Chapter 3

Connections

31. Tangent Connectors

We assume that $r \in \tilde{}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 1..r$ and a C^s bundle $(\mathcal{B}, \tau, \mathcal{M})$ be given. We assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and we put

$$n := \dim \mathcal{M} \quad \text{and} \quad m := \dim \mathcal{B} - \dim \mathcal{M}. \tag{31.1}$$

Then $m = \dim \mathcal{B}_x$ for all $x \in \mathcal{M}$.

Recall that for every bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, we have $\mathrm{ev}_1 \circ \phi(\mathbf{v}) = \tau(\mathbf{v})$ and

$$\phi(\mathbf{v}) = \big(\, z\,,\ \mathrm{ev}_2(\phi(\mathbf{v}))\,\big) \qquad \text{where} \quad z := \tau(\mathbf{v}) \tag{31.2}$$

for all $\mathbf{v} \in \mathrm{Dom}\,\phi$. Moreover, if $\phi, \psi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, it follows easily from (31.2) with ϕ replaced by ψ that

$$(\psi \,\square\, \phi^{\leftarrow})(z, \mathbf{u}) = \big(\, z\,,\ \mathrm{ev}_2((\psi \,\square\, \phi^{\leftarrow})(z, \mathbf{u}))\,\big) \tag{31.3}$$

for all $z \in \mathcal{O}_\phi \cap \mathcal{O}_\psi$ and all $\mathbf{u} \in \mathcal{V}_\phi$. ▮

Now let $\mathbf{b} \in \mathcal{B}$ be fixed and put $x := \tau(\mathbf{b})$. Let $\mathrm{in}_x : \mathcal{B}_x \to \mathcal{B}$ be the inclusion mapping

$$\mathrm{in}_x := \mathbf{1}_{\mathcal{B}_x \subset \mathcal{B}}. \tag{31.4}$$

Consider the following diagram

$$\mathcal{B}_x \ \xrightarrow{\mathrm{in}_x}\ \mathcal{B} \ \xrightarrow{\tau}\ \mathcal{M},$$

the composite $\tau \circ \mathrm{in}_x$ is the constant mapping with value x. Taking the gradient of $(\tau \circ \mathrm{in}_x)$ at $\mathbf{b}$, we get $(\nabla_{\mathbf{b}}\tau)(\nabla_{\mathbf{b}}\mathrm{in}_x) = \mathbf{0}$ and hence $\mathrm{Rng}\,\nabla_{\mathbf{b}}\mathrm{in}_x \subset \mathrm{Null}\,\nabla_{\mathbf{b}}\tau$. Indeed, we have $\mathrm{Rng}\,\nabla_{\mathbf{b}}\mathrm{in}_x = \mathrm{Null}\,\nabla_{\mathbf{b}}\tau$ as to be shown in Prop.1.

<u>**Notation:**</u> *We define the* **projection mapping $\mathbf{P}_{\mathbf{b}}$** *at* **b** *by*

$$\mathbf{P}_{\mathbf{b}} := \nabla_{\mathbf{b}}\tau \in \mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{31.5}$$

and the **injection mapping $\mathbf{I}_{\mathbf{b}}$** *at* **b** *by*

$$\mathbf{I}_{\mathbf{b}} := \nabla_{\mathbf{b}}\mathrm{in}_x \in \mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}_x, \mathrm{T}_{\mathbf{b}}\mathcal{B}). \tag{31.6}$$

Proposition 1: *The projection mapping* $\mathbf{P_b}$ *is surjective, the injection mapping* $\mathbf{I_b}$ *is injective, and we have*

$$\operatorname{Null}\mathbf{P_b} = \operatorname{Rng}\mathbf{I_b} \tag{31.7}$$

i.e.

$$\mathrm{T}_{\mathbf{b}}\mathcal{B}_x \xrightarrow{\mathbf{I_b}} \mathrm{T}_{\mathbf{b}}\mathcal{B} \xrightarrow{\mathbf{P_b}} \mathrm{T}_x\mathcal{M} \tag{31.8}$$

is a short exact sequence.

Proof: Choose a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$. It follows from (31.2) that

$$(\phi \circ \mathrm{in}_x)(\mathbf{d}) = \big(\, x\ ,\ \phi\rfloor_x(\mathbf{d})\,\big) \qquad \text{for all} \quad \mathbf{d} \in \mathcal{B}_x.$$

Using the chain rule and (31.6), we obtain

$$((\nabla_{\mathbf{b}}\phi)\,\mathbf{I_b})\,\mathbf{m} = \Big(\mathbf{0}, \nabla_{\mathbf{b}}\phi\rfloor_x\mathbf{m}\Big) \quad \text{for all} \quad \mathbf{m} \in \mathrm{T}_{\mathbf{b}}\mathcal{B}_x\ . \tag{31.9}$$

Since both $\nabla_{\mathbf{b}}\,\phi$ and $\nabla_{\mathbf{b}}\phi\rfloor_x$ are invertible, it follows that $\operatorname{Null}\mathbf{I_b} = \{\mathbf{0}\}$ and

$$\operatorname{Rng}\mathbf{I_b} = (\nabla_{\mathbf{b}}\phi)^{<}(\{\mathbf{0}\} \times \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi) \quad \text{where} \quad \mathbf{v} := \mathrm{ev}_2(\phi(\mathbf{b})). \tag{31.10}$$

On the other hand, it follows from (31.2) that

$$(\tau \circ \phi^{\leftarrow})(z,\mathbf{u}) = z \qquad \text{for all} \quad z \in \mathcal{O}_\phi$$

and all $\mathbf{u} \in \mathcal{V}_\phi$. Using the chain rule and (31.5) we conclude that

$$\mathbf{P_b}(\nabla_{\mathbf{b}}\phi)^{-1}(\mathbf{t},\mathbf{w}) = \mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M} \tag{31.11}$$

and all $\mathbf{w} \in \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi$. Since $\nabla_{\mathbf{b}}\phi$ is invertible, it follows that $\operatorname{Rng}\mathbf{P_b} = \mathrm{T}_x\mathcal{M}$ and

$$\operatorname{Null}\mathbf{P_b} = ((\nabla_{\mathbf{b}}\phi)^{-1})_{>}(\{\mathbf{0}\} \times \mathrm{T}_{\mathbf{v}}\mathcal{V}_\phi) \quad \text{where} \quad \mathbf{v} := \mathrm{ev}_2(\phi(\mathbf{b})). \tag{31.12}$$

Since $((\nabla_{\mathbf{b}}\phi)^{-1})_{>} = (\nabla_{\mathbf{b}}\phi)^{<}$, comparison of (31.10) with (31.12) shows that (31.7) holds. ∎

Definition: *A linear right-inverse of the projection-mapping* $\mathbf{P_b}$ *will be called a* **right tangent-connector** *at* $\mathbf{b}$, *a linear left-inverse of the injection-mapping* $\mathbf{I_b}$ *will be called a* **left tangent-connector** *at* $\mathbf{b}$. *The sets*

$$\begin{aligned} \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} &:= \mathrm{Riv}(\mathbf{P_b}) \\ \mathrm{Lcon}_{\mathbf{b}}\mathcal{B} &:= \mathrm{Liv}(\mathbf{I_b}) \end{aligned} \tag{31.13}$$

of allright tangent-connectors at $\mathbf{b}$ *and all left tangent-connectors at* $\mathbf{b}$ *will be called the* **right tangent-connector space** *at* $\mathbf{b}$ *and the* **left tangent-connector space** *at* $\mathbf{b}$, *respectively.*

The right tangent connector space $\mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ is a flat in $\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B})$ with direction space

$$\big\{\, \mathbf{I}_{\mathbf{b}}\mathbf{L} \;\big|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\,\big\}, \tag{31.14}$$

and the left tangent connector space $\mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$ is a flat in $\mathrm{Lin}\,(\mathrm{T}_{\mathbf{b}}\mathcal{B}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$ with direction space

$$\big\{\, -\mathbf{L}\mathbf{P}_{\mathbf{b}} \;\big|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\,\big\}. \tag{31.15}$$

Using the identifications

$$\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)\{\mathbf{P}_{\mathbf{b}}\} \cong \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x) \cong \{\mathbf{I}_{\mathbf{b}}\}\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}),$$

we consider $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$ as the external translation space of both $\mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ and $\mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$. Since $\dim \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x) = n\,m$, we have

$$\dim \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} = n\,m = \dim \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}. \tag{31.16}$$

By Prop. 1 of Sect. 14, there is a flat isomorphism

$$\mathbf{\Lambda} : \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} \to \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$$

which assigns to every $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$ an element $\mathbf{\Lambda}(\mathbf{K}) \in \mathrm{Lcon}_{\mathbf{b}}\mathcal{B}$ such that

$$\{\mathbf{0}\} \;\longleftarrow\; \mathrm{T}_{\mathbf{b}}\mathcal{B}_x \;\underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow}\; \mathrm{T}_{\mathbf{b}}\mathcal{B} \;\underset{\mathbf{K}}{\longleftarrow}\; \mathrm{T}_x\mathcal{M} \;\longleftarrow\; \{\mathbf{0}\} \tag{31.17}$$

is again a short exact sequence. We have

$$\mathbf{K}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\mathbf{\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathrm{T}_{\mathbf{b}}\mathcal{B}}. \tag{31.18}$$

Proposition 2: *For each bundle chart* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, *let* $\mathbf{A}^{\phi}_{\mathbf{b}}$ *in* $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B})$ *be defined by*

$$\mathbf{A}^{\phi}_{\mathbf{b}}\,\mathbf{t} := (\nabla_{\mathbf{b}}\phi)^{-1}(\mathbf{t}, \mathbf{0}) \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}\ . \tag{31.19}$$

Then $\mathbf{A}^{\phi}_{\mathbf{b}}$ *is a linear right-inverse of* $\mathbf{P}_{\mathbf{b}}$*; i.e.* $\mathbf{A}^{\phi}_{\mathbf{b}} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$.

Proof : If we substitute $\mathbf{w} := \mathbf{0}$ in (31.11) and use (31.19), we obtain

$$\mathbf{P}_{\mathbf{b}}(\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{t}) = \mathbf{t} \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}$$

which shows that $\mathbf{A}^{\phi}_{\mathbf{b}}$ is a linear right-inverse of $\mathbf{P}_{\mathbf{b}}$. ∎

Proposition 3: *If* $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$*, then* $\mathbf{A}_{\mathbf{b}}^{\psi}$ *and* $\mathbf{A}_{\mathbf{b}}^{\phi}$ *differ by*

$$\begin{aligned} \mathbf{A}_{\mathbf{b}}^{\phi} - \mathbf{A}_{\mathbf{b}}^{\psi} &= \mathbf{I}_{\mathbf{b}}\, \mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi} \\ \mathbf{\Lambda}(\mathbf{A}_{\mathbf{b}}^{\phi}) - \mathbf{\Lambda}(\mathbf{A}_{\mathbf{b}}^{\psi}) &= -\mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi}\mathbf{P}_{\mathbf{b}} \end{aligned} \tag{31.20}$$

where

$$\mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi} := (\nabla_{\mathbf{b}}\psi\rfloor_x)^{-1}\Big(\mathrm{ev}_2 \circ \nabla_x\big((\psi \,\square\, \phi^{\leftarrow})(\cdot, \phi\rfloor_x \mathbf{b})\big)\Big) \tag{31.21}$$

which belongs to $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$*.*

Proof : It follows from (31.2) that

$$\phi(\mathbf{b}) = (x, \phi\rfloor_x \mathbf{b}). \tag{31.22}$$

Using (31.3) and (31.22), we obtain

$$\nabla_{\phi(\mathbf{b})}(\psi \,\square\, \phi^{\leftarrow})(\mathbf{t}, \mathbf{0}) = \Big(\mathbf{t}\ ,\ \mathrm{ev}_2\big(\nabla_x\big((\psi \,\square\, \phi^{\leftarrow})(\cdot, \phi\rfloor_x \mathbf{b})\big)\mathbf{t}\big)\Big) \tag{31.23}$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

In view of (23.16), with x replaced by $\mathbf{b}$, γ by ψ, and χ by ϕ, we have

$$\nabla_{\phi(\mathbf{b})}(\psi \,\square\, \phi^{\leftarrow}) = (\nabla_{\mathbf{b}}\psi)(\nabla_{\mathbf{b}}\phi)^{-1}.$$

If we substitute this formula into (31.23) and use (31.19) and (31.21), we obtain

$$(\nabla_{\mathbf{b}}\psi)(\mathbf{A}_{\mathbf{b}}^{\phi}\mathbf{t}) = \Big(\mathbf{t}\ ,\ \nabla_{\mathbf{b}}\psi\rfloor_x \mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi}\mathbf{t}\Big)$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. Using (31.19) with ψ replaced by ϕ, we conclude that

$$\mathbf{A}_{\mathbf{b}}^{\phi}\mathbf{t} = \mathbf{A}_{\mathbf{b}}^{\psi}\mathbf{t} + (\nabla_{\mathbf{b}}\psi)^{-1}\Big(\mathbf{0}\ ,\ \nabla_{\mathbf{b}}\psi\rfloor_x \mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi}\mathbf{t}\Big)$$

for all $\mathbf{t} \in \mathrm{T}_x\mathcal{M}$. The desired result $(31.20)_1$ now follows from (31.9), with ϕ replaced by ψ and $\mathbf{m} := \mathbf{\Gamma}_{\mathbf{b}}^{\phi,\psi}\mathbf{t}$. Equation $(31.20)_2$ follows from $(31.20)_1$ and Prop. 3 of Sect.14. ∎

<u>**Notation:**</u> *Let* $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ *be given. The mapping*

$$\mathbf{\Gamma}_{\mathbf{b}}^{\phi} : \mathrm{Rcon}_{\mathbf{b}}\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T}_{\mathbf{b}}\mathcal{B}_x)$$

is defined by $\mathbf{\Gamma}_{\mathbf{b}}^{\phi} := \mathbf{\Gamma}^{\mathbf{A}_{\mathbf{b}}^{\phi}}$ *in terms of* (14.10)*; i.e. by*

$$\mathbf{\Gamma}_{\mathbf{b}}^{\phi}(\mathbf{K}) := -\mathbf{\Lambda}(\mathbf{A}_{\mathbf{b}}^{\phi})\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}. \tag{31.24}$$

If $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, we have, by Prop. 6 of Sect. 14,

$$\begin{aligned}\mathbf{A}^{\phi}_{\mathbf{b}} - \mathbf{K} &= \mathbf{I}_{\mathbf{b}}\,\mathbf{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K})\\ \mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) - \mathbf{\Lambda}(\mathbf{K}) &= -\mathbf{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K})\mathbf{P}_{\mathbf{b}}\end{aligned} \tag{31.25}$$

for all $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$. Moreover; if $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, then (31.20) and (31.24) give

$$\mathbf{\Gamma}^{\phi}_{\mathbf{b}}(\mathbf{K}) - \mathbf{\Gamma}^{\psi}_{\mathbf{b}}(\mathbf{K}) = \mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}, \tag{31.26}$$

where $\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}}$ is defined by (31.21). It follows from (31.26) and $\mathbf{\Gamma}^{\psi}_{\mathbf{b}}\big(\mathbf{A}^{\psi}_{\mathbf{b}}\big) = \mathbf{0}$ that $\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{b}} = \mathbf{\Gamma}^{\phi}_{\mathbf{b}}\big(\mathbf{A}^{\psi}_{\mathbf{b}}\big)$ for all $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$.

Convention : Assume that $\mathcal{B}$ is a flat-space bundle. Let $\mathbf{b} \in \mathcal{B}$ be given and put $x := \tau(\mathbf{b})$. The fiber $\mathcal{B}_x$ has the structure of a flat space; the translation space of $\mathcal{B}_x$ is denoted by $\mathcal{U}_x$. We may and will use the identification as described in (23.9) and (23.10); i.e. we identify $\mathrm{T}_{\mathbf{b}}\mathcal{B}_x$ with $\mathcal{U}_x$. Then (31.8) becomes

$$\mathcal{U}_x \xrightarrow{\mathbf{I}_{\mathbf{b}}} \mathrm{T}_{\mathbf{b}}\mathcal{B} \xrightarrow{\mathbf{P}_{\mathbf{b}}} \mathrm{T}_x\mathcal{M}. \tag{31.27}$$

In particular, if $\mathcal{B}$ is a linear-space bundle, we have $\mathcal{U}_x = \mathcal{B}_x$ and (31.27) becomes

$$\mathcal{B}_x \xrightarrow{\mathbf{I}_{\mathbf{b}}} \mathrm{T}_{\mathbf{b}}\mathcal{B} \xrightarrow{\mathbf{P}_{\mathbf{b}}} \mathrm{T}_x\mathcal{M}. \tag{31.28}$$

Remark 1: For every bundle chart ϕ in $\mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, we have

$$\begin{aligned}&\mathbf{P}_{\mathbf{b}} = \mathrm{ev}_1 \circ \nabla_{\mathbf{b}}\phi, && \mathbf{A}^{\phi}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_1,\\ &\mathbf{I}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_2 \circ \nabla_{\mathbf{b}}\phi\big\rfloor_x, && \mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = (\nabla_{\mathbf{b}}\phi\big\rfloor_x)^{-1}(\mathrm{ev}_2 \circ \nabla_{\mathbf{b}}\phi),\end{aligned} \tag{31.29}$$

where ev_i and ins_i, $i \in 2^{\rceil}$, are evaluations and insertions, respectively.

Proof: Let $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. Using (31.9), (31.19) and also observing $\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = \mathbf{1}_{\mathrm{T}_{\mathbf{b}}\mathcal{B}}$, we have

$$\nabla_{\mathbf{b}}\phi = \nabla_{\mathbf{b}}\phi\big(\mathbf{A}^{\phi}_{\mathbf{b}}\mathbf{P}_{\mathbf{b}} + \mathbf{I}_{\mathbf{b}}\mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}})\big) = \big(\,\mathbf{P}_{\mathbf{b}}\,,\,(\nabla_{\mathbf{b}}\phi)\big\rfloor_x\mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}})\big). \tag{31.30}$$

The desired result (31.29) follows from (31.9), (31.19) and (31.30). ∎

If in addition $\phi\big\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$, then we have

$$\mathbf{I}_{\mathbf{b}} = (\nabla_{\mathbf{b}}\phi)^{-1} \circ \mathrm{ins}_2 \qquad \text{and} \qquad \mathbf{\Lambda}(\mathbf{A}^{\phi}_{\mathbf{b}}) = (\mathrm{ev}_2 \circ \nabla_{\mathbf{b}}\phi).$$

Remark 2: For every cross section $\mathbf{s} : \mathcal{M} \to \mathcal{B}$, we have $\tau \circ \mathbf{s} = \mathbf{1}_{\mathcal{M}}$. If $\mathbf{s}$ is differentiable at $x \in \mathcal{M}$, then the gradient of $\mathbf{1}_{\mathcal{M}} = \tau \circ \mathbf{s}$ at x gives

$$\mathbf{1}_{\mathrm{T}_x\mathcal{M}} = \nabla_x(\tau \circ \mathbf{s}) = (\nabla_{\mathbf{s}(x)}\tau)(\nabla_x \mathbf{s}) = \mathbf{P}_{\mathbf{s}(x)}\nabla_x \mathbf{s}. \tag{31.31}$$

We see that $\nabla_x \mathbf{s}$ is a right tangent connector at $\mathbf{s}(x)$; i.e. $\nabla_x \mathbf{s} \in \mathrm{Rcon}_{\mathbf{s}(x)}(\mathcal{B})$. ∎

Remark 3: Let $\mathcal{B}$ be a linear space bundle and let $x \in \mathcal{M}$ be given. Denote the zero of the linear space $\mathcal{B}_x$ by $\mathbf{0}_x$. It follows from (31.21) that $\mathbf{\Gamma}^{\phi,\psi}_{\mathbf{0}_x} = \mathbf{0}$ and then from (31.20) that $\mathbf{A}^{\phi}_{\mathbf{0}_x} = \mathbf{A}^{\psi}_{\mathbf{0}_x}$ for all $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$. This shows that $\left\{ \mathbf{A}^{\phi}_{\mathbf{0}_x} \;\middle|\; \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \right\}$ is a singleton and hence

$$\left\{ \mathbf{A}^{\phi}_{\mathbf{0}_x} \;\middle|\; \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \right\} \;\; \mathrm{Rcon}_{\mathbf{0}_x}\mathcal{B}.$$ ∎

Remark 4: For every $\mathbf{b} \in \mathcal{B}$, we define the **vertical space** $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ **of** $\mathcal{B}$ **at b** by

$$\mathrm{V}_{\mathbf{b}}\mathcal{B} := \mathrm{Null}\,\mathbf{P}_{\mathbf{b}} = \mathrm{Rng}\,\mathbf{I}_{\mathbf{b}} \subset \mathrm{T}_{\mathbf{b}}\mathcal{B}\,. \tag{31.32}$$

Since $\mathbf{I}_{\mathbf{b}}$ is injective, $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ is isomorphic with $\mathrm{T}_{\mathbf{b}}\mathcal{B}_{\tau(\mathbf{b})}$. The sequence

$$\mathrm{V}_{\mathbf{b}}\mathcal{B} \hookrightarrow \mathrm{T}_{\mathbf{b}}\mathcal{B} \xrightarrow{\mathbf{P}_{\mathbf{b}}} \mathrm{T}_{\tau(\mathbf{b})}\mathcal{M} \tag{31.33}$$

is a short exact sequence. For every right tangent connector $\mathbf{K} \in \mathrm{Rcon}_{\mathbf{b}}\mathcal{B}$, the range of $\mathbf{K}$

$$\mathrm{H}^{\mathbf{K}}_{\mathbf{b}}\mathcal{B} := \mathrm{Rng}\mathbf{K} \subset \mathrm{T}_{\mathbf{b}}\mathcal{B} \tag{31.34}$$

is called the **horizontal space** of $\mathcal{B}$ at $\mathbf{b}$ relative to $\mathbf{K}$. It is easily seen that $\mathrm{V}_{\mathbf{b}}\mathcal{B}$ and $\mathrm{H}^{\mathbf{K}}_{\mathbf{b}}\mathcal{B}$ are supplementary in $\mathrm{T}_{\mathbf{b}}\mathcal{B}$. ∎

Notes 31

(1) The convention that we made in this section was first introduced by Noll, in 1974, on the tangent bundle $\mathrm{T}\mathcal{M}$ (see [N3]). This convention plays a central role in our development.

(2) The short exact sequence (31.33) can be found in [Sa].

32. Transfer Isomorphisms, Shift Spaces

We assume that $r \in \tilde{}$ with $r \geq 2$ and a C^r-manifold $\mathcal{M}$ are given. Let a number $s \in 1..r$ be given and let $\mathcal{B}$ be a C^s linear-space bundle over $\mathcal{M}$. We assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim \mathcal{M}$ and $m := \dim \mathcal{B} - \dim \mathcal{M}$. Then

$$m = \dim \mathcal{B}_x \quad \text{for all} \quad x \in \mathcal{M}. \tag{32.1}$$

Now let $x \in \mathcal{M}$ be fixed. We define the **bundle of transfer isomorphisms** of $\mathcal{B}$ from x by

$$\mathrm{Tlis}_x\mathcal{B} := \bigcup_{y \in \mathcal{M}} \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y). \tag{32.2}$$

It is endowed with the natural structure of a C^s-fiber bundle as shown below. The corresponding bundle projection $\pi_x : \mathrm{Tlis}_x\mathcal{B} \to \mathcal{M}$ is given by

$$\pi_x(\mathbf{T}) :\in \left\{ \, y \in \mathcal{M} \; \middle| \; \mathbf{T} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{B}_y) \, \right\} \tag{32.3}$$

and the bundle inclusion $\iota_x : \mathrm{Lis}\,\mathcal{B}_x \to \mathrm{Tlis}_x\,\mathcal{B}$ at x is

$$\iota_x := \mathbf{1}_{\mathrm{Lis}\mathcal{B}_x \subset \mathrm{Tlis}_x\mathcal{B}}. \tag{32.4}$$

For every bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, we define

$$\mathrm{tlis}_x^{\phi} : \mathrm{Tlis}_x(\mathcal{O}_\phi) \to \mathcal{O}_\phi \times \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi) \tag{32.5}$$

by

$$\mathrm{tlis}_x^{\phi}(\mathbf{T}) := \left(\, z \, , \; \phi\rfloor_z \mathbf{T} \, \right), \quad \text{where} \quad z := \pi_x(\mathbf{T}). \tag{32.6}$$

It is easily seen that tlis_x^{ϕ} is invertible and that

$$\mathrm{tlis}_x^{\phi\leftarrow}(z, \mathbf{L}) = (\phi\rfloor_z)^{-1}\mathbf{L} \tag{32.7}$$

for all $z \in \mathcal{O}_\phi$ and all $\mathbf{L} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi)$. Moreover, if $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, it follows easily from (32.7) and (32.6) with ϕ replaced by ψ that

$$\left(\mathrm{tlis}_x^{\psi} \circ \mathrm{tlis}_x^{\phi\leftarrow}\right)(z, \mathbf{L}) = \left(\, z \, , \; (\psi \diamond \phi)(z)\mathbf{L} \, \right) \tag{32.8}$$

for all $z \in \mathcal{O}_\psi \cap \mathcal{O}_\phi$ and all $\mathbf{L} \in \mathrm{Lis}(\mathcal{B}_x, \mathcal{V}_\phi)$ (See (22.7) for the definition of $\psi \diamond \phi$). It is clear that $\mathrm{tlis}_x^{\psi} \circ \mathrm{tlis}_x^{\phi\leftarrow}$ is of class C^s. Since $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ were arbitrary, it follows that $\left\{ \mathrm{tlis}_x^{\alpha} \;\middle|\; \alpha \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M}) \, \right\}$ is a C^s-bundle atlas of $\mathrm{Tlis}_x\mathcal{B}$. We consider $\left(\mathrm{Tlis}_x\mathcal{B} \, , \, \pi_x, \mathcal{M} \right)$ as being endowed with the C^s fiber bundle structure over $\mathcal{M}$ determined by this atlas.

Remark : We may view $\mathrm{Tlis}_x\mathcal{B}$ as a Tran_x-bundle, where Tran_x is the iso-category whose objects are of the form $\mathrm{Lis}(\mathcal{B}_x, \mathcal{V})$ with $\mathcal{V} \in LS$ and whose isomorphisms are of the form

$$(\mathbf{T} \mapsto \mathbf{LT}) : \mathrm{Lis}(\mathcal{B}_x, \mathrm{Dom}\mathbf{L}) \to \mathrm{Lis}(\mathcal{B}_x, \mathrm{Cod}\mathbf{L})$$

with $\mathbf{L} \in \mathrm{LIS}$. ▮

It is easily seen that the mappings π_x and ι_x defined by (32.3) and (32.4) are of class C^s.

We now apply the results of Sect.31 by replacing the ISO-bundle $\mathcal{B}$ there by the bundle $\mathrm{Tlis}_x\mathcal{B}$ and $\mathbf{b} \in \mathcal{B}$ there by $\mathbf{1}_{\mathcal{B}_x} \in \mathrm{Tlis}_x\mathcal{B}$.

Definition: *The* **shift-space** $\mathrm{S}_x\mathcal{B}$ *of* $\mathcal{B}$ *at* $x \in \mathcal{M}$ *is defined to be*

$$\mathrm{S}_x\mathcal{B} := \mathrm{T}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B}. \tag{32.9}$$

We define the **projection mapping** *of* $\mathrm{S}_x\mathcal{B}$ *by*

$$\mathbf{P}_x := \mathbf{P}_{\mathbf{1}_{\mathcal{B}_x}} = \nabla_{\mathbf{1}_{\mathcal{B}_x}}\pi_x \in \mathrm{Lin}\,(\mathrm{S}_x\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{32.10}$$

and the **injection mapping** *of* $\mathrm{S}_x\mathcal{B}$ *by*

$$\mathbf{I}_x := \mathbf{I}_{\mathbf{1}_{\mathcal{B}_x}} = \nabla_{\mathbf{1}_{\mathcal{B}_x}}\iota_x \in \mathrm{Lin}\,(\mathrm{Lin}\mathcal{B}_x, \mathrm{S}_x\mathcal{B}) \tag{32.11}$$

in terms of (31.5) *and* (31.6)*; respectively, where* π_x *and* ι_x *are defined by* (32.3) *and* (32.4).

It is clear from (32.5) that

$$\dim\,(\mathrm{Tlis}_x\mathcal{B}) = \dim\,(\mathrm{S}_x\mathcal{B}) = n + m^2. \tag{32.12}$$

Proposition 1: *The projection mapping* $\mathbf{P}_x$ *is surjective, the injection mapping* $\mathbf{I}_x$ *is injective, and we have*

$$\mathrm{Null}\,\mathbf{P}_x = \mathrm{Rng}\,\mathbf{I}_x \tag{32.13}$$

i.e.

$$\mathrm{Lin}\,\mathcal{B}_x \xrightarrow{\mathbf{I}_x} \mathcal{S}_x\mathcal{B} \xrightarrow{\mathbf{P}_x} \mathrm{T}_x\mathcal{M} \tag{32.14}$$

is a short exact sequence.

Definition: *A linear right-inverse of the projection-mapping* $\mathbf{P}_x$ *will be called a* **right shift-connector** *(or simply* **right connector***) at* x*, a linear left-inverse* *of the injection-mapping* $\mathbf{I}_x$ *will be called a* **left shift-connector** *(or simply* **left connector***) at* x*. The sets*

$$\begin{aligned} \mathrm{Rcon}_x\mathcal{B} &:= \mathrm{Rcon}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B} \\ \mathrm{Lcon}_x\mathcal{B} &:= \mathrm{Lcon}_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{Tlis}_x\mathcal{B} \end{aligned} \tag{32.15}$$

of all right connectors at x *and all left connector at* x *will be called the* **right connector space** *at* x *and the* **left connector space** *at* x*, respectively.*

The right connector space $\mathrm{Rcon}_x\mathcal{B}$ is a flat in $\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathcal{S}_x\mathcal{B})$ with direction space

$$\left\{ \mathbf{I}_x\mathbf{L} \;\middle|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \right\}, \tag{32.16}$$

and the left connector space $\mathrm{Lcon}_x\mathcal{B}$ is a flat in $\mathrm{Lin}\,(\mathcal{S}_x\mathcal{B}, \mathrm{Lin}\,\mathcal{B}_x)$ with direction space

$$\left\{ -\mathbf{L}\mathbf{P}_x \;\middle|\; \mathbf{L} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \right\}. \tag{32.17}$$

Using the identifications

$$\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)\{\mathbf{P}_x\} \cong \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \cong \{\mathbf{I}_x\}\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x),$$

we consider $\mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$ as the external translation space of both $\mathrm{Rcon}_x\mathcal{B}$ and $\mathrm{Lcon}_x\mathcal{B}$. Since $\dim \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) = nm^2$, we have

$$\dim \mathrm{Rcon}_x\mathcal{B} = nm^2 = \dim \mathrm{Lcon}_x\mathcal{B}. \tag{32.18}$$

The flat isomorphism

$$\mathbf{\Lambda} : \mathrm{Rcon}_x\mathcal{B} \to \mathrm{Lcon}_x\mathcal{B}$$

assigns to every $\mathbf{K} \in \mathrm{Rcon}_x\mathcal{B}$ an element $\mathbf{\Lambda}(\mathbf{K}) \in \mathrm{Lcon}_x\mathcal{B}$ such that

$$\mathrm{Lin}\,\mathcal{B}_x \;\underset{\mathbf{\Lambda}(\mathbf{K})}{\longleftarrow}\; \mathcal{S}_x\mathcal{B} \;\underset{\mathbf{K}}{\longleftarrow}\; \mathrm{T}_x\mathcal{M} \tag{32.19}$$

is again a short exact sequence. We have

$$\mathbf{K}\mathbf{P}_x + \mathbf{I}_x\mathbf{\Lambda}(\mathbf{K}) = \mathbf{1}_{\mathcal{S}_x\mathcal{B}} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Rcon}_x\mathcal{B}. \tag{32.20}$$

Convention : Since there is one-to-one correspondence between right connectors and left connectors, we shall only deal with one kind of connectors, say right connectors. If we say "connector", we mean a right connector. The notation

$$\mathrm{Con}_x\mathcal{B} := \mathrm{Rcon}_x\mathcal{B}$$

is also used.

Proposition 2: *For each* $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$*, let* $\mathbf{A}_x^\phi \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathcal{S}_x\mathcal{B})$ *be defined by* $\mathbf{A}_x^\phi := \mathbf{C}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi}$ *in terms of* (31.19)*; i.e.*

$$\mathbf{A}_x^\phi\,\mathbf{t} := (\nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{tlis}_x^\phi)^{-1}(\mathbf{t},\mathbf{0}) \qquad \text{for all} \quad \mathbf{t} \in \mathrm{T}_x\mathcal{M}\ . \tag{32.21}$$

Then $\mathbf{A}_x^\phi$ *is a linear right-inverse of* $\mathbf{P}_x$*, i.e.* $\mathbf{A}_x^\phi \in \mathrm{Con}_x\mathcal{B}$*.*

Let $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. We have the following short exact sequence

$$\mathrm{Lin}\,\mathcal{B}_x \quad \underset{\mathbf{\Lambda}(\mathbf{A}_x^\phi)}{\longleftarrow} \quad \mathcal{S}_x\mathcal{B} \quad \underset{\mathbf{A}_x^\phi}{\longleftarrow} \quad \mathrm{T}_x\mathcal{M} \tag{32.22}$$

and

$$\mathbf{A}_x^\phi\mathbf{P}_x + \mathbf{I}_x\mathbf{\Lambda}(\mathbf{A}_x^\phi) = \mathbf{1}_{\mathcal{S}_x\mathcal{B}}. \tag{32.23}$$

Proposition 3: *If* $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *are given, then*

$$\begin{aligned} \mathbf{A}_x^\phi - \mathbf{A}_x^\psi &= \mathbf{I}_x\,\mathbf{\Gamma}_x^{\phi,\psi} \\ \mathbf{\Lambda}(\mathbf{A}_x^\phi) - \mathbf{\Lambda}(\mathbf{A}_x^\psi) &= -\mathbf{\Gamma}_x^{\phi,\psi}\mathbf{P}_x \end{aligned} \tag{32.24}$$

where $\mathbf{\Gamma}_x^{\phi,\psi} := \mathbf{\Gamma}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi,\mathrm{tlis}_x^\psi}$ *in terms of* (31.21) *is of the form*

$$\mathbf{\Gamma}_x^{\phi,\psi} := (\psi\rfloor_x)^{-1}\big(\nabla_x(\psi \diamond \phi)\big) \circ (\mathbf{1}_{\mathrm{T}_x\mathcal{B}} \times \phi\rfloor_x) \tag{32.25}$$

which belongs to $\mathrm{Lin}\,(\mathrm{T}_x, \mathrm{Lin}\,\mathcal{B}_x)$*. Here, the notation* (22.7) *is used.*

Proof : Applying Prop. 3 in Sect. 32 with ϕ replaced by tlis_x^ϕ and ψ replaced by tlis_x^ψ together with (32.6) and (32.8), we obtain the desired result (32.25). ▮

__Notation:__ *Let* $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *be given. We define the mapping*

$$\mathbf{\Gamma}_x^\phi : \mathrm{Con}_x\mathcal{B} \to \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x)$$

by $\mathbf{\Gamma}_x^\phi := \mathbf{\Gamma}^{\mathbf{A}_x^\phi} = \mathbf{\Gamma}_{\mathbf{1}_{\mathcal{B}_x}}^{\mathrm{tlis}_x^\phi}$ *in terms of* (14.10) *and* (31.24)*; i.e.*

$$\mathbf{\Gamma}_x^\phi(\mathbf{K}) = -\mathbf{\Lambda}(\mathbf{A}_x^\phi)\mathbf{K} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}. \tag{32.26}$$

If $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, then (31.25) reduces to

$$\begin{aligned} \mathbf{A}_x^\phi - \mathbf{K} &= \mathbf{I}_x\,\mathbf{\Gamma}_x^\phi(\mathbf{K}) \\ \mathbf{\Lambda}(\mathbf{A}_x^\phi) - \mathbf{\Lambda}(\mathbf{K}) &= -\mathbf{\Gamma}_x^\phi(\mathbf{K})\mathbf{P}_x \end{aligned} \tag{32.27}$$

for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$. Moreover; if $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$, then

$$\mathbf{\Gamma}_x^{\phi}(\mathbf{K}) - \mathbf{\Gamma}_x^{\psi}(\mathbf{K}) = \mathbf{\Gamma}_x^{\phi,\psi} \quad \text{for all} \quad \mathbf{K} \in \mathrm{Con}_x\mathcal{B}, \tag{32.28}$$

where $\mathbf{\Gamma}_x^{\phi,\psi}$ is defined by (32.25). It follows from (32.28) that $\mathbf{\Gamma}_x^{\psi,\phi} = -\mathbf{\Gamma}_x^{\phi,\psi}$ and from $\mathbf{\Gamma}_x^{\psi}\left(\mathbf{A}_x^{\psi}\right) = \mathbf{0}$ that $\mathbf{\Gamma}_x^{\phi}\left(\mathbf{A}_x^{\psi}\right) = \mathbf{\Gamma}_x^{\phi,\psi}$ for all bundle charts $\psi, \phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$.

For every cross section $\mathbf{H} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ of the bundle $\mathrm{Tlis}_x\mathcal{B}$, the mapping $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{B}$ defined by

$$\mathbf{T}(y) := \mathbf{H}(y)\mathbf{H}^{-1}(x) \qquad \text{for all} \quad y \in \mathcal{M} \tag{32.29}$$

is a cross section of the bundle $\mathrm{Tlis}_x\mathcal{B}$ with $\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}$.

Definition: *A cross section* $\mathbf{T} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ *of the bundle* $\mathrm{Tlis}_x\mathcal{B}$ *such that* $\mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}$ *is called a* ***transport from*** *x.*

For every bundle chart $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$, we see that

$$\left(y \mapsto (\phi\rfloor_y)^{-1}\phi\rfloor_x\right) : \mathcal{O}_\phi \to \mathrm{Tlis}_x\mathcal{B}$$

is a transport from x which is of class C^s.

Remark 1: For every $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$, there is a bundle chart $\phi \in \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$ with $\phi\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$ such that

$$\mathbf{K} = \nabla_x(\phi\rfloor)^{-1} = \mathbf{A}_x^{\phi}. \tag{32.30}$$

Proof: Let $\mathbf{K} \in \mathrm{Con}_x\mathcal{B}$ be given. It is not hard to construct a transport $\mathbf{T} : \mathcal{O} \to \mathrm{Tlis}_x\mathcal{B}$ from x such that (Ask Prof. Noll!!!!!!!!!!!!!!!!!!!!!!)

$$\mathbf{K} = \nabla_x\mathbf{T}. \tag{32.31}$$

There is a bundle chart $\phi : \tau^{<}(\mathcal{O}) \to \mathcal{O} \times \mathcal{B}_x$ induced from $\mathbf{T}$ by

$$\phi(\mathbf{v}) := (\ y\ ,\ \mathbf{T}^{-1}(y)\mathbf{v}) \qquad \text{where} \quad y := \tau(\mathbf{v}) \tag{32.32}$$

for all $\mathbf{v} \in \tau^{<}(\mathcal{O})$. It is easily seen that $(\phi\rfloor)^{-1} = \mathbf{T}$. The first part of (32.30) follows from (32.31). In view of (31.29) we have

$$\begin{aligned} \mathbf{\Lambda}(\mathbf{A}_x^{\phi})\left(\nabla_x(\phi\rfloor)^{-1}\right) &= \left(\mathrm{ev}_2 \circ \nabla_{\mathbf{1}_{\mathcal{B}_x}}\mathrm{tlis}_x^{\phi}\right)\nabla_x(\phi\rfloor)^{-1} \\ &= \mathrm{ev}_2 \circ \nabla_x\left(y \mapsto \mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1})\right). \end{aligned} \tag{32.33}$$

Using (32.6) and ovbserving $\phi\rfloor_y \in \mathrm{Lin}\,(\mathcal{B}_y, \mathcal{B}_x)$, we have

$$\mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1}) = (\ y\ ,\ \phi\rfloor_y(\phi\rfloor_y)^{-1}) = (\ y\ ,\ \mathbf{1}_{\mathcal{B}_x}). \tag{32.34}$$

Taking the gradient of (32.34) at x, we observe that

$$\nabla_x\left(y \mapsto \mathrm{tlis}_x^{\phi}((\phi\rfloor_y)^{-1})\right) = (\mathbf{1}_{\mathrm{T}_x\mathcal{M}}, \mathbf{0}). \tag{32.35}$$

It follows from (32.33) and (32.35) that

$$\mathbf{\Lambda}(\mathbf{A}_x^{\phi})\left(\nabla_x(\phi\rfloor)^{-1}\right) = \mathbf{0}.$$

This can happen only when $\nabla_x(\phi\rfloor)^{-1} = \mathbf{A}_x^{\phi}$. ∎

33. Torsion

Let $r \in \tilde{}$, with $r \geq 2$, and a C^r-manifold $\mathcal{M}$ be given. For every $x \in \mathcal{M}$, we have; as described in Sect. 32 with $\mathcal{B} := T\mathcal{M}$,

$$\mathrm{Tlis}_x T\mathcal{M} := \bigcup_{y \in \mathcal{M}} \mathrm{Lis}(T_x\mathcal{M}, T_y\mathcal{M}). \tag{33.1}$$

We also have the following short exact sequence

$$\mathrm{Lin}\, T_x\mathcal{M} \xrightarrow{\mathbf{I}_x} S_x T\mathcal{M} \xrightarrow{\mathbf{P}_x} T_x\mathcal{M}. \tag{33.2}$$

The short exact sequence (33.2) is of the form (15.1) and hence all of the results in Sect.15 can be used here.

For every manifold chart $\chi \in \mathrm{Ch}\mathcal{M}$, the tangent mapping tgt_χ; as defined in (22.13), is a bundle chart of the tangent bundle $T\mathcal{M}$ such that $\mathrm{ev}_2 \circ \mathrm{tgt}_\chi = \nabla\chi$. Note that not every tangent bundle chart $\phi \in \mathrm{Ch}(T\mathcal{M}, \mathcal{M})$ can be obtained from the gradient of a manifold chart. To avoid complicated notations, we replace all the superscript of $\phi = \mathrm{tgt}_\chi$ by superscript of χ; i.e. we use the following notation

$$\mathbf{A}_x^\chi := \mathbf{A}_x^{\mathrm{tgt}_\chi}, \quad \mathbf{\Gamma}_x^\chi := \mathbf{\Gamma}_x^{\mathrm{tgt}_\chi} \quad \text{and} \quad \mathbf{\Gamma}_x^{\chi,\gamma} := \mathbf{\Gamma}_x^{\mathrm{tgt}_\chi, \mathrm{tgt}_\gamma} \tag{33.3}$$

for all manifold charts $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$. Given $\chi, \gamma \in \mathrm{Ch}\mathcal{M}$. It is easily seen from (32.25) and (23.16) that

$$\mathbf{\Gamma}_x^{\chi,\gamma} := \left((\nabla_x\gamma)^{-1}\nabla_\chi^{(2)}\gamma(x)\right) \circ (\nabla_x\chi \times \nabla_x\chi). \tag{33.4}$$

It follows from the Theorem on Symmetry of Second Gradients (see Sect.612, [FDS]) that $\mathbf{\Gamma}_x^{\chi,\gamma}$ belongs to the subspace $\mathrm{Sym}_2(T_x\mathcal{M}^2, T_x\mathcal{M})$ of $\mathrm{Lin}_2(T_x\mathcal{M}^2, T_x\mathcal{M}) \cong \mathrm{Lin}(T_x\mathcal{M}, \mathrm{Lin}T_x\mathcal{M})$.

Proposition 1: *There is exactly one flat $\mathcal{F}$ in $\mathrm{Con}_x T\mathcal{M}$ with direction space $\{\mathbf{I}_x\}\mathrm{Sym}_2(T_x\mathcal{M}^2, T_x\mathcal{M})$ which contains $\mathbf{A}_x^\chi$ for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, so that*

$$\mathcal{F} = \mathbf{A}_x^\chi + \{\mathbf{I}_x\}\mathrm{Sym}_2(T_x\mathcal{M}^2, T_x\mathcal{M}) \quad \text{for all} \quad \chi \in \mathrm{Ch}_x\mathcal{M}. \tag{33.5}$$

Definition: *The* **shift-bracket** $\mathbf{B}_x \in \mathrm{Skw}_2\left(S_x T\mathcal{M}^2, T_x\mathcal{M}\right)$ *of* $S_x T\mathcal{M}$ *is defined by*

$$\mathbf{B}_x := \mathbf{B}_\mathcal{F} \tag{33.6}$$

where $\mathbf{B}_\mathcal{F}$ *is defined as in* (15.5).

Definition: *The* **torsion-mapping** $\mathbf{T}_x : \mathrm{Con}_x T\mathcal{M} \to \mathrm{Skw}_2\left(T_x\mathcal{M}^2, T_x\mathcal{M}\right)$ *of* $\mathrm{Con}_x T\mathcal{M}$ *is defined by*

$$\mathbf{T}_x := \mathbf{T}_\mathcal{F} \tag{33.7}$$

where $\mathbf{T}_{\mathcal{F}}$ *is defined as in* (15.8).

It follows from Prop.3 of Sect.15 that, for every manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$, we have

$$\mathbf{T}_x = \boldsymbol{\Gamma}_x^{\chi} - \boldsymbol{\Gamma}_x^{\chi\sim} \tag{33.8}$$

where $^{\sim}$ denotes the value-wise switch, so that $\boldsymbol{\Gamma}_x^{\chi\sim}(\mathbf{K})(\mathbf{s},\mathbf{t}) = \boldsymbol{\Gamma}_x^{\chi}(\mathbf{K})(\mathbf{t},\mathbf{s})$ for all $\mathbf{K} \in \mathrm{Con}_x\mathcal{M}$ and all $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

The torsion-mapping $\mathbf{T}_x$ is a surjective flat mapping with $\mathbf{T}_x^{<}(\{\mathbf{0}\}) = \mathcal{F}$ whose gradient

$$\nabla\mathbf{T}_x \in \mathrm{Lin}\big(\ \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})\ ,\ \mathrm{Skw}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})\big) \tag{33.9}$$

is given by

$$(\nabla\mathbf{T}_x)\mathbf{L} = \mathbf{L}^{\sim} - \mathbf{L} \tag{33.10}$$

for all $\mathbf{L} \in \mathrm{Lin}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M})$.

<u>Definition</u>: *We say that a connector* $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$ *is* **torsion-free** (*or* **symmetric**) *if* $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$, *i.e.* $\mathbf{K} \in \mathcal{F}$. *The flat of all symmetric connectors will be denoted by* $\mathrm{Scon}_x\mathcal{M} := \mathbf{T}_x^{<}(\{\mathbf{0}\})$.

The mapping

$$\mathbf{S}_x := (\mathbf{1}_{\mathrm{Con}_x\mathrm{T}\mathcal{M}} + \frac{1}{2}\mathbf{I}_x\mathbf{T}_x)\big|^{\mathrm{Scon}_x\mathcal{M}} \tag{33.11}$$

is the projection of $\mathrm{Con}_x\mathrm{T}\mathcal{M}$ onto $\mathrm{Scon}_x\mathcal{M}$ with

$$\mathrm{Null}\,\nabla\mathbf{S}_x = \mathrm{Skw}_2\,(\mathrm{T}_x\mathcal{M}^2, \mathrm{T}_x\mathcal{M}).$$

If $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$, we call $\mathbf{S}_x(\mathbf{K}) = \mathbf{K} + \frac{1}{2}\mathbf{I}_x\big(\mathbf{T}_x(\mathbf{K})\big)$ the **symmetric part** of $\mathbf{K}$.

Theorem : *A connector* $\mathbf{K} \in \mathrm{Con}_x\mathrm{T}\mathcal{M}$ *is symmetric if and only if* $\mathbf{K} = \mathbf{A}_x^{\chi}$ *for some* $\chi \in \mathrm{Ch}_x\mathcal{M}$. *Thus* $\mathrm{Scon}_x\mathcal{M} = \{\,\mathbf{A}_x^{\chi} \mid \chi \in \mathrm{Ch}_x\mathcal{M}\}$.

Proof: Let $\mathbf{K} \in \mathrm{Con}_x\mathcal{M}$ be given. If $\mathbf{K} = \mathbf{A}_x^{\chi}$ for some $\chi \in \mathrm{Ch}_x\mathcal{M}$, then $\boldsymbol{\Gamma}_x^{\chi}(\mathbf{K}) = \mathbf{0}$ and hence $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$ by (33.8).

Assume now that $\mathbf{T}_x(\mathbf{K}) = \mathbf{0}$. We choose $\gamma \in \mathrm{Ch}_x\mathcal{M}$ and put

$$\mathbf{L} := \nabla_x\gamma\,\boldsymbol{\Gamma}_x^{\gamma}(\mathbf{K}) \circ \big((\nabla_x\gamma)^{-1} \times (\nabla_x\gamma)^{-1}\big)\ . \tag{33.12}$$

It follows from (33.8) that $\mathbf{L}$ is symmetric, i.e. that $\mathbf{L} \in \mathrm{Sym}_2(\mathcal{V}_\gamma^2, \mathcal{V}_\gamma)$. We now define the mapping $\alpha : \mathrm{Dom}\,\gamma \to \mathcal{V}_\gamma$ by

$$\alpha(z) := \gamma(z) + \frac{1}{2}\mathbf{L}\big(\,\gamma(z) - \gamma(x)\,,\,\gamma(z) - \gamma(x)\,\big) \quad \text{for all} \quad z \in \mathrm{Dom}\,\gamma\ .$$

Take the gradient at x, we have $\nabla_x\alpha = \nabla_x\gamma$ i.e. that is $(\nabla_x\alpha)(\nabla_x\gamma)^{-1} = \mathbf{1}_{\mathcal{V}_\gamma}$. It follows from the Local Inversion Theorem that there exist an open subset $\mathcal{N}$ of $\operatorname{Dom}\alpha$ such that $\chi := \alpha|_{\mathcal{N}}^{\alpha_>(\mathcal{N})}$ is a bijection of class C^r. It is easily seen that $\chi \in \mathrm{Ch}_x\mathcal{M}$ and that

$$\nabla_\gamma^{(2)}\chi(x) = \mathbf{L}$$

Using (33.12), (32.25) and $\nabla_x\chi = \nabla_x\gamma$, we conclude that

$$\mathbf{\Gamma}_x^\gamma(\mathbf{K}) = (\nabla_x\chi)^{-1}\nabla_\gamma^{(2)}\chi \circ \left(\nabla_x\gamma \times \nabla_x\gamma\right) = \mathbf{\Gamma}_x^{\gamma,\chi} \ .$$

Hence, by (32.24) and (32.27), we have

$$\mathbf{A}_x^\gamma - \mathbf{A}_x^\chi = \mathbf{I}_x\mathbf{\Gamma}_x^{\gamma,\chi} = \mathbf{I}_x\mathbf{\Gamma}_x^\gamma(\mathbf{K}) = \mathbf{A}_x^\gamma - \mathbf{K} \ ,$$

which gives $\mathbf{K} = \mathbf{A}_x^\chi$. ∎

34. Connections, Curvature

From now on, in this chapter, we assume a linear-space bundle $(\mathcal{B}, \tau, \mathcal{M})$ of class C^s, $s \geq 2$, is given. We also assume that both $\mathcal{M}$ and $\mathcal{B}$ have constant dimensions, and put $n := \dim\mathcal{M}$ and $m := \dim\mathcal{B} - \dim\mathcal{M}$. Then we have, as in (32.1),

$$m = \dim\mathcal{B}_x \quad \text{for all} \quad x \in \mathcal{M}. \tag{34.1}$$

<u>**Definition:**</u> *The* **connector bundle** $\operatorname{Con}\mathcal{B}$ *of* $\mathcal{B}$ *is defined to be the union of all the right-connector spaces*

$$\operatorname{Con}\mathcal{B} := \bigcup_{x\in\mathcal{M}} \operatorname{Con}_x\mathcal{B} \ . \tag{34.2}$$

It is endowed with the structure of a C^{s-1}-flat space bundle over $\mathcal{M}$ as shown below.

If $\mathcal{P}$ is an open subset of $\mathcal{M}$ and $x \in \mathcal{P}$, we can identify $\operatorname{Con}_x\mathcal{A} \cong \operatorname{Con}_x\mathcal{B}$, where $\mathcal{A} := \tau^<(\mathcal{P})$, in the same way as was done for the tangent space. Hence we may regard $\operatorname{Con}\mathcal{A}$ as a subset of $\operatorname{Con}\mathcal{B}$.

Note that the family $(\operatorname{Con}_x\mathcal{B} \,|\, x \in \mathcal{M})$ is disjoint. The bundle projection $\rho : \operatorname{Con}\mathcal{B} \to \mathcal{M}$ is given by

$$\rho(\mathbf{K}) :\in \left\{ \ y \in \mathcal{M} \ \middle| \ \mathbf{K} \in \operatorname{Con}_x\mathcal{B} \ \right\}, \tag{34.3}$$

and, for every $x \in \mathcal{M}$, the bundle inclusion $\mathrm{in}_x : \operatorname{Con}_x\mathcal{B} \to \operatorname{Con}\mathcal{B}$ at x is

$$\mathrm{in}_x := \mathbf{1}_{\operatorname{Con}_x\mathcal{B}\subset\operatorname{Con}\mathcal{B}} \ . \tag{34.4}$$

For every $(\chi,\phi)\in \mathrm{Ch}\mathcal{M}\times \mathrm{Ch}(\mathcal{B},\mathcal{M})$ we define

$$\mathrm{con}^{(\chi,\phi)} : \mathrm{Con}(\mathrm{Dom}\phi)\to (\mathrm{Dom}\,\chi\cap \mathcal{O}_\phi)\times \mathrm{Lin}(\mathcal{V}_\chi,\mathrm{Lin}\mathcal{V}_\phi) \tag{34.5}$$

by

$$\mathrm{con}^{(\chi,\phi)}(\mathbf{H}) := \Big(\ z\ , \phi\big\rfloor_z\ \mathbf{\Lambda}(\mathbf{A}^\phi_z)(\mathbf{H})\ (\nabla_z\chi^{-1}\times \phi\big\rfloor_z^{-1})\ \Big) \quad \text{where} \quad z:=\rho(\mathbf{H}) \tag{34.6}$$

for all $\mathbf{H}\in \mathrm{Con}(\mathrm{Dom}\phi)$. It is easily seen that $\mathrm{con}^{(\chi,\phi)}$ is invertible and

$$\mathrm{con}^{(\chi,\phi)\leftarrow}(z,\mathbf{L}) = \mathbf{A}^\phi_z + \mathbf{I}_z\phi\big\rfloor_z^{-1}\,\mathbf{L}\ (\nabla_z\chi\times\phi\big\rfloor_z) \tag{34.7}$$

for all $z\in(\mathrm{Dom}\chi\cap\mathcal{O}_\phi)$ and all $\mathbf{L}\in \mathrm{Lin}(\mathcal{V}_\chi,\mathrm{Lin}\mathcal{V}_\phi)$. Let $(\chi,\phi),(\gamma,\psi)\in \mathrm{Ch}\mathcal{M}\times \mathrm{Ch}(\mathcal{B},\mathcal{M})$ be given. We easily deduce from (34.7) and (34.6), with (χ,ϕ) replaced by (γ,ψ) and $\mathbf{\Lambda}(\mathbf{A}^\psi_z)(\mathbf{A}^\phi_z) = -\mathbf{\Gamma}^{\psi,\phi}_z = \mathbf{\Gamma}^{\phi,\psi}_z$, that

$$\begin{aligned}&\big(\mathrm{con}^{(\gamma,\psi)}\circ \mathrm{con}^{(\chi,\phi)\leftarrow}\big)(z,\mathbf{L})\\ &\quad = \Big(\ z\ ,\ \psi\big\rfloor_z\,\mathbf{\Gamma}^{\phi,\psi}_z(\nabla_z\gamma^{-1}\times\psi\big\rfloor_z^{-1}) + \kappa(z)\,\mathbf{L}\,\big(\nabla_z\lambda\times\kappa(z)^{-1}\big)\Big)\\ &\qquad \text{where}\ \ \lambda:=\gamma\circ\chi^{\leftarrow}\ \ \text{and}\ \ \ \kappa:=\psi\diamond\phi\ \ (\textit{see}\ (22.7))\end{aligned} \tag{34.8}$$

for all $z\in(\mathrm{Dom}\chi\cap\mathcal{O}_\phi)\cap(\mathrm{Dom}\gamma\cap\mathcal{O}_\psi)$ and $\mathbf{L}\in\mathrm{Lin}(\mathcal{V}_\chi,\mathrm{Lin}\mathcal{V}_\phi)$. It is clear that $\mathrm{con}^{(\gamma,\psi)}\circ\mathrm{con}^{(\chi,\phi)\leftarrow}$ is of class C^{s-1}. Since $(\gamma,\psi),(\chi,\phi)\in\mathrm{Ch}\mathcal{M}\times\mathrm{Ch}(\mathcal{B},\mathcal{M})$ were arbitrary, it follows that $\big\{\,\mathrm{con}^{(\alpha,\phi)}\ \big|\ (\alpha,\phi)\in\mathrm{Ch}\mathcal{M}\times\mathrm{Ch}(\mathcal{B},\mathcal{M})\,\big\}$ is a C^{s-1}-bundle atlas of $\mathrm{Con}\,\mathcal{B}$; it determines the natural structure of a C^{s-1} flat-space bundle over $\mathcal{M}$.

The mappings ρ and in_x defined by (34.3) and (34.4) are easily seen to be of class C^{s-1}.

<u>Definition</u>: *Let $\mathcal{O}$ be an open subset of $\mathcal{M}$. A cross section on $\mathcal{O}$ of the connector bundle* $\mathrm{Con}\,\mathcal{B}$

$$\mathbf{A}:\mathcal{O}\to\mathrm{Con}\,\mathcal{B} \tag{34.9}$$

is called a **connection on** $\mathcal{O}$ **for** *the bundle $\mathcal{B}$. A connection on $\mathcal{M}$ for the bundle $\mathcal{B}$ is simply called a connection for the bundle $\mathcal{B}$. For every bundle chart ϕ in* $\mathrm{Ch}(\mathcal{B},\mathcal{M})$*, the connection $\mathbf{A}^\phi$ on $\mathcal{O}_\phi$ is defined by*

$$\mathbf{A}^\phi(x) := \mathbf{A}^\phi_x \qquad \text{for all}\quad x\in\mathcal{O}_\phi, \tag{34.10}$$

where $\mathbf{A}^\phi_x$ is given by (32.21).

<u>Definition</u>: *The tangent-space of* $\mathrm{Con}\,\mathcal{B}$ *at* $\mathbf{K}$ *is denoted by*

$$\mathrm{T}_{\mathbf{K}}\mathrm{Con}\,\mathcal{B}. \tag{34.11}$$

We define the **projection mapping** *of* $\mathrm{T_K Con}\,\mathcal{B}$ *by*

$$\mathbf{P_K} := \nabla_{\mathbf{K}}\rho \in \mathrm{Lin}\,(\mathrm{T_K Con}\,\mathcal{B}, \mathrm{T}_x\mathcal{M}) \tag{34.12}$$

and the **injection mapping** *of* $\mathrm{T_K Con}\,\mathcal{B}$ *by*

$$\mathbf{I_K} := \nabla_{\mathbf{K}}\mathrm{in}_x \in \mathrm{Lin}\left(\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x), \mathrm{T_K Con}\,\mathcal{B}\right) \tag{34.13}$$

where ρ *and* in_x *are defined by* (34.3) *and* (34.4)*.*

It is clear from (34.5) that

$$\dim\,(\mathrm{Con}\,\mathcal{B}) = \dim\,(\mathrm{T_K Con}\,\mathcal{B}) = n + nm^2. \tag{34.14}$$

Proposition 1: *The projection mapping* $\mathbf{P_K}$ *is surjective, the injection mapping* $\mathbf{I_K}$ *is injective, and we have*

$$\mathrm{Null}\,\mathbf{P_K} = \mathrm{Rng}\,\mathbf{I_K} \tag{34.15}$$

i.e.

$$\mathrm{Lin}(\mathrm{T}_x\mathcal{M}, \mathrm{Lin}\mathcal{B}_x) \xrightarrow{\mathbf{I_K}} \mathrm{T_K Con}\,\mathcal{B} \xrightarrow{\mathbf{P_K}} \mathrm{T}_x\mathcal{M} \tag{34.16}$$

is a short exact sequence.

The short exact sequence (34.16) is of the form (15.1) and hence all of the results in Sect.15 can be used here.

Proposition 2: *For each* $(\chi, \phi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B}, \mathcal{M})$*, let*

$$\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} \in \mathrm{Lin}\,(\mathrm{T}_x\mathcal{M}, \mathrm{T_K Con}\,\mathcal{B})$$

be defined by $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} := \mathbf{A}_{\mathbf{K}}^{\mathrm{con}^{(\chi,\phi)}}$ *in terms of the notation* (32.21)*; i.e.*

$$\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} := \left(\nabla_{\mathbf{K}}\mathrm{con}^{(\chi,\phi)}\right)^{-1} \circ \mathrm{ins}_1. \tag{34.17}$$

Then $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}$ *is a linear right-inverse of* $\mathbf{P_K}$*; i.e.* $\mathbf{P_K}\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)} = \mathbf{1}_{\mathrm{T}_x\mathcal{M}}$.

Proposition 3: *If* $(\gamma,\psi),(\chi,\phi)\in \mathrm{Ch}_x\mathcal{M}\times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$*, with* $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$*, then*

$$\begin{aligned}\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}-\mathbf{A}_{\mathbf{K}}^{(\gamma,\psi)}&=\mathbf{I}_{\mathbf{K}}\,\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\\ \mathbf{\Lambda}(\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)})-\mathbf{\Lambda}(\mathbf{A}_{\mathbf{K}}^{(\gamma,\psi)})&=-\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\mathbf{P}_{\mathbf{K}}\end{aligned}\tag{34.18}$$

where $\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}:=\mathbf{\Gamma}_{\mathbf{K}}^{\mathrm{con}^{(\chi,\phi)},\mathrm{con}^{(\gamma,\psi)}}$ *in terms of the notation* (32.25) *is given by*

$$\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}')=(\psi\rfloor_x)^{-1}\big(\nabla^{(2)}_{\gamma(x)}(\psi\diamond\phi)(\nabla_x\gamma\,\mathbf{t},\nabla_x\gamma\,\mathbf{t}')\big)\phi\rfloor_x\tag{34.19}$$

for all $\mathbf{t},\mathbf{t}'\in \mathrm{T}_x\mathcal{M}$*. We have* $\mathbf{\Gamma}_{\mathbf{K}}^{(\chi,\phi),(\gamma,\psi)}\in \mathrm{Sym}_2(\mathrm{T}_x\mathcal{M}^2,\mathrm{Lin}\mathcal{B}_x)$*. Here, the notation* (22.7) *is used.*

Proof: Let $(\gamma,\psi),(\chi,\phi)\in \mathrm{Ch}_x\mathcal{M}\times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, with $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$, be given. Then, we have $\nabla_x(\psi\diamond\phi)=\mathbf{\Lambda}(\mathbf{A}_x^{\phi})(\mathbf{K})=\mathbf{0}$. It follows from (34.6) that

$$\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K})=\mathbf{0}.\tag{34.20}$$

Using (34.8), (34.20) and (33.25), we obtain

$$\begin{aligned}&\big(\mathrm{con}^{(\gamma,\psi)}\,\square\,\mathrm{con}^{(\chi,\phi)\leftarrow}\big)(z,\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K}))\\ &\qquad=\Big(z\,,\,\nabla_z(\psi\diamond\phi)\big(\nabla_z\gamma^{-1}\times(\phi\rfloor_z\circ\psi\rfloor_z^{-1})\big)\Big).\end{aligned}\tag{34.21}$$

Taking the gradient of (34.21) with respect to z at x and observing $\nabla_x(\psi\diamond\phi)=\mathbf{0}$, we have

$$\begin{aligned}&\mathrm{ev}_2\Big(\nabla_x\big((\mathrm{con}^{(\gamma,\psi)}\,\square\,\mathrm{con}^{(\chi,\phi)\leftarrow})\big(\cdot,\mathrm{con}^{(\chi,\phi)}\rfloor_x(\mathbf{K})\big)\big)\mathbf{t}\Big)\\ &\qquad=\big((\nabla^{(2)}_{\gamma(x)}(\psi\diamond\phi))\nabla_x\gamma\,\mathbf{t}\big)(\mathbf{1}_{\mathcal{V}_\gamma}\times(\phi\rfloor_x\circ\psi\rfloor_x^{-1}))\end{aligned}\tag{34.22}$$

for all $\mathbf{t}\in \mathrm{T}_x\mathcal{M}$. Using (34.22), (34.6) with (χ,ϕ) replaced by (γ,ψ) and applying Prop. 3 in Sect. 32 with ϕ replaced by $\mathrm{con}^{(\chi,\phi)}$ and ψ replaced by $\mathrm{con}^{(\gamma,\psi)}$, we obtain the desired result (34.19). ∎

If $\phi,\psi\in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, with $\mathbf{A}_x^{\phi}=\mathbf{K}=\mathbf{A}_x^{\psi}$, we have $\mathbf{\Gamma}_x^{\phi,\psi}=\mathbf{0}$ by (33.25). It follows from (21.9) that the right hand side of (34.19) does not depend on the manifold charts $\chi,\gamma\in \mathrm{Ch}_x\mathcal{M}$. In particular, when $\psi=\phi$ we have $\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}=\mathbf{A}_{\mathbf{K}}^{(\gamma,\phi)}$ for all manifold charts $\chi,\gamma\in \mathrm{Ch}_x\mathcal{M}$.

By using the definition of the gradient

$$\nabla_x\mathbf{A}^{\phi}=(\nabla_{\mathbf{K}}\mathrm{con}^{\chi,\phi})^{-1}\nabla_{\chi(x)}\big(\mathrm{con}^{\chi,\phi}\,\square\,\mathbf{A}^{\phi}\,\square\,\chi^{\leftarrow}\big)\nabla_x\chi$$

and (34.6), we can easily seen that for every bundle chart $\phi\in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ with $\mathbf{A}_x^{\phi}=\mathbf{K}$

$$\nabla_x\mathbf{A}^{\phi}=\mathbf{A}_{\mathbf{K}}^{(\chi,\phi)}\qquad\text{for all}\quad \chi\in \mathrm{Ch}_x\mathcal{M}.\tag{34.23}$$

for all bundle charts $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ with $\mathbf{A}_x^\phi = \mathbf{K}$.

Proof: The assertion follows from (34.23) together with (34.18) and (34.19). ▮

Definition: *The* **bracket** $\mathbf{B}_\mathbf{K} \in \mathrm{Skw}_2\,(\mathrm{T}_\mathbf{K}\mathrm{Con}\,\mathcal{B}^2, \mathrm{T}_x\mathcal{M})$ *of* $\mathrm{T}_\mathbf{K}\mathrm{Con}\,\mathcal{B}$ *is defined by*

$$\mathbf{B}_\mathbf{K} := \mathbf{B}_{\mathcal{F}_\mathbf{K}} \tag{34.25}$$

where $\mathbf{B}_{\mathcal{F}_\mathbf{K}}$ *is defined as in* (15.5).

Definition: *Let* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ *be a connection which is differentiable at* x. *The* **curvature of A at** x, *denoted by*

$$\mathbf{R}_x(\mathbf{A}) \in \mathrm{Skw}_2\big(\mathrm{T}_x\mathcal{M}^2, \mathrm{Lin}\mathcal{B}_x\big), \tag{34.26}$$

is defined by

$$\mathbf{R}_x(\mathbf{A}) := \mathbf{T}_{\mathcal{F}_{\mathbf{A}(x)}}(\nabla_x\mathbf{A}) \tag{34.27}$$

where $\mathbf{T}_{\mathcal{F}_{\mathbf{A}(x)}}$ *is defined as in* (15.8).

If $\mathbf{A}$ *is differentiable, then the mapping* $\mathbf{R}(\mathbf{A}) : \mathcal{M} \to \mathrm{Skw}_2(\,\mathrm{Tan}\mathcal{M}^2\,,\,\mathrm{Lin}\,\mathcal{B}\,)$ *defined by*

$$\mathbf{R}(\mathbf{A})(x) := \mathbf{R}_x(\mathbf{A}) \qquad \text{for all} \qquad x \in \mathcal{M}$$

is called the **curvature field** *of the connection* **A**.

A fomula for the curvature field $\mathbf{R}(\mathbf{A})$ in terms of covariant gradients will be given in Prop. 5. If the connection $\mathbf{A}$ is of class C^p, with $p \in 1..s-1$, then $\nabla\mathbf{A}$ is of class C^{p-1}, and so is the curvature field $\mathbf{R}(\mathbf{A})$.

More generally, if $\phi, \psi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$, without assuming that $\mathbf{A}_x^\phi = \mathbf{K} = \mathbf{A}_x^\psi$, then Eq. (34.19) must be replaced by

$$\begin{aligned}
&\boldsymbol{\Gamma}_\mathbf{K}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}') \\
&\quad = -\boldsymbol{\Gamma}_x^{\phi,\psi}(\mathbf{t})\boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t}') + \boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})(\mathbf{t}')\boldsymbol{\Gamma}_x^{\phi,\psi}(\mathbf{t}) + \boldsymbol{\Gamma}_x^{\phi}(\mathbf{K})\boldsymbol{\Gamma}_x^{\chi,\gamma}(\mathbf{t},\mathbf{t}') \\
&\qquad - \boldsymbol{\Gamma}_x^{\phi,\psi}(\mathbf{t}')\boldsymbol{\Gamma}_x^{\phi,\psi}(\mathbf{t}) + (\psi\rfloor_x)^{-1}\big(\nabla_\gamma^{(2)}(\psi\diamond\phi)\big)(x)(\nabla_x\gamma\,\mathbf{t}, \nabla_x\gamma\,\mathbf{t}')\phi\rfloor_x
\end{aligned} \tag{34.28}$$

for all $\mathbf{t},\mathbf{t}' \in \mathrm{T}_x\mathcal{M}$. If one of those two bundle charts, say ϕ, satisfies $\mathbf{A}_x^\phi = \mathbf{K}$, then it follows from (34.28), $\boldsymbol{\Gamma}_x^\phi(\mathbf{K}) = \mathbf{0}$ and $-\boldsymbol{\Gamma}_x^{\phi,\psi} = \boldsymbol{\Gamma}_x^\psi(\mathbf{K})$ that

$$\begin{aligned}
&\boldsymbol{\Gamma}_\mathbf{K}^{(\chi,\phi),(\gamma,\psi)}(\mathbf{t},\mathbf{t}') \\
&\quad = -\boldsymbol{\Gamma}_x^{\psi}(\mathbf{K})\mathbf{t}'\boldsymbol{\Gamma}_x^{\psi}(\mathbf{K})\mathbf{t} + (\psi\rfloor_x)^{-1}\big(\nabla_\gamma^{(2)}(\psi\diamond\phi)\big)(x)(\nabla_x\gamma\,\mathbf{t}, \nabla_x\gamma\,\mathbf{t}')\phi\rfloor_x
\end{aligned} \tag{34.29}$$

for all $\mathbf{t},\mathbf{t}' \in \mathrm{T}_x\mathcal{M}$.

Proposition 5: *Let* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\,\mathcal{B}$ *be a connection that is differentiable at* $x \in \mathcal{M}$*. The curvature of* $\mathbf{A}$ *at* x *is given by*

$$\begin{aligned}\big(\mathbf{R}_x(\mathbf{A})\big)(\mathbf{s},\mathbf{t}) &= (\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}))(\mathbf{s},\mathbf{t}) - (\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}))(\mathbf{t},\mathbf{s}) \\ &\quad + \Big(\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{s}\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{t} - \mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{t}\mathbf{\Gamma}_x^{\psi}(\mathbf{A}(x))\mathbf{s}\Big)\end{aligned} \tag{34.30}$$

for all $(\gamma,\psi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *and all* $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$*.*

Proof: Let a bundle chart $(\gamma,\psi) \in \mathrm{Ch}_x\mathcal{M} \times \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ be given. It follows from (42.6) and $\mathbf{\Lambda}(\mathbf{A}_z^{\psi})(\mathbf{A}(z)) = -\mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))$ that

$$\mathrm{con}^{(\gamma,\psi)} \circ \mathbf{A}(z) = \Big(\ z\ , -\psi\big\rfloor_z \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))\ (\nabla_z\gamma^{-1} \times \psi\big\rfloor_z^{-1})\ \Big) \tag{34.31}$$

In view of (32.29), we have

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{A}_{\mathbf{A}(x)}^{(\gamma,\psi)})(\nabla_x\mathbf{A}) &= \mathrm{con}^{(\gamma,\psi)}\big\rfloor_x^{-1}\big(\mathrm{ev}_2 \circ \nabla_{\mathbf{A}(x)}\big(\mathrm{con}^{(\gamma,\psi)}\big)\big)\big(\nabla_x\mathbf{A}\big) \\ &= \mathrm{con}^{(\gamma,\psi)}\big\rfloor_x^{-1}\mathrm{ev}_2 \circ \big(\nabla_x\big(\mathrm{con}^{(\gamma,\psi)} \circ \mathbf{A}\big)\big) \\ &= \nabla_x\Big(z \mapsto -\psi\big\rfloor_x^{-1}\psi\big\rfloor_z\ \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))(\nabla_z\gamma^{-1}\nabla_x\gamma \times \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)\Big)\end{aligned} \tag{34.32}$$

By using

$$\mathbf{A}_x^{\gamma} = \nabla_x(z \mapsto \nabla_z\gamma^{-1}\nabla_x\gamma) \quad , \quad \mathbf{A}_x^{\psi} = \nabla_x(z \to \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)$$

and (42.38), we observe that

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{A}_{\mathbf{A}(x)}^{(\gamma,\psi)})(\nabla_x\mathbf{A}) &= \nabla_x\Big(z \mapsto -\psi\big\rfloor_x^{-1}\psi\big\rfloor_z\ \mathbf{\Gamma}_z^{\psi}(\mathbf{A}(z))(\nabla_z\gamma^{-1}\nabla_x\gamma \times \psi\big\rfloor_z^{-1}\psi\big\rfloor_x)\Big) \\ &= -\big(\Box_x\mathbf{\Gamma}^{\psi}(\mathbf{A})\big)(\mathbf{A}_x^{\gamma},\mathbf{A}_x^{\psi}) \\ &= -\nabla_x^{\gamma,\psi}\mathbf{\Gamma}^{\psi}(\mathbf{A}).\end{aligned}$$

Together with (42.27) and (42.29), we prove (34.12). ∎

Remark : When the linear-space bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$, we have

$$\begin{aligned}\big(\mathbf{R}_x(\mathbf{A})\big)(\mathbf{s},\mathbf{t}) &= (\nabla_x^{\chi}\mathbf{\Gamma}^{\chi}(\mathbf{A}))(\mathbf{s},\mathbf{t}) - (\nabla_x^{\chi}\mathbf{\Gamma}^{\chi}(\mathbf{A}))(\mathbf{t},\mathbf{s}) \\ &\quad + \Big(\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{s}\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{t} - \mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{t}\mathbf{\Gamma}_x^{\chi}(\mathbf{A}(x))\mathbf{s}\Big)\end{aligned} \tag{34.33}$$

for all manifold chart $\chi \in \mathrm{Ch}_x\mathcal{M}$ and all $\mathbf{s},\mathbf{t} \in \mathrm{T}_x\mathcal{M}$.

If a transport $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{M}$ from x is differentiable at y, we define the **connector-gradient**, $\overline{\nabla}_y\ \mathbf{T} \in \mathrm{Lin}\,(\mathcal{T}_y,\mathcal{S}_y)$, of $\mathbf{T}$ at y by

$$\overline{\nabla}_y\ \mathbf{T} := \nabla_y\big(z \mapsto \mathbf{T}(z)\mathbf{T}(y)^{-1}\big). \tag{34.34}$$

Theorem : *A connection* $\mathbf{A} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *is curvature-free if and only if, locally* $\mathbf{A}$ *agrees with* $\mathbf{A}^\phi$ *for some bundle chart* $\phi \in \mathrm{Ch}(\mathcal{B}, \mathcal{M})$*. In other word, for every* $x \in \mathcal{M}$*, there is an open neighbourhood* $\mathcal{N}_x$ *of* x *and a transport* $\mathbf{T} : \mathcal{N}_x \to \mathrm{Tlis}_x\mathcal{M}$ *from* x *such that* $\nabla\mathbf{T} = \mathbf{A}\big|_{\mathcal{N}_x}$

Proof: Ask Prof. Noll!!!!!!!!!!!!!!!!!

35. Parallelisms, Geodesics

Let a connector $\mathbf{K} \in \mathrm{Con}\,\mathcal{B}$ be given and put $x := \rho(\mathbf{K})$.

We now apply the results of Sect. 32 by replacing the ISO-bundle there by the flat-space bundle $\mathrm{Con}\,\mathcal{B}$ and $\mathbf{b} \in \mathcal{B}$ there by $\mathbf{K}$.

<u>**Definition**</u>**:** *The* **shift bundle** $\mathrm{S}\mathcal{B}$ *of* $(\mathcal{B}, \tau, \mathcal{M})$ *is defined to be the union of all the shift spaces of* $\mathcal{B}$:

$$\mathrm{S}\mathcal{B} := \bigcup_{y \in \mathcal{M}} \mathrm{S}_y\mathcal{B}. \tag{35.1}$$

It is endowed with the structure of a C^{r-2}*-manifold.*

We defined the mapping $\sigma : \mathrm{S}\mathcal{B} \to \mathcal{M}$ by

$$\sigma(\mathbf{s}) :\in \left\{ y \in \mathcal{M} \;\middle|\; \mathbf{s} \in \mathrm{S}_y\mathcal{B} \right\}, \tag{35.2}$$

and every $y \in \mathcal{M}$ the mapping $\mathrm{in}_y : \mathrm{S}_y\mathcal{B} \to \mathrm{S}\mathcal{B}$ by

$$\mathrm{in}_y := \mathbf{1}_{\mathrm{S}_y\mathcal{B} \subset \mathrm{S}\mathcal{B}}\ . \tag{35.3}$$

We define the **projection** $\mathbf{P} : \mathrm{S}\mathcal{B} \to \mathrm{T}\mathcal{M}$ by

$$\mathbf{P}(\mathbf{s}) := \mathbf{P}_{\sigma(\mathbf{s})}\mathbf{s} \qquad \text{for all} \qquad \mathbf{s} \in \mathrm{S}\mathcal{B} \tag{35.4}$$

and the **injection** $\mathbf{I} : \mathrm{Lin}\,\mathcal{B} \to \mathrm{S}\mathcal{B}$ by

$$\mathbf{I}(\mathbf{L}) := \mathbf{I}_{\tau^{\mathrm{Ln}}(\mathbf{L})}\,\mathbf{L} \qquad \text{for all} \qquad \mathbf{L} \in \mathrm{Lin}\,\mathcal{B} \tag{35.5}$$

where Ln is the lineon functor (see Sect.13) and

$$\mathrm{Lin}\,\mathcal{B} := \mathrm{Ln}(\mathcal{B}) = \bigcup_{y \in \mathcal{M}} \mathrm{Lin}\,\mathcal{B}_y. \tag{35.6}$$

We have

$$\mathrm{pt}(\mathbf{P}(\mathbf{s})) = \sigma(\mathbf{s}) \qquad \text{for all} \qquad \mathbf{s} \in \mathrm{S}\mathcal{B} \tag{35.7}$$

and

$$\sigma(\mathbf{I}\,\mathbf{L}) = \tau^{\mathrm{Ln}}(\mathbf{L}) \qquad \text{for all} \qquad \mathbf{L} \in \operatorname{Lin}\mathcal{B}. \tag{35.8}$$

It is easily seen that $\mathbf{P}$ and $\mathbf{I}$ are of class C^{r-2}.

We now fix $x \in \mathcal{M}$ and consider the bundle $\mathrm{Tlis}_x\mathcal{B}$ of transfer-isomorphism from x as defined by (32.2). A mapping of the type

$$\mathbf{T} : [0, d] \to \mathrm{Tlis}_x\mathcal{B} \qquad \text{with} \qquad \mathbf{T}(0) = \mathbf{1}_{\mathcal{B}_x} \ , \tag{35.9}$$

where $d \in ^{\times}$, will be called a **transfer-process** of $\mathcal{B}$ from x. If $\mathbf{T}$ is differentiable at a given $t \in [0, d]$, we defined the **shift-derivative** $\mathrm{sd}_t\mathbf{T} \in \mathrm{S}_{\pi_x(\mathbf{T}(t))}\mathcal{B}$ at t of $\mathbf{T}$ by

$$\mathrm{sd}_t\mathbf{T} := \partial_t \left(s \mapsto \mathbf{T}(s)\mathbf{T}(t)^{-1} \right) \ . \tag{35.10}$$

We have

$$\sigma\left(\mathrm{sd}_t\mathbf{T}\right) = \pi_x\left(\mathbf{T}(t)\right) \ , \tag{35.11}$$

when π_x is defined by (32.3). If $\mathbf{T}$ is differentiable, we define the **shift-derivative** (-process) $\mathrm{sd}\mathbf{T} : [0, d] \to \mathrm{S}\mathcal{B}$ by

$$\left(\mathrm{sd}\mathbf{T}\right)(t) := \mathrm{sd}_t\mathbf{T} \qquad \text{for all} \qquad t \in [0, d] \ . \tag{35.12}$$

If $\mathbf{T}$ is of class C^s, $s \in 1..(r-2)$, then $\mathrm{sd}\mathbf{T}$ is of class C^{s-1}.

Proposition 1: *Let* $\mathbf{T} : [0, d] \to \mathrm{Tlis}_x\mathcal{B}$ *be a transfer-process of* $\mathcal{B}$ *from* x *and put*

$$p := \pi_x \circ \mathbf{T} = \sigma \circ (\mathrm{sd}\mathbf{T}) : [0, d] \to \mathcal{M}. \tag{35.13}$$

Then p *is differentiable and*

$$\mathbf{P} \circ (\mathrm{sd}\mathbf{T}) = p^{\cdot} \ . \tag{35.14}$$

Proof: Let $t \in [0, d]$ be given and put $y := p(t)$. Then $\mathbf{T}(s)\mathbf{T}(t)^{-1} \in \mathrm{Tlis}_y\mathcal{B}$ and

$$\pi_y\left(\mathbf{T}(s)\mathbf{T}(t)^{-1}\right) = \pi_x\left(\mathbf{T}(s)\right) = p(s)$$

for all $s \in [0, d]$. Differentiation with respect to s at t, using (35.10), (32.10), and the chain rule, gives $\mathbf{P}_y(\mathrm{sd}_t\mathbf{T}) = p^{\cdot}(t)$. Since $t \in [0, d]$ was arbitrary, (35.14) follows. ∎

Proposition 2: *Let* $\mathbf{T}$ *be a differentiable transfer-process from* x *and let* p *be defined as in Prop. 1. Assume, moreover, that* $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ *is a chart such that* $\mathrm{Rng}\, p \subset \mathcal{O}_\phi$*. If we define* $\mathbf{H} : [0,d] \to \mathrm{Lis}\mathcal{B}_x$ *and* $\mathbf{V} : [0,d] \to \mathrm{Lin}\mathcal{B}_x$ *by*

$$\mathbf{H}(t) := (\phi\rfloor_y)\,\mathbf{T}(t) \tag{35.15}$$

and

$$\mathbf{V}(t) := \phi\rfloor_y \left(\mathbf{\Lambda}(\mathbf{A}^\phi_y)(\mathrm{sd}_t\mathbf{T})\right)(\phi\rfloor_y)^{-1} \tag{35.16}$$

when $y := p(t)$ *and* $t \in [0,d]$*, then*

$$\mathbf{H}^{\cdot} = \mathbf{V}\mathbf{H} \quad , \quad \mathbf{H}(0) = \mathbf{1}_{\mathcal{B}_x} \ . \tag{35.17}$$

Proof: Let $t \in [0,d]$ be given and put $y := p(t)$. Using (32.6) with x replaced by y and $\mathbf{T}$ by $\mathbf{T}(s)\mathbf{T}(t)^{-1}$, we obtain from (35.15) that

$$\mathrm{tlis}^\phi_y(\mathbf{T}(s)\mathbf{T}(t)^{-1}) = \left(p(s)\ ,\ \phi\rfloor_y\mathbf{H}(s)\mathbf{H}(t)^{-1}(\phi\rfloor_y)^{-1}\right) \quad \text{for all} \quad s \in [0,d].$$

In view of (31.30) with ϕ replaced by tlis^ϕ_y and (35.10) we conclude that

$$\left(\nabla_{\mathbf{1}_{\mathrm{T}_y}}\mathrm{tlis}^\phi_y\right)(\mathrm{sd}_t\mathbf{T}) = \left(p^{\cdot}(t)\ ,\ \phi\rfloor_y(\mathbf{H}^{\cdot}\mathbf{H}^{-1})(t)(\phi\rfloor_y)^{-1}\right).$$

Comparing this result with (31.29) and (35.16), and using the injectivity of $\nabla_{\mathbf{1}_{\mathrm{T}_x}}\mathrm{tlis}^\phi_y$, we obtain $(\mathbf{H}^{\cdot}\mathbf{H}^{-1})(t) = \mathbf{V}(t)$. Since $t \in [0,d]$ was arbitrtary, $(35.17)_1$ follows. Since both $\phi\rfloor_x = \mathbf{1}_{\mathcal{B}_x}$ and $\mathbf{T}(0) = \mathbf{1}_{\mathcal{B}_x}$, $(35.17)_2$ is a direct consequence of (35.15). ∎

Theorem on Shift-Processes: *Let* $\mathbf{U} : [0,d] \to \mathrm{S}\mathcal{B}$*, with* $d \in {}^\times$*, be a continuous shift-process of* $\mathcal{B}$ *such that* $p := \sigma \circ \mathbf{U}$ *is differentiable and*

$$\mathbf{P} \circ \mathbf{U} = p^{\cdot} : [0,d] \to \mathrm{Tan}\mathcal{M} \ . \tag{35.18}$$

Then there exists exactly one transfer-process $\mathbf{T} : [0,d] \to \mathrm{Tlis}_x\mathcal{B}$ *of* $\mathcal{B}$ *from* $x := p(0)$*, of class* C^1*, such that* $\mathrm{sd}\mathbf{T} = \mathbf{U}$.

Proof: Assume first that $\phi \in \mathrm{Ch}(\mathcal{B},\mathcal{M})$ can be chosen such that $\mathrm{Rng}\, p \subset \mathrm{Dom}\,\chi$. Define $\overline{\mathbf{V}} : [0,d] \to \mathrm{Lin}\mathcal{V}_\phi$ by

$$\overline{\mathbf{V}}(t) := (\phi\rfloor_y)\left(\mathbf{\Lambda}(\mathbf{A}^\phi_y)\mathbf{U}(t)\right)(\phi\rfloor_y)^{-1} \quad \text{when} \quad y := p(t). \tag{35.19}$$

Since $\mathbf{U}$ is continuous, so is $\overline{\mathbf{V}}$. Let $\overline{\mathbf{H}} : [0,d] \to \mathrm{Lin}\mathcal{V}_\phi$ be the unique solution of the initial value problem

$$?\,\overline{\mathbf{H}} \quad , \quad \overline{\mathbf{H}}^{\cdot} = \overline{\mathbf{V}}\,\overline{\mathbf{H}} \quad , \quad \overline{\mathbf{H}}(0) = \mathbf{1}_{\mathcal{V}_\phi} \ . \tag{35.20}$$

This solution is of class C^1.

Now, if $\mathbf{T}$ is a process that satisfies the conditions, then $\overline{\mathbf{V}}$, as defined by (35.19), coincides with $\mathbf{V}$, as defined by (35.16). Therefore, by Prop. 2, we have $\mathbf{H} = \overline{\mathbf{H}}$ and hence $\mathbf{T}$ must be given by

$$\mathbf{T}(t) = (\phi\rfloor_{p(t)})^{-1}\overline{\mathbf{H}}(t)\phi\rfloor_x \qquad \text{for all} \qquad t \in [0,d]. \tag{35.21}$$

On the other hand, if we *define* $\mathbf{T}$ by (35.21) and then $\mathbf{H}$ and $\mathbf{V}$ by (35.15) and (35.16), we have $\pi_x \circ \mathbf{T} = p$, $\overline{\mathbf{H}} = \mathbf{H}$, and $\overline{\mathbf{V}} = \mathbf{V}$. Thus, using (31.30) with ϕ replaced by tlis_y^ϕ and (35.19), we conclude that

$$\big(\nabla_{\mathbf{1}_{\mathcal{B}_y}}\mathrm{tlis}_y^\phi\big)(\mathrm{sd}_t\mathbf{T}) = \big(\nabla_{\mathbf{1}_{\mathcal{B}_y}}\mathrm{tlis}_y^\phi\big)(\mathbf{U}(t)) \quad \text{when} \quad y := p(t)$$

for all $t \in [0,d]$. Since $\nabla_{\mathbf{1}_{\mathcal{B}_y}}\mathrm{tlis}_y^\phi$ is injective for all $y \in \mathcal{M}$, we conclude that $\mathbf{U} = \mathrm{sd}\mathbf{T}$.

There need not be a single bundle chart $\phi \in \mathrm{Ch}(\mathcal{B},\mathcal{M})$ such that $\mathrm{Rng}\, p \subset \mathrm{Dom}\,\chi$. However, since $\mathrm{Rng}\, p$ is a compact subset of $\mathcal{M}$, we can find a finite set $\mathfrak{F} \subset \mathrm{Ch}\mathcal{M}$ such that

$$\mathrm{Rng}\, p \subset \bigcup_{\chi\in\mathfrak{F}} \mathrm{Dom}\,\chi.$$

We can then determine a strictly isotone list $\big(a_i \mid i \in (m+1)^{[}\,\big)$ in such that $a_0 = 0$, $a_m = d$ and such that, for each $i \in m^{[}$, $p_{>}([a_i, a_{i+1}])$ is included in a single chart belonging to $\mathfrak{F}$. By applying the result already proved, for each $i \in m^{[}$, to the case when $\mathbf{U}$ is replaced by

$$\big(\, t \mapsto \mathbf{U}(a_i + t)\,\big) : [\,0\,,\, a_{i+1} - a_i] \to \mathrm{S}\mathcal{B},$$

one easily sees that the assertion of the theorem is valid in general. ∎

We assume now that a continuous connection $\mathbf{C}$ is prescribed.

Let $d \in {}^{\times}$ and a process $p : [0,d] \to \mathcal{M}$ of class C^1 be given and put $x := p(0)$. We define the shift process $\mathbf{U} : [0,d] \to \mathrm{S}\mathcal{B}$ by

$$\mathbf{U}(t) := \mathbf{C}\big(p(t)\big)p^{\cdot}(t) \quad \text{for all} \quad t \in [0,d]. \tag{35.22}$$

Clearly, $\mathbf{U}$ is continuous and, since $\mathbf{P}_y\mathbf{C}(y) = \mathbf{1}_{\mathrm{T}_y}$ for all $y \in \mathcal{M}$, (35.18) is valid. Hence, by the Theorem on Shift Processes there is a unique transfer process $\mathbf{T} : [0,d] \to \mathrm{Tlis}_x\mathcal{B}$ of class C^1 such that

$$\mathrm{sd}\mathbf{T} = (\mathbf{C} \circ p)p^{\cdot}\ . \tag{35.23}$$

This process is called the **parallelism along** p for the connection $\mathbf{C}$.

Let $\mathbf{H} : [0,d\,] \to \mathbf{\Phi}(\mathcal{B})$ be a process on $\mathbf{\Phi}(\mathcal{B})$ and put $p := \tau \circ \mathbf{H}$. We say that $\mathbf{H}$ is a **parallel process** for $\mathbf{C}$ if $\mathbf{H}(0) \neq \mathbf{0}$ and if

$$\mathbf{H}(t) = \mathbf{\Phi}(\mathbf{T}(t))\mathbf{H}(0) \qquad \text{for all} \quad t \in [0,d\,] \tag{35.24}$$

where $\mathbf{T}$ is the parallelism along p for $\mathbf{C}$.

Let $\mathbf{H} : [0,d\,] \to \mathbf{\Phi}(\mathcal{B})$ be a process on $\mathbf{\Phi}(\mathcal{B})$ and let $\mathbf{T}$ be the parallelism along $p := \tau^{\mathbf{\Phi}} \circ \mathbf{H}$ for the connection $\mathbf{C}$. Given $\phi \in \mathrm{Ch}_x(\mathcal{B},\mathcal{M})$ that satisfies $\mathrm{Rng}\, p \subset \mathcal{O}_\phi$. Define $\big(\mathbf{H}^{\phi\rfloor}\big)^\bullet : [0,d\,] \to \tau^<(\mathrm{Rng}p)$ and $\big(\mathbf{H}^{T}\big)^\bullet : [0,d\,] \to \tau^<(\mathrm{Rng}p)$ by

$$\begin{aligned}\big(\mathbf{H}^{\phi\rfloor}\big)^\bullet(t) &:= \partial_t\big(\ s \mapsto \mathbf{\Phi}(\phi\rfloor_{p(t)}^{-1}\phi\rfloor_{p(s)})\mathbf{H}(s)\ \big)\\ \big(\mathbf{H}^{T}\big)^\bullet(t) &:= \partial_t\big(\ s \mapsto \mathbf{\Phi}(\mathbf{T}(t)\mathbf{T}^{-1}(s))\mathbf{H}(s)\ \big)\end{aligned} \tag{35.25}$$

for all $t \in [0,d\,]$.

Proposition 3: *A process $\mathbf{H} : [0,d] \to \mathbf{\Phi}(\mathcal{B})$ is parallel with respect to $\mathbf{C}$ if and only if $\mathbf{H}$ is of class C^1 and satisfies the differential equation*

$$\mathbf{0} = \big(\mathbf{H}^{T}\big)^\bullet = \big(\mathbf{H}^{\phi\rfloor}\big)^\bullet + \mathbf{\Phi}^\bullet\big(\,\big(\mathbf{\Gamma}^\phi(\mathbf{C})\circ p\big)\,p^\bullet\big)\mathbf{H}. \tag{35.26}$$

We assume now that the linear space bundle $\mathcal{B}$ is the tangent bundle $\mathrm{T}\mathcal{M}$ and that a continuous connection $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathrm{T}\mathcal{M}$ for $\mathrm{T}\mathcal{M}$ is prescribed.

We say that $p : [0,d] \to \mathcal{M}$ is a **geodesic process** for $\mathbf{C}$ if $p^\bullet(0) \neq \mathbf{0}$ and if

$$\mathbf{T}(t)p^\bullet(0) = p^\bullet(t) \quad \text{for all} \quad t \in [0,d], \tag{35.28}$$

where $\mathbf{T}$ is the parallelism along p for $\mathbf{C}$, i.e. $p^\bullet$ is parallel with respect to the parallelism $\mathbf{T}$.

Let $p : [0,d\,] \to \mathcal{M}$ be a process of class C^1 such that $p^\bullet(0) \neq \mathbf{0}$ and given $\chi \in \mathrm{Ch}\mathcal{M}$ that satisfies $\mathrm{Rng}\, p \subset \mathrm{Dom}\,\chi$. Define $\overline{p} : [0,d\,] \to \mathrm{Cod}\,\chi$ by $\overline{p} := \chi \circ p$ and $\overline{\mathbf{\Gamma}} : \mathrm{Cod}\,\chi \to \mathrm{Lin}_2\,(\mathcal{V}_\chi^2, \mathcal{V}_\chi)$ by

$$\overline{\mathbf{\Gamma}}(z) := \nabla_y\chi\,\mathbf{\Gamma}_y^\chi(\mathbf{C}(y)) \circ (\nabla_y\chi^{-1} \times \nabla_y\chi^{-1}) \quad \text{when} \quad y := \chi^{\leftarrow}(z), \tag{35.29}$$

where $\mathbf{\Gamma}_y^\chi$ is defined by (33.3).

Proposition 4: *The process p is a geodedic process if and only if $\overline{p}$ is of class C^2 and satisfies the differential equation*

$$\overline{p}^{\bullet\bullet} + \big(\overline{\mathbf{\Gamma}} \circ \overline{p}\big)\,(\overline{p}^{\,\bullet}, \overline{p}^{\,\bullet}) = \mathbf{0}\ . \tag{35.30}$$

Geodesic Deviations: **Study the derivative of (35.26)???**

36. Holonomy

Let a continuous connection $\mathbf{C} : \mathcal{M} \to \text{Con}\mathcal{B}$ be given. For every C^1 process $p : [0, d_p] \to \mathcal{M}$ there is exactly one parallelism $\mathbf{T}_p : [0, d_p] \to \text{Tlis}_x\mathcal{B}$ from $x := p(0)$ along p for the connection $\mathbf{C}$. The **reverse process** $p^- : [0, d_p] \to \mathcal{M}$ of $p : [0, d_p] \to \mathcal{M}$ is given by

$$p^-(t) := p(d_p - t) \qquad \text{for all} \quad t \in [0, d_p].$$

Proposition 1: *Let $p^- : [0, d_p] \to \mathcal{M}$ be the reverse process of a C^1 process $p : [0, d_p] \to \mathcal{M}$. We have*

$$\mathbf{T}_{p^-}(t) = \mathbf{T}_p(d_p - t)\mathbf{T}_p^{-1}(d_p) \quad \text{for all} \quad t \in [0, d_p]. \tag{36.1}$$

Let C^1 processes $p : [0, d_p] \to \mathcal{M}$ and , $q : [0, d_q] \to \mathcal{M}$ with $q(0) = p(d_p)$ be given. We define the **continuation process** $q * p : [0, d_p + d_q] \to \mathcal{M}$ of p with q by

$$(q * p)(t) := \begin{cases} p(t) & t \in [0, d_p], \\ \\ q(t - d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.2}$$

If in addition that $\overset{\bullet}{q}(0) = \overset{\bullet}{p}(d_p)$, then the continuation process $q * p$ is of class C^1 and

$$\mathbf{T}_{q*p}(t) = \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \\ \mathbf{T}_q(t - d_p)\mathbf{T}_p(d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.3}$$

<u>**Definition**</u>**:** *For every pair of C^1 processes $p : [0, d_p] \to \mathcal{M}$ and , $q : [0, d_q] \to \mathcal{M}$ with $q(0) = p(d_p)$ be given. We define the piecewise parallelism (along $q * p$)*

$$\mathbf{T}_{q*p} : [0, d_p + d_q] \to \text{Tlis}_x\mathcal{B} \quad \text{where} \quad x := p(0)$$

by

$$\mathbf{T}_{q*p}(t) := \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \\ \mathbf{T}_q(t - d_p)\mathbf{T}_p(d_p) & t \in [d_p, d_p + d_q]. \end{cases} \tag{36.4}$$

In view of (36.1), if $q := p^-$ we have $\mathbf{T}_{p^-}(t - d_p)\mathbf{T}_p(d_p) = \mathbf{T}_p(2d_p - t)$ and hence

$$\mathbf{T}_{-p*p}(t) := \begin{cases} \mathbf{T}_p(t) & t \in [0, d_p], \\ \\ \mathbf{T}_p(2d_p - t) & t \in [d_p, 2d_p]. \end{cases} \tag{36.5}$$

In particular, $\mathbf{T}_{p^-*p}(2d_p) = \mathbf{T}_{-p*p}(0) = \mathbf{1}_{\mathcal{B}_x}$.

Let $\mathcal{O}$ be an open neighboorhood of $x \in \mathcal{M}$ and let $\mathcal{L}(\mathcal{O}, x)$ be the set of all piecewise C^1 loops $p : [0, d_p] \to \mathcal{M}$ at x with $\mathrm{Rng}p \subset \mathcal{O}$. It is easily seen that $(\mathcal{L}(\mathcal{O}, x), *)$ is a group. We also use the following notation

$$\mathcal{H}(\mathcal{O}, x) := \{\mathbf{T}_p(d_p) \,|\, p \in \mathcal{L}(\mathcal{O}, x)\}. \tag{36.6}$$

Proposition 3: *For every $q, p \in \mathcal{L}(\mathcal{O}, x)$, we have*

$$\mathbf{T}_{q*p}(d_p + d_q) = \mathbf{T}_q(d_q)\mathbf{T}_p(d_p). \tag{36.7}$$

Hence $\mathcal{H}(\mathcal{O}, x)$ is a subgroup of $\mathrm{Lis}\mathcal{B}_x$*, which is called the* **holonomy group** *on $\mathcal{O}$ of the connection* $\mathbf{C}$ **at** *x.*

Let $\mathbf{T} : \mathcal{M} \to \mathrm{Tlis}_x\mathcal{M}$ be a transport from $x \in \mathcal{M}$ of class C^1. For every differentiable process $\lambda : [0, 1] \to \mathcal{M}$, we see that $\mathbf{T} \circ \lambda : [0, 1] \to \mathrm{Tlis}_x\mathcal{M}$ is a transfer process from x and

$$\mathrm{sd}\mathbf{T} = ((\nabla\mathbf{T}) \circ \lambda)\lambda^{\bullet}.$$

Hence $\mathbf{T} \circ \lambda$ is the parallelism along λ for the connection $\nabla\mathbf{T}$. For every $t \in [0, 1]$, $(\mathbf{T} \circ \lambda)(t) = \mathbf{T}(\lambda(t))$ depends on, of course, only on the point $y := \lambda(t)$, not on the process λ. When λ is closed, beginning and ending at $\lambda(0) = x = \lambda(1)$, then

$$(\mathbf{T} \circ \lambda)(1) = \mathbf{T}(x) = \mathbf{1}_{\mathcal{B}_x}.$$

The following theorem is a immediated consequence of the above discussion and the Theorem of Sect.34.

Theorem : *A continuous connection* $\mathbf{C} : \mathcal{M} \to \mathrm{Con}\mathcal{B}$ *is curvature-free; i.e.* $\mathbf{R}(\mathbf{C}) = \mathbf{0}$ *if and only if locally the holonomy groups are $\mathcal{H}(\mathcal{O}, x) = \{\mathbf{1}_{\mathcal{B}_x}\}$ for some open subset set $\mathcal{O}$ of $\mathcal{M}$ and all $x \in \mathcal{M}$.*

Question ?: Does there exist a connection $\mathbf{C}$ such that $\mathcal{H}(\mathcal{O}, x) = \mathrm{Lis}\mathcal{B}_x$ for some x?