Chapter 4

Classical Spacetimes

So far in this book, we have only considered mathematical structures involving temporal concepts such as precedence and timelapse. In a classical eventworld, we have an "absolute time" in the sense that every event takes place at a particular instant (see §**2.4**), and hence the question, "How long ago did an event take place?" is meaningful.

In this chapter, we consider structures which allow us to unambiguously ask, "At what location did an event take place?" We begin in §**4.1** with structures which involve the concept of distance. In general, we will use the term "spacetime" for a structure that also involves "spatial" concepts such as distance and location. With the introduction of reference frames in §**4.2**, we may define concepts such as "location" and "motion" relative to a reference frame. This will allow us to decide at what location an event takes place. Thus, we see that one does not require a context of "absolute space" in which to discuss classical physics, contrary to some ideas of Newton. In §**4.3**, we consider spacetimes which may be characterized by the singling out of certain reference frames. Finally, at the end of §**4.3**, we discuss the contexts in which the various classical spacetimes presented in this chapter are appropriate.

4.1 Pre-Classical Spacetimes

Let a classical timed eventworld $\mathcal{E}$ be given. We use the notations as given in §**2.4**. It is apparent that our familiar ideas about the world go beyond the consideration of instants and timelapses. We hardly go a day without making or referring to some kind of length measurement. We infuse this concept into that of a classical timed eventworld in the following definition.[1]

4100 Definition: *A* **pre-classical spacetime** *is a classical timed eventworld* $\mathcal{E}$ *(with precedence* $\prec$ *and timelapse* t*), with additional structure given by specifying a function*

$$\mathsf{d} : \mathsf{Gr}(\sim) \to \mathbb{P}$$

such that for each instant $\sigma \in \Gamma$ *the restriction of* d *to* $\sigma \times \sigma$*, i.e., the mapping* $\mathsf{d}|_{\sigma\times\sigma} : \sigma \times \sigma \to \mathbb{P}$*, endows* σ *with the structure of a finite-dimensional genuine Euclidean space with distance function* $\mathsf{d}|_{\sigma\times\sigma}$*, the dimension of which is the same for each* $\sigma \in \Gamma$*. For brevity, we put* $\mathsf{d}_\sigma := \mathsf{d}|_{\sigma\times\sigma}$*.*

Remark: We offer in §**5.1** a definition of a genuine Euclidean space. For now, suffice it to say that a genuine Euclidean space is a space where the usual rules of Euclidean geometry apply. We recall the following fact, which will be useful in the proof of **Thm. 4201**: for all $\sigma \in \Gamma$ and for all $x, y \in \sigma$, we have

$$\mathsf{d}_\sigma(x, y) = 0 \iff x = y.$$

(See the discussion following **Def. 5103**.)

Perhaps it is helpful to imagine how one might measure distance in a classical, Euclidean sense in order to provide some justification for **Def. 4100**. Suppose one wanted to describe the height of a bouncing ball relative to the floor as a function of time. One might set up a camera to take photographs of the ball at regular time intervals, and then take measurements from the photographs (scaling them by an appropriate factor). If the time interval is short enough, one may get a reasonably accurate approximation to this function.

The important observation to make is that the measurements in this case were made *at an instant;* that is, any particular measurement of the height of the ball above the floor was made at some instant during the ball's trajectory.

[1] This definition was first introduced by W. Noll in 1967, but he used "neo-classical" rather than "pre-classical".

In general, it is only possible to measure distance between locations that belong to the same instant. A moment's thought reveals that due to ongoing changes in temperature, humidity or barometric pressure, objects may alter their form, if only slightly, from one instant to the next. If we are to measure the length of a greeting card, for example, we line up one edge of the card with one end of our ruler. When we glance down at the other end of the ruler to record the measurement, we of course presume that the edges of the greeting card and the ruler are still aligned. While in practice satisfactory, it is important to recognize this as merely a presumption; the card may have expanded slightly (as the ruler might have, or both) while we were shifting our glance.

Although the foregoing example may seem a bit contrived, it nevertheless indicates the necessity for measuring distances between events belonging to the same instant.

Now that we have some idea of distance and time in a classical context, how do we get a handle on the concepts of speed and acceleration? Suppose that you and a friend are travelling along different worldpaths "at the same time"; that is, the intersections of any particular instant with your worldpath and your friend's worldpath are either both empty or each a singleton. At any given instant during your travel, then, there is exactly one event on your friend's worldpath which occurs at that instant – only between two such simultaneous events may we calculate the distance. Thus, for any event on your worldpath, you can calculate the distance between that event and the event occurring at the same instant on your friend's worldpath. This motivates the definition given below.

We now assume that a pre-classical spacetime $\mathcal{E}$ (with notations as in **Def. 4100**) and two worldpaths $\mathcal{L}$ and $\mathcal{L}'$ are given. Recall that we may describe the worldpaths $\mathcal{L}$ and $\mathcal{L}'$ with their natural parameterizations $w_{\mathcal{L}}$ and $w_{\mathcal{L}'}$ (see **Def. 1406**). We further assume that $\Lambda_{\mathcal{L}} = \Lambda_{\mathcal{L}'}$, and put $\Lambda := \Lambda_{\mathcal{L}}$.

4101 Definition: *Define the function*

$$\delta : \Lambda \to \mathbb{P}$$

for each $\sigma \in \Lambda$ by

$$\delta(\sigma) := \mathsf{d}(w_{\mathcal{L}}(\sigma), w_{\mathcal{L}'}(\sigma)).$$

$\delta(\sigma)$ is called the **distance from $\mathcal{L}$ to $\mathcal{L}'$ at** *σ, and δ is called the* **distance function**.

Since we now know the distance between two worldpaths at any instant, it is appropriate to ask whether we can determine a relative speed between two worldpaths.

We may formulate the concept of relative speed in a natural way; that is, as a limit of average relative speeds.

4102 Definition: *Let $\sigma \in \Lambda$ be given. Suppose that the quotient*

$$\frac{\delta(\sigma + s) - \delta(\sigma)}{s}$$

has a limit as s tends towards 0. Then we say that δ **is differentiable at** σ, *and we write $\partial_\sigma \delta$ for this limit. $\partial_\sigma \delta$ is called the* **relative speed of $\mathcal{L}$ and $\mathcal{L}'$ at the instant** σ. *If δ is differentiable at σ for each $\sigma \in \Lambda$, then we say that δ is* **differentiable**, *and the function*

$$\nu \, : \, \Lambda \to \mathbb{R},$$

defined by $\nu(\sigma) := \partial_\sigma \delta$ for all $\sigma \in \Lambda$, is called the **relative speed function**.

Analogously, we define the concept of relative acceleration.

4103 Definition: *Assume that δ is differentiable and let ν be the relative speed function. Let $\sigma \in \Lambda$ be given, and suppose that ν is differentiable at σ; i.e., the quotient*

$$\frac{\nu(\sigma + s) - \nu(\sigma)}{s}$$

has a limit as s tends towards 0. Then we call this limit the **relative acceleration of $\mathcal{L}$ and $\mathcal{L}'$ at** σ, *and we write $\partial_\sigma \nu$ for this limit. If ν is differentiable at all $\sigma \in \Lambda$, we call the function*

$$\alpha \, : \, \Lambda \to \mathbb{R},$$

defined for each $\sigma \in \Lambda$ by $\alpha(\sigma) := \partial_\sigma \nu$, the **relative acceleration function**.

4.2 Reference Frames and Newtonian Spacetimes

As promised in §**1.4**, we now make precise the concept of "location". We begin with an excerpt from Einstein's *Relativity,* [**1**, pp. 9-10]

> It is not clear what is to be understood here by "position" and "space". I stand at the window of a railway carriage which is travelling uniformly, and drop a stone on the embankment, without throwing it. Then, disregarding the influence of the air resistance, I see the stone descend in a straight line. A pedestrian who observes the misdeed from the footpath notices that the stone falls to Earth in a parabolic curve. I now ask: Do the "positions" traversed by the stone lie "in reality" on a straight line or on a parabola? Moreover, what is meant here by motion "in space"?...With the aid of this example it is clearly seen that there is no such thing as an independently existing trajectory, but only a trajectory relative to a particular body of reference.

Thus, we see that even in the classical case, we can describe motion only as relative to some reference frame. For example, when we say that we are driving at 90 km/h, we actually mean that we are travelling at 90 km/h with respect to the Earth. We often make measurements (like speed) relative to some reference frame without making the frame explicit. It is, however, occasionally done: altitude is usually measured in meters above sea level.

In order to make such relativity in classical physics explicit, we formalize the concept of a *reference frame;* that is, some set of references relative to which we make all of our measurements. Usually, scientists make measurements relative to the walls of the laboratory, while astronomers make measurements relative to the "fixed stars"; that is, certain stars which, as far as can be observed during a person's lifetime, do not move in relation to each other or to the Sun. Yet it is just as legitimate to make measurements relative to some other standard reference frame.

Our approach in describing a reference frame is essentially to elucidate what it means to be "at rest". If we know what a particle "at rest" looks like (*e.g.,* a fixed star), then we might be able to describe "motion" *via* speeds relative to such a particle. In particular, two particles "at rest" should be "at rest" relative to each other; that is, the relative distance between two particles "at rest" should be constant.

It is with these ideas in mind that we propose the following definition. For the remainder of this section, we assume that $\mathcal{E}$ has the structure of a pre-classical spacetime.

4200 Definition: *A* **reference frame** *is a collection* F *of worldlines, called* **locations relative to** F, *with the following properties:*

- (F_1) *The relative distance function of any two worldlines in* F *is constant;*

- (F_2) *For every* $x \in \mathcal{E}$, *there is a worldline in* F *which contains* x. *In other words,* $\bigcup \mathsf{F} = \mathcal{E}$.

Let a reference frame F be given.

Given $x \in \mathcal{E}$, then by (F_2), we may find some location $\mathcal{P}$ relative to F such that $x \in \mathcal{P}$. Since $\mathcal{P}$ is a material worldline, we may employ its natural parameterization $w_{\mathcal{P}} : \Gamma \to \mathcal{E}$ with $\mathsf{Rng}\, w_{\mathcal{P}} = \mathcal{P}$ (see **Def. 1406**); hence, we may find $\sigma \in \Gamma$ such that $x = w_{\mathcal{P}}(\sigma)$. Thus, we may assign to x a worldline $\mathcal{P} \in \mathsf{F}$ and an instant $\sigma \in \Gamma$. Since we may make such assignments to each event in $\mathcal{E}$, this suggests that we might represent $\mathcal{E}$ by $\mathsf{F} \times \Gamma$, assigning to each event in $\mathcal{E}$ a location relative to F and an instant in Γ. That this assignment can always be carried out unambiguously is demonstrated in the following Theorem.

4201 Theorem: F *is a partition of* $\mathcal{E}$. *Moreover, given* $x \in \mathcal{E}$, *there is exactly one* $\mathcal{P} \in \mathsf{F}$ *and exactly one* $\sigma \in \Gamma$ *such that* $x = w_{\mathcal{P}}(\sigma)$. *We say that* $\mathcal{P}$ *is the* **location of** x **relative to** F. *We also note that* σ *is the time of* x.

Hence, the mapping

$$\Phi : \mathsf{F} \times \Gamma \to \mathcal{E}$$

given by

$$\Phi(\mathcal{P}, \sigma) := w_{\mathcal{P}}(\sigma)$$

for all $\mathcal{P} \in \mathsf{F}$ *and* $\sigma \in \Gamma$ *is a bijection.*

Proof: Since we know that $\bigcup \mathsf{F} = \mathcal{E}$, recall that to show that F is a partition of $\mathcal{E}$, it is sufficient to show that for all $\mathcal{P}, \mathcal{Q} \in \mathsf{F}$, we have

$$\mathcal{P} \cap \mathcal{Q} \neq \emptyset \implies \mathcal{P} = \mathcal{Q}$$

(see the Note for p. 19 in Appendix A concerning partitions). To this end, let $\mathcal{P}, \mathcal{Q} \in \mathsf{F}$ be given, and assume that $\mathcal{P} \cap \mathcal{Q} \neq \emptyset$. Then we may choose $x \in \mathcal{P} \cap \mathcal{Q}$, and hence we may determine $\sigma, \tau \in \Gamma$ such that $x = w_{\mathcal{P}}(\sigma) = w_{\mathcal{Q}}(\tau)$. Since $w_{\mathcal{P}}(\sigma) \in \sigma$ and $w_{\mathcal{Q}}(\tau) \in \tau$, it follows from the fact that Γ is a partition of $\mathcal{E}$ that $\sigma = \tau$.

Now since F is a reference frame, it follows that the relative distance function $\delta : \Gamma \to \mathbb{P}$ from $\mathcal{P}$ to $\mathcal{Q}$ must be constant. Since

$$\delta(\sigma) = \mathsf{d}_\sigma(w_{\mathcal{P}}(\sigma), w_{\mathcal{Q}}(\sigma)) = \mathsf{d}_\sigma(x, x) = 0$$

(see the remark following **Def. 4100**), then δ is constant and equal to zero.

Now let $\gamma \in \Gamma$ be given. Since $\delta(\gamma) = \mathsf{d}_\gamma(w_{\mathcal{P}}(\gamma), w_{\mathcal{Q}}(\gamma)) = 0$ and d_γ is the distance function of a genuine Euclidean space, we must have $w_{\mathcal{P}}(\gamma) = w_{\mathcal{Q}}(\gamma)$ (see the remark following **Def. 4100**), and hence $\mathcal{P} \cap \gamma = \mathcal{Q} \cap \gamma$ (see **Def. 1406**). Since this is true for all $\gamma \in \Gamma$, and since Γ is a partition of $\mathcal{E}$, it follows that $\mathcal{P} = \mathcal{Q}$. As $\mathcal{P}, \mathcal{Q}$ were arbitrary in F, we see that F is a partition of $\mathcal{E}$.

The remainder of the proof follows easily from the fact that F is a partition of $\mathcal{E}$. The details are left as an Exercise. $\Diamond$

One motivation for calling elements of F "locations" relative to F is that we may give F the natural structure of a genuine Euclidean space as shown in the following result.

4202 Theorem: F *has exactly one structure of a Euclidean space with distance function*

$$\tilde{\mathsf{d}} : \mathsf{F} \times \mathsf{F} \to \mathbb{R}$$

satisfying

$$\tilde{\mathsf{d}}(\mathcal{P}, \mathcal{Q}) = \mathsf{d}(w_{\mathcal{P}}(\sigma), w_{\mathcal{Q}}(\sigma)) \tag{42.1}$$

for all $\mathcal{P}, \mathcal{Q} \in \mathsf{F}$ *and* $\sigma \in \Gamma$.

Proof: Assume that F has been given such a Euclidean structure, and choose $\sigma \in \Gamma$. It follows from **Thm. 4201** and (42.1) that the mapping

$$(\mathcal{P} \mapsto w_{\mathcal{P}}(\sigma)) \,:\, \mathsf{F} \to \sigma \tag{42.2}$$

must be a Euclidean-space isomorphism. Hence the Euclidean structure of F must be the one that is uniquely determined by the Euclidean structure of σ. Conversely, if we define a Euclidean structure on F *via* the bijection (42.2), one can prove that (42.1) holds. $\Diamond$

Remark: We see from the previous Theorem that F has the natural structure of a Euclidean space, and hence also the natural structure of a flat space. We see from Example 1 in §**3.1** that Γ has a natural flat-space structure. Hence it follows from the Example 4 of §**3.1** that $\mathsf{F} \times \Gamma$ also has a flat-space structure. Hence $\mathcal{E}$ may be given the structure of a flat space *via* the bijection Φ described in **Thm. 4201**. If we denote by Z the translation space of F, we see that $\mathsf{Z} \times \mathbb{R}$ is an (external) translation space of $\mathsf{F} \times \Gamma$.

We may characterize a worldpath $\mathcal{L}$ *via* F in a fashion analogous to that given in **Def. 1406**. In the following Proposition, we see that the mapping L assigns to each instant in Λ not an *event* in $\mathcal{E}$ (see **Def. 1406**) but a *location* relative to F. This assignment may be done in a natural way *via* $w_{\mathcal{L}}$. The proof is left as an Exercise.

4203 Proposition: *A subset $\mathcal{L}$ of $\mathcal{E}$ is a worldpath if and only if there is a genuine interval Λ in Γ and a function*

$$L \,:\, \Lambda \to \mathsf{F}$$

such that

$$\mathcal{L} = \bigcup \{L(\sigma) \cap \sigma \,|\, \sigma \in \Lambda\}.$$

If this is the case, then $\Lambda = \Lambda_{\mathcal{L}}$, and

$$w_{\mathcal{L}}(\sigma) = \Phi(L(\sigma), \sigma)$$

for all $\sigma \in \Lambda$ (where Φ is as in **Thm. 4201***). (We interpret $L(\tau)$, roughly, as the location at the instant τ of the particle whose worldpath is $\mathcal{L}$.)*

Having developed the concept of a reference frame, we are now in a position to describe the kind of spacetime considered by Newton.

> Absolute space, in its own nature, without relation to anything external, remains always similar and immovable. Relative space is some movable dimension or measure of the absolute spaces; which our senses determine by its position to bodies; and which is commonly taken for immovable space; such is the dimension of a subterraneous, an aerial, or celestial space, determined by its position in respect of the earth. [**4**, p. 8]

Note that Newton had formulated a concept of relativity in a classical context. He also, however, believed that there was an "absolute space". This motivates the following.

4204 Definition: *A* **Newtonian spacetime** *is a pre-classical spacetime endowed with additional structure by the prescription of a distinguished reference frame, called* **(absolute) space***.*

In a Newtonian spacetime, we speak of unqualified locations; these are, in the terminology of pre-classical spacetimes, "locations relative to absolute space".

4.3 Galilean Spacetimes

In §**4.1**, we introduced the concept of "length" into that of a classical timed eventworld. Here, we go further in order to include the ideas presented in §**3.3**.

4300 Definition: *A* **Galilean spacetime** *is a pre-classical spacetime which also has the structure of a timed flat eventworld such that the distance function is translation-invariant.*

Before comparing the structure of a Galilean spacetime with other structures developed in the previous section, we wish to investigate the structure of the precedence relation in a Galilean spacetime. The following Theorem gives us a complete description of the direction cone of such a precedence relation.

4301 Theorem: *Let a timed flat eventworld $\mathcal{E}$ (see* **Def. 3300***) be given, and assume that $\mathcal{E}$ is classical (see* **Def. 2400***). Let $\mathcal{V}$ be the translation* *space of $\mathcal{E}$, let Γ be the set of all instants, let $\mathcal{F}$ be the future cone, and put $n := \dim \mathcal{E}$. (It follows from* **Def. 2400***(3) that $n \geq 1$.) Let τ be as described in* **Thm. 3302***, and let $\mathcal{F}_1$ be as given in* **Not. 3408***. Then there is exactly one $(n-1)$-dimensional subspace $\mathcal{U}$ of $\mathcal{V}$ with the following properties:*

(1) *Γ consists of all flats with direction space $\mathcal{U}$, so that $\sigma - \sigma = \mathcal{U}$ for all $\sigma \in \Gamma$;*

(2) *The future cone $\mathcal{F}$ (see* **Def. 3203***) of $\mathcal{E}$ is a closed half-space with boundary $\mathcal{U}$;*

(3) *For all $x \in \mathcal{E}$, we have $\mathsf{Past}(x) = x - (\mathcal{F} \setminus \mathcal{U})$, $\mathsf{Pres}(x) = x + \mathcal{U}$, and $\mathsf{Fut}(x) = x + (\mathcal{F} \setminus \mathcal{U})$.*

Proof: We begin by showing the validity of (2). Since the Intermediate Event Inequality becomes equality (see **Def. 2400**(1)), this implies that **Thm. 3302**(2) also becomes equality. Moreover, since $\prec$ is total (being classical), we see that $\mathcal{F} \cup (-\mathcal{F}) = \mathcal{V}$. These observations, along with **Thm. 3302**(3), imply that we may uniquely extend τ to a linear mapping $\bar{\tau} : \mathcal{V} \to \mathbb{R}$ such that $\bar{\tau}(\mathbf{v}) = \tau(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{F}$. Because of **Thm. 3302**(1), one may easily show that $\bar{\tau}(\mathbf{v}) = \bar{\mathfrak{t}}(x, x+\mathbf{v})$ for all $x \in \mathcal{E}$ and $\mathbf{v} \in \mathcal{V}$. Hence, it follows from **Def. 2102** that

$$\bar{\tau}(\mathbf{v}) \geq 0 \iff \mathbf{v} \in \mathcal{F}.$$

This implies, however, that $\mathcal{F} = \bar{\tau}^{<}(\mathbb{P})$, and hence $\mathcal{F}$ is a closed half-space with boundary $\mathcal{U} := \tau^{<}(\{0\})$. With $\mathcal{U}$ thus defined, one may proceed to verify the remainder of the Theorem. Details are left to the Exercises. $\Diamond$

We now assume that a Galilean spacetime $\mathcal{E}$ with translation space $\mathcal{V}$ is given, and put $n := \dim \mathcal{E}$. The set of all instants will be denoted by Γ (as in **Thm. 4301**). Since $\mathcal{E}$ has the structure of a flat eventworld, there are many reference frames (see **Def. 4200**) which may be described in a simple way.

4302 Theorem: *Let $\mathcal{U}$ be described as in* **Thm. 4301***, and suppose that $\mathbf{v} \in \mathcal{F}_1$ is given. Then*

$$\mathsf{F}_{\mathbf{v}} := \{q + \mathbb{R}\mathbf{v} \mid q \in \sigma\}$$

is independent of $\sigma \in \Gamma$, and is a reference frame in $\mathcal{E}$.

Proof: To show that $\{q + \mathbb{R}\mathbf{v} \mid q \in \sigma\}$ is independent of σ is left as an Exercise.

We proceed to show that $\mathsf{F}_\mathbf{v}$ is a reference frame in $\mathcal{E}$ (see **Def. 4200**). The fact that $\bigcup \mathsf{F}_\mathbf{v} = \mathcal{E}$ follows from that fact that $\mathbb{R}\mathbf{v} + \mathcal{U} = \mathcal{V}$. It remains to show that the distance function between any two worldlines in $\mathsf{F}_\mathbf{v}$ is constant.

To this end, let $\mathcal{P}, \mathcal{Q} \in \mathsf{F}_\mathbf{v}$ be given. Since $\{q + \mathbb{R}\mathbf{v} \mid q \in \sigma\}$ is independent of $\sigma \in \Gamma$, we may choose $\sigma \in \Gamma$ and $q, q' \in \sigma$ such that $\mathcal{P} = q + \mathbb{R}\mathbf{v}$ and $\mathcal{Q} = q' + \mathbb{R}\mathbf{v}$. Let $\delta : \Gamma \to \mathbb{R}$ be the distance function from $\mathcal{P}$ to $\mathcal{Q}$ (see **Def. 4101**).

Now let $\sigma' \in \Gamma$ be given. Since $\mathcal{P}$ is a worldline (being in $\mathsf{F}_\mathbf{v}$), we may determine $\mathbf{w} \in \mathbb{R}\mathbf{v}$ such that $q + \mathbf{w} \in \sigma'$. Since $\prec$ is translation-invariant and $q \sim q'$, we have $q + \mathbf{w} \sim q' + \mathbf{w}$, and hence $q' + \mathbf{w} \in \sigma'$. Since d is translation-invariant, we have

$$\begin{aligned} \delta(\sigma') &= \mathsf{d}_{\sigma'}(q + \mathbf{w}, q' + \mathbf{w}) \\ &= \mathsf{d}_{\sigma}(q, q') \\ &= \delta(\sigma). \end{aligned}$$

Since $\sigma' \in \Gamma$ was arbitrary, then δ must be constant. As $\mathcal{P}, \mathcal{Q} \in \mathsf{F}_\mathbf{v}$ were arbitrary, we see that $\mathsf{F}_\mathbf{v}$ is a reference frame in $\mathcal{E}$. $\Diamond$

Remark: Let $\mathcal{U}$ be as in **Thm. 4301**, let $q \in \mathcal{E}$ be given, and let $\mathbf{v} \in \mathcal{F}_1$ be given. Then for each $x \in \mathcal{E}$, we may determine $(\mathbf{u}_x, t_x) \in \mathcal{U} \times \mathbb{R}$ such that

$$x = q + \mathbf{u}_x + t_x\mathbf{v}.$$

We define the mapping $C_\mathbf{v} : \mathcal{E} \to \mathcal{U} \times \mathbb{R}$ by $C_\mathbf{v}(x) := (\mathbf{u}_x, t_x)$ for all $x \in \mathcal{E}$. It is left as an Exercise to show that for each $x \in \mathcal{E}$, we have

$$\begin{aligned} \mathbf{u}_x &= (x - q) - \mathsf{t}(q, x)\mathbf{v}, \\ t_x &= \mathsf{t}(q, x). \end{aligned}$$

We may think of $C_\mathbf{v}$ as a means of "locating" events relative to the reference frame $\mathsf{F}_\mathbf{v}$. Note that the location of x relative to $\mathsf{F}_\mathbf{v}$ is the worldline $q + \mathbf{u}_x + \mathbb{R}\mathbf{v}$, and the time of x is the instant $t_x\mathbf{v} + \mathcal{U}$, so that with the notations given in **Thm. 4201** (with $\mathsf{F} := \mathsf{F}_\mathbf{v}$), we have

$$\Phi(q + \mathbf{u}_x + \mathbb{R}\mathbf{v}, t_x\mathbf{v} + \mathcal{U}) = x.$$

Suppose that $\mathbf{v}' \in \mathcal{F}_1$ is also given. Then one can show that there is exactly one mapping

$$\mathbf{G} : \mathcal{U} \times \mathbb{R} \to \mathcal{U} \times \mathbb{R}$$

such that

$$\mathbf{G} \circ C_{\mathbf{v}} = C_{\mathbf{v}'}.$$

$\mathbf{G}$ is called the *Galilean transformation* from the reference frame $\mathsf{F}_{\mathbf{v}}$ to the reference frame $\mathsf{F}_{\mathbf{v}'}$. It is left as an Exercise to show that for all $(\mathbf{u}, t) \in \mathcal{U} \times \mathbb{R}$, we have

$$\mathbf{G}(\mathbf{u}, t) = (\mathbf{u} + t(\mathbf{v} - \mathbf{v}'), t).$$

This is the formula for a Galilean transformation which is most often seen in the literature.

One may also show that when $\mathbf{v}, \mathbf{v}' \in \mathcal{F}_1$, then $\mathbf{v} - \mathbf{v}' \in \mathcal{U}$. Since instants are genuine Euclidean spaces and $\mathcal{U}$ is the common direction space of instants in Γ, then we may induce a genuine inner-product space structure (see §**5.1**), and hence a magnitude $|\cdot|$ on $\mathcal{U}$. In doing so, we find that the relative speed function between an arbitrary worldline in $\mathsf{F}_{\mathbf{v}}$ and an arbitrary worldline in $\mathsf{F}_{\mathbf{v}'}$ is constant and equal to $|\mathbf{v} - \mathbf{v}'|$. Details are left to the Exercises.

The following result shows that a Newtonian spacetime structure induces exactly one Galilean spacetime structure. (The converse, however, is not true; a Galilean spacetime structure is compatible with infinitely many Newtonian spacetime structures.)

4303 Theorem: *Let $\mathcal{E}$ be a Newtonian spacetime with absolute space* F *as described in* **Def. 4204**. *Then $\mathcal{E}$ has exactly one structure of a Galilean spacetime, with translation space $\mathcal{V}$, such that $\mathsf{F} = \mathsf{F}_{\mathbf{v}}$ for some $\mathbf{v} \in \mathcal{F}_1$, where $\mathsf{F}_{\mathbf{v}}$ is defined as in* **Thm. 4302**.

Proof: Let $\mathcal{E}$ be as described with timelapse t and distance function d.

Clearly, $\mathcal{E}$ is a pre-classical spacetime (see **Def. 4100**). Moreover, $\mathcal{E}$ has the structure of a flat space with external translation space $\mathsf{Z} \times \mathbb{R}$ (see the Remark following the proof of **Thm. 4202**). By examination of **Defs. 4300** and **3300**, to show that $\mathcal{E}$ has the structure of a Galilean spacetime requires us to show that

(1) $\prec$ *is translation-invariant,*

(2) $\prec$ *is connected,*

(3) t *is translation-invariant, and*

(4) d *is translation-invariant.*

Note that since $\mathcal{E}$ is classical, we have $\mathsf{t}_{\mathcal{L}}(x,y) = \mathsf{t}(x,y)$ for all worldpaths $\mathcal{L}$ and events $x, y \in \mathcal{L}$; hence **Def. 3300**(2) is immediately satisfied.

We show (4) here; the demonstration of the others is left as an Exercise.

To see that d is translation-invariant, let $x, y \in \mathcal{E}$ be given such that $x \sim y$, and let $\mathbf{v} := (\mathbf{z}, \alpha) \in \mathsf{Z} \times \mathbb{R}$ be given. Because we may identify $\mathcal{E}$ with $\mathsf{F} \times \Gamma$, and since $x \sim y$, we may determine $\mathcal{P}, \mathcal{Q} \in \mathsf{F}$ and $\sigma \in \Gamma$ such that

$$x = \Phi(\mathcal{P}, \sigma), \quad y = \Phi(\mathcal{Q}, \sigma),$$

where Φ is as described in **Thm. 4201**. Then we also have that

$$x + \mathbf{v} = \Phi(\mathcal{P} + \mathbf{z}, \sigma + \alpha), \quad y + \mathbf{v} = \Phi(\mathcal{Q} + \mathbf{z}, \sigma + \alpha).$$

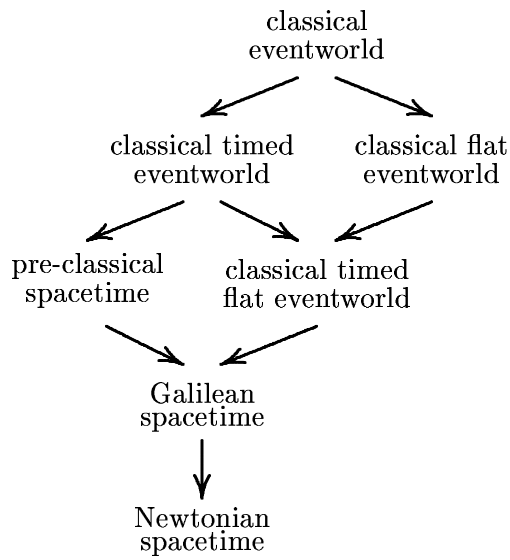
Now $x \sim y$; hence we see from (1) that $x + \mathbf{v} \sim y + \mathbf{v}$. We then have from the definition of Φ, the definition of $\widetilde{\mathsf{d}}$ (see **Thm. 4202**), and the fact that $\widetilde{\mathsf{d}}$ is translation-invariant (as it is the distance function of a genuine Euclidean space), that

$$\begin{aligned}
\mathsf{d}(x + \mathbf{v}, y + \mathbf{v}) &= \mathsf{d}(w_{\mathcal{P}+\mathbf{z}}(\sigma + \alpha), w_{\mathcal{Q}+\mathbf{z}}(\sigma + \alpha)) \\
&= \widetilde{\mathsf{d}}(\mathcal{P} + \mathbf{z}, \mathcal{Q} + \mathbf{z}) \\
&= \widetilde{\mathsf{d}}(\mathcal{P}, \mathcal{Q}) \\
&= \mathsf{d}(w_{\mathcal{P}}(\sigma), w_{\mathcal{Q}}(\sigma)) \\
&= \mathsf{d}(x, y).
\end{aligned}$$

Since $x, y \in \mathcal{E}$ and $\mathbf{v} \in \mathsf{Z} \times \mathbb{R}$ were arbitrary, we see that d is translation-invariant.

Finally, let $\mathcal{P} \in \mathsf{F}$ be given, and choose $x, y \in \mathcal{P}$ such that $\tau(y - x) = 1$; that is, $y - x \in \mathcal{F}_1$. With $\mathbf{v} := y - x$, one may show that $\mathsf{F} = \mathsf{F}_{\mathbf{v}}$. Details are left to the Exercises. $\Diamond$

We summarize the hierarchy of classical eventworlds and spacetimes discussed thus far in Figure 43a.



HIERARCHICAL STRUCTURE OF CLASSICAL EVENTWORLDS

Figure 43a

An arrow pointing downward indicates that the type of structure at the head of the arrow may be obtained from that at the tail of the arrow by introducing additional structure. Said in another way, the type of structure at the tail of an arrow may be obtained by "forgetting" some structural aspect of the type of structure at the head of that arrow.

As it happens, some physical theories are traditionally modelled using Newtonian spacetimes when other spacetimes are more appropriate. One such example occurs in the field of continuum mechanics. An important concept in this field is that of "frame-indifference".[2] This principle is necessary in order to "compensate" for dealing with continuum mechanics in the context

[2]It was first made explicit by Walter Noll with the name "principle of objectivity"; see "A Mathematical Theory of the Mechanical Behavior of Continuous Media", *Archive for Rational Mechanics and Analysis,* Volume 2, pp. 197-226 (1958), p. 209.

of Newtonian spacetime (*i.e.*, with an "absolute" reference frame), when pre-classical spacetime would be more appropriate. A similar phenomenon occurs in classical particle mechanics, where classical spacetime is used when Galilean spacetime would be more appropriate. The principle of "Galilean invariance" is used to compensate for the use of Newtonian spacetime.

Exercises

Exercises, I

1. Complete the proof of **Thm. 4201**.

2. Complete the proof of **Thm. 4202**.

3. Prove **Prop. 4203**.

4. Complete the proof of **Thm. 4301**.

5. Complete the proof of **Thm. 4302**.

6. Show that for each $x \in \mathcal{E}$, we have

$$\begin{aligned} \mathbf{u}_x &= (x - q) - \mathsf{t}(q, x)\mathbf{v}, \\ t_x &= \mathsf{t}(q, x) \end{aligned}$$

   (see the remark following **Thm. 4302**).

7. Show that for all $(\mathbf{u}, t) \in \mathcal{U} \times \mathbb{R}$, we have

$$\mathbf{G}(\mathbf{u}, t) = (\mathbf{u} + t(\mathbf{v} - \mathbf{v}'), t)$$

   (see the remark following **Thm. 4302**).

8. Let a Galilean spacetime $\mathcal{E}$ (with translation space $\mathcal{V}$) be given, and let $\mathcal{U}$ be as described in **Thm. 4301**. Let $|\cdot|$ be the magnitude on $\mathcal{U}$ (as described in the Remark following the proof of **Thm. 4302**). Let $\mathbf{v}, \mathbf{v}' \in \mathcal{F}_1$ be given. Show that $\mathbf{v} - \mathbf{v}' \in \mathcal{U}$, and that the relative speed function between an arbitrary worldline in $\mathsf{F}_{\mathbf{v}}$ (see **Thm. 4302**) and an arbitrary worldline in $\mathsf{F}_{\mathbf{v}'}$ is constant and equal to $|\mathbf{v} - \mathbf{v}'|$.

9. Complete the proof of **Thm. 4303**.

EXERCISES, II

1. Let $\mathcal{L}$, $\mathcal{L}'$, and Λ be as described immediately before **Def. 4101**, and assume that $\Lambda = \Gamma$. Let $\gamma \in \Gamma$ be given, and define $\bar{\delta} : \mathbb{R} \to \mathcal{E}$ by
$$\bar{\delta}(r) := \delta(\gamma + r)$$
for all $r \in \mathbb{R}$.
   (a) Show that if δ is differentiable at each $\tau \in \Gamma$, then $\nu(\tau) = \bar{\delta}^{\bullet}(\tau - \gamma)$ for all $\tau \in \Gamma$.
   (b) Assume that δ is differentiable and that ν is defined as in **Def. 4102**. Show that if ν is differentiable at each $\tau \in \Gamma$, then $\alpha(\tau) = \bar{\delta}^{\bullet\bullet}(\tau - \gamma)$ for all $\tau \in \Gamma$.

EXERCISES, III

1. Consider $\mathcal{E} := \mathbb{R}^3$ with $\prec$, $\bar{\mathsf{t}}$, and d given as follows:
$$(a, b, c) \prec (d, e, f) \;:\Longleftrightarrow\; a \le d,$$
$$\bar{\mathsf{t}}((a, b, c), (d, e, f)) := d - a$$
for all $(a, b, c), (d, e, f) \in \mathbb{R}^3$ (note that $\bar{\mathsf{t}}$ is defined on all pairs in $\mathbb{R}^3 \times \mathbb{R}^3$ since $\prec$ is total), and
$$\mathsf{d}((a, b, c), (d, e, f)) := \sqrt{(e - b)^2 + (f - c)^2}$$
for all $((a, b, c), (d, e, f)) \in \mathsf{Gr}(\sim)$.
   (a) Show that $\prec$, $\bar{\mathsf{t}}$, and d give $\mathcal{E}$ the structure of a Galilean spacetime.
   (b) Show that $\{(a, 0, 0) \mid a \in \mathbb{R}\}$ is a worldline in $\mathcal{E}$.

2. Let $\mathcal{E}$ be given as in the previous Exercise, and let $\kappa \in \mathbb{P}^{\times}$ be given. Consider the parameterization $p : \mathbb{R} \to \mathcal{E}$ of the straight worldline $\mathcal{L} := \{(a, 0, 0) \mid a \in \mathbb{R}\}$ given by
$$p(s) := (s, 0, 0)$$
for all $s \in \mathbb{R}$. Let $p' : \mathbb{R} \to \mathcal{E}$ be a time-parameterization of a worldline $\mathcal{M}$ such that $p(s) \sim p'(s)$ for all $s \in \mathbb{R}$, and suppose that the distance function of $\mathcal{L}$ and $\mathcal{M}$, δ, satisfies
$$\delta(p(s), p'(s)) = \kappa$$
for all $s \in \mathbb{R}$. One can show that $\mathcal{M} := \mathsf{Rng}\, p'$ must be confined to some geometrical figure in $\mathcal{E}$. Describe this figure.

EXERCISES, IV

1. Let $\mathcal{E}$ be a Galilean spacetime, and suppose that $\mathcal{L}$ is a straight world-line in $\mathcal{E}$. Show that there is exactly one $\mathbf{v} \in \mathcal{F}_1$ such that $\mathcal{L}$ is a location in the reference frame $\mathsf{F}_{\mathbf{v}}$.