\mathbb{C}}\mathbf{H}$. By Prop.2 of Sect.82, $\mathcal{U}_\sigma := \operatorname{Sps}_{\mathbf{H}}(\sigma)$ is $\mathbf{G}$-invariant for each $\sigma \in Z$. Hence we may consider

$$\mathbf{D}_\sigma := \mathbf{G}_{|\mathcal{U}_\sigma} \quad \text{for all} \quad \sigma \in Z.$$

It is clear that $\mathbf{D}_\sigma \in \operatorname{Sym}^{\mathbb{C}}\mathcal{U}_\sigma$ for all $\sigma \in Z$. Put $Q_\sigma := \operatorname{Spec}\mathbf{D}_\sigma = \operatorname{Spec}^{\mathbb{C}}\mathbf{D}_\sigma$ and $\mathcal{W}_{(\sigma,\tau)} := \operatorname{Sps}_{\mathbf{D}_\sigma}(\tau)$ for all $\sigma \in Z$ and all $\tau \in Q_\sigma$. It is easily seen that $\mathcal{W}_{(\sigma,\tau)}$ is $\mathbf{N}$-invariant and that

$$\mathbf{N}_{|\mathcal{W}_{(\sigma,\tau)}} = \sigma\mathbf{1}_{\mathcal{W}_{(\sigma,\tau)}} + \mathrm{i}\,\tau\mathbf{1}_{\mathcal{W}_{(\sigma,\tau)}} = (\sigma + \mathrm{i}\,\tau)\mathbf{1}_{\mathcal{W}_{(\sigma,\tau)}}$$

for all $\sigma \in Z$ and all $\tau \in Q_\sigma$. Also, the family $(\mathcal{W}_{(\sigma,\tau)} \mid \sigma \in Z, \tau \in Q_\sigma)$ is an orthogonal decomposition of $\mathcal{V}$ whose terms are $\mathbb{C}$-subspaces of $\mathcal{V}$. It follows from Prop.3 that $\operatorname{Spec}^{\mathbb{C}}\mathbf{N} = \{\sigma + \mathrm{i}\,\tau \mid \sigma \in Z, \tau \in Q_\sigma\}$ and that $\operatorname{Sps}_{\mathbf{N}}(\zeta) = \mathcal{W}_{(\operatorname{Re}\zeta, \operatorname{Im}\zeta)}$ for all $\zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{N}$.

Assume now that $\mathbf{L} \in \operatorname{Lin}^{\mathbb{C}}\mathcal{V}$ is such that $(\operatorname{Sps}_{\mathbf{L}}(\zeta) \mid \zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{L})$ is an orthogonal decomposition of $\mathcal{V}$. We can apply Prop.2 and conclude that (810.6) holds. By Prop.4 of Sect.83, each of the idempotents $\mathbf{E}_\zeta$, $\zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{L}$, belongs to $\operatorname{Sym}^{\mathbb{C}}\mathcal{V}$. Using this fact and (89.21) we derive from (810.6) that

$$\mathbf{L}^\top = \sum_{\zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{L}} \overline{\zeta}\mathbf{E}_\zeta$$

and hence, by (81.6), that

$$\mathbf{L}\mathbf{L}^\top = \sum_{\zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{L}} (\zeta\overline{\zeta})\mathbf{E}_\zeta = \mathbf{L}^\top\mathbf{L},$$

i.e. that $\mathbf{L}$ is normal. ■

The $\mathbb{C}$-spectrum of a normal $\mathbb{C}$-lineon is related to its pair-spectrum (see Sect.88) by

$$\operatorname{Pspec}\mathbf{N} = \{(\operatorname{Re}\zeta, |\operatorname{Im}\zeta|) \mid \zeta \in \operatorname{Spec}^{\mathbb{C}}\mathbf{N}\}. \tag{810.8}$$

A normal $\mathbb{C}$-lineon is symmetric, skew, or unitary depending on whether its $\mathbb{C}$-spectrum consists only of real numbers, imaginary numbers, or numbers of absolute value 1, respectively. The angle-spectrum of a unitary lineon $\mathbf{U}$ is related to its $\mathbb{C}$-spectrum by

$$\operatorname{Aspec}\mathbf{U} = \{\varphi \in \,]0,\pi[\; \mid \; e^{\mathrm{i}\varphi} \in \operatorname{Spec}^{\mathbb{C}}\mathbf{U} \quad \text{or} \quad e^{-\mathrm{i}\varphi} \in \operatorname{Spec}^{\mathbb{C}}\mathbf{U}\}. \tag{810.9}$$

The Spectral Theorem for Normal $\mathbb{C}$-lineons has two corollaries that are analogues of Cors.1 and 2 to the Spectral Theorem in Sect.84. We leave the formulation of these corollaries to the reader.

811 Problems for Chapter 8

(1) Let $\mathcal{V}$ be a linear space, let $(\mathcal{U}_i \mid i \in I)$ be a finite family of subspaces of $\mathcal{V}$, let Π be a partition of I (see Sect.01) and put

$$\mathcal{W}_J := \textstyle\sum_{j\in J}\mathcal{U}_j \quad \text{for all} \quad J \in \Pi \tag{P8.1}$$

(see (07.14)). Prove that the family $(\mathcal{U}_i \mid i \in I)$ is disjunct [a decomposition of $\mathcal{V}$] if and only if the family $(\mathcal{W}_J \mid J \in \Pi)$ is disjunct [a decomposition of $\mathcal{V}$] and, for each $J \in \Pi$, the family $(\mathcal{U}_j \mid j \in J)$ is disjunct.

(2) Let $(\mathbf{E}_i \mid i \in I)$ be a finite family of lineons on a given linear space $\mathcal{V}$.

(a) Show: If (81.5) and (81.6) hold, then $(\operatorname{Rng}\mathbf{E}_i \mid i \in I)$ is a decomposition of $\mathcal{V}$ and $(\mathbf{E}_i \mid i \in I)$ is the family of idempotents associated with this decomposition.

(b) Prove: If (81.5) alone holds and if $\mathbf{E}_i$ is idempotent for every $i \in I$, then $(\operatorname{Rng}\mathbf{E}_i \mid i \in I)$ is already a decomposition of $\mathcal{V}$ and hence (81.6) follows. (Hint: Apply Prop.4 of Sect.81.)

(3) Let $\mathcal{V}$ be a genuine inner-product space and let $\mathbf{S}$ and $\mathbf{T}$ be symmetric lineons on $\mathcal{V}$.

(a) Show that $\mathbf{ST}$ is symmetric if and only if $\mathbf{S}$ and $\mathbf{T}$ commute.

(b) Prove: If $\mathbf{S}$ and $\mathbf{T}$ commute then

$$\operatorname{Spec}(\mathbf{ST}) \subset \{\sigma\tau \mid \sigma \in \operatorname{Spec}\mathbf{S}, \tau \in \operatorname{Spec}\mathbf{T}\}. \tag{P8.2}$$

(4) Let $\mathcal{V}$ be a genuine inner-product space and let $\mathbf{S}$ be a symmetric lineon on $\mathcal{V}$. Show that the operator norm of $\mathbf{S}$, as defined by (52.19), is given by

$$||\mathbf{S}|| = \max\{|\sigma| \mid \sigma \in \operatorname{Spec}\mathbf{S}\}. \tag{P8.3}$$

(5) Let a genuine Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$, a point $q \in \mathcal{E}$, and a strictly positive symmetric lineon $\mathbf{S} \in \operatorname{Pos}^+\mathcal{V}$ be given. Then the set

$$\mathcal{S} := \{x \in \mathcal{E} \mid \overline{\mathbf{S}}(x-q) = 1\}$$

is called an **ellipsoid** centered at q, and the spectral spaces of $\mathbf{S}$ are called the **principal directions** of the ellipsoid. Put$\mathbf{T} := (\sqrt{\mathbf{S}})^{-1}$ (see Sect.85). The spectral values of $\mathbf{T}$ are called the **semi-axes** of $\mathcal{S}$.

(a) Show that

$$\mathcal{S} = q + \mathbf{T}_{>}(\operatorname{Usph}\mathcal{V}). \tag{P8.4}$$

(see (42.9)).

(b) Show: If $\mathbf{e}$ is a spectral unit vector of $\mathbf{S}$, then

$$\mathcal{S} \cap (q + \mathbb{R}\mathbf{e}) = \{q - a\mathbf{e}, q + a\mathbf{e}\}, \tag{P8.5}$$

where a is a semi-axis of $\mathcal{S}$.

(c) Let $\mathfrak{e}$ be an orthonormal basis-set whose terms are spectral vectors of $\mathbf{S}$ (see Cor.2 to the Spectral Theorem). Let Γ be the Cartesian coordinate system with origin q which satisfies $\{\nabla c \mid c \in \Gamma\} = \mathfrak{e}$ (see Sect.74). Show that

$$\mathcal{S} = \left\{ x \in \mathcal{E} \;\middle|\; .\sum_{c\in\Gamma} \tfrac{c(x)^2}{a_c^2} = 1 \right\}, \tag{P8.6}$$

where $(a_c \mid c \in \Gamma)$ is a family whose terms are the semi-axes of the ellipsoid $\mathcal{S}$.

(d) Let $p \in \mathcal{E}$ and $\rho \in \mathbb{P}^{\times}$ be given and let $\alpha : \mathcal{E} \to \mathcal{E}$ be an invertible flat mapping. Show that the image $\alpha_{>}(\mathrm{Sph}_{p,\rho}\mathcal{E})$ under α of the sphere $\mathrm{Sph}_{p,\rho}\mathcal{E}$ (see (46.11)) is an ellipsoid centered at $\alpha(p)$ whose semi-axes form the set

$$\left\{\rho\sqrt{\sigma} \;\middle|\; \sigma \in \mathrm{Spec}\left((\nabla\alpha)^{\top}\nabla\alpha\right)\right\}.$$

(6) Let $\mathbf{S}$ be a symmetric lineon on a genuine inner-product space $\mathcal{V}$ and let $(\mathbf{E}_\sigma \mid \sigma \in \mathrm{Spec}\,\mathbf{S})$ be the spectral resolution of $\mathbf{S}$.

(a) Show that, for all $\sigma \in \mathrm{Spec}\,\mathbf{S}$, we have

$$\int_0^1 \exp_{\mathcal{V}} \circ (\iota(\mathbf{S} - \sigma \mathbf{1}_{\mathcal{V}})) = \mathbf{E}_\sigma + \sum_{\tau \in \mathrm{Spec}\,\mathbf{S}\setminus\{\sigma\}} \tfrac{\mathrm{e}^{\tau-\sigma}-1}{\tau-\sigma}\mathbf{E}_\tau. \qquad \text{(P8.7)}$$

(b) Show that, for every $\mathbf{M} \in \mathrm{Lin}\mathcal{V}$, every $\sigma \in \mathrm{Spec}\,\mathbf{S}$, and every $\mathbf{v} \in \mathrm{Sps}_{\mathbf{S}}(\sigma)$,

$$((\nabla_{\mathbf{S}} \exp_{\mathcal{V}})\mathbf{M})\mathbf{v} = \left(\mathrm{e}^{\sigma}\mathbf{E}_\sigma + \sum_{\tau \in \mathrm{Spec}\,\mathbf{S}\setminus\{\sigma\}} \tfrac{\mathrm{e}^{\tau}-\mathrm{e}^{\sigma}}{\tau-\sigma}\mathbf{E}_\tau\right)\mathbf{M}\mathbf{v}. \qquad \text{(P8.8)}$$

(c) Show that $\exp_{\mathcal{V}}$ is locally invertible near $\mathbf{S}$.

(d) Prove that the lineonic logarithm $\log_{(\mathcal{V})}$ defined in Sect.85 is of class C^1.

(7) Consider $L \in \mathrm{Lin}\,\mathbb{R}^3$ and $T \in \mathrm{Sym}\,\mathbb{R}^3$ as given by

$$L := \frac{1}{5}\begin{bmatrix} 10 & 0 & 5 \\ 3 & 8 & 6 \\ 4 & -6 & 8 \end{bmatrix}, \quad T := \begin{bmatrix} 5 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 5 \end{bmatrix}.$$

(a) Determine the spectrum and the spectral idempotents of T.

(b) Determine the right and left polar decompositions of L (see Sect.86).

(c) Find the spectrum and angle-spectrum of $R := \mathrm{or}(L) \in \mathrm{Orth}\,\mathbb{R}^3$ (see Sect.88).

(8) Let $\mathbf{L}$ be an invertible lineon on a genuine inner-product space $\mathcal{V}$, and let $(\mathbf{R}, \mathbf{S})$ be the right polar decomposition and $(\mathbf{R}, \mathbf{T})$ the left polar decomposition of $\mathbf{L}$. Show that the following are equivalent:

(i) $\mathbf{L} \in \operatorname{Skew}\mathcal{V}$,

(ii) $\mathbf{R}$ is a perpendicular turn that commutes with $\mathbf{S}$ or with $\mathbf{T}$,

(iii) $\mathbf{S} = \mathbf{T}$ and $\mathbf{R} \in \operatorname{Skew}\mathcal{V}$,

(iv) $\operatorname{Sps}_{\mathbf{S}}(\kappa) = \operatorname{Qsps}_{\mathbf{L}}(\kappa)$ for all $\kappa \in \operatorname{Spec}\mathbf{S}$.

(9) Let $\mathcal{V}$ be a genuine inner-product space, let $I \in \operatorname{Sub}\mathbb{R}$ be a genuine interval, and let $\mathbf{F} : I \to \operatorname{Lis}\mathcal{V}$ be a lineonic process of class C^1 with invertible values. Define $\mathbf{L} : I \to \operatorname{Lin}\mathcal{V}$ by

$$\mathbf{L}(t) := \mathbf{F}^{\bullet}(t)\mathbf{F}(t)^{-1} \quad \text{for all} \quad t \in I, \tag{P8.9}$$

and define $\mathbf{D} : I \to \operatorname{Sym}\mathcal{V}$ and $\mathbf{W} : I \to \operatorname{Skew}\mathcal{V}$ by the requirements that $(\mathbf{D}(t), \mathbf{W}(t))$ be the additive decomposition of $\mathbf{L}(t)$ for all $t \in I$ (see Sect.86). For each $t \in I$, define $\mathbf{F}_{(t)} : I \to \operatorname{Lis}\mathcal{V}$ by

$$\mathbf{F}_{(t)}(s) := \mathbf{F}(s)\mathbf{F}(t)^{-1} \quad \text{for all} \quad s \in I \tag{P8.10}$$

and put $\mathbf{R}_{(t)} := \operatorname{or} \circ \mathbf{F}_{(t)}$, $\mathbf{U}_{(t)} := \operatorname{rp} \circ \mathbf{F}_{(t)}$, $\mathbf{V}_{(t)} := \operatorname{lp} \circ \mathbf{F}_{(t)}$ (see Sect.86). Prove that

$$\mathbf{R}^{\bullet}_{(t)}(t) = \mathbf{W}(t), \;\; \mathbf{U}^{\bullet}_{(t)}(t) = \mathbf{V}^{\bullet}_{(t)}(t) = \mathbf{D}(t) \quad \text{for all} \quad t \in I \tag{P8.11}$$

(Hint: Use Prop.1 of Sect.86).

(10) Let $\mathcal{V}$ be a genuine inner-product space. Prove: If $\mathbf{N} \in \operatorname{Lin}\mathcal{V}$ is normal and satisfies $\mathbf{N}^2 = -\mathbf{1}_{\mathcal{V}}$, then $\mathbf{N}$ is a perpendicular turn (see Def.1 of Sect.87 and Def.1 of Sect.88).

(11) Let $\mathbf{S}$ be a lineon on a genuine inner product space $\mathcal{V}$. Prove that the following are equivalent:

(i) $\mathbf{S} \in \operatorname{Sym}\mathcal{V} \cap \operatorname{Orth}\mathcal{V}$,

(ii) $\mathbf{S} \in \operatorname{Sym}\mathcal{V}$ and $\operatorname{Spec}\mathbf{S} \subset \{1, -1\}$,

(iii) $\mathbf{S} \in \operatorname{Orth}\mathcal{V}$ and $\mathbf{S}^2 = \mathbf{1}_{\mathcal{V}}$,

(iv) $\mathbf{S} = \mathbf{E} - \mathbf{F}$ for some symmetric idempotents $\mathbf{E}$ and $\mathbf{F}$ that satisfy $\mathbf{EF} = \mathbf{0}$ and $\mathbf{E} + \mathbf{F} = \mathbf{1}_{\mathcal{V}}$,

(v) $\mathbf{S} \in \operatorname{Orth}\mathcal{V}$ and $\operatorname{Aspec}\mathbf{S} = \emptyset$.

(12) Let $\mathbf{U}$ be a skew lineon on a genuine inner product space. Show that $\mathbf{U}$ is a perpendicular turn if and only if $\operatorname{Null}\mathbf{U} = \{\mathbf{0}\}$ and $\operatorname{Aspec}\mathbf{U} = \left\{\frac{\pi}{2}\right\}$.

(13) Let $\mathbf{A}$ be a skew lineon on a genuine inner-product space and let $\mathbf{R} := \exp_{\mathcal{V}}(\mathbf{A})$.

(a) Show that $\mathbf{R}$ is orthogonal. (Hint: Use the Corollary to Prop.2 of Sect.66.)

(b) Show that the angle-spectrum of $\mathbf{R}$ is given, in terms of the quasi-spectrum of $\mathbf{A}$, by

$$\operatorname{Aspec}\mathbf{R} = \left\{\kappa - \pi\lfloor\tfrac{\kappa}{\pi} \;\middle|\; \kappa \in \operatorname{Qspec}\mathbf{A}\right\}, \tag{P8.12}$$

where the **floor** $\lfloor a$ of $a \in \mathbb{R}$ is defined by $\lfloor a := \max\{n \in \mathbb{Z} \mid n \le a\}$. (Hint: Use the Structure Theorems of Sects.87 and 88 and the results of Problem 9 of Chap.6)

Remark: The assertion of Part (a) is valid even if the inner-product space $\mathcal{V}$ is not genuine. ■

(14) Let $\mathcal{V}$ be a linear space and define $\mathbf{J} \in \operatorname{Lin}\mathcal{V}^2$ by

$$\mathbf{J}(\mathbf{v}_1, \mathbf{v}_2) = (-\mathbf{v}_2, \mathbf{v}_1) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}^2. \tag{P8.13}$$

(a) Show that $\mathbf{J}^2 = -\mathbf{1}_{\mathcal{V}^2}$, and hence that $\mathcal{V}^2$ has the natural structure of a complex space with complexor $\mathbf{J}$ (see Def.1 of Sect.89). The space $\mathcal{V}^2$, with this complex-space structure, is called the **complexification** of $\mathcal{V}$.

(b) Show that

$$(\xi + \mathrm{i}\,\eta)\mathbf{v} = (\xi\mathbf{v}_1 - \eta\mathbf{v}_2,\ \eta\mathbf{v}_1 + \xi\mathbf{v}_2) \tag{P8.14}$$

for all $\xi, \eta \in \mathbb{R}$ and all $\mathbf{v} \in \mathcal{V}^2$.

(c) Prove: If $\mathbf{L}, \mathbf{M} \in \operatorname{Lin}\mathcal{V}$, then the cross-product $\mathbf{L} \times \mathbf{M} \in \operatorname{Lin}\mathcal{V}^2$ (see (04.18)) is $\mathbb{C}$-linear if and only if $\mathbf{M} = \mathbf{L}$ and it is conjugate-linear if and only if $\mathbf{M} = -\mathbf{L}$ (see Sect.89).

(d) Let $\mathbf{L} \in \operatorname{Lin}\mathcal{V}$ and $\zeta \in \mathbb{C}$ be given. Show that $\zeta \in \operatorname{Spec}^{\mathbb{C}}(\mathbf{L} \times \mathbf{L})$ if and only if either $\operatorname{Im}\zeta = 0$ and $\zeta \in \operatorname{Spec}\mathbf{L}$ or else $\operatorname{Im}\zeta \ne 0$ and $(\operatorname{Re}\zeta, |\operatorname{Im}\zeta|) \in \operatorname{Pspec}\mathbf{L}$ (see Def.2 of Sect.88). Conclude that

$$\zeta \in \mathrm{Spec}^{\mathbb{C}}(\mathbf{L} \times \mathbf{L}) \iff \overline{\zeta} \in \mathrm{Spec}^{\mathbb{C}}(\mathbf{L} \times \mathbf{L}). \tag{P8.15}$$

(e) Let $\mathbf{L} \in \mathrm{Lin}\mathcal{V}$ and $\zeta \in \mathrm{Spec}^{\mathbb{C}}(\mathbf{L} \times \mathbf{L})$ be given and put $\xi := \mathrm{Re}\,\zeta$, $\eta := \mathrm{Im}\,\zeta$, so that $\zeta = \xi + \mathrm{i}\,\eta$. Show that

$$\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}\times\mathbf{L}}(\zeta) = (\mathrm{Sps}_{\mathbf{L}}(\xi))^2 \quad \text{if} \quad \eta = 0 \tag{P8.16}$$

while

$$\mathrm{Sps}^{\mathbb{C}}_{\mathbf{L}\times\mathbf{L}}(\zeta) = \left\{(\mathbf{v}, \frac{1}{\eta}(\xi\mathbf{1}_{\mathcal{V}} - \mathbf{L})\mathbf{v}) \;\middle|\; \mathbf{v} \in \mathrm{Psps}_{\mathbf{L}}(\xi, |\eta|)\right\}$$

$$\text{if } \eta \neq 0. \tag{P8.17}$$

(15) Let $\mathcal{V}$ be an inner-product space and let $\mathcal{V}^2$ be the complexification of $\mathcal{V}$ as described in Problem 14. Recall that $\mathcal{V}^2$ carries the natural structure of an inner-product space (see Sect.44).

(a) Show that the complexor $\mathbf{J} \in \mathrm{Lin}\mathcal{V}^2$ defined by (P8.14) is a perpendicular turn and hence endows $\mathcal{V}^2$ with the structure of a unitary space.

(b) Show that the unitary product of $\mathcal{V}^2$ is given by

$$\langle \mathbf{u} \,|\mathbf{v}\rangle = \mathbf{u}_1 \cdot \mathbf{v}_1 + \mathbf{u}_2 \cdot \mathbf{v}_2 + \mathrm{i}\,(\mathbf{u}_2 \cdot \mathbf{v}_1 - \mathbf{u}_1 \cdot \mathbf{v}_2) \tag{P8.18}$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}^2$.

(c) Show that $\mathbf{H} \in \mathrm{Lin}\mathcal{V}^2$ is a symmetric $\mathbb{C}$-lineon on $\mathcal{V}^2$ if and only if $\mathbf{H} = \mathbf{T} \times \mathbf{T} + \mathbf{J} \circ (\mathbf{A} \times \mathbf{A})$ for some $\mathbf{T} \in \mathrm{Sym}\mathcal{V}$ and some $\mathbf{A} \in \mathrm{Skew}\mathcal{V}$.