Chapter 5

Minkowskian Spacetimes

In this chapter, we discuss the last of the major structural ingredients used in our presentation of a theory of special relativity: the inner product. We offer general definitions and results in §**5.1**, while results germane to a theory of special relativity are relegated to §**5.2**. We are then in a position to define a Minkowskian spacetime in §**5.3**, where we bring together many of the concepts presented thus far. In §**5.4**, we discuss spacetime decompositions, with some applications presented in §**5.5**. We follow with a discussion of parameterizations of worldpaths in the context of a Minkowskian spacetime in §**5.6**, and consider in §**5.7** parameterizations of worldpaths in light of the discussion in §**5.4**. Finally, in §**5.8**, we apply these principles to an analysis of interstellar travel.

5.1 Inner-Product Spaces

Although the structure of a timed flat relativistic eventworld is very rich, there is one concept that we cannot easily describe within this structure; namely, "orthogonality". Before we see how this concept is used in the theory of special relativity, we introduce some necessary background.

5100 Definition: *An* **inner-product space** *is a linear space* $\mathcal{V}$ *(with scalar multiplication* sm*) with an additional ingredient*

$$\mathsf{ip} : \mathcal{V} \times \mathcal{V} \to \mathbb{R},$$

called the **inner product on** $\mathcal{V}$, *which satisfies the following rules, where* $\mathsf{ip}(\mathbf{u},\mathbf{v})$ *is written* $\mathbf{u}\cdot\mathbf{v}$ *for convenience, and scalar multiplication is indicated by juxtaposition:*

$$\begin{array}{llcll}
(I_1) & \mathbf{u}\cdot\mathbf{v} & = & \mathbf{v}\cdot\mathbf{u} & \text{for all } \mathbf{u},\mathbf{v}\in\mathcal{V},\\
(I_2) & \mathbf{w}\cdot(\mathbf{u}+\mathbf{v}) & = & \mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v} & \text{for all } \mathbf{u},\mathbf{v},\mathbf{w}\in\mathcal{V},\\
(I_3) & (\xi\mathbf{u})\cdot\mathbf{v} & = & \xi(\mathbf{u}\cdot\mathbf{v}) & \text{for all } \xi\in\mathbb{R},\mathbf{u},\mathbf{v}\in\mathcal{V},\\
(I_4) & (\mathbf{u}\cdot\mathbf{v}=0 \text{ for all } \mathbf{v}\in\mathcal{V}) & \Longrightarrow & \mathbf{u}=0 & \text{for all } \mathbf{u}\in\mathcal{V}.
\end{array}$$

If (I_4) *is replaced by* (I_4'),

$$(I_4') \quad \mathbf{u}\cdot\mathbf{u}>0 \text{ for all } \mathbf{u}\in\mathcal{V}^\times,$$

we say that the inner-product space $\mathcal{V}$ *is* **genuine**.

Note that (I_4') implies (I_4). We are most familiar with genuine inner-product spaces. But for our discussion of special relativity, certain kinds of non-genuine inner-product spaces are of central importance. More will be said about this in §**5.2**.

Let $\mathcal{V}$ be an inner-product space.

5101 Notation: *Given any subset* $\mathcal{U}$ *of* $\mathcal{V}$, *we use the notation*

$$\mathcal{U}^\perp := \{\mathbf{v}\in\mathcal{V} \mid \mathbf{u}\cdot\mathbf{v}=0 \text{ for all } \mathbf{u}\in\mathcal{U}\}.$$

If $\mathcal{U}_1$ and $\mathcal{U}_2$ are given subsets of $\mathcal{V}$ such that $\mathcal{U}_1\subset\mathcal{U}_2$, we clearly have $\mathcal{U}_2^\perp\subset\mathcal{U}_1^\perp$. The proof of the following Proposition is left as an Exercise.

5102 Proposition: *Let* $\mathcal{U}\subset\mathcal{V}$ *be given. Then* $\mathcal{U}^\perp$ *is a subspace of* $\mathcal{V}$. *Moreover, if* $\mathcal{U}$ *is a subspace of* $\mathcal{V}$, *then* $(\mathcal{U}^\perp)^\perp=\mathcal{U}$ *and*

$$\dim\mathcal{U}+\dim\mathcal{U}^\perp=\dim\mathcal{V}.$$

——Euclidean Spaces

We now give a formal definition of a Euclidean space, as alluded to in §**4.1**.

5103 Definition: *A* **Euclidean space** *is a flat space $\mathcal{E}$ with additional structure given by specifying an inner product on the translation space of $\mathcal{E}$. If the inner product is genuine, we say that the Euclidean space is* **genuine**.

Given a genuine Euclidean space $\mathcal{E}$ (with translation space $\mathcal{V}$), we may induce a natural distance function on $\mathcal{E}$, $\mathsf{d} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$, given by

$$\mathsf{d}(x, y) := \sqrt{(y - x) \cdot (y - x)}$$

for all $x, y \in \mathcal{E}$. Then d satisfies

(E_1) *For all $x, y \in \mathcal{E}$, $\mathsf{d}(x, y) = 0 \implies x = y$.*

(E_2) *For all $x, y \in \mathcal{E}$, $\mathsf{d}(x, y) = \mathsf{d}(y, x)$, and*

(E_3) *For all $x, y, z \in \mathcal{E}$, $\mathsf{d}(x, z) \leq \mathsf{d}(x, y) + \mathsf{d}(y, z)$.*

—**Signatures**

As remarked earlier, we are familiar with the concept of a genuine inner-product space; that is, an inner-product space where (I_4') is valid. In this section, we consider **non-genuine** inner-product spaces; that is, inner-product spaces where (I_4') is not valid. The analysis of such spaces is more subtle than that of genuine inner-product spaces. However, the structure of non-genuine inner-product spaces can be described *via* certain properties of subspaces; hence, it is useful to develop some terminology to facilitate such a description.

Let $\mathcal{V}$ be an inner-product space.

5104 Definition: *A subspace $\mathcal{U}$ of $\mathcal{V}$ is* **regular** *if $\mathcal{U} \cap \mathcal{U}^\perp = \{\mathbf{0}\}$. We say that $\mathcal{U}$ is* **positive-regular** [**negative-regular**] *if $\mathbf{u} \cdot \mathbf{u} > 0$ [$\mathbf{u} \cdot \mathbf{u} < 0$] for all $\mathbf{u} \in \mathcal{U}^\times = \mathcal{U} \setminus \{\mathbf{0}\}$. We say that $\mathcal{U}$ is* **singular** *if $\mathcal{U}$ is not regular, and* **totally singular** *if $\mathbf{u} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in \mathcal{U}$.*

We note that positive-regular and negative-regular subspaces are indeed regular, and totally singular subspaces are likewise singular. However, there can be regular subspaces that are neither positive-regular nor negative-regular.

5105 Definition: *The greatest among the dimensions of all positive-regular [negative-regular] subspaces of $\mathcal{V}$ is denoted by $\mathsf{sig}^+\mathcal{V}$ [$\mathsf{sig}^-\mathcal{V}$], and the pair $(\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}) \in \mathbb{N} \times \mathbb{N}$ is called the* **signature of** $\mathcal{V}$.

One may recall that if $\mathcal{V}$ is a genuine inner-product space, then for each subspace $\mathcal{U}$ of $\mathcal{V}$, $\mathcal{U}$ and $\mathcal{U}^\perp$ are supplementary. However, this need not be the case if $\mathcal{V}$ is non-genuine. In fact, in §**5.2**, we will discuss a subspace $\mathcal{U}$ of a two-dimensional non-genuine inner-product space such that $\mathcal{U} = \mathcal{U}^\perp$! That $\mathcal{U}$ is regular, however, is necessary and sufficient to guarantee that $\mathcal{U}$ and $\mathcal{U}^\perp$ be supplementary.

5106 Proposition: *A subspace $\mathcal{U}$ of $\mathcal{V}$ is regular if and only if $\mathcal{U}$ and $\mathcal{U}^\perp$ are supplementary subspaces of $\mathcal{V}$.*

Proof: Assume that $\mathcal{U}$ is a regular subspace of $\mathcal{V}$. We see from **Prop. 5102** that $\mathsf{dim}\ \mathcal{U} + \mathsf{dim}\ \mathcal{U}^\perp = \mathsf{dim}\ \mathcal{V}$. Since $\mathcal{U}$ is regular, we have $\mathcal{U} \cap \mathcal{U}^\perp = \{\mathbf{0}\}$ (see **Def. 5104**). We apply **Prop. D09** of Appendix D to conclude that $\mathcal{U} + \mathcal{U}^\perp = \mathcal{V}$. Hence, $\mathcal{U}$ and $\mathcal{U}^\perp$ are supplementary subspaces of $\mathcal{V}$.

The proof of the reverse implication is analogous. ◊

The following Theorem has many consequences which are important to the study of special relativity.

5107 Theorem: (Inner-Product Signature Theorem): *Let $\mathcal{U}$ be a positive-regular [negative-regular] subspace of $\mathcal{V}$. Then the following are equivalent:*

(1) $\mathsf{dim}\ \mathcal{U} = \mathsf{sig}^+\mathcal{V}$ $[\mathsf{dim}\ \mathcal{U} = \mathsf{sig}^-\mathcal{V}]$,

(2) *$\mathcal{U}$ is maximal among all positive-regular [negative-regular] subspaces of $\mathcal{V}$; that is, no positive-regular [negative-regular] subspace of $\mathcal{V}$ strictly includes $\mathcal{U}$, and*

(3) *$\mathcal{U}^\perp$ is negative-regular [positive-regular].*

Remark: We note that condition (2) does not imply that $\mathcal{U}$ includes every positive-regular subspace of $\mathcal{V}$. In general, there is no such "maximum" positive-regular subspace.

Proof: To prove this Theorem, we proceed in three steps; we show successively that (1) $\Longrightarrow$ (2), that (2) $\Longrightarrow$ (3), and that (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2) follows directly from the definition of $\mathsf{sig}^+\mathcal{V}$.

To show that (2) $\Longrightarrow$ (3), we proceed by contraposition. So assume that $\mathcal{U}^\perp$ is not negative-regular. Our goal is to show that $\mathcal{U}$ is not maximal among positive-regular subspaces of $\mathcal{V}$.

Since $\mathcal{U}^\perp$ is not negative-regular, we may choose $\mathbf{w} \in (\mathcal{U}^\perp)^\times$ such that $\mathbf{w}\cdot\mathbf{w} \geq 0$. Now if $\mathbf{w}\cdot\mathbf{v} = 0$ for all $\mathbf{v} \in \mathcal{U}^\perp$, we would have $\mathbf{w} \in \mathcal{U}^{\perp\perp} = \mathcal{U}$, and hence $\mathbf{0} \neq \mathbf{w} \in \mathcal{U}\cap\mathcal{U}^\perp$, which is impossible since $\mathcal{U}$ is regular (and hence $\mathcal{U}\cap\mathcal{U}^\perp = \{\mathbf{0}\}$). Therefore, we may choose $\mathbf{v} \in \mathcal{U}^\perp$ such that $\alpha := \mathbf{w}\cdot\mathbf{v} \neq 0$. Since $\mathbf{w}\cdot\mathbf{w} \geq 0$, we have

$$(\xi\mathbf{w}+\mathbf{v})\cdot(\xi\mathbf{w}+\mathbf{v}) = \xi^2\mathbf{w}\cdot\mathbf{w} + 2\xi\alpha + \mathbf{v}\cdot\mathbf{v} \geq 2\xi\alpha + \mathbf{v}\cdot\mathbf{v}$$

for all $\xi \in \mathbb{R}$. If we use this fact with $\xi := \frac{1}{2\alpha}(1-\mathbf{v}\cdot\mathbf{v})$ and put $\mathbf{w}' := \xi\mathbf{w}+\mathbf{v}$, we find that

$$\mathbf{w}'\cdot\mathbf{w}' \geq 2\xi\alpha + \mathbf{v}\cdot\mathbf{v} = 2\alpha\left(\frac{1}{2\alpha}(1-\mathbf{v}\cdot\mathbf{v})\right) + \mathbf{v}\cdot\mathbf{v} = 1.$$

Since $\mathbf{w},\mathbf{v} \in \mathcal{U}^\perp$, we have $\mathbf{w}' \in \mathcal{U}^\perp$. Thus, for all $\mathbf{u} \in \mathcal{U}$ and $\lambda \in \mathbb{R}$,

$$\begin{aligned}(\mathbf{u}+\lambda\mathbf{w}')\cdot(\mathbf{u}+\lambda\mathbf{w}') &= \mathbf{u}\cdot\mathbf{u} + 2\lambda\mathbf{u}\cdot\mathbf{w}' + \lambda^2\mathbf{w}'\cdot\mathbf{w}' \\ &\geq \mathbf{u}\cdot\mathbf{u} + \lambda^2 \\ &\geq 0.\end{aligned}$$

Since $\mathcal{U}$ is positive-regular, equality can hold only if $\mathbf{u} = \mathbf{0}$ and $\lambda = 0$. Hence, we see that $\mathcal{U} + \mathbb{R}\mathbf{w}'$ is a positive-regular subspace of $\mathcal{V}$. If $\mathbf{w}' \in \mathcal{U}$, then $\mathbf{w}' \in \mathcal{U}\cap\mathcal{U}^\perp = \{\mathbf{0}\}$ (since $\mathcal{U}$ is regular) would imply that $\mathbf{w}' = \mathbf{0}$, which is impossible since $\mathbf{w}'\cdot\mathbf{w}' \geq 1$. Hence, $\mathbf{w}' \notin \mathcal{U}$, and so $\mathcal{U} + \mathbb{R}\mathbf{w}'$ strictly includes $\mathcal{U}$. Thus, $\mathcal{U}$ is not maximal among positive-regular subspaces of $\mathcal{V}$.

We now show that (3) $\Longrightarrow$ (1). So assume that $\mathcal{U}^\perp$ is negative-regular. By **Def. 5105**, we may choose a positive-regular subspace $\mathcal{W}$ of $\mathcal{V}$ such that $\mathsf{dim}\ \mathcal{W} = \mathsf{sig}^+\mathcal{V}$. It is clear that $\mathcal{W}\cap\mathcal{U}^\perp = \{\mathbf{0}\}$, and thus (using $\mathcal{U}_1 := \mathcal{W}$, $\mathcal{U}_2 := \mathcal{U}^\perp$, and $\mathcal{V} := \mathcal{W}+\mathcal{U}^\perp$ in **Prop. D09** of Appendix D)

$$\mathsf{sig}^+\mathcal{V} + \mathsf{dim}\ \mathcal{U}^\perp = \mathsf{dim}\ (\mathcal{W}+\mathcal{U}^\perp) \leq \mathsf{dim}\ \mathcal{V}.$$

Since $\mathsf{dim}\ \mathcal{V} = \mathsf{dim}\ \mathcal{U} + \mathsf{dim}\ \mathcal{U}^\perp$ (by **Prop. 5102**), it follows that $\mathsf{sig}^+\mathcal{V} \leq \mathsf{dim}\ \mathcal{U}$ and hence $\mathsf{sig}^+\mathcal{V} = \mathsf{dim}\ \mathcal{U}$ by **Def. 5105**. $\Diamond$

5108 Corollary: *There are positive-regular subspaces $\mathcal{U}$ of $\mathcal{V}$ such that $\mathcal{U}^\perp$ is negative-regular. If $\mathcal{U}$ is such a subspace, then*

$$\mathsf{dim}\ \mathcal{U} = \mathsf{sig}^+\mathcal{V} \quad \textit{and} \quad \mathsf{dim}\ \mathcal{U}^\perp = \mathsf{sig}^-\mathcal{V}.$$

Moreover, we have

$$\mathsf{sig}^+\mathcal{V} + \mathsf{sig}^-\mathcal{V} = \mathsf{dim}\ \mathcal{V}.$$

We now prove a useful result which follows easily from **Cor. 5108**.

5109 Proposition: *Let $\mathcal{U}$ be a totally singular subspace of $\mathcal{V}$. Then*

$$\mathsf{dim}\ \mathcal{U} \leq \min\{\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}\}.$$

Proof: By **Cor. 5108**, we may choose a positive-regular subspace $\mathcal{W}$ of $\mathcal{V}$ such that $\mathsf{dim}\ \mathcal{W} = \mathsf{sig}^+\mathcal{V}$. Clearly, we must have $\mathcal{U} \cap \mathcal{W} = \{\mathbf{0}\}$. By **Prop. D08** of Appendix D, we see that $\mathsf{dim}\ \mathcal{U} + \mathsf{dim}\ \mathcal{W} \leq \mathsf{dim}\ \mathcal{V}$. As a result of **Cor. 5108**, it follows that

$$\begin{aligned} \mathsf{dim}\ \mathcal{U} &\leq \mathsf{dim}\ \mathcal{V} - \mathsf{dim}\ \mathcal{W} \\ &= \mathsf{dim}\ \mathcal{V} - \mathsf{sig}^+\mathcal{V} \\ &= \mathsf{sig}^-\mathcal{V}. \end{aligned}$$

We may similarly show that $\mathsf{dim}\ \mathcal{U} \leq \mathsf{sig}^+\mathcal{V}$, yielding

$$\mathsf{dim}\ \mathcal{U} \leq \min\{\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}\}.$$

As $\mathcal{U}$ was arbitrary, the Proposition is proved. $\Diamond$

Remark: One can prove that there are totally singular subspaces $\mathcal{U}$ of $\mathcal{V}$ such that $\mathsf{dim}\ \mathcal{U} = \min\{\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}\}$, and that there are many such subspaces when $\min\{\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}\} > 0$.

Finally, we give a result which follows from the Inner-Product Signature Theorem which will be useful in Chapter 7. The proof is left as an Exercise.

5110 Proposition: *Let $\mathcal{V}$ be an inner-product space with signature $(p, m) \in \mathbb{N} \times \mathbb{N}$, and put $n := p + m$. Then we may find a list-basis (see* **Def. D05** *in Appendix D) $\mathbf{b} = (\mathbf{b}_i \,|\, i \in 1..n) \in \mathcal{V}^n$ such that*

1. $\mathbf{b}_i \cdot \mathbf{b}_i = -1$ *for all* $i \in 1..m$,

2. $\mathbf{b}_i \cdot \mathbf{b}_i = 1$ *for all* $i \in (m+1)..n$, *and*

3. $\mathbf{b}_i \cdot \mathbf{b}_j = 0$ *for all* $i, j \in 1..n$ *such that* $i \neq j$.

Such a list-basis is called an **orthonormal list-basis of** $\mathcal{V}$.

5.2 Inner-Product Spaces with $\mathsf{sig}^{-}\mathcal{V} = 1$

Now that some basic definitions and results have been presented, we look at some applications which will be relevant to our study of special relativity. We will be concerned with a particular class of non-genuine inner-product spaces; namely, those for which the signature is of the form $(n-1, 1)$, where $n \in \mathbb{N}$ and $n \geq 2$.

Let an inner-product space $\mathcal{V}$ with $\mathsf{sig}^{-}\mathcal{V} = 1$ be given.

Notation: We use the following notational conventions:

$$\begin{aligned}
\mathcal{N} &:= \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{v} \leq 0\}, \\
\mathcal{V}^{-} &:= \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{v} < 0\}, \\
\mathcal{V}^{+} &:= \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{v} > 0\}, \\
\mathcal{V}^{0} &:= \{\mathbf{v} \in \mathcal{V} \,|\, \mathbf{v} \cdot \mathbf{v} = 0\}.
\end{aligned}$$

Note that $\mathcal{N} = \mathcal{V}^{-} \cup \mathcal{V}^{0}$. We call members of $\mathcal{V}^{-}$ **timelike vectors**, members of $\mathcal{V}^{+}$ **spacelike vectors**, and members of $\mathcal{V}^{0}$ **signal vectors**.

5200 Proposition: *Let* $\mathbf{u} \in (\mathcal{V}^{-})^{\times} = \mathcal{V}^{-} \setminus \{\mathbf{0}\}$ *be given. Then* $\{\mathbf{u}\}^{\perp}$ *is positive-regular. Moreover,* $\mathbb{R}\mathbf{u}$ *and* $\{\mathbf{u}\}^{\perp}$ *are supplementary subspaces of* $\mathcal{V}$.

Proof: Clearly, $\mathbb{R}\mathbf{u}$ is negative-regular. Since $\mathsf{sig}^{-}\mathcal{V} = 1$ and $\dim \mathbb{R}\mathbf{u} = 1$, it follows from **Thm. 5107** that $\{\mathbf{u}\}^{\perp}$ is positive-regular. Since $\mathbb{R}\mathbf{u}$ is regular, it follows from **Prop. 5106** that $\mathbb{R}\mathbf{u}$ and $\{\mathbf{u}\}^{\perp}$ are supplementary. ◊

5201 Theorem: (Reverse Inner-Product Inequality): *If* $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ *are given and* $\mathbf{u} \in \mathcal{V}^-$, *then*

$$(\mathbf{u} \cdot \mathbf{v})^2 \geq (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}). \tag{52.1}$$

Equality holds if and only if one of $\mathbf{u}$ *and* $\mathbf{v}$ *is a multiple of the other.*

Proof: First, we assume that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent and also put $\mathbf{w} := (\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}$. It is not hard to see that $\mathbf{w} \cdot \mathbf{u} = 0$, and hence $\mathbf{w} \in \{\mathbf{u}\}^\perp$. The linear independence of $\{\mathbf{u}, \mathbf{v}\}$ implies that $\mathbf{w} \neq \mathbf{0}$. Since $\{\mathbf{u}\}^\perp$ is positive-regular (see **Prop. 5200**), we have that

$$0 < \mathbf{w} \cdot \mathbf{w} = ((\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}) \cdot ((\mathbf{u} \cdot \mathbf{v})\mathbf{u} - (\mathbf{u} \cdot \mathbf{u})\mathbf{v}),$$

and hence

$$0 < (\mathbf{u} \cdot \mathbf{u})((\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) - (\mathbf{u} \cdot \mathbf{v})^2).$$

Since $\mathbf{u} \cdot \mathbf{u} < 0$, we conclude that

$$(\mathbf{u} \cdot \mathbf{v})^2 > (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}).$$

Assume, on the other hand, that one of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of the other. Since $\mathbf{u} \neq \mathbf{0}$, we then have $\mathbf{v} = \alpha\mathbf{u}$ for some $\alpha \in \mathbb{R}$. It is not difficult to show that (52.1) then holds with equality. ◊

Remark: The reader may recall that if $\mathcal{V}$ is a genuine inner-product space, then we have

$$(\mathbf{u} \cdot \mathbf{v})^2 \leq (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}$. It is because of the familiarity with this inequality that the terminology "Reverse Inner-Product Inequality" is used.

5202 Proposition: *Let* $\mathbf{u}, \mathbf{v} \in \mathcal{N}^\times = \mathcal{N} \setminus \{\mathbf{0}\}$ *be given. If* $\mathbf{u} \cdot \mathbf{v} = 0$, *then* $\mathbf{u}, \mathbf{v} \in \mathcal{V}^0$ *and one of* $\mathbf{u}$ *and* $\mathbf{v}$ *is a multiple of the other.*

Proof: Suppose that $\mathbf{u} \cdot \mathbf{v} = 0$. Now if $\mathbf{u} \notin \mathcal{V}^0$, then we would have $\mathbf{u} \in \mathcal{V}^-$; since $\mathbf{v} \in \{\mathbf{u}\}^\perp$, since $\mathbf{v} \neq \mathbf{0}$, and since $\{\mathbf{u}\}^\perp$ is positive-regular (**Prop. 5200**), we would have $\mathbf{v} \cdot \mathbf{v} > 0$, contradicting $\mathbf{v} \in \mathcal{N}^\times$. Hence $\mathbf{u} \in \mathcal{V}^0$, and by a similar argument, $\mathbf{v} \in \mathcal{V}^0$. Since $\mathbf{u} \cdot \mathbf{v} = 0$, we must have $\mathbf{w} \cdot \mathbf{w} = 0$ for all $\mathbf{w} \in \mathsf{Lsp}\,\{\mathbf{u}, \mathbf{v}\}$ (see **Prop. D03** of Appendix D); *i.e.*, $\mathsf{Lsp}\,\{\mathbf{u}, \mathbf{v}\} \subset \mathcal{V}^0$. By **Prop. 5109**, $\dim \mathsf{Lsp}\,\{\mathbf{u}, \mathbf{v}\} \leq 1$; *i.e.*, one of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of the other. ◊

Pitfall: In view of the previous Proposition, one may be tempted to restate **Thm. 5201** as follows: If $\mathbf{u}, \mathbf{v} \in \mathcal{V}$ are given and $\mathbf{u} \in \mathcal{N}$, then $(\mathbf{u}\cdot\mathbf{v})^2 \geq (\mathbf{u}\cdot\mathbf{u})(\mathbf{v}\cdot\mathbf{v})$, with equality if and only if one of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of the other. This restatement, however, is false (see the Exercises).

We now include a result related to **Prop. 5106** in the case that $\mathsf{sig}^{-}\mathcal{V} = 1$.

5203 Proposition: *Let $\mathcal{U}$ be a regular subspace of $\mathcal{V}$. Then either $\mathcal{U}$ or $\mathcal{U}^{\perp}$ is positive-regular.*

Proof: For all $\mathbf{u} \in \mathcal{U}$ and $\mathbf{v} \in \mathcal{U}^{\perp}$, we have

$$(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}.$$

Since $\mathsf{sig}^{-}\mathcal{V} = 1$ and $\mathcal{U} + \mathcal{U}^{\perp} = \mathcal{V}$, then either $\mathcal{U}$ or $\mathcal{U}^{\perp}$ must contain a member of $\mathcal{V}^{-}$; without loss, suppose that $\mathbf{u} \in \mathcal{U} \cap \mathcal{V}^{-}$. Since $\mathbf{u} \in \mathcal{U}$, then $\mathcal{U}^{\perp} \subset \{\mathbf{u}\}^{\perp}$; but we know from **Prop. 5200** that $\{\mathbf{u}\}^{\perp}$ is positive-regular. Hence $\mathcal{U}^{\perp}$ must be positive-regular. ◊

The following Theorem gives some insight into why an inner-product space with $\mathsf{sig}^{-}\mathcal{V} = 1$ is an appropriate model for a theory of special relativity. We see that $\mathcal{N}^{\times}$ may be partitioned into linear cones (see **Def. 3201**) which we will later interpret as the "future cone" and the "past cone".

5204 Theorem: *$\mathcal{N}^{\times}$ has exactly one doubleton partition whose pieces are linear cones. For each $\mathbf{u} \in \mathcal{V}^{-}$, this partition is given by*

$$\Big\{\{\mathbf{v} \in \mathcal{N}^{\times} \,|\, \mathbf{v}\cdot\mathbf{u} < 0\}, \{\mathbf{v} \in \mathcal{N}^{\times} \,|\, \mathbf{v}\cdot\mathbf{u} > 0\}\Big\}.$$

Hence, if $\widehat{\mathcal{F}}$ is one piece of the partition, then $-\widehat{\mathcal{F}}$ is the other.

Remark: We use the symbol "$\widehat{\mathcal{F}}$" here as we reserve the symbol "$\mathcal{F}$" for later use (see **Prop. 5206** and **Thm. 5300**).

Proof: Let $\mathbf{u} \in \mathcal{V}^-$ be given. It is left as an Exercise to show that

$$\Big\{\{\mathbf{v} \in \mathcal{N}^\times \,|\, \mathbf{v} \cdot \mathbf{u} < 0\}, \{\mathbf{v} \in \mathcal{N}^\times \,|\, \mathbf{v} \cdot \mathbf{u} > 0\}\Big\}$$

is indeed a doubleton partition of $\mathcal{N}^\times$ whose pieces are linear cones.

Now suppose that $\{\widehat{\mathcal{F}}, \widehat{\mathcal{F}}'\}$ is another partition of $\mathcal{N}^\times$ whose pieces are linear cones, and assume without loss that $\widehat{\mathcal{F}}$ is the piece of the partition to which $\mathbf{u}$ belongs. Since $\mathbf{u} \cdot \mathbf{u} < 0$ and hence $\{\mathbf{u}\}^\perp$ is positive-regular (by **Prop. 5200**), it follows that $\{\mathbf{u}\}^\perp \cap \mathcal{N}^\times = \emptyset$. Since $\widehat{\mathcal{F}} \subset \mathcal{N}^\times$, we have $\{\mathbf{u}\}^\perp \cap \widehat{\mathcal{F}} = \emptyset$.

Let $\mathbf{v} \in \widehat{\mathcal{F}}$ be given. Since $\{\mathbf{u}\}^\perp \cap \widehat{\mathcal{F}} = \emptyset$, we must have $\mathbf{v} \notin \{\mathbf{u}\}^\perp$; *i.e.,* $\mathbf{v} \cdot \mathbf{u} \neq 0$. Suppose that $\mathbf{v} \cdot \mathbf{u} > 0$. Then $\lambda := -\frac{\mathbf{u}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{u}} > 0$ because $\mathbf{u} \cdot \mathbf{u} < 0$. Since $\widehat{\mathcal{F}}$ is a linear cone and $\mathbf{u}, \mathbf{v} \in \widehat{\mathcal{F}}$, we conclude that $\mathbf{u} + \lambda\mathbf{v} \in \widehat{\mathcal{F}}$. But $(\mathbf{u} + \lambda\mathbf{v}) \cdot \mathbf{u} = 0$; *i.e.,* $(\mathbf{u} + \lambda\mathbf{v}) \in \{\mathbf{u}\}^\perp$, which contradicts $\widehat{\mathcal{F}} \cap \{\mathbf{u}\}^\perp = \emptyset$. We conclude that $\mathbf{v} \cdot \mathbf{u} < 0$. Since $\mathbf{v} \in \widehat{\mathcal{F}}$ was arbitrary, we have $\widehat{\mathcal{F}} \subset \mathcal{N}_- := \{\mathbf{v} \in \mathcal{N}^\times \,|\, \mathbf{v} \cdot \mathbf{u} < 0\}$.

Since $-\mathbf{u} \in \mathcal{N}^\times$ and $(-\mathbf{u}) \cdot \mathbf{u} = -\mathbf{u} \cdot \mathbf{u} > 0$, it follows that $-\mathbf{u}$ must belong to the other piece $\widehat{\mathcal{F}}'$ of the partition. By reasoning similar to that given above, we conclude that $\widehat{\mathcal{F}}' \subset \mathcal{N}_+ := \{\mathbf{v} \in \mathcal{N}^\times \,|\, \mathbf{v} \cdot \mathbf{u} > 0\}$. Since $\{\mathbf{u}\}^\perp \cap \mathcal{N}^\times = \emptyset$, it follows that $\{\mathcal{N}_-, \mathcal{N}_+\}$, with $\mathcal{N}_-$, $\mathcal{N}_+$ as defined above, is a partition of $\mathcal{N}^\times$. Hence, we must have $\widehat{\mathcal{F}} = \mathcal{N}_-$ and $\widehat{\mathcal{F}}' = \mathcal{N}_+$, showing that $\{\mathcal{N}_-, \mathcal{N}_+\}$ is the only candidate for the desired partition. Furthermore, it is clear that $\mathcal{N}_- = -\mathcal{N}_+$, and hence $\widehat{\mathcal{F}}' = -\widehat{\mathcal{F}}$. ◊

5205 Corollary: *Let $\widehat{\mathcal{F}}$ be one of the linear cones of the partition of $\mathcal{N}^\times$ described in the previous Theorem, and let $\mathbf{u}, \mathbf{v} \in \widehat{\mathcal{F}}$ be given. Then $\mathbf{u}\cdot\mathbf{v} \leq 0$, with equality if and only if $\mathbf{u}, \mathbf{v} \in \mathcal{V}^0$ and one of $\mathbf{u}$ and $\mathbf{v}$ is a multiple of the other.*

Proof: If either $\mathbf{u} \cdot \mathbf{u} < 0$ or $\mathbf{v} \cdot \mathbf{v} < 0$, then $\mathbf{u} \cdot \mathbf{v} < 0$ by the previous Theorem. So, assume that $\mathbf{u} \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{v} = 0$. Since $\widehat{\mathcal{F}}$ is a linear cone (and hence stable under addition), $\mathbf{u} + \mathbf{v} \in \widehat{\mathcal{F}} \subset \mathcal{N}^\times$; thus we have

$$0 \geq (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} = 2\mathbf{u} \cdot \mathbf{v}.$$

Hence, $\mathbf{u} \cdot \mathbf{v} \leq 0$. The remainder of the Corollary is an immediate consequence of **Prop. 5202**. ◊

We now include a Proposition which will be useful in §**5.3**.

5206 Proposition: *Let $\widehat{\mathcal{F}}$ be one of the linear cones of the partition of $\mathcal{N}^{\times}$ described in* **Thm. 5204**, *and put $\mathcal{F} := \widehat{\mathcal{F}} \cup \{\mathbf{0}\}$. Then $\mathcal{V} = \mathcal{F} - \mathcal{F}$.*

Proof: Let $\mathbf{v} \in \mathcal{V}$ be given. Now if $\mathbf{v} \in \mathcal{N} = \mathcal{F} \cup (-\mathcal{F})$, then $\mathbf{v} \in \mathcal{F} - \mathcal{F}$ is immediate since $\mathbf{0} \in \mathcal{F}$. So suppose that $\mathbf{v} \notin \mathcal{N}$, *i.e.*, $\mathbf{v} \in \mathcal{V}^{+}$. Then we may choose $\mathbf{u} \in \{\mathbf{v}\}^{\perp} \cap \mathcal{V}^{-}$; assume without loss that $\mathbf{u} \in \mathcal{F}$ (otherwise replace $\mathbf{u}$ by $-\mathbf{u}$). Put $\alpha := \sqrt{-\mathbf{v} \cdot \mathbf{v}/\mathbf{u} \cdot \mathbf{u}} > 0$. Then

$$(\mathbf{v} + \alpha\mathbf{u}) \cdot (\mathbf{v} + \alpha\mathbf{u}) = \mathbf{v} \cdot \mathbf{v} + \alpha^2 \mathbf{u} \cdot \mathbf{u} = 0.$$

Moreover,

$$(\mathbf{v} + \alpha\mathbf{u}) \cdot \mathbf{u} = \mathbf{v} \cdot \mathbf{u} + \alpha\mathbf{u} \cdot \mathbf{u} < 0.$$

These relations imply that $\mathbf{v} + \alpha\mathbf{u} \in \mathcal{F}$, so that

$$\mathbf{v} = (\mathbf{v} + \alpha\mathbf{u}) - \alpha\mathbf{u} \in \mathcal{F} - \mathcal{F}.$$

As $\mathbf{v} \in \mathcal{V}$ was arbitrary, we see that $\mathcal{V} = \mathcal{F} - \mathcal{F}$. ◊

—**An Example**

Put $\mathcal{V} := \mathbb{R}^2$; if $a \in \mathcal{V}$, we write $a = (a_1, a_2)$, and put $\mathbf{0} := (0, 0)$. Instead of the usual inner product in $\mathbb{R}^2$, we let $k \in \mathbb{P}^{\times}$ be given, and define

$$a \cdot b := k^2 a_1 b_1 - a_2 b_2$$

for all $a, b \in \mathcal{V}$. We claim that this definition gives an inner product on $\mathcal{V}$. We leave it to the reader to show that Axioms (I_1)–(I_3) are satisfied.

To see that (I_4) is valid, let $u \in \mathcal{V}$ be given, and assume that $u \cdot v = 0$ for all $v \in \mathcal{V}$. Then for all $v \in \mathcal{V}$, we have $k^2 u_1 v_1 - u_2 v_2 = 0$. In particular, we must have $u \cdot v = 0$ when $v := (u_1, -u_2)$; *i.e.*, $k^2 u_1 u_1 - u_2(-u_2) = k^2 u_1^2 + u_2^2 = 0$. Since $k > 0$, this can occur only if $u_1 = u_2 = 0$; that is, $u = \mathbf{0}$. Since $u \in \mathcal{V}$ was arbitrary, (I_4) holds.

What is the signature of $\mathcal{V}$? We note that for all $\alpha \in \mathbb{R}^{\times}$, we have

$$(\alpha, 0) \cdot (\alpha, 0) = k^2 \alpha^2 > 0.$$

Hence, $\mathbb{R}(1,0)$ is a positive-regular subspace of $\mathcal{V}$. Similarly, we see that $\mathbb{R}(0,1)$ is a negative-regular subspace of $\mathcal{V}$. Hence, by the definitions of $\mathsf{sig}^+\mathcal{V}$ and $\mathsf{sig}^-\mathcal{V}$, we must have $\mathsf{sig}^+\mathcal{V} \geq 1$ and $\mathsf{sig}^-\mathcal{V} \geq 1$. But by **Cor. 5108**, we must also have $\mathsf{sig}^+\mathcal{V} + \mathsf{sig}^-\mathcal{V} = \mathsf{dim}\ \mathcal{V} = 2$. This forces $\mathsf{sig}^+\mathcal{V} = 1 = \mathsf{sig}^-\mathcal{V}$. Hence, $\mathcal{V}$ has signature $(1,1)$.

We now ask the following questions: Are there other positive-regular or negative-regular subspaces of $\mathcal{V}$? Are there subspaces of $\mathcal{V}$ which are neither positive-regular nor negative-regular? What do such subspaces look like?

To answer the first question, suppose that $\mathcal{U}$ is a positive-regular subspace of $\mathcal{V}$ different from $\{\mathbf{0}\}$. We must have $\mathsf{dim}\ \mathcal{U} = 1$, and hence we may choose $u \in \mathcal{V}$ such that $\mathcal{U} = \mathbb{R}u$. Then

$$0 < (u_1, u_2) \cdot (u_1, u_2) = k^2 u_1^2 - u_2^2,$$

and hence $k|u_1| > |u_2|$. Indeed, we may characterize *all* positive-regular subspaces of $\mathcal{V}$ in this way: if $u \in \mathcal{V}$ is such that $k|u_1| > |u_2|$, then $\mathbb{R}u$ is a positive-regular subspace of $\mathcal{V}$. Similarly, it can be shown that if $u \in \mathcal{V}$ is such that $k|u_1| < |u_2|$, then $\mathbb{R}u$ is a negative-regular subspace of $\mathcal{V}$.

What happens if $u \in \mathcal{V}$ is such that $k|u_1| = |u_2|$? Suppose we are given such a $u \in \mathcal{V}$. Then $u \cdot u = k^2 u_1^2 - u_2^2 = 0$. As long as $u \neq \mathbf{0}$, $\mathbb{R}u$ is a one-dimensional subspace of $\mathcal{V}$ such that $v \cdot v = 0$ for all $v \in \mathbb{R}u$. Thus, $\mathbb{R}u$ is totally singular. It is not difficult to show that there are exactly two such subspaces of $\mathcal{V}$; namely, $\mathbb{R}(1,k)$ and $\mathbb{R}(-1,k)$.

What do these subspaces look like in $\mathcal{V} = \mathbb{R}^2$ when $\mathbb{R}^2$ is represented by a coordinatized plane? We see $\mathbb{R}(1,k)$ and $\mathbb{R}(-1,k)$ as the oblique lines in Figure 52a. Moreover, if a subspace lies within the shaded area of Figure 52a, then it must be negative-regular. Analogously, if a subspace lies outside of the shaded area and its boundary, then it must be positive-regular.

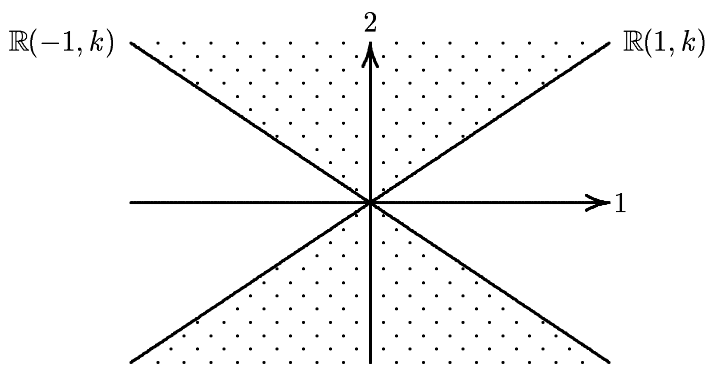

Figure 52a

Suppose that $\mathcal{U}$ is a one-dimensional subspace of $\mathcal{V}$. What does $\mathcal{U}^{\perp}$ look like? We first choose $u \in \mathcal{V}^{\times}$ such that $\mathcal{U} = \mathbb{R}u$. Then if $v \in \mathcal{U}^{\perp}$, we have

$$0 = u \cdot v = k^2 u_1 v_1 - u_2 v_2,$$

i.e., $k^2 u_1 v_1 = u_2 v_2$.

Assume that $u_2 \neq 0$. Then $v_2 = \frac{u_1}{u_2} k^2 v_1$, and hence $v = \left(v_1, \frac{u_1}{u_2} k^2 v_1\right) = \frac{v_1}{u_2}(u_2, k^2 u_1)$. Thus, we see that v is a multiple of $(u_2, k^2 u_1)$. It is not difficult to show that we must have $\mathcal{U}^{\perp} = \mathbb{R}(u_2, k^2 u_1)$. If $u_2 = 0$, then we have $k^2 u_1 v_1 = u_2 v_2 = 0$. But $u_1 \neq 0$, since $u \neq \mathbf{0}$. Hence, v_1 must be zero, and hence $v = (0, v_2) = \frac{v_2}{k^2 u_1}(0, k^2 u_1) = \frac{v_2}{k^2 u_1}(u_2, k^2 u_1)$. In an analogous way, we see that $\mathcal{U}^{\perp}$ must be $\mathbb{R}(u_2, k^2 u_1)$.

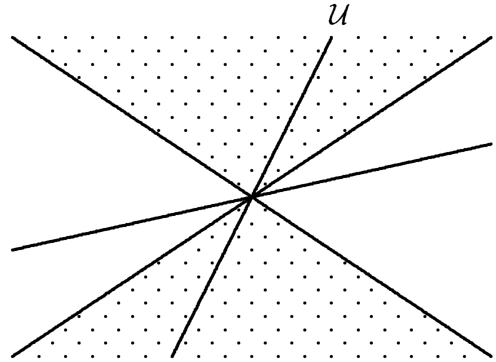

Figure 52b

Thus, in either case, we have $\mathcal{U}^\perp = \mathbb{R}(u_2, k^2u_1)$. From **Thm. 5107**, we see that if $\mathcal{U}$ is positive-regular, then $\mathcal{U}^\perp$ is negative-regular, and if $\mathcal{U}$ is negative-regular, then $\mathcal{U}^\perp$ is positive-regular.

What if $\mathcal{U}$ is neither positive-regular nor negative-regular? Then $\mathcal{U}$ is totally singular, and must be either $\mathbb{R}(1,k)$ or $\mathbb{R}(-1,k)$. In either case, the fact that $\mathcal{U}^\perp = \mathbb{R}(u_2, k^2u_1)$ implies that $\mathcal{U} = \mathcal{U}^\perp$.

—**Intrinsic Geometry of Spacetime Diagrams**

It will often be useful to draw diagrams of physical phenomena in order to facilitate an understanding of them. We now analyze the important geometrical principles which underlie the construction of such diagrams. The principles involved are similar to those just discussed, but are applied in a "coordinate-free" and hence more geometrically concrete setting.

Let a Euclidean space $\mathcal{E}$ be given (see **Def. 5103**) with $\dim \mathcal{E} = 2$. Assume that the translation space $\mathcal{V}$ of $\mathcal{E}$ satisfies $\mathrm{sig}^-\mathcal{V} = \mathrm{sig}^+\mathcal{V} = 1$. Let $\widehat{\mathcal{F}}$ be one of the linear cones described in **Thm. 5204**, and denote by $\widehat{\mathcal{F}}_1$ the set of vectors $\mathbf{v} \in \widehat{\mathcal{F}}$ with $\mathbf{v}\cdot\mathbf{v} = -1$. We begin with a useful Proposition.

5207 Proposition: *Let* $\mathbf{d} \in \widehat{\mathcal{F}}_1$ *be given. Then there are exactly two vectors* $\mathbf{u}$ *in* $\mathcal{V}$ *satisfying*

$$\mathbf{u}\cdot\mathbf{u} = 1 \quad \textit{and} \quad \mathbf{u}\cdot\mathbf{d} = 0.$$

If $\mathbf{u}$ *is one, then* $-\mathbf{u}$ *is the other. Moreover,* $\{\mathbf{d}, \mathbf{u}\}$ *is a basis of* $\mathcal{V}$.

Proof: From **Prop. 5200**, we see that $\mathcal{U} := \{\mathbf{d}\}^{\perp}$ is positive-regular. Since $\mathrm{sig}^{-}\mathcal{V} = \mathrm{sig}^{+}\mathcal{V} = 1$, we must have $\dim \mathcal{U} = 1$. Hence there are exactly two vectors $\mathbf{u}$ in $\mathcal{U}$ satisfying $\mathbf{u} \cdot \mathbf{u} = 1$, one being the opposite of the other. This proves the first part of the Proposition.

It is easily seen that $\{\mathbf{d}, \mathbf{u}\}$ is linearly independent and hence a basis of $\mathcal{V}$. ◊

We are now ready to prove the main result of this section.

5208 Theorem: *Let* $\mathbf{d} \in \widehat{\mathcal{F}}_1$ *be given. Then there is exactly one doubleton* $\{\mathbf{a}, \mathbf{b}\} \subset \mathcal{V}^0$ *such that* $\mathbf{d} = \mathbf{a} + \mathbf{b}$*. Moreover,* $\{\mathbf{a}, \mathbf{b}\}$ *is a basis of* $\mathcal{V}$ *and* $\mathcal{V}^0 = \mathbb{R}\mathbf{a} \cup \mathbb{R}\mathbf{b}$.

Proof: Suppose that a doubleton $\{\mathbf{a}, \mathbf{b}\} \subset \mathcal{V}^0$ is given such that $\mathbf{d} = \mathbf{a}+\mathbf{b}$. Then

$$\mathbf{d} \cdot (\mathbf{a} - \mathbf{b}) = (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - \mathbf{b} \cdot \mathbf{b} = 0,$$

and hence $\mathbf{a} - \mathbf{b} \in \{\mathbf{d}\}^{\perp}$. Also, we have

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = -2\mathbf{a} \cdot \mathbf{b} = -(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = 1.$$

By **Prop. 5207**, there are exactly two elements $\mathbf{v}$ in $\{\mathbf{d}\}^{\perp}$ such that $\mathbf{v} \cdot \mathbf{v} = 1$; let $\mathbf{u}$ be one of them. Then $\mathbf{a} - \mathbf{b} = \mathbf{u}$ or $\mathbf{a} - \mathbf{b} = -\mathbf{u}$, and hence $2\mathbf{a} = \mathbf{d} + \mathbf{u}$ or $2\mathbf{a} = \mathbf{d} - \mathbf{u}$. Interchanging $\mathbf{u}$ with $-\mathbf{u}$ only interchanges $\mathbf{a}$ and $\mathbf{b}$, and hence the doubleton $\{\mathbf{a}, \mathbf{b}\}$ is determined by $\mathbf{d}$. On the other hand, if we *define* $\mathbf{a}$ and $\mathbf{b}$ by

$$\mathbf{a} := \frac{1}{2}(\mathbf{d} + \mathbf{u}), \quad \mathbf{b} := \frac{1}{2}(\mathbf{d} - \mathbf{u}),$$

then $\{\mathbf{a}, \mathbf{b}\} \subset \mathcal{V}^0$ and $\mathbf{d} = \mathbf{a} + \mathbf{b}$. Since $\{\mathbf{d}, \mathbf{u}\}$ is a basis of $\mathcal{V}$ (see **Prop. 5207**), so is $\{\mathbf{a}, \mathbf{b}\}$.

We now show that $\mathcal{V}^0 = \mathbb{R}\mathbf{a} \cup \mathbb{R}\mathbf{b}$. To this end, let $\mathbf{f} \in \mathcal{V}^0$ be given. Since $\{\mathbf{a}, \mathbf{b}\}$ is a basis for $\mathcal{V}$, we may determine $\xi, \eta \in \mathbb{R}$ such that $\mathbf{f} = \xi\mathbf{a} + \eta\mathbf{b}$. Since $\mathbf{f} \in \mathcal{V}^0$, we have

$$0 = \mathbf{f} \cdot \mathbf{f} = \xi^2\mathbf{a} \cdot \mathbf{a} + 2\xi\eta\mathbf{a} \cdot \mathbf{b} + \eta^2\mathbf{b} \cdot \mathbf{b} = 2\xi\eta\mathbf{a} \cdot \mathbf{b}.$$

From **Cor. 5205**, it follows that $\mathbf{a} \cdot \mathbf{b} \neq 0$. Hence, either $\xi = 0$ or $\eta = 0$; *i.e.,* $\mathbf{f}$ is a scalar multiple of either $\mathbf{a}$ or $\mathbf{b}$. ◊

Illustration: We use this result to discuss representations of $\mathcal{N}$ and $\widehat{\mathcal{F}}$ in the plane. Let a basis $\{\mathbf{a}, \mathbf{b}\}$ for $\mathcal{V}$ be given such that $\{\mathbf{a}, \mathbf{b}\} \subset \mathcal{V}^0$. That we may choose such a basis follows from the previous Theorem; simply choose $\mathbf{d} \in \widehat{\mathcal{F}}_1$, and let $\{\mathbf{a}, \mathbf{b}\}$ be the basis of $\mathcal{V}$ described in that Theorem. We represent $\mathcal{V}^0 = \mathbb{R}\mathbf{a} \cup \mathbb{R}\mathbf{b}$ by the two lines in Figure 52c(1) and divide $\mathcal{V} \setminus \mathcal{V}^0$ into four quadrants as labelled in that Figure. It is easy to see that any vector $\mathbf{v}$ represented in quadrant I can be written as $\mathbf{v} = \alpha\mathbf{a} + \beta\mathbf{b}$, where $\alpha, \beta \in \mathbb{P}^\times$. For such $\mathbf{v}$, we have

$$\mathbf{v} \cdot \mathbf{v} = \alpha^2 \mathbf{a} \cdot \mathbf{a} + 2\alpha\beta \mathbf{a} \cdot \mathbf{b} + \beta^2 \mathbf{b} \cdot \mathbf{b} = 2\alpha\beta \mathbf{a} \cdot \mathbf{b}.$$

Suppose that $\mathbf{a} \cdot \mathbf{b} < 0$; we then have $\mathbf{v} \cdot \mathbf{v} < 0$. Thus, each vector represented in quadrant I belongs to $\mathcal{N}$. By similar analysis, it can be shown that those vectors represented in quadrant III also belong to $\mathcal{N}$, while those represented in quadrants II and IV do not belong to $\mathcal{N}$.

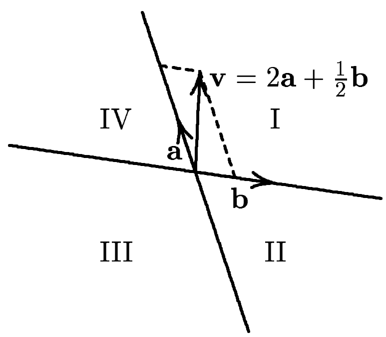

Figure 52c(1)

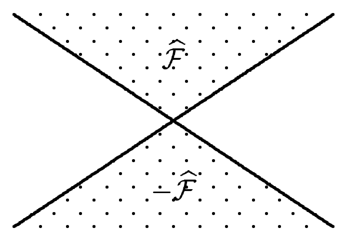

Figure 52c(2)

On the other hand, suppose that $\mathbf{a} \cdot \mathbf{b} > 0$. Then it follows that those vectors represented in quadrants II and IV belong to $\mathcal{N}$, while those represented in quadrants I and III do not.

These two cases are exhaustive; it follows from **Cor. 5205** that $\mathbf{a} \cdot \mathbf{b}$ cannot be zero.

It is customary to represent $\mathcal{N}$ by quadrants which open up towards the top of the page and down towards the bottom of the page (see Figure 52c(2)). It is also customary to forego representing $\mathbf{a}$ and $\mathbf{b}$ explicitly, since $\mathcal{V}^0 = \mathbb{R}\mathbf{a} \cup \mathbb{R}\mathbf{b}$ does not depend on that particular choice of $\mathbf{a}$ and $\mathbf{b}$; only the lines representing $\mathcal{V}^0$ are drawn. Note that points on these lines also represent vectors in $\mathcal{N}$, as $\mathcal{V}^0 \subset \mathcal{N}$.

The two quadrants shaded in Figure 52c(2), along with the lines which form their respective boundaries but excluding the intersection of these lines, represent the partition $\{\widehat{\mathcal{F}}, -\widehat{\mathcal{F}}\}$ of $\mathcal{N}^{\times}$ as described in **Thm. 5204**. It is customary to represent $\widehat{\mathcal{F}}$ by the quadrant opening up, and $-\widehat{\mathcal{F}}$ by the one opening down. Sometimes $\widehat{\mathcal{F}}$ is represented alone, being described by the upper half of Figure 52c(2).

Illustration: Let $\mathbf{d} \in \widehat{\mathcal{F}}_1$ be given. If $\mathbf{a}$ and $\mathbf{b}$ are sides of a parallelogram with diagonal $\mathbf{d}$ and whose sides are parallel to the lines representing the boundary of $\widehat{\mathcal{F}}$, then $\{\mathbf{a}, \mathbf{b}\}$ is that basis of $\mathcal{V}$ described in **Thm. 5208**. The vectors $\mathbf{a} - \mathbf{b}$ and $\mathbf{b} - \mathbf{a}$ (as described in the proof of **Thm. 5208**) are represented in Figure 52d.

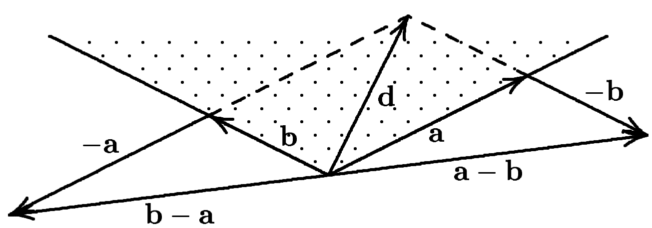

Figure 52d

Let $\mathbf{u} \in \mathcal{V}$ and $\mathbf{e} \in \{\mathbf{a} - \mathbf{b}, \mathbf{b} - \mathbf{a}\}$ be given, and determine $u_1, u_2 \in \mathbb{R}$ such that $\mathbf{u} = u_1\mathbf{d} + u_2\mathbf{e}$. If $\mathbf{u} \in \widehat{\mathcal{F}}_1$, then relative to the basis $\{\mathbf{d}, \mathbf{e}\}$ of $\mathcal{V}$, we have that

$$-1 = \mathbf{u} \cdot \mathbf{u} = (u_1\mathbf{d} + u_2\mathbf{e}) \cdot (u_1\mathbf{d} + u_2\mathbf{e}) = -u_1^2 + u_2^2,$$

or equivalently,

$$u_1^2 - u_2^2 = 1.$$

Thus, the members of $\widehat{\mathcal{F}}_1$ form a branch of a hyperbola since they exhaust solutions to the equations

$$u_1^2 - u_2^2 = 1, \quad u_1 > 0.$$

For example, if we put $\mathbf{u} := \sqrt{2}\mathbf{d} + \mathbf{e}$, we have $u_1^2 - u_2^2 = 1$, and hence $\mathbf{u} \in \widehat{\mathcal{F}}_1$ is on this branch of the hyperbola (see Figure 52e).

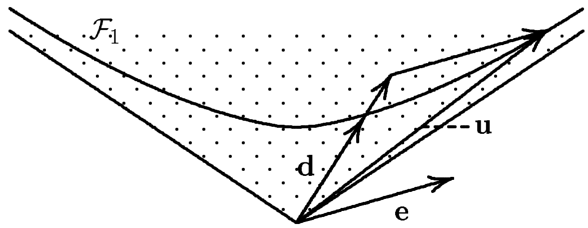

Figure 52e

The asymptotes of the hyperbola are formed by the members of $\mathcal{V}^0$. It can easily be shown, as above, that if $\mathbf{u} \in \mathcal{V}^0$, then $u_1^2 - u_2^2 = 0$; *i.e.,* $u_1 = \pm u_2$.

We conclude this section with some results which will be important in §**7.2**, but which are valid in the context of this section. We assume, as in the beginning of this section, that an inner-product space $\mathcal{V}$ with $\operatorname{sig}^-\mathcal{V} = 1$ is given.

5209 Proposition: *Let $\mathcal{U}$ be a singular subspace of $\mathcal{V}$ and put $\mathcal{T} := \mathcal{U} \cap \mathcal{U}^\perp$. Then $\mathcal{T}$ is the only totally singular subspace of $\mathcal{U}$. We have $\dim \mathcal{T} = 1$ and $\mathcal{U} \setminus \mathcal{T} \subset \mathcal{V}^+$. Moreover, if $\dim \mathcal{U} = 2$, then $\{\mathbf{u}\}^\perp \cap \mathcal{U} = \mathcal{T}$ for all $\mathbf{u} \in \mathcal{U} \setminus \mathcal{T}$.*

Proof: Since $\mathcal{U}$ is not regular, it follows from **Def. 5104** that $\mathcal{T}$ is not the zero space. It is clear from **Def. 5104** that $\mathcal{T}$ is totally singular. By **Prop. 5109** we have $\dim \mathcal{T} \leq \operatorname{sig}^-\mathcal{V} = 1$ and hence $\dim \mathcal{T} = 1$. Hence we may choose $\mathbf{t} \in \mathcal{V}^0$ such that $\mathbf{t} \neq \mathbf{0}$ and $\mathcal{T} = \mathbb{R}\mathbf{t}$.

Now let $\mathbf{u} \in \mathcal{U} \setminus \mathcal{T}$ be given. Since $\mathbf{t} \in \mathcal{T} \subset \mathcal{U}^\perp$, we have $\mathbf{t} \cdot \mathbf{u} = 0$. We cannot have $\mathbf{u} \cdot \mathbf{u} < 0$, *i.e.,* $\mathbf{u} \in \mathcal{V}^-$, because if this were the case, it would follow from **Thm. 5204** that $\mathbf{t} \cdot \mathbf{u} > 0$ or $\mathbf{t} \cdot \mathbf{u} < 0$; *i.e.,* $\mathbf{t} \cdot \mathbf{u} \neq 0$. We cannot have $\mathbf{u} \cdot \mathbf{u} = 0$ because then $\mathbf{u}, \mathbf{t} \in \mathcal{N}^\times$ and it would follow from **Prop. 5205** that $\mathbf{u}$ is a multiple of $\mathbf{t}$; *i.e.,* $\mathbf{u} \in \mathcal{T}$. We conclude that $\mathbf{u} \cdot \mathbf{u} > 0$. Since $\mathbf{u} \in \mathcal{U} \setminus \mathcal{T}$ was arbitrary, we have $\mathcal{U} \setminus \mathcal{T} \subset \mathcal{V}^+$. This inclusion also shows that the only non-zero signal vectors in $\mathcal{U}$ are also in $\mathcal{T}$; *i.e.,* that $\mathcal{T}$ is the only totally singular subspace of $\mathcal{U}$.

Now assume that $\dim \mathcal{U} = 2$ and let $\mathbf{u} \in \mathcal{U} \setminus \mathcal{T}$ be given. Since $\{\mathbf{u}\} \subset \mathcal{U}$ and hence $\mathcal{U}^\perp \subset \{\mathbf{u}\}^\perp$, it follows that $\mathcal{T} = \mathcal{U}^\perp \cap \mathcal{U} \subset \{\mathbf{u}\}^\perp \cap \mathcal{U}$. It follows from $\mathcal{U} \setminus \mathcal{T} \subset \mathcal{V}^+$ that $\mathbf{u} \cdot \mathbf{u} > 0$. Since $\dim \mathcal{U} = 2$, $\{\mathbf{t}, \mathbf{u}\}$ must be a basis of $\mathcal{U}$.

Let $\mathbf{v} \in \{\mathbf{u}\}^\perp \cap \mathcal{U}$ be given. We can then determine $\alpha, \beta \in \mathbb{R}$ such that $\mathbf{v} = \alpha\mathbf{t} + \beta\mathbf{u}$. Since $\mathbf{t} \in \mathcal{U}^\perp$, we have $\mathbf{u} \cdot \mathbf{t} = 0$ and since $\mathbf{v} \in \{\mathbf{u}\}^\perp$, we have $\mathbf{v} \cdot \mathbf{u} = 0$. Hence $0 = \mathbf{u} \cdot \mathbf{v} = \alpha\mathbf{t} \cdot \mathbf{u} + \beta\mathbf{u} \cdot \mathbf{u} = \beta\mathbf{u} \cdot \mathbf{u}$. Since $\mathbf{u} \cdot \mathbf{u} > 0$, it follows that $\beta = 0$ and hence $\mathbf{v} = \alpha\mathbf{t} \in \mathcal{T}$. We conclude that $\{\mathbf{u}\}^\perp \subset \mathcal{T}$ and hence $\{\mathbf{u}\}^\perp \cap \mathcal{U} = \mathcal{T}$. ◊

5210 Corollary: *Every subspace of $\mathcal{V}$ of dimension three or more includes a two-dimensional positive-regular subspace.*

Proof: Let $\mathcal{U}$ be a subspace of $\mathcal{V}$ with $\dim \mathcal{U} \geq 3$. If $\mathcal{U}$ is singular, we can put $\mathcal{T} := \mathcal{U} \cap \mathcal{U}^\perp$ as in **Prop. 5209** and choose a supplement $\mathcal{W}$ of $\mathcal{T}$ in $\mathcal{U}$. Since $\mathcal{W}^\times \subset \mathcal{U} \setminus \mathcal{T}$, it follows from **Prop. 5209** that $\mathcal{W}^\times \subset \mathcal{V}^+$, which means that $\mathcal{W}$ is positive-regular. Since $\dim \mathcal{T} = 1$, we have $\dim \mathcal{W} \geq 3 - 1 = 2$, and hence every two-dimensional subspace of $\mathcal{W}$ is positive-regular.

On the other hand, if $\mathcal{U}$ is regular, then we can apply **Thm. 5107** to $\mathcal{U}$ and choose a positive-regular subspace $\mathcal{W}$ of $\mathcal{U}$ with $\dim \mathcal{W} = \operatorname{sig}^+\mathcal{U} = \dim \mathcal{U} - 1 \geq 2$. Thus, any two-dimensional subspace of $\mathcal{W}$ is positive-regular. ◊

5.3 Minkowskian Spacetimes

In the following Theorem, we bring together many of the results seen in previous chapters. With this Theorem, we set the stage for what follows in the remainder of the book.

Let a Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ such that $\dim \mathcal{V} \geq 2$ and $\operatorname{sig}^-\mathcal{V} = 1$ be given. Select one of the linear cones of the decomposition described in **Thm. 5204**, call it $\widehat{\mathcal{F}}$, and put $\mathcal{F} := \widehat{\mathcal{F}} \cup \{\mathbf{0}\}$. Define the relation $\prec$ on $\mathcal{E}$ by

$$x \prec y :\Longleftrightarrow y - x \in \mathcal{F}$$

for all $x, y \in \mathcal{E}$, the mapping $\tau : \mathcal{F} \to \mathbb{P}$ by

$$\tau(\mathbf{u}) := \sqrt{-\mathbf{u} \cdot \mathbf{u}}$$

for all $\mathbf{u} \in \mathcal{F}$, and the function $\mathsf{t} : \mathsf{Gr}(\prec) \to \mathbb{P}$ by

$$\mathsf{t}(x, y) := \tau(y - x)$$

for all $(x, y) \in \mathsf{Gr}(\prec)$.

5300 Theorem: *The space $\mathcal{E}$ with future cone $\mathcal{F}$ and timelapse t as given above is a genuine (in the sense of* **Def. 3205***) timed flat eventworld. Moreover, the precedence $\prec$ is relativistic and $\mathcal{F}$ is closed (see* §**3.4***). Finally, if $x, y, z \in \mathcal{E}$ are such that $x \prec y \prec z$, then*

$$\mathsf{t}(x, z) = \mathsf{t}(x, y) + \mathsf{t}(y, z)$$

if and only if $y \in [x, z]$.

Proof: We begin with the following Lemma.

5301 Lemma: *The function τ is superadditive in the sense that*

$$\tau(\mathbf{u} + \mathbf{v}) \geq \tau(\mathbf{u}) + \tau(\mathbf{v})$$

for all $\mathbf{u}, \mathbf{v} \in \mathcal{F}$, with equality if and only if either $\mathbf{u}$ or $\mathbf{v}$ is a positive scalar multiple of the other.

Proof: Let $\mathbf{u}, \mathbf{v} \in \mathcal{F}$ be given. If either $\mathbf{u}$ or $\mathbf{v}$ is $\mathbf{0}$, then the assertion is trivial, since $\tau(\mathbf{0}) = 0$. In this case, either $\mathbf{u}$ or $\mathbf{v}$ is a 0-multiple of the other.

So assume that $\mathbf{u}, \mathbf{v} \in \mathcal{F}^\times = \widehat{\mathcal{F}}$. Then

$$\tau(\mathbf{u} + \mathbf{v})^2 = -(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) = \tau(\mathbf{u})^2 - 2\mathbf{u} \cdot \mathbf{v} + \tau(\mathbf{v})^2.$$

By the Reverse Inner-Product Inequality (**Thm. 5201**), we have

$$(\mathbf{u} \cdot \mathbf{v})^2 \geq (\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) = \tau(\mathbf{u})^2 \tau(\mathbf{v})^2,$$

with equality if and only if either $\mathbf{u}$ or $\mathbf{v}$ is a strictly positive scalar multiple of the other. That the scalar multiple is strictly positive follows from the fact that $\mathbf{u}, \mathbf{v} \in \mathcal{F}^\times$.

Since $\mathbf{u} \cdot \mathbf{v} \leq 0$ (**Cor. 5205**), it follows that $-\mathbf{u} \cdot \mathbf{v} \geq \tau(\mathbf{u})\tau(\mathbf{v})$, and hence

$$\tau(\mathbf{u} + \mathbf{v})^2 \geq \tau(\mathbf{u})^2 + 2\tau(\mathbf{u})\tau(\mathbf{v}) + \tau(\mathbf{v})^2 = (\tau(\mathbf{u}) + \tau(\mathbf{v}))^2.$$

Since $\mathsf{Rng}\ \tau \subset \mathbb{P}$, we conclude that

$$\tau(\mathbf{u}+\mathbf{v}) \geq \tau(\mathbf{u}) + \tau(\mathbf{v}).$$

Moreover, we see from the preceding argument that equality holds if and only if either $\mathbf{u}$ or $\mathbf{v}$ is a strictly positive scalar multiple of the other. $\Diamond$

It is easy to see that since $\widehat{\mathcal{F}}$ is a linear cone, then so is $\mathcal{F}$. We see from **Prop. 5206** that $\mathcal{V} = \mathcal{F} - \mathcal{F}$. It therefore follows from **Prop. 3204** that $\mathcal{E}$ with precedence $\prec$ has the structure of a flat eventworld.

That $\mathcal{E}$ is genuine is left as an Exercise. It follows immediately from the definition of τ that $\tau(\alpha\mathbf{u}) = \alpha\tau(\mathbf{u})$ for all $\mathbf{u} \in \mathcal{F}$ and $\alpha \in \mathbb{P}$. With these observations and **Lemma 5301**, we conclude from **Thm. 3302** that $\mathcal{E}$ has the structure of a *timed* flat eventworld.

That the precedence is relativistic and $\mathcal{F}$ is closed is left as an Exercise. The last statement in the Theorem follows directly from **Lemma 5301**. $\Diamond$

5302 Definition: *When the precedence $\prec$ and the timelapse* t *are related to $\mathcal{F}$ and the inner product as in* **Thm. 5300**, *we say that $\mathcal{E}$ is a* **Minkowskian spacetime**. *In this case, we have (see* **Not. 3408**)

$$\mathcal{F}_1 = \{\mathbf{u} \in \mathcal{F} \,|\, \tau(\mathbf{u}) = 1\} = \{\mathbf{u} \in \mathcal{F} \,|\, \mathbf{u}\cdot\mathbf{u} = -1\},$$

and we call members of $\mathcal{F}_1$ **world-directions**. *When $\mathbf{u} \in \mathcal{V}^+ \cup \{\mathbf{0}\}$, we put*

$$|\mathbf{u}| := \sqrt{\mathbf{u}\cdot\mathbf{u}}.$$

Remark: The previous Theorem tells us that given a non-genuine Euclidean space $\mathcal{E}$ with $\mathsf{dim}\ \mathcal{E} \geq 2$ and $\mathsf{sig}^-\mathcal{V} = 1$, we may endow $\mathcal{E}$ with the structure of a relativistic timed flat eventworld by choosing a piece $\widehat{\mathcal{F}}$ of the linear-cone decomposition described in **Thm. 5204**, and then defining $\prec$ and t as described in **Thm. 3302**. In essence, the only assumptions made about $\mathcal{E}$ are that $\mathsf{dim}\ \mathcal{E} \geq 2$ and $\mathsf{sig}^-\mathcal{V} = 1$.

What about a converse of this Theorem? We pose the following question: suppose that $\mathcal{E}$ and $\mathcal{E}'$ are both Minkowskian spacetimes with precedences $\prec$ and $\prec'$ respectively. If $\varphi : \mathcal{E} \to \mathcal{E}'$ is such that φ is invertible and $x \prec y \iff \varphi(x) \prec' \varphi(y)$ for all $x, y \in \mathcal{E}$ (that is, φ is an *order-isomorphism*), what may we say about φ in terms of the flat space structures of $\mathcal{E}$ and $\mathcal{E}'$? In other words, if we know that the orders on $\mathcal{E}$ and $\mathcal{E}'$ and related isomorphically by φ, how are the flat space structures of $\mathcal{E}$ and $\mathcal{E}'$ related by φ? Remarkably, when $\dim \mathcal{E}, \dim \mathcal{E}' \geq 3$, the answer to this question is that φ is a flat isomorphism (see [**8**] and the literature cited there). That is, there is a linear mapping $\mathbf{L} : \mathcal{V} \to \mathcal{V}'$ (where $\mathcal{V} = \mathcal{E} - \mathcal{E}$ and $\mathcal{V}' = \mathcal{E}' - \mathcal{E}'$) such that $\varphi(x) = q + \mathbf{L}(x - q)$ for all $x \in \mathcal{E}$ and $q \in \mathcal{E}$. Moreover, $\mathbf{L}$ is a positive scalar multiple of an orthogonal (inner-product preserving) transformation. One can conclude that the precedence relation alone determines the flat space structure in a Minkowskian spacetime whose dimension is three or more. Also note that it was not necessary to assume anything about timelapse functions! In fact, the timelapse function is determined by the precedence relation to within a strictly positive scale factor. The reader is encouraged to investigate the case when $\dim \mathcal{E} = \dim \mathcal{E}' = 2$ to see that, in this case, there are infinitely many flat space structures compatible with the precedence relation.

We now consider $\mathcal{E}$ to be a Minkowskian spacetime in the sense of **Def. 5302**.

Since $\{\mathcal{F}, -\mathcal{F}\}$ is determined by the inner product on $\mathcal{V}$ (see **Def. 5302** and **Thm. 5204**), we see that the precedence relation is determined by the inner product to within reversal. We may also describe the signal relation $\rightharpoonup$ (as given in **Def. 1501**) by way of the inner product as follows.

5303 Theorem: *Let $x, y \in \mathcal{E}$ be given such that $x \prec y$. Then the following are equivalent:*

(1) $x \rightharpoonup y$.

(2) $(y - x) \cdot (y - x) = 0$.

(3) $[\![x, y]\!] = [x, y]$.

Proof: (1) $\Longrightarrow$ (2). Since $x \prec y$, we must have $(y - x) \cdot (y - x) \leq 0$; so suppose that $(y - x) \cdot (y - x) < 0$. Then we may choose $\mathbf{e} \in \{y - x\}^{\perp}$

such that $\mathbf{e} \cdot \mathbf{e} = 1$ and $\alpha \in \mathbb{P}^\times$ such that $\alpha < \frac{1}{2}\tau(y - x)$. With m as the midpoint of $[x, y]$, define

$$q_+ := m + \alpha\mathbf{e}, \quad q_- := m - \alpha\mathbf{e}.$$

One may easily show that $q_+, q_- \in [\![x, y]\!]$. But since $[\![x, y]\!]$ is totally ordered by $\prec$, we must have either $q_+ \prec q_-$ or $q_- \prec q_+$, and thus

$$0 \geq (q_+ - q_-) \cdot (q_+ - q_-) = 4\alpha^2 > 0.$$

As this is a contradiction, we must have $(y - x) \cdot (y - x) = 0$.

(2) $\implies$ (3). Since the relation $\prec$ is connected (see **Defs. 3200** and **3203**), we have that $[x, y] \subset [\![x, y]\!]$. To show that $[\![x, y]\!] \subset [x, y]$, let $z \in [\![x, y]\!]$ be given. Then $x \prec z \prec y$, and since

$$\tau(y - x)^2 = -(y - x) \cdot (y - x) = 0,$$

we have

$$\mathsf{t}(x, z) + \mathsf{t}(z, y) \leq \mathsf{t}(x, y) = \tau(y - x) = 0.$$

Since $\mathsf{Rng}\ \mathsf{t} \subset \mathbb{P}$, this inequality must actually be equality, and by **Thm. 5300**, $z \in [x, y]$. As z was arbitrary, we see that $[\![x, y]\!] \subset [x, y]$. Since $[x, y] \subset [\![x, y]\!] \subset [x, y]$, it follows that $[\![x, y]\!] = [x, y]$.

(3) $\implies$ (1). It follows from (3) that $[\![x, y]\!]$ is totally ordered, and hence from **Def. 1501** that $x \rightharpoonup y$. $\Diamond$

The following Proposition will be illustrated while discussing the emission and reception of electromagnetic signals in §**5.8**. We remark that if $\mathcal{L}$ is a straight worldline, then it follows from **Def. 3300**(2) that $\mathsf{t}_\mathcal{L} = \mathsf{t}|_\mathcal{L}$.

5304 Proposition: *Let $\mathcal{L}$ be a straight material worldline, and let $e \in \mathcal{E}$ be given. Then there is exactly one $x \in \mathcal{L}$ such that $x \rightharpoonup e$ and exactly one $y \in \mathcal{L}$ such that $e \rightharpoonup y$. In addition, there is exactly one $z \in \mathcal{L}$ such that $e - z \in \{\mathbf{u}\}^\perp$. This z is the midpoint of $[x, y]$, so that $z - x = y - z$. Moreover, $|e - z| = \frac{1}{2}\mathsf{t}(x, y) = \frac{1}{2}\mathsf{t}_\mathcal{L}(x, y)$.*

Proof: Since $\mathcal{L}$ is a straight material worldline, we may choose $q \in \mathcal{L}$ and $\mathbf{u} \in \mathcal{F}_1$ such that $\mathcal{L} = q + \mathbb{R}\mathbf{u}$. Since $\mathbb{R}\mathbf{u}$ and $\{\mathbf{u}\}^\perp$ are supplementary (**Prop. 5200**), we may determine $\eta \in \mathbb{R}$ and $\mathbf{w} \in \{\mathbf{u}\}^\perp$ such that $e - q = \eta\mathbf{u} + \mathbf{w}$.

Put $x := e - \mathbf{w} - |\mathbf{w}|\mathbf{u}$. Then we have $x = q + (\eta - |\mathbf{w}|)\mathbf{u} \in \mathcal{L}$ and

$$\begin{aligned}(e-x)\cdot(e-x) &= (\mathbf{w}+|\mathbf{w}|\mathbf{u})\cdot(\mathbf{w}+|\mathbf{w}|\mathbf{u})\\ &= |\mathbf{w}|^2 - |\mathbf{w}|^2\\ &= 0.\end{aligned}$$

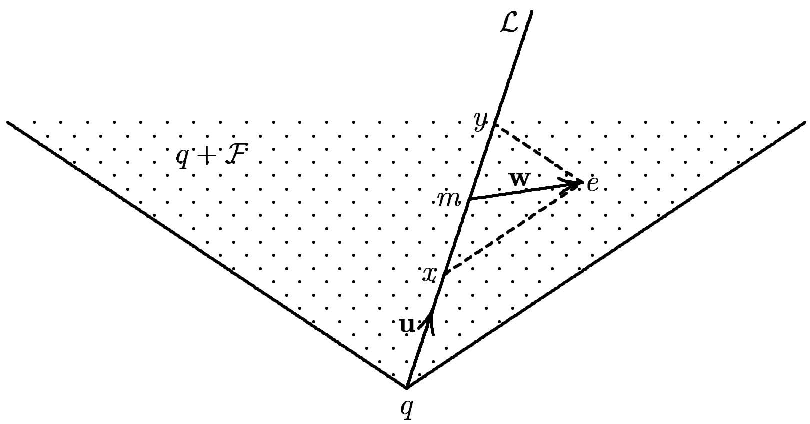

Figure 53a

We also have that

$$\begin{aligned}(e-x)\cdot\mathbf{u} &= (\mathbf{w}+|\mathbf{w}|\mathbf{u})\cdot\mathbf{u}\\ &= -|\mathbf{w}| < 0,\end{aligned}$$

and hence $e - x \in \mathcal{F}$. It follows from **Thm. 5303** that $x \rightharpoonup e$. It is left as an Exercise that there can be no other event in $\mathcal{L}$ with this property.

The proof that there is exactly one $y \in \mathcal{L}$ such that $e \rightharpoonup y$ is similar.

Since $\mathbf{u} \in \mathcal{F}_1$, we have $\mathbb{R}\mathbf{u} + \{\mathbf{u}\}^{\perp} = \mathcal{V}$ and $\mathbb{R}\mathbf{u} \cap \{\mathbf{u}\}^{\perp} = \{\mathbf{0}\}$. Hence $\mathcal{L} \cap (e + \{\mathbf{u}\}^{\perp})$ is a singleton; we let z be the member of this singleton. Clearly, z is the only event in $\mathcal{L}$ satisfying $e - z \in \{\mathbf{u}\}^{\perp}$.

We know that $x = e - \mathbf{w} - |\mathbf{w}|\mathbf{u}$; we can analogously show that y must satisfy $y = e - \mathbf{w} + |\mathbf{w}|\mathbf{u}$. From this, we see that the midpoint m of $[x, y]$ is given by

$$\begin{aligned}m &= x + \tfrac{1}{2}(y - x)\\ &= e - \mathbf{w} - |\mathbf{w}|\mathbf{u} + \tfrac{1}{2}(2|\mathbf{w}|\mathbf{u})\\ &= e - \mathbf{w}.\end{aligned}$$

Hence, $e - m = \mathbf{w} \in \{\mathbf{u}\}^\perp$. Since $m \in [x, y] \subset \mathcal{L}$, we find that $m \in \mathcal{L}$. Since z is the only event in $\mathcal{L}$ satisfying $e - z \in \{\mathbf{u}\}^\perp$, we see that in fact, $m = z$.

Finally, we see that

$$\begin{aligned} \mathsf{t}(x, y)^2 &= -(y - x) \cdot (y - x) \\ &= -(2|\mathbf{w}|\mathbf{u}) \cdot (2|\mathbf{w}|\mathbf{u}) \\ &= 4|\mathbf{w}|^2 \\ &= 4|e - z|^2. \end{aligned}$$

The desired result follows immediately. ◊

5305 Definition: *Let $\mathcal{L}$ be a straight material worldline, and let $e \in \mathcal{E}$ be given. We define the* **distance from** e **to** $\mathcal{L}$ *by*

$$\mathsf{dst}(e, \mathcal{L}) := \frac{1}{2}\mathsf{t}(x, y),$$

where x and y are determined by e and $\mathcal{L}$ as in the previous Proposition.

The careful reader may have noticed that in this definition, distance is defined in terms of the timelapse along a worldpath. How is this possible? In special relativity, there is no distinction between "distance" and "timelapse".

Distances and timelapses are specified relative to physical units, such as second ("s"), meter ("m"), year ("yr"), inch ("in"), or mile ("mi"). These various units are related by conversion factors. For example, the unit "s" (second) is related to "m" (meter) by

$$1s = 299,792,458m.$$

This conversion is exact, having been settled upon in 1986 by international agreement. This number is the figure usually given for the "speed of light". The exact conversion $1in = 2.54cm$ has been "legalized" in the U.S. since 1959, and in the United Kingdom since 1963.

When giving distances in terms of units such as "second" or "year", it is customary to use the term "light-second" or "light-year". Then the "speed of light" is the dimensionless number, 1, as in "1 light-year per year".

The definition of distance given by **Def. 5305** is the one that is the conceptual basis for distance measurements by radar and laser beams. It contrasts with the pre-classical notion of distance (see §**4.1**) which is the conceptual basis for distance measurements by rulers and measuring tapes.

5.4 Spacetime Decompositions

In this section, we consider how one might decompose a Minkowskian spacetime relative to a given world-direction (in the sense of **Def. 5302**). We may imagine this world-direction as representing an "observer" whose worldpath is a straight line, the direction of which is the given world-direction. We may also think of this world-direction as generating a reference frame, the locations of which are all straight worldlines (*i.e.,* "observers"), the direction of each being the given world-direction.

For the remainder of this section, let a Minkowskian spacetime $\mathcal{E}$ be given, with all notations as introduced in §**5.3**.

Let $\mathbf{d} \in \mathcal{F}_1$ be given. Upon examining **Def. 4200**, it seems plausible that a candidate for the reference frame determined by $\mathbf{d}$ is $\{x + \mathbb{R}\mathbf{d} \mid x \in \mathcal{E}\}$.

But what of instants and the bijection as described in **Thm. 4201**? Given the decomposition of $\mathcal{E}$ as described in that Theorem, it seems reasonable to use the decomposition $(\mathbb{R}\mathbf{d}, \{\mathbf{d}\}^{\perp})$ of $\mathcal{V}$ for an analogous decomposition of $\mathcal{E}$. One is immediately led to guess that $\{x + \{\mathbf{d}\}^{\perp} \mid x \in \mathcal{E}\}$ would form a suitable collection of instants. Moreover, a natural timelapse on the set of instants is apparent: simply require that the timelapse between two instants be that multiple of $\mathbf{d}$ which separates them. Such a timelapse can naturally be extended to all of $\mathcal{E}$ as suggested by **Prop. 2402**.

We base a formal development of the reference frame relative to $\mathbf{d}$ upon the foregoing observations. The result is the following definition of relative precedence.

5400 Definition: *The* **precedence relative to d**, *denoted by* $\prec_{\mathbf{d}}$, *is defined by*

$$x \prec_{\mathbf{d}} y \;:\Longleftrightarrow\; (y - x) \cdot \mathbf{d} \leq 0$$

for all $x, y \in \mathcal{E}$. *The* **timelapse relative to d**, $\mathsf{t}_{\mathbf{d}} : \mathsf{Gr}(\prec_{\mathbf{d}}) \to \mathbb{P}$, *is defined by*

$$\mathsf{t}_{\mathbf{d}}(x, y) := -(y - x) \cdot \mathbf{d}$$

for $(x, y) \in \mathsf{Gr}(\prec_{\mathbf{d}})$. *The* **signed timelapse relative to d**, $\bar{\mathsf{t}}_{\mathbf{d}} : \mathcal{E} \times \mathcal{E} \to \mathbb{R}$, *is given by the same formula for all* $x, y \in \mathcal{E}$.

Note that in Figure 54a, we have $x \prec_{\mathbf{d}} y$, $y - x = t\mathbf{d} + \alpha\mathbf{e}$, and $\mathsf{t}_{\mathbf{d}}(x, y) = t$.

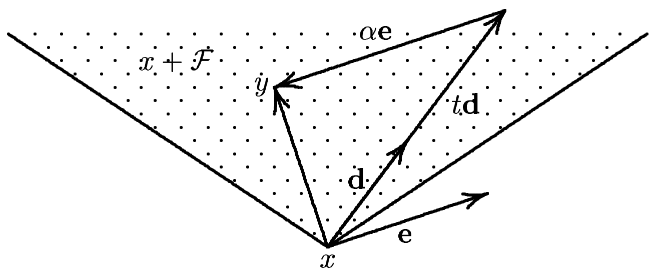

Figure 54a

It is left as an Exercise to verify that $\prec_{\mathbf{d}}$ is reflexive, transitive, and total (and hence classical). We may define (as in **Def. 1201**) **simultaneity relative to d** by

$$x \sim_{\mathbf{d}} y :\Longleftrightarrow x \prec_{\mathbf{d}} y \text{ and } y \prec_{\mathbf{d}} x$$

for all $x, y \in \mathcal{E}$. We have that $\sim_{\mathbf{d}}$ is an equivalence relation (see **Thm. 1202**), and that if $x, y \in \mathcal{E}$, then $x \sim_{\mathbf{d}} y$ if and only if $(y-x)\cdot \mathbf{d} = 0$. As in **Def. 1205**, we define the **past**, **present**, and **future relative to d**; and we therefore have, for all $x \in \mathcal{E}$,

$$\begin{aligned} \mathsf{Past}_{\mathbf{d}}(x) &:= \{z \in \mathcal{E} \mid (z-x)\cdot \mathbf{d} > 0\}, \\ \mathsf{Pres}_{\mathbf{d}}(x) &:= \{z \in \mathcal{E} \mid (z-x)\cdot \mathbf{d} = 0\}, \\ \mathsf{Fut}_{\mathbf{d}}(x) &:= \{z \in \mathcal{E} \mid (z-x)\cdot \mathbf{d} < 0\}. \end{aligned}$$

It follows that the partition of $\mathcal{E}$ determined by $\sim_{\mathbf{d}}$; that is, the set of all instants relative to $\mathbf{d}$, is given by

$$\Gamma_{\mathbf{d}} = \{\mathsf{Pres}_{\mathbf{d}}(x) \mid x \in \mathcal{E}\}.$$

Given $x \in \mathcal{E}$, we have

$$\begin{aligned} \mathsf{Pres}_{\mathbf{d}}(x) &= \{z \in \mathcal{E} \mid (z-x)\cdot \mathbf{d} = 0\} \\ &= \{z \in \mathcal{E} \mid z - x \in \{\mathbf{d}\}^{\perp}\} \\ &= x + \{\mathbf{d}\}^{\perp}. \end{aligned}$$

Hence

$$\Gamma_{\mathbf{d}} = \{x + \{\mathbf{d}\}^{\perp} \mid x \in \mathcal{E}\},$$

which validates our earlier conjecture. In addition, the relation $\tilde{\prec}_{\mathbf{d}}$ on $\Gamma_{\mathbf{d}}$, defined for all $\sigma, \sigma' \in \Gamma_{\mathbf{d}}$ by $\sigma \tilde{\prec}_{\mathbf{d}} \sigma' :\Longleftrightarrow x \prec_{\mathbf{d}} x'$ for all $x \in \sigma$, $x' \in \sigma'$, is a total order (see **Prop. 1403**).

It is also left as an Exercise to verify that the precedence $\prec_{\mathbf{d}}$ and timelapse $\bar{\mathsf{t}}_{\mathbf{d}}$ give $\mathcal{E}$ the structure of a classical timed eventworld. Moreover, we may define $\mathsf{dst}_{\mathbf{d}} : \mathsf{Gr}(\sim_{\mathbf{d}}) \to \mathbb{P}$ by

$$\mathsf{dst}_{\mathbf{d}}(x, y) := |y - x|$$

for all $(x, y) \in \mathsf{Gr}(\sim_{\mathbf{d}})$. This makes sense, since if $x \sim_{\mathbf{d}} y$, then $y - x \in \{\mathbf{d}\}^{\perp} \subset \mathcal{V}^{+} \cup \{\mathbf{0}\}$. This structural ingredient gives $\mathcal{E}$ (with $\prec_{\mathbf{d}}$ and $\mathsf{t}_{\mathbf{d}}$) the structure of a pre-classical spacetime.

It is left as an Exercise to show that the set of all worldlines whose direction space is $\mathbb{R}\mathbf{d}$; that is,

$$\mathsf{F}_{\mathbf{d}} := \{q + \mathbb{R}\mathbf{d} \mid q \in \mathcal{E}\},$$

is a reference frame (as in **Def. 4200**). Thus, the pre-classical spacetime $\mathcal{E}$ becomes a Newtonian spacetime by singling out the reference frame $\mathsf{F}_{\mathbf{d}}$. Members of $\mathsf{F}_{\mathbf{d}}$ are called **locations relative to d**; moreover, we see that $\mathsf{F}_{\mathbf{d}}$ has the natural structure of a Euclidean space (see **Thm. 4202**). It is left as an Exercise to verify that $\tilde{\mathsf{d}}_{\mathbf{d}}$ (the distance function for $\mathsf{F}_{\mathbf{d}}$; see **Thm. 4202**) satisfies

$$\tilde{\mathsf{d}}_{\mathbf{d}}(x + \mathbb{R}\mathbf{d}, y + \mathbb{R}\mathbf{d}) = \sqrt{(y - x) \cdot (y - x) + ((y - x) \cdot \mathbf{d})^2} \qquad (54.1)$$

for all $x, y \in \mathcal{E}$.

How does the precedence (relative to $\mathbf{d}$) compare to relativistic precedence? It is easy to verify that for all $x, y \in \mathcal{E}$, we have $x \prec y \implies x \prec_{\mathbf{d}} y$, and thus $\prec_{\mathbf{d}}$ is a coarser relation on $\mathcal{E}$ than $\prec$. It is left as an Exercise to show that for $x, y \in \mathcal{E}$, $x \prec y$ if and only if we have $x \prec_{\mathbf{d}} y$ for all $\mathbf{d} \in \mathcal{F}_1$. We also have, given $x \in \mathcal{E}$, that $\mathsf{Fut}(x) \subset \mathsf{Fut}_{\mathbf{d}}(x)$ and $\mathsf{Past}(x) \subset \mathsf{Past}_{\mathbf{d}}(x)$.

We conclude this section with an important result. Given two different world-directions (which we may imagine to represent two different "observers"), it is natural to ask how the classical structure determined by one of them is "perceived" by the other. The rest of this chapter is concerned to a large extent with applications of this Theorem, and so we defer an interpretation to subsequent sections.

5401 Theorem: *Let* $\mathbf{d}_1$ *and* $\mathbf{d}_2$ *be two distinct world-directions. Also, put* $\mathcal{D}_1 := \mathbb{R}\mathbf{d}_1$, $\mathcal{D}_2 := \mathbb{R}\mathbf{d}_2$, $\mathcal{W}_1 := \{\mathbf{d}_1\}^\perp$, *and* $\mathcal{W}_2 := \{\mathbf{d}_2\}^\perp$. *Then* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are positive-regular and* $\mathcal{W}_1$ *and* $\mathcal{W}_2$ *are negative-regular. Moreover,*

(1) *There is exactly one* $\mu \in \mathbb{R}$, *exactly one* $\nu \in \mathbb{P}^\times$, *and exactly one* $\mathbf{e}_1 \in \mathcal{W}_1$ *with* $|\mathbf{e}_1| = 1$ *such that*

$$\mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1).$$

(2) *We have* $\nu \in]0,1[$ *and* $\mu \in 1 + \mathbb{P}^\times$; *moreover,* μ *and* ν *are related by*

$$\mu = \frac{1}{\sqrt{1-\nu^2}} \quad \text{and} \quad \nu = \sqrt{1 - \frac{1}{\mu^2}}.$$

(3) *With* μ *and* ν *as determined in (1), there is exactly one* $\mathbf{e}_2 \in \mathcal{W}_2$ *with* $|\mathbf{e}_2| = 1$ *such that*

$$\mathbf{d}_1 = \mu(\mathbf{d}_2 - \nu\mathbf{e}_2).$$

(4) *We have*

$$\mathbf{e}_2 = \mu(\mathbf{e}_1 + \nu\mathbf{d}_1)$$

and

$$\mathbf{e}_1 = \mu(\mathbf{e}_2 - \nu\mathbf{d}_2).$$

(5) *We have*

$$-\mathbf{d}_1 \cdot \mathbf{d}_2 = \mu = \mathbf{e}_1 \cdot \mathbf{e}_2$$

and

$$-\mathbf{d}_1 \cdot \mathbf{e}_2 = \mu\nu = \mathbf{d}_2 \cdot \mathbf{e}_1.$$

Proof: It is immediate that $\mathcal{D}_1$ and $\mathcal{D}_2$ are negative-regular. That $\mathcal{W}_1$ and $\mathcal{W}_2$ are positive-regular follows from **Prop. 5200**.

(1). Since $\mathcal{D}_1$ and $\mathcal{W}_1$ are supplementary, there is exactly one $\mu \in \mathbb{R}$ and exactly one $\mathbf{f} \in \mathcal{W}_1$ such that $\mathbf{d}_2 = \mu\mathbf{d}_1 + \mathbf{f}$. Since neither $\mathbf{d}_1$

nor $\mathbf{d}_2$ is a multiple of the other, we must have $\mathbf{f} \neq \mathbf{0}$. Since $\mathcal{W}_1$ is positive-regular and $\mathcal{D}_2$ is negative-regular, it follows that $\mathbf{d}_2 \neq \mathbf{f}$ and hence $\mu \neq 0$. Hence, since $\mathcal{W}_1$ is positive-regular, there is exactly one $\nu \in \mathbb{P}^\times$ and exactly one $\mathbf{e}_1 \in \mathcal{W}_1$ such that $|\mathbf{e}_1| = 1$ and $\mathbf{f} = \mu\nu\mathbf{e}_1$. Thus, $\mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1)$.

(2). From (1), we see that $\mathbf{d}_2 \cdot \mathbf{d}_1 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1) \cdot \mathbf{d}_1 = -\mu$, and hence $\mu = -\mathbf{d}_1 \cdot \mathbf{d}_2$. From the Reverse Inner-Product Inequality (**Thm. 5201**), we have

$$(\mathbf{d}_1 \cdot \mathbf{d}_2)^2 > (\mathbf{d}_1 \cdot \mathbf{d}_1)(\mathbf{d}_2 \cdot \mathbf{d}_2) = 1,$$

as neither $\mathbf{d}_1$ nor $\mathbf{d}_2$ is a multiple of the other. From **Cor. 5205**, it follows that $\mathbf{d}_1 \cdot \mathbf{d}_2 < 0$, and hence $\mathbf{d}_1 \cdot \mathbf{d}_2 < -1$. Thus, $\mu \in 1 + \mathbb{P}^\times$.

From (1), it follows that

$$-1 = \mathbf{d}_2 \cdot \mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1) \cdot \mu(\mathbf{d}_1 + \nu\mathbf{e}_1) = -\mu^2 + \mu^2\nu^2.$$

Since $\mu, \nu \in \mathbb{P}^\times$ and $\mu > 1$, this relationship results in

$$\nu = \sqrt{1 - \frac{1}{\mu^2}}.$$

Thus, since $\mu > 1$, we must have $\nu \in]0, 1[$. It follows that

$$\mu = \frac{1}{\sqrt{1 - \nu^2}}.$$

(3). Since $\mu = -\mathbf{d}_1 \cdot \mathbf{d}_2$, we have $(\mathbf{d}_1 - \mu\mathbf{d}_2) \cdot \mathbf{d}_2 = 0$, and hence $\mathbf{d}_1 - \mu\mathbf{d}_2 \in \mathcal{W}_2$. Thus, there is exactly one $\lambda \in \mathbb{P}^\times$ and exactly one $\mathbf{e}_2 \in \mathcal{W}_2$ such that $|\mathbf{e}_2| = 1$ and $\mathbf{d}_1 - \mu\mathbf{d}_2 = -\lambda\mathbf{e}_2$. Then

$$\lambda^2 = (-\lambda\mathbf{e}_2) \cdot (-\lambda\mathbf{e}_2) = (\mathbf{d}_1 - \mu\mathbf{d}_2) \cdot (\mathbf{d}_1 - \mu\mathbf{d}_2) = \mu^2 - 1.$$

Hence, since $\lambda \in \mathbb{P}^\times$,

$$\lambda = \sqrt{\mu^2 - 1} = \mu\sqrt{1 - \frac{1}{\mu^2}} = \mu\nu.$$

Hence, we have $\mathbf{d}_1 - \mu\mathbf{d}_2 = -\mu\nu\mathbf{e}_2$, or $\mathbf{d}_1 = \mu(\mathbf{d}_2 - \nu\mathbf{e}_2)$.

(4) and (5) are easy consequences of (1), (2), and (3). ◊

With the notations of the previous Theorem, μ is called the **time-dilation between $\mathbf{d}_1$ and $\mathbf{d}_2$**, ν is called the **relative speed between $\mathbf{d}_1$ and $\mathbf{d}_2$**, and $\mathbf{e}_1$ is called the **direction of motion of $\mathbf{d}_2$ relative to $\mathbf{d}_1$** (with an analogous nomenclature for $\mathbf{e}_2$). The appropriateness of these terms will be discussed in detail in the remainder of this chapter.

Remark: With the notations of the previous Theorem, let $q \in \mathcal{E}$ be given, and put $\mathcal{H} := q + \mathsf{Lsp}\,\{\mathbf{d}_1, \mathbf{d}_2\} = q + \mathsf{Lsp}\,\{\mathbf{d}_1, \mathbf{e}_1\} = q + \mathsf{Lsp}\,\{\mathbf{d}_2, \mathbf{e}_2\}$. We may define

$$x, t \,:\, \mathcal{H} \to \mathbb{R} \text{ and } x', t' \,:\, \mathcal{H} \to \mathbb{R}$$

so that for all $e \in \mathcal{H}$, we have

$$e = x(e)\mathbf{d}_1 + t(e)\mathbf{e}_1 = x'(e)\mathbf{d}_2 + t'(e)\mathbf{e}_2.$$

Then it follows directly from (3) and the second equation in (4) of **Thm. 5401** that

$$\begin{aligned} x' &= \mu\,(x - \nu t) \\ t' &= \mu\,(t - \nu x). \end{aligned}$$

This is the form which **Thm. 5401** usually takes in the literature. The transformation by which x' and t' are expressed in terms of x and t is often called a *Lorentz transformation.*

5.5 Some Applications

As promised in the previous section, we offer a few examples as applications of **Thm. 5401**. Let a Minkowskian spacetime $\mathcal{E}$ be given.

—"Addition" of relative speeds

Consider the following scenario: we have a moving train, and a person running on the train in the direction in which the train is moving. We assume, for simplicity, that the worldpaths of the Earth, train, and person are all straight (and thus each moves with constant relative speed with respect to each of the others). We also assume that the directions of motion are such that we are able to confine our discussion to a two-dimensional spacetime.

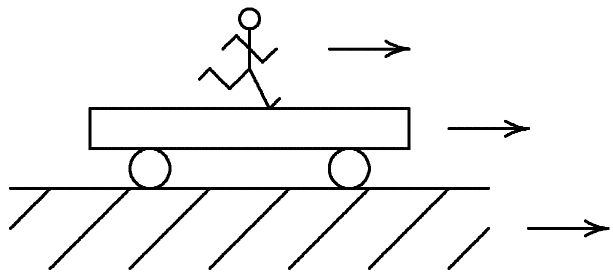

Figure 55(a)

If the world-direction of the person, train, and Earth are $\mathbf{d}_p$, $\mathbf{d}_t$, and $\mathbf{d}_e$, respectively, and ν_1 is the speed of the person relative to the train, ν_2 is the speed of the the train relative to the Earth, and ν is the speed of the person relative to the Earth, how may we relate ν_1 and ν_2 to ν?

In a classical world, we would have $\nu = \nu_1 + \nu_2$; that is, relative speeds would add. But this cannot be the case in a relativistic world, for if $\nu_1 > \frac{1}{2}$ and $\nu_2 > \frac{1}{2}$, we would have $\nu > 1$; but recall from **Thm. 5401**(2) that a relative speed greater than 1 is impossible. So something is awry with regard to our classical intuitions. We proceed to investigate this matter in greater detail using the tools of the previous section.

We may draw a spacetime diagram of the situation as follows,

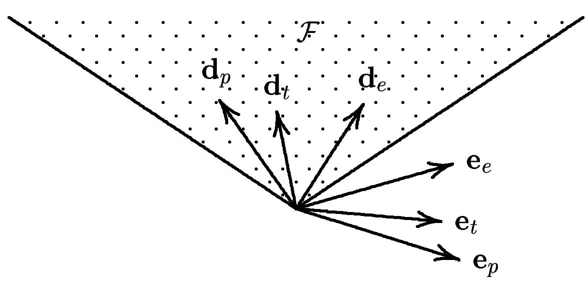

Figure 55b

where $\mathbf{e}_p$, $\mathbf{e}_t$, and $\mathbf{e}_e$ take on the obvious interpretation. That $\mathbf{e}_p$, $\mathbf{e}_t$, and $\mathbf{e}_e$ are all on the "same side" of the diagram is a consequence of the fact that the person, train, and Earth are all moving, roughly, in the same "direction".

From **Thm. 5401** (with subscripts "1" and "2" replaced with "p" and "t", respectively), we must have

$$\mathbf{d}_t = \mu_1(\mathbf{d}_p + \nu_1 \mathbf{e}_p) \quad \text{and} \quad \mathbf{e}_t = \mu_1(\mathbf{e}_p + \nu_1 \mathbf{d}_p),$$

where μ_1 and ν_1 are related as in **Thm. 5401**(2). We must also have

$$\mathbf{d}_e = \mu_2(\mathbf{d}_t + \nu_2 \mathbf{e}_t),$$

where μ_2 and ν_2 are related in the same way as μ_1 and ν_1.

We now substitute in the previous equation the expressions for $\mathbf{d}_t$ and $\mathbf{e}_t$, resulting in

$$\begin{aligned} \mathbf{d}_e &= \mu_2(\mu_1(\mathbf{d}_p + \nu_1 \mathbf{e}_p) + \nu_2 \mu_1(\mathbf{e}_p + \nu_1 \mathbf{d}_p)) \\ &= \mu_1\mu_2((1 + \nu_1\nu_2)\mathbf{d}_p + (\nu_1 + \nu_2)\mathbf{e}_p) \\ &= \mu_1\mu_2(1 + \nu_1\nu_2)\left(\mathbf{d}_p + \frac{\nu_1 + \nu_2}{1 + \nu_1\nu_2}\mathbf{e}_p\right). \end{aligned}$$

Thus, we see that the relative speed between the person and the Earth is given by

$$\nu = \frac{\nu_1 + \nu_2}{1 + \nu_1\nu_2}.$$

The reader may verify that in defining

$$\mu := \mu_1\mu_2(1 + \nu_1\nu_2),$$

we have that ν and μ are related as in **Thm. 5401**(2).

Thus, the relative speed ν is not simply the sum of ν_1 and ν_2, but a "scaled version" of this. One may easily verify that if $\nu_1, \nu_2 \in \,]0, 1[$, then we also have $\nu \in \,]0, 1[$.

We illustrate the previous results with a few calculations. If the speed of the train is $\nu_2 = 100\,km/h = 0.0000000927$ relative to the Earth, and the speed of the person relative to the train is $\nu_1 = 8\,km/h = 0.00000000741$, one finds that the speed of the person relative to the Earth is given by

$$\nu \approx 0.1000692285594455 \approx 107.999999999999926\,km/h.$$

Thus, we see that for ordinary calculations, the usual classical formula; *i.e.*, $\nu = \nu_1 + \nu_2$, is a good approximation.

Now suppose (rather unrealistically) that our train is travelling at $\nu_2 = 0.2 = 216,000,000\,km/h$ relative to the Earth, and that the person is moving at $\nu_1 = 0.1 = 108,000,000\,km/h$ relative to the train. In this case, our analysis gives us that the person is travelling at a speed of $\nu = 0.29412 = 317,647,059\,km/h$ relative to the Earth, which is a bit less than 0.3. Thus, we see that very great relative speeds are necessary for the relativistic formula to differ even slightly from the classical formula.

One may also analyze cases where the direction of motion of the person is opposite that of the train. Such variations are treated in the Exercises.

—**Lorentz-Fitzgerald contraction**

We now consider the path of a rigid rod; that is, an object whose ends may be described by paths which are parallel straight worldlines.

So let $\mathcal{L}$ and $\mathcal{L}'$ be two distinct parallel straight worldlines with world-direction $\mathbf{u}$. It easily follows that we may determine $\mathbf{w} \in \{\mathbf{u}\}^{\perp}$ with $|\mathbf{w}| = 1$ and $\alpha \in \mathbb{P}^{\times}$ such that $\mathcal{L}' = \mathcal{L} + \alpha\mathbf{w}$. We say that α is the **length** of the rigid rod whose ends are described by $\mathcal{L}$ and $\mathcal{L}'$.

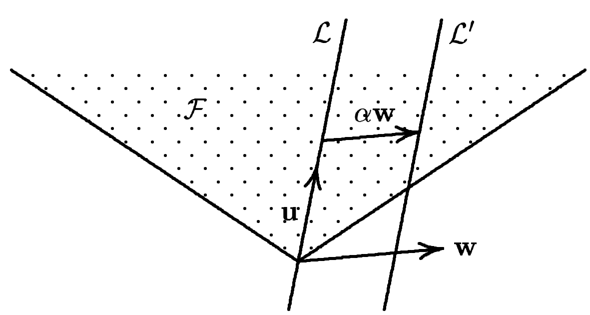

Figure 55c

Remark: We have $\alpha = \widetilde{\mathsf{d}}_{\mathbf{u}}(\mathcal{L}, \mathcal{L}')$, where $\widetilde{\mathsf{d}}_{\mathbf{u}}$ is the distance function for the frame $\mathsf{F}_{\mathbf{u}}$ in which $\mathcal{L}$ and $\mathcal{L}'$ can be considered as "locations" (see §**5.4** and (54.1)).

How does such an object appear to an "observer" with a world-direction other than $\mathbf{u}$? Let $\mathbf{d} \in \mathcal{F}_1$ be given such that $\mathbf{d} \neq \mathbf{u}$, and let $\mathbf{e} \in \{\mathbf{d}\}^\perp$ and $\alpha' \in \mathbb{P}^\times$ be determined such that $|\mathbf{e}| = 1$ and $\mathcal{L}' = \mathcal{L} + \alpha'\mathbf{e}$. One may easily show that if a given $q \in \mathcal{L}$ and a given $q' \in \mathcal{L}'$ are such that $q \sim_{\mathbf{d}} q'$, then $\mathsf{dst}_{\mathbf{d}}(q, q') = \alpha'$. Hence we call α' the *length of the rod relative to* $\mathbf{d}$.

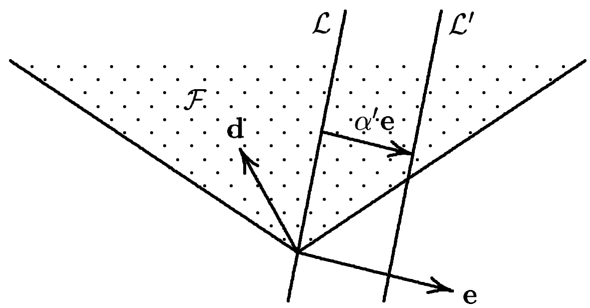

Figure 55d

From **Thm. 5401** (with $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{e}_1$, and $\mathbf{e}_2$ replaced by $\mathbf{d}$, $\mathbf{u}$, $\mathbf{e}$, and $\mathbf{w}$, respectively), we may find $\mu \in 1 + \mathbb{P}^\times$ and $\nu \in]0,1[$ such that

$$\mathbf{d} = \mu(\mathbf{u} - \nu\mathbf{w}) \quad \text{and} \quad \mathbf{u} = \mu(\mathbf{d} + \nu\mathbf{e}).$$

Now for every $x \in \mathcal{L}$, we have that $x + \alpha\mathbf{w}, x + \alpha'\mathbf{e} \in \mathcal{L}'$, and so it follows that

$$\alpha'\mathbf{e} - \alpha\mathbf{w} = (x + \alpha'\mathbf{e}) - (x + \alpha\mathbf{w}) \in \mathbb{R}\mathbf{u}.$$

Thus, since $\mathbf{w} \in \{\mathbf{u}\}^\perp$, we must have

$$0 = (\alpha'\mathbf{e} - \alpha\mathbf{w}) \cdot \mathbf{w} = \alpha'\mathbf{e} \cdot \mathbf{w} - \alpha.$$

But we see from the selection of μ and ν that $\mathbf{e} \cdot \mathbf{w} = \mu$ (see **Thm. 5401**(5), with $\mathbf{e}_1$ and $\mathbf{e}_2$ replaced by $\mathbf{e}$ and $\mathbf{w}$, respectively), and hence it follows that

$$\alpha' = \frac{1}{\mu}\alpha = \sqrt{1 - \nu^2}\,\alpha < \alpha.$$

Thus, the length of the rod relative to $\mathbf{d}$ is strictly less than the length of the rod. This phenomenon is often referred to as "Lorentz-Fitzgerald contraction".

Remark: Although the rod "appears" shorter relative to $\mathbf{d}$, this does not imply that the rod *actually* shrinks. Indeed, the rod is "at rest" in the frame $\mathsf{F}_{\mathbf{u}}$ and the observer with world-direction $\mathbf{d}$ can be viewed as running past the rod. The rod would not "shrink" merely as a result of an observer running by!

5.6 Worldpaths of Particles

Let a Minkowskian spacetime $\mathcal{E}$ be given.

It is an easy Exercise to see that the inner product on $\mathcal{V}$ is continuous; it then follows that τ is continuous. Thus, all results about smooth parameterizations in §**3.4** are applicable in a Minkowskian spacetime. We freely use this fact for the remainder of this chapter, and recall the notation

$$\mathcal{F}_1 = \{\mathbf{u} \in \mathcal{F} \,|\, \tau(\mathbf{u}) = 1\}$$

for the set of all world-directions.

Now let a material worldpath $\mathcal{L}$, a genuine interval I in $\mathbb{R}$, and a smooth time-parameterization $p : I \to \mathcal{E}$ of $\mathcal{L}$ be given. We wish to interpret $\mathcal{L}$ as the worldpath of a "material particle" (which will be defined precisely in Chapter 6).

Put $\mathbf{d} := p^{\bullet}$, and let $s \in I$ be given. Since $\mathbf{d}(s) \in \mathcal{F}_1$ (see **Prop. 3409**), we say that $\mathbf{d}(s)$ is the **world-direction of the particle at** s. It is inappropriate to think of $\mathbf{d}(s)$ as a "velocity". We may, however, offer the following: at s, the particle is "instantaneously" at rest with respect to the reference frame $\mathsf{F}_{\mathbf{d}(s)}$ (see §**5.4**).

Assume that $p^{\bullet}$ is differentiable. Then differentiating $p^{\bullet} \cdot p^{\bullet} = -1$ yields $p^{\bullet} \cdot p^{\bullet\bullet} = 0$, and hence we see that

$$p^{\bullet\bullet}(s) \in \{p^{\bullet}(s)\}^{\perp} \subset \mathcal{V}^{+} \cup \{\mathbf{0}\}. \tag{56.1}$$

The relativistic version of "Newton's law of motion" states that $p^{\bullet\bullet}$ is the force per unit mass (see §**6.1** for a precise definition) exerted on the particle. It follows that $|p^{\bullet\bullet}(s)|$ is the "g-force" experienced by the particle. For example, if we imagine $\mathcal{L}$ to be the worldpath of a spaceship, then $|p^{\bullet\bullet}(s)|$ is the force per unit mass exerted on a person sitting in the spaceship. If, at time s, a person's experience on the spaceship was that of being in a room on Earth; that is, the person walked about with no more or less difficulty that he or she would walk about the surface of the Earth, then we would say that the person experienced a g-force of one Earth gravity, or $1g$, at time s. Thus, we use "g" as a unit of force per unit mass. A unit of $1g$ is related to the unit yr^{-1} by

$$1g = \frac{9.8m}{s^2} \times \frac{1s}{3.0 \times 10^8 m} \times \frac{3.16 \times 10^7 s}{yr} \approx 1.031 yr^{-1}.$$

Remark: In classical mechanics, the force $p^{\bullet\bullet}(s)$ is equal to the acceleration at time s, which is a vector lying in a fixed three-dimensional vector space. In special relativity, the three-dimensional space $\{\mathbf{d}(s)\}^{\perp}$ may vary along the worldpath. Thus, it is inappropriate to think of $p^{\bullet\bullet}(s)$ as an *intrinsic* acceleration. However, we may interpret $p^{\bullet\bullet}(s)$ as an acceleration relative to the reference frame $\mathsf{F}_{\mathbf{d}(s)}$ at time s (see §**5.7**).

If $\mathcal{L}$ is the worldpath of a spaceship, then a g-force may be produced by firing rockets. We consider this example in §**6.3**. If p describes the worldpath of an electromagnetically charged particle, then the g-force is proportional to $\mathbf{F}(p(s))\mathbf{d}(s)$, where $\mathbf{F}$ is the electromagnetic field. This example will be considered in §**7.3**.

5.7 Relative Parameterizations

Let a Minkowskian spacetime $\mathcal{E}$ and a world-direction $\mathbf{d} \in \mathcal{F}_1$ be given.

As an application of **Thm. 5401**, we consider the following question: How do worldpaths with respect to $\prec$ compare to worldpaths with respect to $\prec_{\mathbf{d}}$? Since a worldpath is defined using a specific precedence relation, what is a worldpath with respect to one precedence relation may not be a worldpath with respect to another. For clarity and brevity, we call a worldpath with respect to $\prec$ a **$\prec$-worldpath**, and we call a worldpath with respect to $\prec_{\mathbf{d}}$ a **$\prec_{\mathbf{d}}$-worldpath**. It can be shown that if $\mathcal{L}$ is a $\prec$-worldpath, then $\mathcal{L}$ is also a $\prec_{\mathbf{d}}$-worldpath (the proof of this is left as an Exercise). But the converse is *not* necessarily true – $\mathcal{L}$ may be a $\prec_{\mathbf{d}}$-worldpath, but fail to be a $\prec$-worldpath. We now proceed to investigate under which conditions a $\prec_{\mathbf{d}}$-worldpath is a $\prec$-worldpath. To this end, let a $\prec_{\mathbf{d}}$-worldpath $\mathcal{L}$ be given.

Our first goal will be to determine a time-parameterization (with respect to $\mathsf{F}_{\mathbf{d}}$) of $\mathcal{L}$. We proceed as in the proof of **Thm. 2306**.

To this end, we fix $q \in \mathcal{L}$. As a result of **Thm. 2401** and **Def. 5400**, we see that the mapping $(\bar{\mathsf{t}}_{\mathbf{d}})^q_{\mathcal{L}} : \mathcal{L} \to \mathbb{R}$ (see **Not. 2302**) is given by

$$(\bar{\mathsf{t}}_{\mathbf{d}})^q_{\mathcal{L}}(x) = \bar{\mathsf{t}}_{\mathbf{d}}(q, x) = -(x - q) \cdot \mathbf{d}$$

for all $x \in \mathcal{L}$. For brevity, we put $\bar{\mathsf{t}}^q_{\mathbf{d}} := (\bar{\mathsf{t}}_{\mathbf{d}})^q_{\mathcal{L}}$, as this timelapse function depends not on the particular events in $\mathcal{L}$, but on the instants to which they belong. As in the proof of **Thm. 2306**, we put $J := \mathsf{Rng}\ \bar{\mathsf{t}}^q_{\mathbf{d}}$, and define $p_{\mathbf{d}} : J \to \mathcal{L}$ to be such that $\bar{\mathsf{t}}^q_{\mathbf{d}}(p_{\mathbf{d}}(s)) = s$ for all $s \in J$. We see that $p_{\mathbf{d}}$ is the desired time-parameterization (with respect to $\mathsf{F}_{\mathbf{d}}$) of $\mathcal{L}$, and we call it the **time-parameterization relative** to the world-direction $\mathbf{d}$.

Put $\mathcal{V}_\perp := \{\mathbf{d}\}^\perp$ and $\mathcal{E}_\perp := q + \mathcal{V}_\perp$, which is a flat in $\mathcal{E}$ with direction space $\mathcal{V}_\perp$. Since $\mathcal{V}_\perp$ is a positive-regular subspace of $\mathcal{V}$, $\mathcal{E}_\perp$ has the natural structure of a *genuine* Euclidean space. Recall that there is a natural bijection between $\mathsf{F}_{\mathbf{d}} \times \Gamma_{\mathbf{d}}$ and $\mathcal{E}$ (see the remark following **Thm. 4202**). We wish to use this bijection to "decompose" $p_{\mathbf{d}}$. For convenience, however, we identify the reference frame $\mathsf{F}_{\mathbf{d}}$ with $\mathcal{E}_\perp$ *via* the Euclidean isomorphism which assigns to each event $z \in \mathcal{E}_\perp$ the location $z + \mathbb{R}\mathbf{d}$ in $\mathsf{F}_{\mathbf{d}}$. We also identify $\Gamma_{\mathbf{d}}$ with $\mathbb{R}$ *via* t^*_γ, where γ is the instant to which q belongs (see **Def. 2405**). Thus, we may consider Φ (see **Thm. 4201**) to be the mapping $\Phi : \mathcal{E}_\perp \times \mathbb{R} \to \mathcal{E}$ given by

$$\Phi(z, \xi) := z + \xi\mathbf{d}$$

for all $(z, \xi) \in \mathcal{E}_\perp \times \mathbb{R}$.

As a result, we may determine a mapping $p_\perp : J \to \mathcal{E}_\perp$ which satisfies

$$p_{\mathbf{d}}(s) = \Phi(p_\perp(s), s) = p_\perp(s) + s\mathbf{d} \tag{57.1}$$

for all $s \in J$. If $p_{\mathbf{d}}$ is smooth, then so is $p_\perp$; in this case, we put $\mathbf{v} := p_\perp^\bullet$. We illustrate a two-dimensional example in Figure 57a.

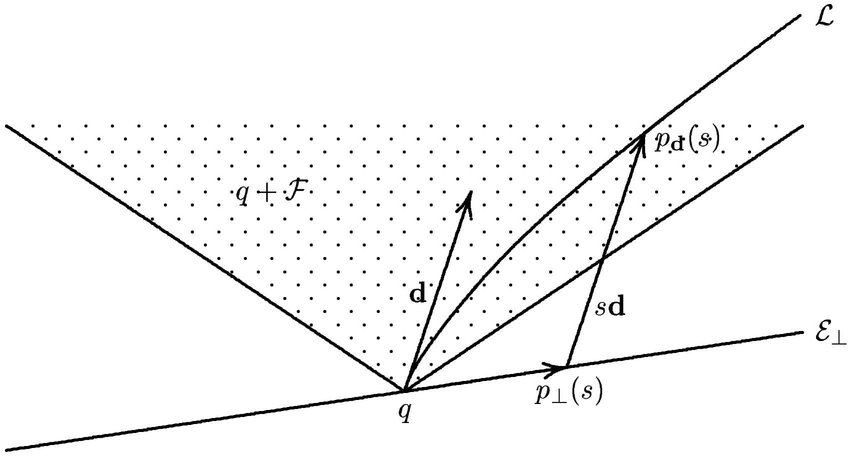

Figure 57a

Armed with this time-parameterization of $\mathcal{L}$, we are in a position to return to our original question of when $\prec_{\mathbf{d}}$-worldpaths are $\prec$-worldpaths. We have the following Theorem, where by a **smooth** worldpath we mean a worldpath with a smooth time-parameterization.

5700 Theorem: *Let $\mathbf{d} \in \mathcal{F}_1$ and a smooth $\prec_{\mathbf{d}}$-worldpath $\mathcal{L}$ be given. Let $p_{\mathbf{d}} : J \to \mathcal{E}$ and $p_\perp : J \to \mathcal{E}_\perp$ be as described above. Then $\mathcal{L}$ is a smooth material $\prec$-worldpath if and only if $|p_\perp^\bullet(s)| < 1$ for all $s \in J$.*

Proof: Assume that $\mathcal{L}$ is a smooth material $\prec$-worldpath, and let a genuine interval I in $\mathbb{R}$ and a mapping $p : I \to \mathcal{E}$ be such that p is a smooth time-parameterization of $\mathcal{L}$. Since $\mathsf{Rng}\ p_{\mathbf{d}} = \mathsf{Rng}\ p = \mathcal{L}$, we may find a smooth bijection $\alpha : I \to J$ such that

$$p(t) = p_{\mathbf{d}}(\alpha(t)),$$

and hence (see (57.1) and recall that $\mathbf{v} = p^{\bullet}_{\perp}$)

$$\begin{aligned} p^{\bullet}(t) &= p^{\bullet}_{\mathbf{d}}(\alpha(t))\alpha^{\bullet}(t) \\ &= \alpha^{\bullet}(t)(\mathbf{d} + \mathbf{v}(\alpha(t))) \end{aligned} \tag{57.2}$$

for all $t \in I$. From **Prop. 3409**, we know that $\mathsf{Rng}\, p^{\bullet} \subset \mathcal{F}_1$, and hence it follows from (57.2) and **Cor. 5205** that

$$0 > \mathbf{d} \cdot p^{\bullet}(t) = -\alpha^{\bullet}(t),$$

and hence $\alpha^{\bullet}(t) > 0$ for all $t \in I$. Using this fact and taking the inner product of each side of (57.2) with itself yields that $|\mathbf{v}(\alpha(t))| < 1$ for all $t \in I$. Since $\alpha : I \to J$ is a bijection, the first half of the Theorem is proved.

The reverse implication is left as an Exercise. ◊

Remark: We note that with $\mu : I \to 1 + \mathbb{P}$ and $\nu : I \to [0, 1[$ given by

$$\mu(t) := \alpha^{\bullet}(t)$$

and

$$\nu(t) := |\mathbf{v}(\alpha(t))|$$

for all $t \in I$, we have

$$\nu(t) = \sqrt{1 - \frac{1}{\mu(t)^2}}$$

and

$$p^{\bullet}(t) = \mu(t)(\mathbf{d} + \mathbf{v}(\alpha(t))) \tag{57.3}$$

for all $t \in I$ (compare this with **Thm. 5401**).

How may we interpret this result? $p_{\perp}$ describes a path in the genuine Euclidean space $\mathcal{E}_{\perp}$. This gives the interpretation that, for $t \in I$, $\mathbf{v}(\alpha(t))$ is the **velocity** of $\mathcal{L}$ at $p(t)$ with respect to $\mathsf{F}_{\mathbf{d}}$, and that $\nu(t)$ is the **speed** of $\mathcal{L}$ at $p(t)$ with respect to $\mathsf{F}_{\mathbf{d}}$. Hence we may interpret the last statement of the previous Theorem as follows: $\mathcal{L}$ is a smooth material $\prec$-worldpath if and only if the speed of $\mathcal{L}$ with respect to $\mathsf{F}_{\mathbf{d}}$ is strictly less than 1. In the literature on relativity, this result is often described by the phrase, "Material particles must travel at a speed that is strictly less than the 'speed of light'. " In the framework used here, the "speed of light" is the same as the number one (see the end of §**5.3**).

Now that we have an interpretation of the velocity of $\mathcal{L}$ with respect to $\mathsf{F}_{\mathbf{d}}$, it is natural to ask about the rate of timelapse along $\mathcal{L}$ with respect to $\mathsf{F}_{\mathbf{d}}$. Hence, we consider the mapping

$$(t \mapsto \bar{\mathsf{t}}_{\mathbf{d}}(q, p(t))) = \bar{\mathsf{t}}^{q}_{\mathbf{d}} \circ p \,:\, I \to \mathbb{R}.$$

Note that for $t \in I$, we have

$$(\bar{\mathsf{t}}^{q}_{\mathbf{d}} \circ p)(t) = \bar{\mathsf{t}}_{\mathbf{d}}(q, p(t)) = -(p(t) - q) \cdot \mathbf{d}.$$

Hence, we see from (57.2) that

$$(\bar{\mathsf{t}}^{q}_{\mathbf{d}} \circ p)^{\bullet} = -p^{\bullet} \cdot \mathbf{d} = \alpha^{\bullet} = \mu.$$

Thus, μ gives the rate at which time elapses along $\mathcal{L}$ with respect to $\mathsf{F}_{\mathbf{d}}$. In the case that μ is constant, the reader may verify that

$$\bar{\mathsf{t}}_{\mathbf{d}}(x, y) = \mu\, \bar{\mathsf{t}}_{\mathcal{L}}(x, y) \tag{57.4}$$

for all $x, y \in \mathcal{L}$. In this case, μ is called the **time-dilation** for $\mathcal{L}$ with respect to $\mathsf{F}_{\mathbf{d}}$ (see the paragraph following the proof of **Thm. 5401**).

Finally, we return to the discussion about acceleration relative to the reference frame $\mathsf{F}_{\mathbf{d}}$ begun at the end of §**5.6**. To this end, let $t \in I$ be given, and put $\mathbf{d} := p^{\bullet}(t)$. It is left as an Exercise to verify that we have

$$p^{\bullet\bullet}(t) = \mathbf{v}^{\bullet}(\alpha(t)),$$

and hence $p^{\bullet\bullet}(t)$ may be considered as the acceleration relative to the frame $\mathsf{F}_{\mathbf{d}}$. That frame, of course, may vary with t.

5.8 Interstellar Travel

In this section, we present a detailed example which incorporates several of the ideas discussed in this chapter. To this end, suppose that Dick and Jane are twins. At the given event q, the event of their 20^{th} birthday which they celebrate together, Jane decides to take an interstellar journey, never to return to Earth. She decides to travel so that the g-force γ experienced by her spaceship is approximately one Earth gravity, so as to guarantee a comfortable ride. A value of $\gamma = 1yr^{-1} \approx 0.97g$ would not be a bad choice. She also decides to travel "on a straight path away" from Earth, so that the worldpath of her journey, althought not straight, lies in a two-dimensional flat in $\mathcal{E}$.

Thus, if $p_J : \mathbb{P} \to \mathcal{E}$ were a smooth time-parameterization of Jane's worldpath, and we put $\mathbf{d} := p_J^\bullet$, the above description of Jane's worldpath would be equivalent to:

1. $p_J(0) = q$,
2. $\mathsf{dim}\ \mathsf{Rng}\ p_J = 2$, and
3. $|p_J^{\bullet\bullet}| = |\mathbf{d}^\bullet| = \gamma$.

Our immediate goal is to find a worldpath whose time-parameterization p satisfies these conditions.

In view of (56.1), we have $\mathbf{d}^\bullet(s) \in \{\mathbf{d}(s)\}^\perp \subset \mathcal{V}^+ \cup \{\mathbf{0}\}$, and hence $|p_J^{\bullet\bullet}(s)| = |\mathbf{d}^\bullet(s)| = \sqrt{\mathbf{d}^\bullet(s) \cdot \mathbf{d}^\bullet(s)}$ makes sense for all $s \in \mathbb{P}$.

Put $\mathbf{d}_0 := \mathbf{d}(0)$ and $\mathbf{e} := \frac{1}{\gamma}\mathbf{d}^\bullet(0)$. Then $\mathbf{d}_0 \cdot \mathbf{d}_0 = -1$, $\mathbf{e} \cdot \mathbf{e} = 1$, and $\mathbf{d}_0 \cdot \mathbf{e} = 0$. Since $\mathsf{dim}\ \mathsf{Rng}\ p = 2$, $(\mathbf{d}_0, \mathbf{e})$ must be a list-basis of $\mathsf{Lsp}\ \mathsf{Rng}\ \mathbf{d}$, and hence there are continuously differentiable functions $\alpha, \beta : \mathbb{P} \to \mathbb{R}$ such that

$$\mathbf{d} = \alpha\mathbf{d}_0 + \beta\mathbf{e},$$

and hence

$$\mathbf{d}^\bullet = \alpha^\bullet\mathbf{d}_0 + \beta^\bullet\mathbf{e}.$$

By the definition of $\mathbf{d}_0$ and $\mathbf{e}$, we have

$$\alpha(0) = 1,\ \beta(0) = 0,\ \alpha^\bullet(0) = 0,\ \text{and}\ \beta^\bullet(0) = \gamma. \tag{58.1}$$

The condition $\mathbf{d} \cdot \mathbf{d} = -1$ gives

$$\alpha^2 - \beta^2 = 1, \tag{58.2}$$

and the condition $\mathbf{d}^\bullet \cdot \mathbf{d}^\bullet = \gamma^2$ gives

$$\beta^{\bullet 2} - \alpha^{\bullet 2} = \gamma^2. \tag{58.3}$$

Using elementary calculus, one finds that the relations (58.1)–(58.3) are satisfied if and only if for all $s \in \mathbb{P}$, we have

$$\alpha(s) = \cosh(\gamma s)$$

and

$$\beta(s) = \sinh(\gamma s),$$

so that

$$\mathbf{d}(s) = \cosh(\gamma s)\mathbf{d}_0 + \sinh(\gamma s)\mathbf{e} \tag{58.4}$$

for all $s \in \mathbb{P}$. By **Cor. 3410**, the mapping $p_J : \mathbb{P} \to \mathcal{E}$ defined by

$$p_J(s) = q + \int_0^s \mathbf{d} = q + \frac{1}{\gamma}(\sinh(\gamma s)\mathbf{d}_0 + (\cosh(\gamma s) - 1)\mathbf{e}) \tag{58.5}$$

for each $s \in \mathbb{P}$ is the time-parameterization of a material worldpath. The range of this worldpath is in a two-dimensional flat in $\mathcal{E}$; namely, the planar flat $\mathcal{P} := q + \mathsf{Lsp}\,\{\mathbf{d}_0, \mathbf{e}\}$. Note that if we define $y, x : \mathbb{P} \to \mathbb{R}$ by

$$y(s) := \frac{1}{\gamma}\sinh(\gamma s) \quad \text{and} \quad x(s) := \frac{1}{\gamma}(\cosh(\gamma s) - 1)$$

for all $s \in \mathbb{P}$, we see that x and y satisfy the equation

$$(1 + \gamma x)^2 - (\gamma y)^2 = 1,$$

which shows that the worldpath parameterized by p_J is a branch of a hyperbola in the planar flat $\mathcal{P}$ (see Figure 58a).

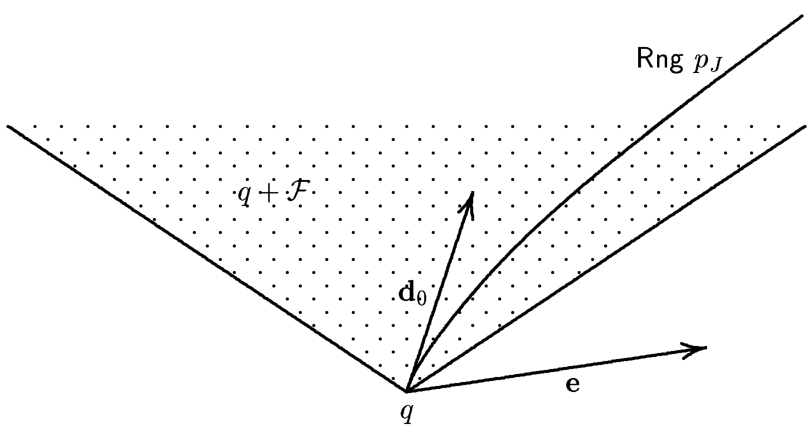

Figure 58a

We also note that (58.4) may be rewritten as

$$\mathbf{d}(s) = \cosh(\gamma s)(\mathbf{d}_0 + \tanh(\gamma s)\mathbf{e}) \tag{58.6}$$

for all $s \in \mathbb{P}$, which the reader may compare with **Thm. 5401**(1). A complete discussion of this relationship is provided below.

Remark: In a two-dimensional genuine Euclidean space, the analogous problem would yield an arc-length parameterization of a circle. The details are left as an Exercise.

Hence, we see that Jane's worldpath $\mathcal{J}$ may be parameterized by $p_J : \mathbb{P} \to \mathcal{E}$ as given in (58.5). We assume that Dick's worldpath is straight (we ignore the gravitational effects of the Earth and the gravitational forces between the Earth, Moon, and Sun — consideration of these effects lies in the domain of general relativity). Hence, Dick's worldpath $\mathcal{D}$ has the time parameterization $p_D : \mathbb{P} \to \mathcal{E}$, given by

$$p_D(t) := q + t\mathbf{d}_0 \quad \text{for all } t \in \mathbb{P}. \tag{58.7}$$

Now suppose that at his time t_e, Dick emits a signal (*i.e.*, sends a message) to Jane. When will Jane receive the message?

We begin by finding the worldpath of the signal that Dick is sending. It is easy to see from our choices of $\mathbf{d}_0$ and $\mathbf{e}$ that $\mathbf{d}_0 + \mathbf{e}$ is a signal vector; *i.e.*, $(\mathbf{d}_0+\mathbf{e})\cdot(\mathbf{d}_0+\mathbf{e}) = 0$. Since the event of the message being sent is $p_D(t_e)$, the worldpath of the signal is included in the half-line $\mathcal{M} := p_D(t_e) + \mathbb{P}(\mathbf{d}_0 + \mathbf{e})$ (see Figure 58b, where the worldpath of the signal is represented by the dashed line). Hence, if Jane receives Dick's message at her time s, then we must have $p_J(s) \in \mathcal{M}$, and hence $(p_J(s) - p_D(t_e)) \cdot (\mathbf{d}_0 + \mathbf{e}) = 0$; *i.e.*,

$$\left(\left(\frac{1}{\gamma}\sinh(\gamma s) - t_e\right)\mathbf{d}_0 + \frac{1}{\gamma}(\cosh(\gamma s) - 1)\mathbf{e}\right) \cdot (\mathbf{d}_0 + \mathbf{e}) = 0.$$

Since $\mathbf{d}_0 \cdot \mathbf{e} = 0$, $\mathbf{d}_0 \cdot \mathbf{d}_0 = -1$, and $\mathbf{e} \cdot \mathbf{e} = 1$, this is equivalent to

$$\frac{1}{\gamma}\sinh(\gamma s) - t_e = \frac{1}{\gamma}(\cosh(\gamma s) - 1),$$

from which we obtain by an easy calculation

$$t_e = \frac{1}{\gamma}(1 - e^{-\gamma s}).$$

Hence, if Dick sends his message at his time t_e, then Jane receives the message at her time $s = -\gamma^{-1}\log(1-\gamma t_e)$. Since the domain of our parameterization is $\mathbb{P}$, we must have $t_e \in [0, \gamma^{-1}[$ since $s \in \mathbb{P}$. In other words, Jane can receive a message from Dick only if he sends it *before* his time γ^{-1}. Hence, if we select $\gamma = 1yr^{-1}$ (if Jane weighed 130 lbs. on Earth, she would weigh 126 lbs. during her trip), then Jane can never know what happens to Dick after his 21^{st} birthday (since she can receive no messages from Dick), even if she lives forever! This is equivalent to the observation that $q + \mathbf{d}_0 + \mathbb{P}(\mathbf{d}_0 + \mathbf{e})$ is an asymptote to Jane's hyperbolic worldpath.

Now suppose that at her time s, Jane receives Dick's message and immediately responds. When will Dick receive the response? Let t_r be the time that Dick receives the response. In order for Jane's signal to reach Earth, it must be sent in the direction $\mathbf{d}_0 - \mathbf{e}$, and hence the worldpath of such a signal must be included in the half-line $\mathcal{P} := p_J(s) + \mathbb{P}(\mathbf{d}_0 - \mathbf{e})$ (see Figure 58b). Hence, analogous to the previous situation, we must have $p_D(t_r) \in \mathcal{P}$, and hence $(p_D(t_r) - p_J(s)) \cdot (\mathbf{d}_0 - \mathbf{e}) = 0$. This results in the relationship

$$t_r = \frac{1}{\gamma}(e^{\gamma s} - 1).$$

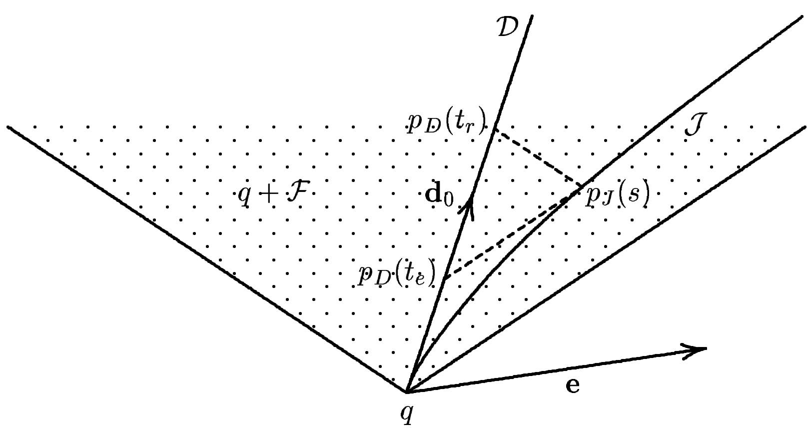

Figure 58b

So, if Dick wanted to send Jane messages so that she would receive them on her birthdays, he would do so according to the following chart (recall that

Jane left Earth when she and Dick both turned 20): if Dick sent a birthday greeting at his time t_e after Jane left, Jane would receive it at her $(20+s)^{\text{th}}$ birthday, and Dick would receive a "Thank you" at age $20 + t_r$ years. γ is taken to be $1yr^{-1}$ in the calculations in the following chart. Any message sent at time $t_e \geq 1$ would never be received by Jane.

s	t_e months	t_e years	t_r
1.0	7.59	0.632	1.72
2.0	10.38	0.865	6.39
4.0	11.78	0.982	53.60
5.0	11.92	0.993	147.41

Remark: We may consider the previous example (and Figure 58b) as an illustration of **Prop. 5304**, with $\mathcal{L}$ taken to be the worldline that includes the halfline $\mathcal{D}$ and "e" replaced with "$p_J(s)$". With this interpretation, we have that "z" is given by $p_D(\frac{t_e+t_r}{2})$. In addition, we see that

$$\mathsf{dst}(p_J(s), \mathcal{L}) = \frac{t_r - t_e}{2}.$$

Consider now the scenario where Jane makes a journey to a distant star, and then returns to Earth. She leaves, as before, on the occasion of her and Dick's 20^{th} birthday. When she returns (say 15 years later, her time), she finds that Dick has become a grandfather and has already retired! In fact, she is not even half Dick's age. How can this be?

Let q be the event of Jane's departure, and r be the event of Jane's reunion with Dick. Then the events of Jane's voyage form a worldpath from q to r, as do the events in Dick's life from the time his sister left to the time she returned. We again assume that Dick's worldpath is straight (we ignore gravitational effects), and we also assume that Jane travels so as to maintain a constant g-force γ inside her spaceship. As we will see later on, it takes Jane *less* time to get from q to r than Dick, and thus Jane's worldpath is not straight. In fact, the magnitude of the g-force is related to the "deviation" of Jane's worldpath from a straight worldpath.

Let $\mathcal{D}$ be the worldpath of Dick on Earth and $\mathcal{S}$ be the worldpath of the star to which Jane travels. We assume that these worldpaths are parallel so as to keep our discussion in two dimensions. We assume that Jane travels directly towards the star on her journey out and directly towards the Earth on her return in the following manner: she burns her rockets so that she travels towards the star, then halfway along the way, she turns her ship around and continues to thrust her rockets so as to maintain a constant g-force γ inside her spaceship. This has the effect of "slowing down" her ship so that Jane does not crash into the star. She then makes her journey back to Earth in an analogous way. Thus, Jane's worldpath $\mathcal{J}$ can be modelled by piecing together four hyperbolic sections of the worldpath described above, as shown in Figure 58c.

Let h_J be the event that is half-way in Jane's travel to the distant star. Then up to this point, we may parameterize Jane's worldpath as in (58.5). As before, we parameterize Dick's worldpath as in (58.7). Then $\mathbf{d}_0$ is the world-direction of Dick's worldpath; let h_D be the event on $\mathcal{D}$ such that $h_D - h_J \in \{\mathbf{d}_0\}^{\perp}$, let t_J be the time that it takes Jane to get to h_J, let t_D be the time that it takes Dick to get to h_D, and let d be the distance from Jane to the Earth when Jane is at h_J (see Figure 58c).

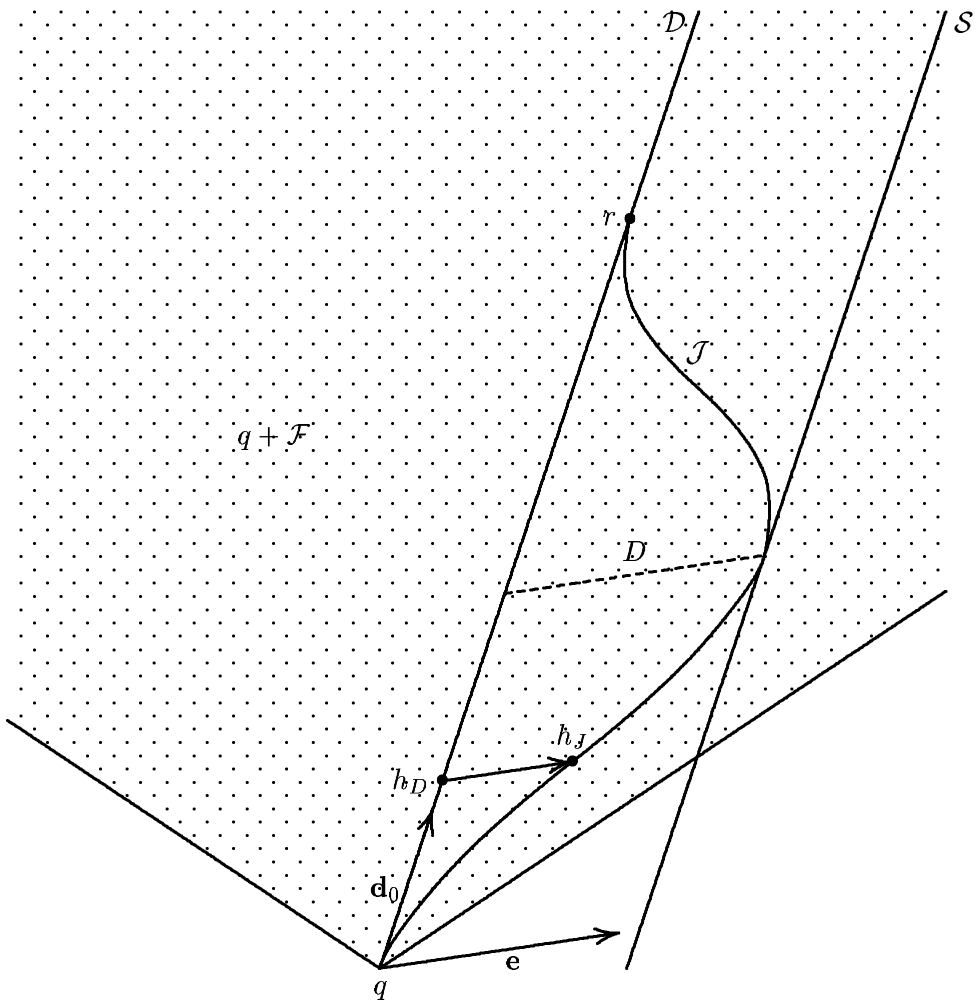

Figure 58c

Given the parameterizations of Jane and Dick as in (58.5) and (58.7), we find that

$$h_J = p_J(t_J) = q + \frac{1}{\gamma}(\sinh(\gamma t_J)\mathbf{d}_0 + (\cosh(\gamma t_J) - 1)\mathbf{e})$$

and

$$h_D = p_D(t_D) = q + t_D\mathbf{d}_0.$$

Since $h_J = h_D + d\mathbf{e}$, we have

$$t_D = \frac{1}{\gamma}\sinh(\gamma t_J) \tag{58.8}$$

and

$$d = \frac{1}{\gamma}(\cosh(\gamma t_J) - 1). \tag{58.9}$$

We now assume that the distance from the Earth to the star to which Jane travels; *i.e.*, the distance between the parallel worldlines $\mathcal{S}$ and $\mathcal{D}$, is D light-years, and the duration of the trip is T_J years for Jane and T_D years for Dick.

We use (58.8) and (58.9) to conclude that

$$t_D = \sqrt{\left(d + \frac{1}{\gamma}\right)^2 - \frac{1}{\gamma^2}};$$

and, since $D = 2d$ and $T_D = 4t_D$, that

$$T_D = 4\sqrt{\left(\frac{D}{2} + \frac{1}{\gamma}\right)^2 - \frac{1}{\gamma^2}}.$$

From (58.8), we have $t_J = \frac{1}{\gamma}\text{arc sinh}(\gamma t_D)$, and since $T_J = 4t_J$ and $T_D = 4t_D$,

$$T_J = \frac{4}{\gamma}\text{arc sinh}\left(\frac{\gamma T_D}{4}\right).$$

For $\gamma = 1yr^{-1}$ and some specific values of D, the corresponding values of T_J and T_D are given in the chart below (all data are in years (or light-years) and are approximate).

Star system	D	T_J	T_D
α Centauri	4.3	7.3	11.9
Sirius	8.7	9.4	21.0
Vega	26.5	13.4	58.9
Center of Milky Way	30,000	41.2	60,000

In order to give an interpretation of this chart, suppose that Dick and Jane's 20th birthday takes place on 1 January 5000. If Jane took a trip to α Centauri and back, she would be 27 upon her return, while Dick would be almost 32. If she made a trip to Vega and back, she would be 33 when she returned, but Dick would be almost 80! This phenomenon is sometimes referred to as the *twin paradox*.

The next chart shows how the duration of Jane's trip varies with different values of γ. (The various values of γ are for comparison only; Jane could not survive a journey with $\gamma = 10yr^{-1}$ because she would be crushed weighing 1260 lbs. all the time, while weighing only 130 lbs. on Earth.) We tabulate this data for a trip to α Centauri; hence $D = 4.3$ in all calculations.

γ	D	T_J	T_D
0.5	4.3	10.9	14.5
1.0	4.3	7.3	11.9
3.0	4.3	3.6	9.8
10.0	4.3	1.5	9.0

Finally, we apply some results from §**5.7**, and we use the notation developed therein. Comparison of (58.6) with (57.3) (where we recall that $\mathbf{d} := p^\bullet$ in (58.6)) yields

$$\mu(s) = \cosh(\gamma s)$$

for all $s \in \mathbb{P}$, and consequently

$$\nu(s) = \tanh(\gamma s)$$

for all $s \in \mathbb{P}$. Thus, we may calculate the speed of Jane relative to Dick for $s \in \mathbb{P}$. Note that as s gets large, $\nu(s)$ approaches 1. Although μ is not constant (as in (57.4)), it follows from (58.8) and the fact that p_J is a time-parameterization that

$$\bar{\mathsf{t}}_{\mathbf{d}_0}(q, p_J(s)) = \frac{\sinh(\gamma s)}{\gamma s}\,\bar{\mathsf{t}}_{\mathcal{J}}(q, p_J(s))$$

for all $s \in \mathbb{P}^{\times}$.

Exercises

EXERCISES, I

1. Prove **Prop. 5102**.
2. Complete the proof of **Prop. 5106**.
3. Prove **Cor. 5108**.
4. Prove **Prop. 5110**.
5. Show that the proposed restatement of **Thm. 5201** (as given in the Pitfall following the proof of **Prop. 5202**) is false.
6. Complete the proof of **Thm. 5204**.
7. Complete the proof of **Thm. 5300**.
8. Complete the proof of **Prop. 5304**.
9. Verify that $\prec_{\mathbf{d}}$ (see **Def. 5400**) is reflexive, transitive, and total.
10. Given a Minkowskian spacetime $\mathcal{E}$ and $\mathbf{d} \in \mathcal{F}_1$, show that $\prec_{\mathbf{d}}$ and $\bar{\mathsf{t}}_{\mathbf{d}}$ (see **Def. 5400**) give $\mathcal{E}$ the structure of a classical timed eventworld.
11. Given a Minkowskian spacetime $\mathcal{E}$ and $\mathbf{d} \in \mathcal{F}_1$, show that

$$\mathsf{F}_{\mathbf{d}} = \{q + \mathbb{R}\mathbf{d} \mid q \in \mathcal{E}\}$$

is a reference frame (see **Def. 4200**).

12. With $\mathsf{F_d}$ as in the previous Exercise, show that the distance function $\widetilde{\mathsf{d}}_{\mathbf{d}}$ for $\mathsf{F_d}$ (see **Thm. 4202**) satisfies (54.1).

13. Show that the precedence relation $\prec$ in a Minkowskian spacetime is relativistic and the future cone $\mathcal{F}$ is closed (see **Thm. 5300**).

14. Given a Minkowskian spacetime $\mathcal{E}$, show that for all $x, y \in \mathcal{E}$, $x \prec y$ if and only if we have $x \prec_{\mathbf{d}} y$ for all $\mathbf{d} \in \mathcal{F}_1$ (see **Def. 5400**).

15. Consider the scenario at the beginning of §**5.5**, but suppose now the direction of motion of the person is *opposite* that of the train, so that the diagram in Figure 55b is as follows.

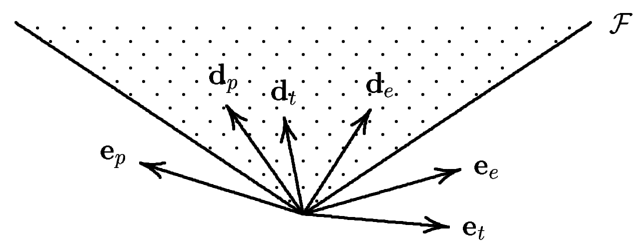

Figure Ex5(a)

Find an expression for ν in terms of ν_1 and ν_2.

16. Show that the inner product on $\mathcal{V}$ is continuous.

17. Let $\mathcal{E}$ be a Minkowskian spacetime. Show that if $\mathcal{L}$ is a $\prec$-worldpath, then $\mathcal{L}$ is a $\prec_{\mathbf{d}}$-worldpath for all $\mathbf{d} \in \mathcal{F}_1$.

18. Complete the proof of **Thm. 5700**.

19. Verify (57.4) in the event that μ is constant.

20. Show that $p^{\bullet\bullet}(t) = \mathbf{v}^{\bullet}(\alpha(t))$ (see the end of §**5.7**).

21. State and solve the problem alluded to in the Remark following (58.6).

Exercises, II

1. Let $\mathcal{V}$ be a non-genuine inner-product space (with arbitrary signature). Show that $\mathcal{U}$ is totally singular (see **Def. 5104**) if and only if $\mathcal{U} \subset \mathcal{U}^{\perp}$.

2. Let $\mathcal{V}$ be an inner-product space. Show that knowledge of $\mathbf{u} \cdot \mathbf{u}$ for all $\mathbf{u} \in \mathcal{V}$ is sufficient to determine $\mathbf{u} \cdot \mathbf{v}$ for every $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

3. Let $\mathcal{V}$ be an inner-product space such that $\operatorname{sig}^{-}\mathcal{V} = 1$ and $\dim \mathcal{V} \geq 2$.

   (a) Show that for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{V}^{-}$, we have
   $$(\mathbf{u} \cdot \mathbf{v})(\mathbf{v} \cdot \mathbf{w})(\mathbf{w} \cdot \mathbf{u}) < 0.$$
   (Hint: Consider $\mathbf{z} := (\mathbf{v} \cdot \mathbf{w})\mathbf{u} - (\mathbf{u} \cdot \mathbf{w})\mathbf{v}$ and use **Prop. 5200**.)

   (b) Define the relation $\sim$ on $\mathcal{V}^{-}$ by
   $$\mathbf{u} \sim \mathbf{v} \;:\Longleftrightarrow\; \mathbf{u} \cdot \mathbf{v} < 0$$
   for all $\mathbf{u}, \mathbf{v} \in \mathcal{V}^{-}$. Show that $\sim$ is an equivalence relation on $\mathcal{V}^{-}$, and that the corresponding partition of $\mathcal{V}^{-}$ is given by
   $$\{\widehat{\mathcal{F}} \setminus \mathcal{V}^{0}, (-\widehat{\mathcal{F}}) \setminus \mathcal{V}^{0}\},$$
   where $\widehat{\mathcal{F}}$ is one of the linear cones in the decomposition described in **Thm. 5204**.

   (c) Define the relation $\sim$ on $(\mathcal{V}^{-} \cup \mathcal{V}^{0})^{\times}$ by
   $$\mathbf{u} \sim \mathbf{v} \;:\Longleftrightarrow\; \mathbf{u} \cdot \mathbf{v} \leq 0$$
   for all $\mathbf{u}, \mathbf{v} \in (\mathcal{V}^{-} \cup \mathcal{V}^{0})^{\times}$. Decide whether or not $\sim$ is an equivalence relation.

4. Let $\mathcal{E}$ be a Minkowskian spacetime. Show that knowledge of the world-directions; *i.e.,* members of $\mathcal{F}_1$, is sufficient to determine $\mathbf{u} \cdot \mathbf{v}$ for every $\mathbf{u}, \mathbf{v} \in \mathcal{V}$.

5. Let a basis $\{b_1, b_2\}$ (which is not necessarily orthogonal) of $\mathbb{R}^2$ be given. Assume that an inner product $(u, v) \mapsto u \cdot v$ on $\mathbb{R}^2$ is given. Show that there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that
   $$u \cdot v = \alpha u_1 v_1 + \beta(u_1 v_2 + u_2 v_1) + \gamma u_2 v_2$$
   for all $u, v \in \mathbb{R}^2$, and find expressions for α, β, and γ in terms of b_1 and b_2. In addition, show that the inner-product space $\mathbb{R}^2$ has signature $(1, 1)$ if and only if $\beta^2 > \alpha\gamma$.

In the remaining Exercises for this section, let $\mathcal{E}$ be a Minkowskian spacetime with translation space $\mathcal{V}$.

6. Let $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{F}_1$ be given. Since $\mu := -\mathbf{d}_1 \cdot \mathbf{d}_2 \geq 1$ (see **Thm. 5401**), we may define
$$\alpha := \operatorname{arc}\cosh \mu.$$
We call α the **pseudo-angle between $\mathbf{d}_1$ and $\mathbf{d}_2$.** If ν is the relative speed between $\mathbf{d}_1$ and $\mathbf{d}_2$, show that $\nu = \tanh \alpha$.

7. Suppose that $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3 \in \mathcal{F}_1$ all lie in the same plane, as in the following diagram.

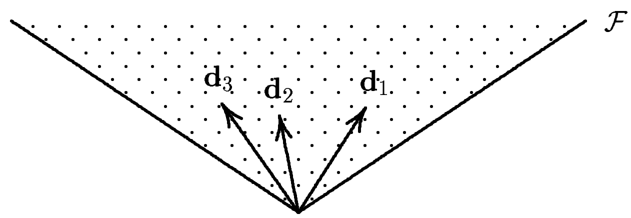

Figure Ex5(b)

Further, suppose that α is the pseudo-angle between $\mathbf{d}_1$ and $\mathbf{d}_2$ and β is the pseudo-angle between $\mathbf{d}_2$ and $\mathbf{d}_3$. Show that $\alpha + \beta$ is the pseudo-angle between $\mathbf{d}_1$ and $\mathbf{d}_3$.

8. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two straight material worldlines. Show that the nonempty sets $\{\mathsf{dst}(q, \mathcal{L}_2) \mid q \in \mathcal{L}_1\}$ and $\{\mathsf{dst}(q, \mathcal{L}_1) \mid q \in \mathcal{L}_2\}$ have a minimum, and that
$$\min_{q \in \mathcal{L}_1} \mathsf{dst}(q, \mathcal{L}_2) = \min_{q \in \mathcal{L}_2} \mathsf{dst}(q, \mathcal{L}_1).$$
If $q_1, q_2 \in \mathcal{E}$ and $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{F}_1$ are such that $\mathcal{L}_1 = q_1 + \mathbb{R}\mathbf{d}_1$ and also $\mathcal{L}_2 = q_2 + \mathbb{R}\mathbf{d}_2$, find this common distance in terms of q_1, q_2, $\mathbf{d}_1$, and $\mathbf{d}_2$.

9. Suppose that p is a time-parameterization of a material worldpath, $\mathcal{L}$. Show that $p + p^{\bullet}$ is a time-parameterization of a material worldpath (say, $\mathcal{M}$) if and only if $p^{\bullet\bullet} = \mathbf{0}$. If this is the case, show that $\mathcal{L}$ and $\mathcal{M}$ are straight and parallel, and determine $\mathbf{d} \in \mathcal{F}_1$ such that $\mathcal{M} = \mathcal{L} + \mathbf{d}$.

10. Using the notation of §**5.7**, show that $\alpha(s) = s$ for all $s \in I$ if and only if $p_\perp$ is constant; that is, $p_\perp(s) = p_\perp(t)$ for all $s, t \in I$. Show also that in this case, $\mathcal{L}$ is a straight worldpath with world-direction $\mathbf{d}$.

11. Assume that $\dim \mathcal{E} = 2$, and let $p \in \mathcal{E}$ be given. For each $\mathbf{d} \in \mathcal{F}_1$, define $\gamma_{\mathbf{d}} : \mathcal{F} \to \mathbb{P}$ by requiring that for every $\mathbf{v} \in \mathcal{F}$, $\gamma_{\mathbf{d}}(\mathbf{v})$ be the g-force necessary to travel from p to $p + \mathbf{v}$ along a path of constant g-force such that the initial world-direction is $\mathbf{d}$. Let $\mathbf{v} \in \mathcal{F}$ be given.

    (a) Show that
    $$\gamma_{\mathbf{d}}(\mathbf{v}) = \frac{2\mu\nu}{\tau(\mathbf{v})},$$
    where
    $$\mu := -\frac{\mathbf{v}}{\tau(\mathbf{v})} \cdot \mathbf{d} \quad \text{and} \quad \nu := \sqrt{1 - \frac{1}{\mu^2}}.$$

    (b) Show that the timelapse along this path of constant g-force from p to $p + \mathbf{v}$ is given by
    $$\frac{2}{\gamma_{\mathbf{d}}(\mathbf{v})} \operatorname{arc\,sinh} \frac{\tau(\mathbf{v})\gamma_{\mathbf{d}}(\mathbf{v})}{2}.$$

    (c) Suppose that $\mathbf{e} \in \mathcal{V}$ is such that $\mathbf{e} \cdot \mathbf{d} = 0$ and $\mathbf{e} \cdot \mathbf{e} = 1$. Define $\sigma_{\mathbf{e}} : \mathcal{V} \to \mathbb{R}$ by
    $$\sigma_{\mathbf{e}}(\mathbf{v}) := \begin{cases} 1 & \text{if } \mathbf{v} \cdot \mathbf{e} > 0, \\ 0 & \text{if } \mathbf{v} \cdot \mathbf{e} = 0, \\ -1 & \text{if } \mathbf{v} \cdot \mathbf{e} < 0 \end{cases}$$
    for all $\mathbf{v} \in \mathcal{V}$, and also define $\bar{\gamma}_{\mathbf{d}} : \mathcal{F} \to \mathbb{R}$ by
    $$\bar{\gamma}_{\mathbf{d}}(\mathbf{v}) := \sigma_{\mathbf{e}}(\mathbf{v})\tau(\mathbf{v})^2\gamma_{\mathbf{d}}(\mathbf{v})$$
    for all $\mathbf{v} \in \mathcal{F}$. Find $\mathbf{w} \in \mathcal{V}$ such that $\bar{\gamma}_{\mathbf{d}}(\mathbf{v}) = \mathbf{w} \cdot \mathbf{v}$ for all $\mathbf{v} \in \mathcal{F}$.

    (d) Discuss in detail the analogous problem in a two-dimensional genuine Euclidean space.

Exercises, III

1. Consider the hyperbola in $\mathbb{R}^2$ defined by the set of all pairs $(x, y) \in \mathbb{R}^2$ which satisfy
$$x^2 - k^2y^2 = 1,$$
where $k \in \mathbb{P}^\times$ is given. Show that if (x_0, y_0) is any point on the hyperbola (*i.e.*, (x_0, y_0) satisfies $x_0^2 - k^2y_0^2 = 1$) and if the vector (x_1, y_1) is tangent to the hyperbola at (x_0, y_0), then
$$x_0x_1 - k^2y_0y_1 = 0.$$
Remark: Minkowski [**3**] used this idea as a geometric interpretation of orthogonality in a non-genuine Euclidean space.

2. In a two-dimensional Euclidean plane $\mathcal{E}$ with translation space $\mathcal{V}$, let a hyperbola $\mathcal{H}$ be given with center c. Let $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ and $p \in \mathcal{H}$ be given such that $\mathbb{R}\mathbf{v}$ and $\mathbb{R}\mathbf{w}$ are lines parallel to the asymptotes of $\mathcal{H}$ and such that $p - c = \mathbf{v} + \mathbf{w}$. Show that $\mathbf{v} - \mathbf{w}$ is tangent to $\mathcal{H}$ at p.

3. Let an inner product on $\mathbb{R}^2$ be defined by
$$(u_1, u_2) \cdot (v_1, v_2) := u_1v_1 - 4u_2v_2$$
for all $(u_1, u_2), (v_1, v_2) \in \mathbb{R}^2$.
   (a) What is $\mathsf{sig}^+\mathbb{R}^2$? $\mathsf{sig}^-\mathbb{R}^2$?
   (b) Describe all positive-regular and negative-regular subspaces of $\mathbb{R}^2$.
   (c) Find all subspaces of $\mathbb{R}^2$ such that $\mathcal{U} = \mathcal{U}^\perp$, if any.

4. Let an inner product on $\mathbb{R}^3$ be defined by
$$(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) := u_1v_1 - 4u_2v_2 + u_3v_3$$
for all $(u_1, u_2, u_3), (v_1, v_2, v_3) \in \mathbb{R}^3$.
   (a) What is $\mathsf{sig}^+\mathbb{R}^3$? $\mathsf{sig}^-\mathbb{R}^3$?
   (b) Describe all positive-regular and negative-regular subspaces of $\mathbb{R}^3$.
   (c) Find all subspaces of $\mathbb{R}^3$ such that $\mathcal{U} = \mathcal{U}^\perp$, if any.

5. Let $\mathcal{E}$ be a two-dimensional Minkowskian spacetime, and suppose that $\{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathcal{V}$ such that $\mathbf{b}_1 \cdot \mathbf{b}_1 = 1$ and $\mathbf{b}_2 \cdot \mathbf{b}_2 = -1$. (Do not assume that $\mathbf{b}_1 \cdot \mathbf{b}_2 = 0$.) Let $\mathbf{v} \in \mathcal{V}$ be given, and suppose that $v_1, v_2 \in \mathbb{R}$ are such that $\mathbf{v} = v_1\mathbf{b}_1 + v_2\mathbf{b}_2$.

   (a) Show that

   $$v_1 = \frac{(\mathbf{b}_1 \cdot \mathbf{b}_2)(\mathbf{b}_2 \cdot \mathbf{v}) + \mathbf{b}_1 \cdot \mathbf{v}}{(\mathbf{b}_1 \cdot \mathbf{b}_2)^2 + 1},$$

   and find an analogous expression for v_2.

   (b) Show that $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if

   $$v_1 = v_2\left(-\beta + \sqrt{\beta^2 + 1}\right) \quad \text{or} \quad v_1 = v_2\left(-\beta - \sqrt{\beta^2 + 1}\right),$$

   where $\beta := \mathbf{b}_1 \cdot \mathbf{b}_2$.

6. Consider the example given in Chapter 2, Exercise II,2, with $k := 1$, and put $\mathcal{R} := \mathbb{P}\{(1,1)\}$. Show that if $\mathbf{d} \in \mathcal{F}_1$ is given, then we may choose $\mathbf{e} \in \{\mathbf{d}\}^\perp$ (with $|\mathbf{e}| = 1$) such that relative to the physical spacetime diagram, $\mathbf{e}$ and $\mathbf{d}$ are of the same apparent length and make the same apparent angle with $\mathcal{R}$.

   Remark: That the apparent lengths and angles of $\mathbf{d}$ and $\mathbf{e}$ are the same has *no physical significance.* This is merely coincidental and depends only on the choice of units (*i.e.,* measuring "time" in seconds and "distance" in light-seconds). A different choice of units would have the effect of widening or narrowing the apparent future cone in our spacetime diagram. We adopt the convention of graphically depicting the future cone at an angle other than a right angle so as to remind the reader of the intrinsic geometry of spacetime diagrams; the extrinsic geometry of such diagrams (*i.e.,* apparent length and angle) offers little insight into the intrinsic geometry of spacetime.

For the remainder of the Exercises in this section, assume that $\mathcal{E}$ is a Minkowskian spacetime with translation space $\mathcal{V}$.

7. Let $q \in \mathcal{E}$ and $\mathbf{v} \in \mathcal{F} \cap \mathcal{V}^-$ be given. (Note: We do *not* assume that $\mathbf{v} \cdot \mathbf{v} = -1$.) Find a time-parameterization of the worldpath $q + \mathbb{P}\mathbf{v}$.

8. Let $\mathcal{L}$ be a straight material worldpath with world-direction $\mathbf{u} \in \mathcal{F}_1$, and let $e \in \mathcal{E}$ be given. Show that

   (a) $\alpha := (e-q)^{\cdot 2} + ((e-q)\cdot \mathbf{u})^2$ is independent of $q \in \mathcal{L}$,

   (b) $\alpha \geq 0$, and

   (c) $\mathsf{dst}(e, \mathcal{L}) = \sqrt{\alpha}$.

9. Let $\mathbf{d}_1$ and $\mathbf{d}_2$ be two distinct world-directions, and let $\mu, \nu \in \mathbb{P}^{\times}$ and $\mathbf{e}_1 \in \{\mathbf{d}_1\}^{\perp}$, $\mathbf{e}_2 \in \{\mathbf{d}_2\}^{\perp}$, with $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$, be determined such that

$$\mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1) \quad \text{and} \quad \mathbf{d}_1 = \mu(\mathbf{d}_2 + \nu\mathbf{e}_2).$$

   Let $\mathbf{v} \in \mathcal{V}$ be given.

   (a) Show that there is exactly one combination of $\tau_1, \tau_2, \delta_1, \delta_2 \in \mathbb{R}$, $\mathbf{f}_1 \in \{\mathbf{d}_1, \mathbf{e}_1\}^{\perp}$, and $\mathbf{f}_2 \in \{\mathbf{d}_2, \mathbf{e}_2\}^{\perp}$ such that

$$\mathbf{v} = \tau_1\mathbf{d}_1 + \delta_1\mathbf{e}_1 + \mathbf{f}_1 = \tau_2\mathbf{d}_2 + \delta_2\mathbf{e}_2 + \mathbf{f}_2.$$

   (b) Show that $\mathbf{f}_1 = \mathbf{f}_2$.

   (c) Find formulas that express τ_2 and δ_2 in terms of μ, ν, τ_1, and δ_1.

10. Let two different world-directions $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{F}_1$ be given. Also, let two events $x, y \in \mathcal{E}$ with $x \prec y$ be given. Denote the timelapses between x and y relative to $\mathbf{d}_1$ and $\mathbf{d}_2$, respectively, by

$$\tau_1 := \mathsf{t}_{\mathbf{d}_1}(x, y) \quad \text{and} \quad \tau_2 := \mathsf{t}_{\mathbf{d}_2}(x, y).$$

   Let $\mathbf{e}_1 \in \{\mathbf{d}_1\}^{\perp}$ and $\mathbf{e}_2 \in \{\mathbf{d}_2\}^{\perp}$ with $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$, $\nu \in]0,1[$, and $\mu \in 1 + \mathbb{P}^{\times}$ be determined such that

$$\mathbf{d}_2 = \mu(\mathbf{d}_1 + \nu\mathbf{e}_1) \quad \text{and} \quad \mathbf{d}_1 = \mu(\mathbf{d}_2 - \nu\mathbf{e}_2).$$

   Put

$$\delta_1 := \mathbf{e}_1 \cdot (y - x) \quad \text{and} \quad \delta_2 := \mathbf{e}_2 \cdot (y - x).$$

   (a) Find a formula for τ_2 in terms of τ_1, δ_1, and ν.

   (b) Find a formula for δ_2 in terms of τ_1, δ_1, and ν.

11. Calculate $p_{\perp}$ and α (as in §**5.7**) when p is the parameterization given in (58.5).

12. For every $t \in \mathbb{P}$, we denote by $r(t)$ the event on Jane's worldpath that is simultaneous with $p_D(t)$ relative to $\mathbf{d}_0$. With the notations in §**5.8**, find the "relative speed" between Dick and Jane at a time t along Dick's worldpath by the following two methods:

    (a) Find the function $\delta : \mathbb{P} \to \mathbb{P}$ defined by

    $$\delta(t) := \mathsf{dst}_{\mathbf{d}_0}(p_D(t), r(t))$$

    for all $t \in \mathbb{P}$ and determine the "relative speed" function $\delta^\bullet$.

    (b) Given $t \in \mathbb{P}$, let $\mathbf{u}(t)$ be the world-direction of Jane's worldpath at the event $r(t)$. Using **Thm. 5401**, determine the relative speed $\nu(t)$ between the world-directions $\mathbf{d}_0$ and $\mathbf{u}(t)$.

    Are $\nu(t)$ and $\delta^\bullet(t)$ equal for all $t \in \mathbb{P}$? If not, explain why. Show that these expressions tend to 1 as t gets arbitrarily large.

13. Let two spaceships A and B having different straight worldpaths $\mathcal{L}_A$ and $\mathcal{L}_B$, respectively, be given. Assume that $\mathcal{L}_A$ and $\mathcal{L}_B$ have some event in common.

    Now assume that spaceship A sends out a radar signal that is reflected by B. When the reflected signal is received by A, a second signal is sent out immediately by A which is again reflected by B. Let σ_1 [σ_2] be the timelapse between the emission of the first [second] signal by A and the reception of the corresponding reflected signal by A. What is the relative speed of A and B?

EXERCISES, IV

1. Let $\mathcal{V}$ be an inner-product space, and define $\mathsf{ind}\, \mathcal{V} := \min\{\mathsf{sig}^+\mathcal{V}, \mathsf{sig}^-\mathcal{V}\}$. Consider the following Lemma.

    5800 Lemma: *For every totally singular subspace $\mathcal{U}$ of $\mathcal{V}$ such that $\dim \mathcal{U} < \mathsf{ind}\, \mathcal{V}$, there is some totally singular subspace $\mathcal{W}$ of $\mathcal{V}$ different from $\mathcal{U}$ such that $\mathcal{U} \subset \mathcal{W} \subset \mathcal{U}^\perp$.*

    (a) Define $\mathsf{sig}^0\mathcal{V}$ to be the maximum among the dimensions of all totally singular subspaces of $\mathcal{V}$. Using the Lemma, show that $\mathsf{sig}^0\mathcal{V} = \mathsf{ind}\, \mathcal{V}$.

(b) Using the Lemma, show that if $\mathcal{U} = \mathcal{U}^{\perp}$, then $\mathcal{U}$ is totally singular and $\mathsf{dim}\ \mathcal{U} = \mathsf{ind}\ \mathcal{V}$. In addition, show that in this case we have $\mathsf{sig}^{+}\mathcal{V} = \mathsf{sig}^{-}\mathcal{V} = \mathsf{ind}\ \mathcal{V}$.

(c) Give an example in an inner-product space to show that the converse of (b) is not necessarily true; *i.e.*, exhibit a totally singular subspace $\mathcal{U}$ of $\mathcal{V}$ such that $\mathsf{dim}\ \mathcal{U} = \mathsf{ind}\ \mathcal{V}$ and $\mathcal{U} \subset \mathcal{U}^{\perp}$, but $\mathcal{U} \neq \mathcal{U}^{\perp}$.

(d) Use (a) to show that if $\mathsf{sig}^{-}\mathcal{V} = 1$ and $\mathbf{u}, \mathbf{v} \in \mathcal{V}^{0}$ are such that neither $\mathbf{u}$ nor $\mathbf{v}$ is a multiple of the other, then $\mathbf{u} \cdot \mathbf{v} \neq 0$.

For the next two Exercises in this section, assume that $\mathcal{E}$ is a Minkowskian spacetime with translation space $\mathcal{V}$.

2. Suppose that $\mathsf{dim}\ \mathcal{E} = 2$. Let $e \in \mathcal{E}$ and $q \in \mathsf{Past}(e)$ be given. Show that there exists an event $x \in \mathcal{E}$ with the following property: There exists $\mathbf{u} \in \mathcal{F}_1$ such that $e \prec_{\mathbf{u}} x$ and there exists $\mathbf{v} \in \mathcal{F}_1$ such that $x \prec_{\mathbf{v}} q$. You may show this geometrically (that is, graphically), but be sure that your diagram is drawn accurately!

   The $\prec_{\mathbf{u}}$-worldpath $[e, x]$ and the $\prec_{\mathbf{v}}$-worldpath $[x, q]$ would involve speeds "faster than the speed of light" (see §**5.7**). Thus, motions "faster than the speed of light" would enable one to reach events in one's past, which is absurd; one could, for example, kill one's mother before one was born!

3. Suppose that $x, y \in \mathcal{E}$ are given such that $x \prec y$. Describe the set of all $z \in \mathcal{E}$ such that $x \rightharpoonup z \rightharpoonup y$.

4. The results of Exercise I,17 above might tempt us to conjecture that for every precedence relation $\prec'$ on $\mathcal{E}$ that is coarser than $\prec$, every $\prec$-worldpath is also a $\prec'$-worldpath. Show that this is false with a counterexample.