3. Manifolds in a Euclidean Space

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In this chapter, a (genuine) Euclidean space $\mathcal{E}$ with translation space $\mathcal{V}$ is assumed given. The dimension of $\mathcal{E}$ and $\mathcal{V}$ is denoted by $n := \dim \mathcal{E} = \dim \mathcal{V}$.

31. Basic definitions.

Definition 1. *A non-empty subset $\mathcal{S}$ of $\mathcal{E}$ is called a* **manifold of class** C^r *($r \in \mathbb{N}^\times$, $r = \infty$, or $r = \omega$) imbedded in $\mathcal{E}$ if, for every $x \in \mathcal{S}$ one can find a subspace $\mathcal{T}_x$ of $\mathcal{V}$ and a mapping π_x with the following properties:*

(S$_1$) *π_x is a mapping of class C^r whose codomain is $\mathcal{E}$ and whose domain $\mathcal{N}_x$ is an open neighborhood of $\mathbf{0}$ in $\mathcal{T}_x$.*

(S$_2$) $\pi_x(\mathbf{0}) = x, \quad \operatorname{Rng} \nabla_{\mathbf{0}}\pi_x \subset \mathcal{T}_x$.

(S$_3$) $\pi_x(\mathbf{u}) - (x + \mathbf{u}) \in {\mathcal{T}_x}^\perp \quad \text{for all} \quad \mathbf{u} \in \mathcal{N}_x$.

(S$_4$) *There is a neighborhood $\mathcal{G}_x$ of x in $\mathcal{E}$ such that*

$$\operatorname{Rng} \pi_x = \mathcal{S} \cap \mathcal{G}_x.$$

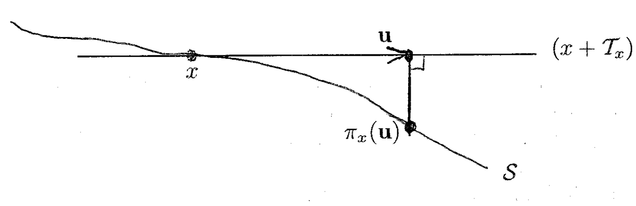

Figure 1: tangent space and local curving

Figure 1 illustrates the geometric meaning of the conditions (S$_1$)–(S$_4$). Roughly, π_x "bends" a neighborhood $\mathcal{N}_x$ of $\mathbf{0}$ in $\mathcal{T}_x$ in such a way that the "flat piece" $\mathcal{N}_x$ becomes a "curved piece" $\operatorname{Rng} \pi_x$ of $\mathcal{S}$.

Taking the gradient of $\mathbf{u} \mapsto \pi_x(\mathbf{u}) - (x + \mathbf{u}) : \mathcal{N}_x \longrightarrow \mathcal{E}$ at $\mathbf{0} \in \mathcal{T}_x$, we see that (S$_2$) and (S$_3$) imply

$$\nabla_{\mathbf{0}}\, \pi_x = \mathbf{1}_{\mathcal{T}_x \subset \mathcal{V}}. \tag{31.1}$$

If $\mathbf{E}_x \in \mathrm{Sym}\,\mathcal{V}$ denotes the symmetric idempotent with $\mathrm{Rng}\,\mathbf{E}_x = \mathcal{T}_x$ (see Prop. 4 of Sect.41 in Vol.I), then (S_3) is equivalent to

$$\mathbf{u} = \mathbf{E}_x(\pi_x(\mathbf{u}) - x) \quad \text{for all} \quad \mathbf{u} \in \mathcal{N}_x = \mathrm{Dom}\,\pi_x. \tag{31.2}$$

We now assume that a manifold $\mathcal{S}$, as defined above, is given.

Proposition 1: *Let $\mathcal{P}$ be an open subset of a flat space $\mathcal{F}$ and let $\varphi : \mathcal{P} \to \mathcal{E}$ be a mapping of class C^1 such that $\mathrm{Rng}\,\varphi \subset \mathcal{S}$. Let $p \in \mathcal{P}$ be given and put $x := \varphi(p) \in \mathcal{S}$. Then*

$$\varphi(q) = \pi_x(\mathbf{E}_x(\varphi(q) - x)) \tag{31.3}$$

for all q in some neighborhood of p in $\mathcal{P}$ and

$$\mathrm{Rng}\,\nabla_p\,\varphi \subset \mathcal{T}_x. \tag{31.4}$$

Proof: Choose $\mathcal{G}_x$ according to (S_4) of Def.1. Since φ is continuous, $\mathcal{R} := \varphi^{<}(\mathcal{G}_x)$ is a neighborhood of p in $\mathcal{F}$. Since $\varphi_{>}(\mathcal{R}) \subset \mathrm{Rng}\,\varphi \subset \mathcal{S}$, it follows from ($S_4$) that $\varphi_{>}(\mathcal{R}) \subset \mathrm{Rng}\,\pi_x$. Hence, for every $q \in \mathcal{R}$ there is a $\mathbf{u} \in \mathcal{N}_x$ such that $\varphi(q) = \pi_x(\mathbf{u})$. Using (31.2), we find

$$\varphi(q) = \pi_x(\mathbf{E}_x(\varphi(q) - x)) \quad \text{for all} \quad q \in \mathcal{R},$$

which proves (31.3). Taking the gradient of φ at p and using (31.2), the Chain Rule, and (31.1), we obtain

$$\nabla_p\,\varphi = \nabla_{\mathbf{0}}\pi_x(\mathbf{E}_x|^{\mathcal{T}_x})\nabla_p\,\varphi = \mathbf{E}_x\nabla_p\,\varphi$$

which shows that $\mathrm{Rng}\,\nabla_p\,\varphi \subset \mathrm{Rng}\,\mathbf{E}_x = \mathcal{T}_x$. ∎

Proposition 2: *If $(\mathcal{T}_x, \pi_x)$ and $(\mathcal{T}'_x, \pi'_x)$ satisfy (S_1)–(S_4), then $\mathcal{T}'_x = \mathcal{T}_x$ and $\pi'_x|_{\mathcal{R}_x} = \pi_x|_{\mathcal{R}_x}$ for some neighborhood $\mathcal{R}_x$ of $\mathbf{0}$ in $\mathcal{T}_x$.*

Proof: Put $\mathcal{N}'_x := \mathrm{Dom}\,\pi'_x$, so that $\mathcal{N}'_x \subset \mathcal{T}'_x$. We apply Prop.1 to the case when $\mathcal{F}$ is replaced by $\mathcal{T}'_x$, $\mathcal{P}$ by $\mathcal{N}'_x$, and φ by π_x. The conclusion (31.4) then becomes $\mathrm{Rng}\,\nabla_{\mathbf{0}}\pi'_x \subset \mathcal{T}_x$. Since π'_x satisfies (31.1), this amounts to $\mathcal{T}'_x = \mathrm{Rng}\,\mathbf{1}_{\mathcal{T}'_x \subset \mathcal{V}} \subset \mathcal{T}_x$. Interchanging the role of $(\mathcal{T}_x, \pi_x)$ and $(\mathcal{T}'_x, \pi'_x)$ we also obtain $\mathcal{T}_x \subset \mathcal{T}'_x$ and hence we conclude that $\mathcal{T}_x = \mathcal{T}'_x$.

The first assertion of Prop.1, when applied to $\varphi := \pi'_x$, shows that there is a neighborhood $\mathcal{R}_x \subset \mathcal{N}'_x = \mathrm{Dom}\,\pi'_x$ of $\mathbf{0}$ in $\mathcal{T}_x = \mathcal{T}'_x$ such that for every $\mathbf{u}' \in \mathcal{R}_x$ we have

$$\pi'_x(\mathbf{u}') = \pi_x(\mathbf{E}_x(\pi'_x(\mathbf{u}') - x)).$$

Writing (31.2) for π'_x and $\mathbf{u}'$ we get $\mathbf{u}' = \mathbf{E}_x(\pi'_x(\mathbf{u}') - x)$. Therefore, we have $\pi'_x(\mathbf{u}') = \pi_x(\mathbf{u}')$ for all $\mathbf{u}' \in \mathcal{R}_x$, which is the desired result. ∎

Prop. 2 proves that the space $\mathcal{T}_x$ is uniquely determined by $\mathcal{S}$ and x. We call $\mathcal{T}_x$ the **tangent space of $\mathcal{S}$ at** x. A mapping π_x that satisfies (S_1)–(S_4) will be called a **local curving for $\mathcal{S}$ at** x. The local curvings at x are locally unique in the sense that any two such agree on some neighborhood of $\mathbf{0}$ in $\mathcal{T}_x$.

The **dimension of $\mathcal{S}$ at** x is defined to be the dimension of $\mathcal{T}_x$. We say that $\mathcal{S}$ has **dimension** m if $m = \dim\mathcal{T}_x$ for all $x \in \mathcal{S}$.

The topology of $\mathcal{E}$ induces a topology on the manifold $\mathcal{S}$ imbedded in $\mathcal{E}$. It follows from (S_4) that the neighborhoods in $\mathcal{S}$ of a point $x \in \mathcal{S}$ are exactly those subsets of $\mathcal{S}$ that contains a set of the form $\pi_{x>}(\mathcal{R}_x)$, where $\mathcal{R}_x$ is a neighborhood of $\mathbf{0}$ in $\mathcal{T}_x$.

If $\mathcal{P}$ is a subset of $\mathcal{S}$ that is open in $\mathcal{S}$, then $\mathcal{P}$ is again a manifold imbedded in $\mathcal{E}$. In fact, the tangent space of $\mathcal{P}$ at x concides with the tangent space of $\mathcal{S}$ at x, and every local curving for $\mathcal{P}$ at x is also a local curving for $\mathcal{S}$ at x.

Proposition 3: *A subset $\mathcal{S}$ of $\mathcal{E}$ is a manifold of class C^r if and only if, for every $x \in \mathcal{S}$, one can find a subspace $\mathcal{T}_x$ of $\mathcal{V}$ and a mapping $\overline{\pi}_x$ with the following properties:*

(C_1) *$\overline{\pi}_x$ is a mapping of class C^r whose codomain is $\mathcal{E}$ and whose domain $\overline{\mathcal{N}}_x$ is an open neighborhood of $\mathbf{0}$ in $\mathcal{V}$ of the form*

$$\overline{\mathcal{N}}_x = (\overline{\mathcal{N}}_x \cap \mathcal{T}_x) + (\overline{\mathcal{N}}_x \cap \mathcal{T}_x^{\perp}).$$

(C_2) $\operatorname{Rng} \overline{\pi}_x$ *is open in $\mathcal{E}$ and $\overline{\pi}_x|^{\operatorname{Rng}}$ is invertible with an inverse of class C^r.*

(C_3) $\overline{\pi}_x(\mathbf{0}) = x$, $\nabla_{\mathbf{0}} \overline{\pi}_x = \mathbf{1}_{\mathcal{V}}$.

(C_4) *Denoting the symmetric idempotent whose range is $\mathcal{T}_x$ by $\mathbf{E}_x$ (see Prop. 4 of Sect.41 in Vol.I), we have $\mathbf{u} = \mathbf{E}_x(\overline{\pi}_x(\mathbf{u}) - x)$* for all *$\mathbf{u} \in \overline{\mathcal{N}}_x \cap \mathcal{T}_x$.*

(C_5) $\operatorname{Rng} \overline{\pi}_x \cap \mathcal{S} = \overline{\pi}_{x>}(\overline{\mathcal{N}}_x \cap \mathcal{T}_x)$.

(C_6) $\overline{\pi}_x(\mathbf{u} + \mathbf{w}) = \overline{\pi}_x(\mathbf{u}) + \mathbf{w}$ for all $\mathbf{u} \in \overline{\mathcal{N}}_x \cap \mathcal{T}_x$, $\mathbf{w} \in \overline{\mathcal{N}}_x \cap \mathcal{T}_x^{\perp}$.

If $\overline{\pi}_x$ is a mapping that satisfies (C_1)–(C_6), then $\pi_x := \overline{\pi}_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x}$ is a local curving for $\mathcal{S}$ at x.

Proof: Assume that $\mathcal{S}$ is a manifold of class C^r, that $\mathcal{T}_x$ is its tangent space at x, and that π_x is a local curving at x. Putting $\mathcal{N}_x := \operatorname{Dom} \pi_x$, we first define $\widetilde{\pi}_x : \mathcal{N}_x + \mathcal{T}_x^{\perp} \to \mathcal{E}$ by

$$\widetilde{\pi}_x(\mathbf{u} + \mathbf{w}) := \pi_x(\mathbf{u}) + \mathbf{w} \quad \text{for all} \quad \mathbf{u} \in \mathcal{N}_x, \mathbf{w} \in \mathcal{T}_x^{\perp}.$$

It follows from (S_1) and (31.1) that $\widetilde{\pi}_x$ is of class C^r and that $\nabla_{\mathbf{0}} \widetilde{\pi}_x = \mathbf{1}_{\mathcal{V}}$. By the Local Inversion Theorem (see Sect.68 in Vol.I), $\widetilde{\pi}_x$ is locally invertiable near $\mathbf{0} \in \mathcal{V}$, i.e. there is an open neighborhood $\overline{\mathcal{N}}_x$ of $\mathbf{0}$ in $\mathcal{V}$ such that $\overline{\pi}_x := \widetilde{\pi}_x|_{\overline{\mathcal{N}}_x}$ satisfies the condition (C_2). Since $\widetilde{\pi}_x$ is continuous, we may assume that $\overline{\mathcal{N}}_x \subset \widetilde{\pi}_x^{<}(\mathcal{G}_x)$, i.e. $\operatorname{Rng} \overline{\pi}_x \subset \mathcal{G}_x$. Also, we may assume that $\overline{\mathcal{N}}_x$ is of the form $\overline{\mathcal{N}}_x = (\overline{\mathcal{N}}_x \cap \mathcal{T}_x) + (\overline{\mathcal{N}}_x \cap \mathcal{T}_x^{\perp})$. Since $\overline{\mathcal{N}}_x \subset (\mathcal{N}_x + \mathcal{T}_x^{\perp})$, we then have $\overline{\mathcal{N}}_x \cap \mathcal{T}_x \subset \mathcal{N}_x$ and hence $\overline{\pi}_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x} = \pi_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x}$.

It is evident that $\overline{\pi}_x$ thus defined satisfies (C_1)–(C_4) and (C_6). To prove (C_5), we use (S_4) to obtain

$$\operatorname{Rng} \pi_x \cap \operatorname{Rng} \overline{\pi}_x = \mathcal{S} \cap (\mathcal{G}_x \cap \operatorname{Rng} \overline{\pi}_x).$$

Since $\operatorname{Rng} \overline{\pi}_x \subset \mathcal{G}_x$ and $\operatorname{Rng} \overline{\pi}_x \cap \operatorname{Rng} \pi_x = \overline{\pi}_{>}(\mathcal{N}_x \cap \mathcal{T}_x)$, this is the desired result (C_5).

It is easily seen that if $(\mathcal{T}_x, \overline{\pi}_x)$ satisfies the conditions (C_1)–(C_6), then $(\mathcal{T}_x, \overline{\pi}_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x})$ satisfies conditions (S_1)–(S_4). For example, (C_5) becomes (S_4) if we put $\pi_x := \overline{\pi}_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x}$ and $\mathcal{G}_x := \operatorname{Rng} \overline{\pi}_x$. Hence, if ($C_1$)–($C_6$) hold, then $\mathcal{T}_x$ is the tangent space of $\mathcal{S}$ at x and $\pi_x := \overline{\pi}_x|_{\overline{\mathcal{N}}_x \cap \mathcal{T}_x}$ is a local curving at x. ∎

A mapping $\overline{\pi}_x$ satisfying (C_1)–(C_6) is an extension of a local curving at x from a neighborhood of $\mathbf{0}$ in $\mathcal{T}_x$ to a neighborhood of $\mathbf{0}$ in $\mathcal{V}$. We call such a mapping $\overline{\pi}_x$ an **extended local curving** for $\mathcal{S}$ at x. The extended local curvings at x are locally unique in the sense that any two such agree on some neighborhood of $\mathbf{0}$ in $\mathcal{V}$.

In the future, if we use local curvings $\pi_x : \mathcal{N}_x \to \mathcal{E}$ and extended local curvings $\overline{\pi}_x : \overline{\mathcal{N}}_x \to \mathcal{E}$, we will assume that

$$\pi_x = \overline{\pi}_x\big|_{\mathcal{N}_x} \quad , \quad \mathcal{N}_x = \overline{\mathcal{N}}_x \cap \mathcal{T}_x. \tag{31.5}$$

We note that (C_6) then is equivalent to

$$\overline{\pi}_x(\mathbf{v}) = \pi_x(\mathbf{E}_x\mathbf{v}) + (\mathbf{1}_{\mathcal{V}} - \mathbf{E}_x)\mathbf{v} \quad \text{for all} \quad \mathbf{v} \in \overline{\mathcal{N}}_x.$$

32. Parametric representation of manifolds.

In may cases, the points of a manifold $\mathcal{S}$ imbedded in $\mathcal{E}$ can be labelled by a "parameter" that varies in an open subset $\mathcal{P}$ of some "parameter-space". More precisely, this means that $\mathcal{S}$ is the range of some injective mapping $\varphi : \mathcal{P} \to \mathcal{E}$, i.e. that $\mathcal{S}$ is the "range manifold" of φ.

Range-Manifold Theorem: *Let $\mathcal{P}$ be an open subset of a Euclidean space $\mathcal{F}$ and let $\varphi : \mathcal{P} \to \mathcal{E}$ be a mapping of class C^r with the following property: There is a number $\kappa \in \mathbb{P}^\times$ such that*

$$|\varphi(p) - \varphi(q)| \geq \kappa \; |p - q| \quad \text{for all} \quad p, q \in \mathcal{P}. \tag{32.1}$$

Then $\mathcal{S} := \operatorname{Rng}\varphi$ is a manifold of class C^r imbedded in $\mathcal{E}$. The tangent space $\mathcal{T}_x$ at $x := \varphi(p)$ is given by

$$\mathcal{T}_x := \operatorname{Rng}\nabla_p\,\varphi \tag{32.2}$$

and the dimension of $\mathcal{S}$ is $\dim\mathcal{F}$.

Proof: It follows from (32.1) that φ is injective and that $|(\nabla_p\varphi)\mathbf{z}| \geq \kappa|\mathbf{z}|$ for all $p \in \mathcal{P}$ and all $\mathbf{z}$ in the translation space $\mathcal{Z}$ of $\mathcal{F}$. Hence we have

$$\operatorname{Null}\nabla_p\varphi = \{\mathbf{0}\} \quad \text{for all} \quad p \in \mathcal{P}. \tag{32.3}$$

We now let $x := \varphi(p) \in \mathcal{S}$ be given and construct $\mathcal{T}_x$ and π'_x as follows: We put $\mathcal{T}_x := \operatorname{Rng}\nabla_p\varphi$ and define $\psi_x : \mathcal{P} \to \mathcal{T}_x$ by

$$\psi_x(q) := \mathbf{E}_x(\varphi(q) - x) \quad \text{for all} \quad q \in \mathcal{P}, \tag{32.4}$$

where $\mathbf{E}_x \in \operatorname{Sym}\mathcal{V}$ is the symmetric idempotent with $\operatorname{Rng}\mathbf{E}_x = \mathcal{T}_x$. Taking the gradient of ψ_x at $p \in \mathcal{P}$, we obtain from (32.4):

$$\nabla_p\psi_x = \mathbf{E}_x|^{\mathcal{T}_x}\nabla_p\,\varphi = \nabla_p\,\varphi|^{\operatorname{Rng}} \in \operatorname{Lin}(\mathcal{Z}, \mathcal{T}_x). \tag{32.5}$$

It follows from (32.3) that $\nabla_p\,\varphi|^{\operatorname{Rng}}$ is invertible. Hence $\nabla_p\,\psi_x$ is invertible and we can apply the Local Inversion Theorem (see Sect.68 in Vol.I) to obtain an open neighborhood $\mathcal{N}'_x$ of $\mathbf{0}$ in $\mathcal{T}_x$ and a mapping $\chi_x : \mathcal{N}'_x \to \mathcal{P}$ of class C^r such that

$$\psi_x(\chi_x(\mathbf{u})) = \mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{N}'_x. \tag{32.6}$$

Finally, we define $\pi'_x : \mathcal{N}'_x \to \mathcal{E}$ by

$$\pi'_x := \varphi \circ \chi_x. \tag{32.7}$$

It is easily seen that the $\mathcal{T}_x$ and π'_x thus constructed satisfy (S_1)–(S_3). Indeed, (S_1) and (S_2) follow immediately from (32.5), (32.6), and (32.7) by application of the Chain Rule. To Prove (S_3), we substitute $q := \chi_x(\mathbf{u})$ into (32.4) and use (32.6) and (32.7) to show that (31.2), which is equivalent to (S_3), is valid.

In general, the π'_x constructed above will not satisfy (S$_4$). In order to satisfy (S$_4$) we must replace π'_x by its restriction $\pi_x := \pi'_x|_{\mathcal{N}_x}$ to a bounded open neighborhood $\mathcal{N}_x$ of $\mathbf{0}$ in $\mathcal{T}_x$ such that the closure of $\mathcal{N}_x$ is included in $\mathcal{N}'_x$. Since $\chi_x|^{\mathrm{Rng}}$ is a homeomorphism, the closure of $\chi_{x>}(\mathcal{N}_x)$ is then compact subset of $\mathrm{Rng}\,\chi_x$. Since $\mathrm{Rng}\,\chi_x$ is open in $\mathcal{F}$, we may apply Prop.6 of Sect.58 in Vol.I and choose a $\delta \in \mathbb{P}^\times$ such that

$$\chi_{x>}(\mathcal{N}_x) + \delta\,\mathrm{Ubl}\,\mathcal{Z} \subset \mathrm{Rng}\,\chi_x. \tag{32.8}$$

We now put

$$\mathcal{G}_x := \mathrm{Rng}\,\pi_x + \kappa\delta\,\mathrm{Ubl}\,\mathcal{T}_x^\perp, \tag{32.9}$$

where κ is the number occuring in condition (32.1). To prove that $\mathcal{S} \cap \mathcal{G}_x \subset \mathrm{Rng}\,\pi_x$, we let $y \in \mathcal{S} \cap \mathcal{G}_x$ be given. Then, by (32.9), we may choose $q \in \mathcal{P}$, $\mathbf{u} \in \mathcal{N}_x$ and $\mathbf{w} \in \kappa\delta\,\mathrm{Ubl}\,\mathcal{T}_x^\perp$ such that $y = \varphi(q) = \pi_x(\mathbf{u}) + \mathbf{w}$. Putting $r := \chi_x(\mathbf{u})$, so that $\varphi(r) = \pi_x(\mathbf{u})$, it follows from (32.7) that

$$\varphi(q) - \varphi(r) = \mathbf{w} \in \kappa\delta\,\mathrm{Ubl}\,\mathcal{T}_x^\perp \subset \mathcal{T}_x^\perp = \mathrm{Null}\,\mathbf{E}_x. \tag{32.10}$$

Using (32.1), it follows that

$$\kappa\delta \geq |\varphi(q) - \varphi(r)| \geq \kappa(q - r)$$

and hence that $q - r \in \delta\mathrm{Ubl}\,\mathcal{Z}$. Since $r \in \chi_{x>}(\mathcal{N}_x)$ we can conclude from (32.8) that $q \in \mathrm{Rng}\,\chi_x$, i.e. that $q = \chi_x(\mathbf{u}')$ for some $\mathbf{u}' \in \mathcal{N}'_x$. Applying $\mathbf{E}_x$ to (32.10) and observing (32.4), we get $\psi(q) - \psi(r) = \mathbf{0}$ and hence $\psi_x(\chi_x(\mathbf{u}')) = \psi_x(\chi_x(\mathbf{u}))$. By (32.6) this means that $\mathbf{u} = \mathbf{u}'$ and hence that $y = \varphi(q) = \varphi_x(\chi_x(\mathbf{u})) = \pi_x(\mathbf{u})$. Since $y \in \mathcal{S} \cap \mathcal{G}_x$ was arbitrary $\mathcal{S} \cap \mathcal{G}_x \subset \mathrm{Rng}\,\pi_x$is valid. The inclusion $\mathcal{S} \cap \mathcal{G}_x \supset \mathrm{Rng}\,\pi_x$ is obvious from the definition (32.9) of $\mathcal{G}_x$. Hence π_x satisfies all of (S$_1$)–(S$_4$), which proves that $\mathcal{S} = \mathrm{Rng}\,\varphi$ is a manifold of class C^r imbedded in $\mathcal{E}$. ∎

Let $\mathcal{S}$ be a manifold in $\mathcal{E}$. Any mapping φ that satisfies the conditions of the Range-Manifold Theorem and whose range is $\mathcal{S}$ is called **parametric representation** of $\mathcal{S}$. Of course, parametric representation are not unique. Not every (imbedded) manifold in $\mathcal{E}$ has parametric representations. For example, a sphere in a space of dimension 2 or greater has none. However, each sphere is the union of two open (in the sphere) subsets that have parametric representations.

Remark 1: The hypotheses of the Range-Manifold Theorem are satisfied when φ is replaced by a local curving π_x for some manifold. Indeed, it follows from (S$_3$) that for all $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{N}_x$ there is a $\mathbf{w} \in \mathcal{T}_x^\perp$ such that

$$\pi_x(\mathbf{u}_1) - \pi_x(\mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2) + \mathbf{w}.$$

Since $(\mathbf{u}_1 - \mathbf{u}_2) \cdot \mathbf{w} = 0$, it follows that $|\pi_x(\mathbf{u}_1) - \pi(\mathbf{u}_2)| \geq |\mathbf{u}_1 - \mathbf{u}_2|$ and hence that the condition (32.1) is satisfied with $\kappa = 1$. Hence π_x is a parametric representation of an open (in $\mathcal{S}$) subset $\mathrm{Rng}\,\pi_x$ of $\mathcal{S}$. ∎

The Range-manifold Theorem tells us that the manifold structure of $\mathrm{Rng}\,\pi_x$ is completely determined by $\pi_x : \mathcal{N}_x \to \mathcal{E}$. In particular, we have

$$\mathcal{T}_{\pi_x(\mathbf{u})} = \mathrm{Rng}\,\nabla_{\mathbf{u}}\,\pi_x \quad \text{for all} \quad \mathbf{u} \in \mathcal{N}_x. \tag{32.11}$$

The construction given in the proof above can be used to construct local curvings $\pi_{\pi_x(\mathbf{u})}$ from π_x for all $\mathbf{u} \in \mathcal{N}_x$.

Remark 2: In convential classical differential geometry, a manifold is most often defined as the range of a parametric represatation $\varphi : \mathcal{P} \to \mathcal{E}$. Also the parameter set $\mathcal{P}$ is usually assumed to be an open subset of a Cartesian space $\mathbf{R}^m$, $m \in \mathbb{N}^\times$. If $(\delta_i \mid i \in m^]) $ denotes the standard basis of $\mathbf{R}^m$ and if $p \in \mathcal{P}$, then

$$\varphi_{,i}(p) := (\nabla_p\,\varphi)\delta_i \qquad \text{for all} \qquad i \in m^]$$

are the values of partial derivatives of φ with respect to the coordinate-parameters. For each $p \in \mathcal{P}$, the family $(\varphi_{,i}(p) \mid im^{]})$ is a basis of the tangent space $\mathcal{T}_{\varphi(p)}$. ∎

33. The curving tensor

Let $\mathcal{S}$ be a manifold of class C^2 imbedded in $\mathcal{E}$ and a point $x \in \mathcal{S}$ be given. Denote the tangent space of $\mathcal{S}$ at x by $\mathcal{T}_x$, as before.

Definition 1. *Choose a local curving π_x for $\mathcal{S}$ at x. The* **curving tensor** *of $\mathcal{S}$ at x is then defined by*

$$\mathbf{\Lambda}_x := \nabla^{(2)}_{\mathbf{0}}\pi_x \in \mathrm{Lin}\,(\mathcal{T}_x, \mathrm{Lin}\,(\mathcal{T}_x, \mathcal{V})) \cong \mathrm{Lin}\,_2(\mathcal{T}_x^2, \mathcal{V}). \tag{33.1}$$

It follows from the Theorem on Symmetry of Second Gradients (see Sect. 611 in Vol.I) that

$$\mathbf{\Lambda}_x \in \mathrm{Sym}\,_2(\mathcal{T}_x^2, \mathcal{V}). \tag{33.2}$$

In view of the local uniqueness of π_x (Prop.2 of Sect.31), $\mathbf{\Lambda}_x$ does not depend on the particular π_x chosen and hence is determined by $\mathcal{S}$ and x alone.

Proposition 1: *The range of $\mathbf{\Lambda}_x$, regarded as a mapping from $\mathcal{T}_x^2$ into $\mathcal{V}$, is included in $\mathcal{T}_x^{\perp}$:*

$$\mathrm{Rng}\,\mathbf{\Lambda}_x \subset \mathcal{T}_x^{\perp}. \tag{33.3}$$

Proof: The axiom (S_3) states that the range of the mapping

$$(\mathbf{u} \mapsto \pi_x(\mathbf{u}) - (x + \mathbf{u})) : \mathcal{N}_x \to \mathcal{V},$$

where $\mathcal{N}_x := \mathrm{Dom}\,\pi_x$, is included in $\mathcal{T}_x^{\perp}$. Hence the gradients of this mapping, of any order, have values that are multilinear mappings with ranges included in $\mathcal{T}_x^{\perp}$. This applies, in particular, to the second gradient at $\mathbf{0} \in \mathcal{T}_x^{\perp}$, which is just $\mathbf{\Lambda}_x$. ∎

Let $\mathcal{S}$ be a manifold of class C^r imbedded in $\mathcal{E}$ and let $\mathcal{E}'$ be Euclidean space (the same as $\mathcal{E}$ or not), with translation space $\mathcal{V}'$ be given.

Definition 2. *We say that a given mapping $\psi : \mathcal{S} \to \mathcal{E}'$ is* **of class** *C^r if, for every local curving π_x of $\mathcal{S}$, $\psi \circ \pi_x\big|^{\mathcal{S}} : \mathcal{N}_x \to \mathcal{E}'$ is of class C^r. We then define the* **gradient** *of ψ at x by*

$$\nabla_x\psi := \nabla_{\mathbf{0}}(\psi \circ \pi_x\big|^{\mathcal{S}}) \in \mathrm{Lin}\,(\mathcal{T}_x, \mathcal{V}')\ . \tag{33.4}$$

Since any two local curvings have the same values in some neighborhood of $\underline{0}$ in $\mathcal{T}_x$ by Prop. 2 of Sect.31, it is clear that $\nabla_x\psi$ as defined above does not depend on the choice of π_x. If $\mathcal{D}$ is an open subset of $\mathcal{E}$ that includes $\mathcal{S}$ and if $\phi : \mathcal{D} \to \mathcal{E}'$ is of class C^r, then $\phi|_{\mathcal{S}}$ is of class C^r and

$$\nabla_x(\phi|_{\mathcal{S}}) = (\nabla_x\phi)|_{\mathcal{T}_x} \quad \text{for all} \quad x \in \mathcal{S},$$

as one verifies easily from the definition (33.4) and (S_2) of Sect.31.

Definition 3. *The mapping*

$$\mathbf{E} : \mathcal{S} \to \mathrm{Lin}\,\mathcal{V}$$

which associates, with each $x \in \mathcal{S}$, the symmetric idempotent $\mathbf{E}_x$ whose range is the tangent space $\mathcal{T}_x$ the **idempotent mapping** of $\mathcal{S}$.

Idempotent Mapping Theorem : *If $\mathcal{S}$ is a manifold of class C^r in $\mathcal{E}$, then the idempotent mapping $\mathbf{E}$ of $\mathcal{S}$ is of class C^{r-1}. Moreover, if $r \geq 2$, then the gradient (defined according to (33.4))*

$$\nabla_x \mathbf{E} \in \mathrm{Lin}\,(\mathcal{T}_x, \mathrm{Lin}\,\mathcal{V}) \cong \mathrm{Lin}\,_2(\mathcal{T}_x \times \mathcal{V}, \mathcal{V})$$

of $\mathbf{E}$ at a given $x \in \mathcal{S}$, regarded as a bilinear mapping from $\mathcal{T}_x \times \mathcal{V}$ into $\mathcal{V}$, is related to the curving tensor $\mathbf{\Lambda}_x$ by

$$\nabla_x \mathbf{E}\big|_{\mathcal{T}_x^2} = \mathbf{\Lambda}_x. \tag{33.5}$$

Proof: For every $x \in \mathcal{S}$ define $\Gamma_x : \mathrm{Linj}(\mathcal{T}_x, \mathcal{V}) \to \mathrm{Lin}\,\mathcal{V}$ according to Def.X of Sect.2X, i.e.,

$$\Gamma_x(\mathbf{F}) := \text{symmetric idempotent with } \mathrm{Rng}\,\Gamma(\mathbf{F}) = \mathrm{Rng}\,\mathbf{F} \tag{33.6}$$

By (32.11) we have, for every $x \in \mathcal{S}$,

$$\mathbf{E} \circ \pi_x\big|^{\mathcal{S}} = \Gamma_x \circ \nabla \pi_x\big|^{\mathrm{Linj}(\mathcal{T}_x, \mathcal{V})},$$

Since $\nabla \pi_x$ is of class C^{r-1}, it follows from Prop.X of Sect.2X and the Chain Rule that $\mathbf{E} \circ \pi_x\big|^{\mathcal{S}}$ is of class C^{r-1} for all $x \in \mathcal{S}$, which means, by Def.2, that $\mathbf{E}$ is of class C^{r-1}.

In view of (32.11) we have

$$\mathbf{E}_{\pi_x(\mathbf{u})} \nabla_{\mathbf{u}} \pi_x = \nabla_{\mathbf{u}} \pi_x \quad \text{for all} \quad \mathbf{u} \in \mathrm{Dom}\,\pi_x.$$

If $\mathcal{S}$ is of class C^2, one can take the gradient of the above with respect to $\mathbf{u}$ at $\mathbf{0} \in \mathcal{T}_x$. Using the Product Rule, we obtain

$$((\nabla_x \mathbf{E})\mathbf{u}_1)\nabla_{\mathbf{0}} \pi_x + \mathbf{E}_x(\nabla_{\mathbf{0}}^{(2)} \pi_x)\mathbf{u}_1 = \nabla_{\mathbf{0}}^{(2)} \pi_x \mathbf{u}_1 \quad \text{for all} \quad \mathbf{u}_1 \in \mathcal{T}_x.$$

In view of (31.1) and (33.1) this is equivalent to

$$\nabla_x \mathbf{E}(\mathbf{u}_1, \mathbf{u}_2) + \mathbf{E}_x \mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) \quad \text{for all} \quad (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{T}_x^2.$$

It follows from Prop.1 that $\mathbf{E}_x \mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{0}$, which proves (33.5). ∎

Proposition 2: *Let $\mathcal{P}$ be an open subset of a flat space $\mathcal{F}$ with translation space $\mathcal{Z}$ and let $\varphi : \mathcal{P} \to \mathcal{E}$ be a mapping of class C^2 such that $\mathrm{Rng}\,\varphi \subset \mathcal{S}$. Let $p \in \mathcal{P}$ be given and put $x := \varphi(p) \in \mathcal{S}$. Then $\nabla_p \varphi \in \mathrm{Lin}\,(\mathcal{Z}, \mathcal{V})$, $\mathrm{Rng} \nabla_p \varphi \subset \mathcal{T}_x$ and*

$$(\mathbf{1}_{\mathcal{V}} - \mathbf{E}_x)\nabla_p^{(2)} \varphi = \mathbf{\Lambda}_x \circ (\nabla_p \varphi \times \nabla_p \varphi). \tag{33.7}$$

Proof: Taking the gradient of (31.3) with respect to $q \in \mathcal{P}$ and using the Chain Rule, we obtain

$$\nabla_q \varphi = (\nabla_{\mathbf{u}} \pi_x)(\mathbf{E}_x\big|^{\mathcal{T}_x})(\nabla_q \varphi) \quad \text{where} \quad \mathbf{u} := \mathbf{E}_x(\varphi(q) - x).$$

Taking the gradient with respect to q again and evaluating the result at $q := p$, we obtain, using the Chain Rule and Product Rule again,

$$(\nabla_p^{(2)}\varphi)\mathbf{z}_1 = (\nabla_{\mathbf{0}}^{(2)}\pi_x)\mathbf{E}_x(\nabla_p\varphi)\mathbf{z}_1\mathbf{E}_x\big|^{\mathcal{T}_x}(\nabla_p\varphi) + \nabla_{\mathbf{0}}\pi_x\mathbf{E}_x\big|^{\mathcal{T}_x}(\nabla_p^{(2)}\varphi)\ \mathbf{z}_1 \qquad \text{for all} \quad \mathbf{z}_1 \in \mathcal{Z}.$$

If we observe (31.1), the definition (33.1) of $\mathbf{\Lambda}_x$ and (31.4), we obtain

$$(\nabla_p^{(2)}\varphi)(\mathbf{z}_1, \mathbf{z}_2) = \mathbf{\Lambda}_x(\nabla_p\varphi\ \mathbf{z}_1, \nabla_p\varphi\ \mathbf{z}_2) + \mathbf{E}_x(\nabla_p^{(2)}\varphi)(\mathbf{z}_1, \mathbf{z}_2)$$

for all $(\mathbf{z}_1, \mathbf{z}_2) \in \mathcal{Z}^2$, which is equivalent to the desired result (33.6). ▮

Prop.2 applies, in particular, when φ is a parametric representation of $\mathcal{S}$, in which case $\nabla_p\varphi\big|^{\mathcal{T}_x}$ is invertible and $\mathbf{E}_x = \Gamma^x(\nabla_x\varphi)$. Hence (33.7) shows how, in this case, the curving tensor $\mathbf{\Lambda}_x$ can be obtained from $\nabla_p\varphi$ and $\nabla_p{}^{(2)}\varphi$.

Remark 1: Let $\varphi : \mathcal{P} \to \mathcal{E}$, with $\mathcal{P} \subset \mathbf{R}^m$, be a coordinate representation of $\mathcal{S}$ as described in Remark 2 of Sec.32. Then

$$\varphi_{,j}(p) = \nabla_x\varphi\ \delta_j, \quad \varphi_{,j,k}(p) = (\nabla_p^{(2)}\varphi)(\delta_j, \delta_j) \qquad \text{for all} \quad i, k \in m^{]}\ ,$$

where $(\delta_i \mid i \in m^{]})$ denotes the standard basis of $\mathbf{R}^m$, and where $\varphi_{,j}$ and $\varphi_{,j,k}$ are the first and second partial derivatives of φ with respect to the coordinates. In this case (33.7) yields

$$(\mathbf{1}_{\mathcal{V}} - \mathbf{E}_x)\varphi_{,j,k}(p) = \mathbf{\Lambda}_x(\varphi_{,j}(p), \varphi_{,j}(p)) \quad \text{for all} \quad i, k \in m^{]} \tag{33.8}$$

It is not difficult to express $\mathbf{E}_x$ in terms of $\varphi_{,j}(p) = \nabla_x\varphi\ \delta_j,\ \ j \in m^{]}$, and hence (33.8) yields an explicit formula for the components of $\mathbf{\Lambda}_x$. ▮

34. Paths, geodesics

Let $I \in \operatorname{Sub}\mathbf{R}$ be an interval and let $f : I \to \mathcal{E}$ be a process of class C^r, $r \in \mathbb{N}^\times$ (see Sect.61 of Vol.I). We say that f is a **regular process** if $\mathbf{0} \notin \operatorname{Rng} f^\bullet$, i.e. if $f^\bullet(t) \neq \mathbf{0}$ for all $t \in I$.

Definition 1. *We say that two processes $f : I \to \mathcal{E}$ and $g : J \to \mathcal{E}$ are* **equivalent** *and we write $f \sim g$ if there is a strictly isotone bijection $\alpha : I \to J$ such that $g \circ \alpha = f$. (It is easily seen that $\sim$ is an equivalent relation on the set of all processes.) We call the resulting equivalent classes* **paths**. *If $\mathfrak{p}$ is a path and $f \in \mathfrak{p}$, we say that $\mathfrak{p}$ is the path of the process f, or that f is a* **parametrization** *of $\mathfrak{p}$. Notes that the range* $\operatorname{Rng} f$ *depends only on the path $\mathfrak{p}$ of f, and hence that it is meaningful to put*

$$\operatorname{Rng}\mathfrak{p} := \operatorname{Rng} f \quad \text{for all} \quad f \in \mathfrak{p}. \tag{34.1}$$

We say that a path $\mathfrak{p}$ is of class C^r [regular] if at least one $f \in \mathfrak{p}$ is of class C^r [regular].

We say that a path $\mathfrak{p}$ is **compact** if for some $f \in \mathfrak{p}$, $\operatorname{Dom} f$ is a compact interval. If this is the case, then $\operatorname{Dom} f$ is a compact interval for every $f \in \mathfrak{p}$. If $\operatorname{Dom} f = [a, b]$ with $a, b \in \mathbf{R}$ with $a < b$, then the points $x := f(a)$ and $y := f(b)$ depends only on the path $\mathfrak{p}$, not on the particular choice of $f \in \mathfrak{p}$. We say that x is the **starting point** and y the **endpoint** of the path $\mathfrak{p}$, and that $\mathfrak{p}$ is a **path from** x **to** y. We say that $\mathfrak{p}$ is a **closed path** if $x = y$.

Let $\mathfrak{p}$ be a path of class C^1. We say that $p \in \mathfrak{p}$ is an **arclength-parametrization** of $\mathfrak{p}$ if $|p^\bullet| = 1$. Physically, such a process p can be interpreted as a motion with constant speed 1 along the path $\mathfrak{p}$.

Proposition 1: *A path $\mathfrak{p}$ has an arclength-parametrization p of class C^r if and only if $\mathfrak{p}$ is regular and of class C^r, $r \in \mathbb{N}^\times$. If $p : J \to \mathcal{E}$ and $q : K \to \mathcal{E}$ are both arclength-parametrizations of $\mathfrak{p}$ then there is a number $d \in \mathbb{R}$ such that $K = d + J$ and $p(s) = q(d + s)$ for all $s \in J$.*

Proof: Assume that $\mathfrak{p}$ is regular and of class C^r. Then we may choose $f \in \mathfrak{p}$ with $f : I \to \mathcal{E}$ regular and of class C^r. We choose $c \in I$ and define $\sigma : I \to J := \operatorname{Rng}\sigma$ by

$$\sigma(t) := \int_c^t |f^\bullet| \quad \text{for all} \quad t \in I. \tag{34.2}$$

Since $\sigma^\bullet = |f^\bullet| > 0$, it follows that σ is strictly isotone bijection of class C^r and that the same is true for the inverse $\sigma^{\leftarrow} : J \to I$. Hence $p := f \circ \sigma^{\leftarrow} : J \to \mathcal{E}$ is a process of class C^r whose path is $\mathfrak{p}$. Moreover, we have $p^\bullet = \frac{f^\bullet}{\sigma^\bullet} \circ \sigma^{\leftarrow} = \frac{f^\bullet}{|f^\bullet|} \circ \sigma^{\leftarrow}$ and hence $|p^\bullet| = 1$, which shows that p is an arclength-parametrization.

Let p and q be arclength-parametrizations of $\mathfrak{p}$. Since $p \sim q$, we can determine a strictly isotone bijection $\alpha : J \to K$ such $q \circ \alpha = p$. Hence, for every $s \in J$ and for every $t \in (J - s)\backslash\{0\}$, we have

$$\left| \frac{q(\alpha(s+t)) - q(\alpha(s))}{\alpha(s+t) - \alpha(s)} \right| \frac{\alpha(s+t) - \alpha(s)}{t} = \left| \frac{p(s+t) - p(s)}{t} \right|.$$

Taking the limit $t \to 0$ and observing that $|p^\bullet(s)| = |q^\bullet(\alpha(s))| = 1$ we see that $\alpha^\bullet(s)$ exists and is equal to 1 for all $s \in J$. It follows that $\alpha^\bullet = 1$ and hence that $\alpha(s) = d + s$ for some $d \in \mathbb{R}$ and all $s \in J$. ∎

It follows from Prop.1 that every regular path has exactly one arclength parametrization p such that $\operatorname{Dom} p = [0, \lambda]$ with $\lambda \in \mathbb{P}$. We call this parametrization the **standard parametrization** of the path and λ the **length** of the path. If $f : [a, b] \to \mathcal{E}$ is any regular parametrization of the path, then

$$\lambda = \int_a^b |f^\bullet|, \quad p\Big(\int_a^t |f^\bullet|\Big) = f(t) \quad \text{for all} \quad t \in [a, b]. \tag{34.3}$$

Let $\mathfrak{p}$ be a path and let $f \in \mathfrak{p}$ be given. Put $I := \operatorname{Dom} f$. We define a process $g : -I \to \mathcal{E}$ by $g(t) := f(-t)$. The path of this process depends only on the path $\mathfrak{p}$ of f. We call the path of g the **reverse** of $\mathfrak{p}$ and denote it by $-\mathfrak{p}$. If $\mathfrak{p}$ is a compact path from x to y, then $-\mathfrak{p}$ is a compact path that goes from y to x.

If $\mathfrak{p}$ is a compact path from x to y with an injective standard parametrization $p : [a, b] \to \mathcal{E}$. One can then show that

$$p_{>}(]0, \lambda[) = \operatorname{Rng}\mathfrak{p}\backslash\{x, y\}$$

is a one-dimensional manifold in $\mathcal{E}$. A manifold obtained in this way is called an **arc** with end-points x and y. If $\mathcal{C}$ is an arc of class C^1, then there are exactly two paths whose range is $\mathcal{C}$, and one of these is the reverse of the other.

One can show that, if $\mathcal{C}$ is a connected one-dimensional manifold of class C^r, then there are regular paths of class C^r whose range is $\mathcal{C}$. In general, there will be infinitely many such paths, even if the closure of $\mathcal{C}$ is compact. For example, a circle is the range of infinitely many regular paths.

Consider a regular path of class C^2 with arclength parametrization $p : J \to \mathcal{E}$. Since $|p^\bullet| = 1$, we have $p^\bullet \cdot p^\bullet = 1$ and hence

$$p^{\bullet\bullet} \cdot p^\bullet = 0 \tag{34.4}$$

We say that $p^\bullet(s)$ is the **unit tangent** to the path $\mathfrak{p}$ at s. The function $\kappa : J \to \mathbb{R}$ defined by $\kappa := |p^{\bullet\bullet}|$ is called the **curvature** of the path $\mathfrak{p}$. If $\kappa = 0$, then the range of the path is an interval on a straight line in $\mathcal{E}$. Let $s \in J$ be given. If $\kappa(s) \neq 0$, one can form the unit vector $\mathbf{v}(s) := \dfrac{1}{\kappa(s)} p^{\bullet\bullet}(s)$ in the direction of $p^{\bullet\bullet}(s)$.

The unit vector $\mathbf{v}(s)$ is called the **principal normal** to the path $\mathfrak{p}$ at s. The reciprocal $\rho(s) := \dfrac{1}{\kappa(s)}$ of the curvature is called the **radius of curvature** and the **circle of curvature** is defined to be the circle of radius $\rho(s)$ whose center is $p(s) + \rho(s)\mathbf{v}(s)$ and which is tangent to the given path at $p(s)$. Any path whose range is this circle has, at $p(s)$, the same curvature and pricipal normal as the given path. Intuitively, the circle of curvature at $p(s)$ is that circle which best fits the given path at s.

Remark: Additional analyses of paths will be given in Sect.38. ▮

We now assume that a manifold $\mathcal{S}$ of class C^2 imbedded in $\mathcal{E}$ is given . We say that a path $\mathfrak{p}$ is a **path on** $\mathcal{S}$ if $\mathrm{Rng}\,\mathfrak{p} \subset \mathcal{S}$. We say that the manifold is **connected** if, for all $x, y \in \mathcal{S}$, there is a path of class C^2 from x to y.

Consider a regular path $\mathfrak{p}$ of class C^2 on $\mathcal{S}$, with arclength parametrization p. Let $\mathbf{E}$ be the idempotent mapping of $\mathcal{S}$ as defined by Def.2. If we apply Prop.1 of Sect.1 to the case when φ is replaced by p, we see that $p^\bullet(s) \in \mathcal{T}_{p(s)}$ and hence

$$\mathbf{E}_{p(s)}\, p^\bullet(s) \in \mathcal{T}_{p(s)} \quad \text{for all} \quad s \in J. \tag{34.5}$$

The components of $p^{\bullet\bullet}(s)$ in $\mathcal{T}_{p(s)}$ and $\mathcal{T}_{p(s)}{}^\perp$ are $\mathbf{E}_{p(s)}\, p^{\bullet\bullet}(s)$ and $(\mathbf{1}_\mathcal{V} - \mathbf{E}_{p(s)})p^{\bullet\bullet}(s)$, respectively. If we apply Prop.2 of Sect.3 to $p : J \to \mathcal{E}$, we see that

$$(\mathbf{1}_\mathcal{V} - \mathbf{E}_{p(s)})p^{\bullet\bullet}(s) = \mathbf{\Lambda}_{p(s)}(p^\bullet(s), p^\bullet(s)) \quad \text{for all} \quad s \in J. \tag{34.6}$$

which shows that the component of $p^{\bullet\bullet}(s)$ in $\mathcal{T}_{p(s)}{}^\perp$ is already determined by the unit tangent $p^\bullet(s)$. We define the **normal curvature** $\kappa_N : J \to \mathbf{R}$ by

$$\kappa_N(s) := |(\mathbf{1}_\mathcal{V} - \mathbf{E}_{p(s)})p^{\bullet\bullet}(s)| = |\mathbf{\Lambda}_{p(s)}(p^\bullet(s), p^\bullet(s))| \tag{34.7}$$

for all $s \in J$ and the **geodesic curvature** $\kappa_G : J \to \mathbf{R}$ by

$$\kappa_G(s) := |\mathbf{E}_{p(s)}\, p^{\bullet\bullet}(s)| \quad \text{for all} \quad s \in J. \tag{34.8}$$

It is clear that κ_N and κ_G are continuous and that the curvature κ of p is related to κ_N and κ_G by

$$\kappa = \sqrt{\kappa_N{}^2 + \kappa_G{}^2}. \tag{34.9}$$

Definition 2. *We say that a path on $\mathcal{S}$ with arclength parametrization p is a* **geodesic** *if one of the following two equivalent conditions hold:*

$$\kappa_G(s) = |\mathbf{E}_{p(s)}\, p^{\bullet\bullet}(s)| = 0 \quad \text{for all} \qquad s \in J. \tag{34.10}$$

$$p^{\bullet\bullet}(s) = \mathbf{\Lambda}_{p(s)}(p^\bullet(s), p^\bullet(s)) \qquad \text{for all} \qquad s \in J. \tag{34.11}$$

Geodesic Theorem: *Let $\mathcal{S}$ be a manifold of class C^2 and let $x, y \in \mathcal{S}$ be given. Let $\mathfrak{p}$ be a regular path on $\mathcal{S}$ of class C^2 from x to y of minimum length. Then $\mathfrak{p}$ is a geodesic.*

Proof: Let $p : [0, \lambda] \to \mathcal{E}$ be the standrad parametrization of $\mathfrak{p}$. We must show that $\kappa_G = 0$. Assume that this is not the case. Since κ_G is continuous, we then may choose $t \in]0, \lambda[$ such that $\kappa_G(t) \neq 0$ and hence (see (34.8))

$$\mathbf{u} := \mathbf{E}_{p(t)}\, p^{\bullet\bullet}(t) \neq \mathbf{0}\ . \tag{34.12}$$

Put $z := p(t)$ and let $\overline{\pi}_z : \overline{\mathcal{N}}_z \to \mathcal{E}$ be an extended local curving for $\mathcal{S}$ at z. (See Prop.3 of Sect.31.) Since $\mathrm{Rng}\,\overline{\pi}_z$ is an open neighborhood of z in $\mathcal{E}$ and since p is continuous, there is an open interval $J \in]0, \lambda[$ such that $t \in J$ and $p_>(J) \subset \operatorname{Rng} \overline{\pi}_z$. In view of condition ($C_2$) of Prop.3 in Sect.31, we can define a mapping $\mathbf{k} : J \to \overline{\mathcal{N}}_z$ of class C^2 by

$$\mathbf{k} := \left(\overline{\pi}_z|^{\operatorname{Rng}}\right)^{\leftarrow} \circ p|_J^{\operatorname{Rng} \overline{\pi}_z}.$$

Using the assumption (31.5), we then have

$$\mathbf{k}(t) = \underline{0}, \quad p(s) = \pi_z(\mathbf{k}(s)) \quad \text{for all} \quad s \in J, \tag{34.13}$$

and $\operatorname{Rng}\mathbf{k} \subset \mathcal{N}_z$.

We now consider the function $\gamma : J \to \mathbf{R}$ defined by

$$\gamma(s) := p^{\bullet\bullet}(s) \cdot (\nabla_{\mathbf{k}(s)} \pi_z)\mathbf{u} \quad \text{for all} \quad s \in J. \tag{34.14}$$

It follows from (34.14), (31.1), and (34.12) that

$$\gamma(t) = p^{\bullet\bullet} \cdot \mathbf{u} = p^{\bullet\bullet} \cdot \mathbf{E}_{p(t)}\mathbf{u} = (\mathbf{E}_{p(t)} p^{\bullet\bullet}) \cdot \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{u} > 0$$

. Since γ is continuous, there is an open interval J' such that

$$t \in J', \quad \operatorname{Clo} J' \subset J, \quad \gamma|_{J'} > 0. \tag{34.15}$$

It is easy to construct a function $\alpha : J \to \mathbf{R}$ of class C^2 such that

$$\alpha \neq 0, \quad \alpha \geq 0, \quad \alpha|_{J \setminus J'} = 0. \tag{34.16}$$

and we assume that this has been done. Since $\mathcal{N}_z$ is an open neighborhood of $\mathbf{0} \in \mathcal{T}_z$, one can find a $\delta \in \mathbb{P}^\times$ such that $\mathbf{k}(s) + \xi\alpha(s)\mathbf{u} \in \mathcal{N}_z$ for all $s \in J$ and all $\xi \in\ ]-\delta, \delta[$. Therefore, we can define a mapping $f : [0, \lambda] \times\ ]-\delta, \delta[\to \mathcal{E}$ by

$$f(s, \xi) := \left\{ \begin{array}{lll} p(s) & \text{if} & s \in [0, \lambda] \setminus J \\ \pi_z(\mathbf{k}(s) + \xi\alpha(s)\mathbf{u}) & \text{if} & s \in J \end{array} \right\} \tag{34.17}$$

It is easily seen that f is of class C^2, that $\operatorname{Rng} f \subset \mathcal{S}$, and that

$$f(\cdot, 0) = p, \quad f(0, \cdot) = p(0) = x, \quad f(\lambda, \cdot) = p(\lambda) = y. \tag{34.18}$$

For each $\xi \in]-\delta, \delta[$, the function $f(\cdot, \xi) : [0, \lambda] \to \mathcal{E}$ is a parametrization of a path from x to y. Thus, f describes an imbededding of a given path $\mathfrak{p}$ into a one-parameter family of paths from x to y. It follows from (34.17) and the Chain Rule that

$$f_{,2}(s, 0) := \left\{ \begin{array}{lll} 0 & \text{if} & s \in [0, \lambda] \setminus J \\ \alpha(s)((\nabla_{\mathbf{k}(s)} \pi_z)\mathbf{u}) & \text{if} & s \in J \end{array} \right\}. \tag{34.19}$$

Now by (34.3), the length $\overline{\lambda}(\xi)$ of the path defined by $f(\cdot, \xi)$ is

$$\overline{\lambda}(\xi) = \int_0^\lambda |f_{,1}(\cdot, \xi)|. \tag{34.20}$$

Since $f_{,1}(\cdot, \xi)$ is of class C^1, we can apply the Differentiation Theorem for Intergral Representations (see Sect. 610 in Vol.I) to conclude that the function $\overline{\lambda} :]-\delta, \delta[\to \mathbf{R}$ defined by (34.19) is of class C^1 and that

$$\overline{\lambda}^\bullet(0) = \int_0^\lambda p^\bullet \cdot f_{,1,2}(\cdot, 0). \tag{34.21}$$

Since $f_{,1,2} = f_{,2,1}$ by the Corollary of the Theorem on the Interchange of Partial Gradients (see Sect.611 in Vol.I), it follows from the Product Rule that

$$p^\bullet \cdot f_{,1,2}(\cdot, 0) = p^\bullet \cdot (f_{,2}(\cdot, 0))^\bullet = (p^\bullet \cdot f_{,2}(\cdot, 0))^\bullet - p^{\bullet\bullet} \cdot f_{,2}(\cdot, 0). \tag{34.22}$$

Substituting this in to (34.21), using the Fundamental Theorem of Calculus and then (34.18), (34.19), and (34.14), we obtain

$$\overline{\lambda}^\bullet(0) = -\int_0^\lambda p^{\bullet\bullet} \cdot f_{,2}(\cdot, 0) = -\int_J \alpha\gamma. \tag{34.23}$$

Now, since the path $\mathfrak{p}$ is assumed to have the minimum length $\lambda = \overline{\lambda}(0)$, it follows from the Extremum Theorem (see Sect.08 in Vol.I) that $\overline{\lambda}^\bullet(0) = 0$. On the other hand, by (34.15) and (34.16) we have $\alpha\gamma \geq 0$ and $\alpha\gamma \neq 0$, which implies $\int_J \alpha\gamma > 0$. Thus (34.22) gives the desired condition. ∎

35. Surfaces,

We say that a manifold $\mathcal{S}$ imbedded in $\mathcal{E}$ is a **surface** (or hypersurface) if it has dimension $n-1$. This means that, for each $x \in \mathcal{S}$,

$$\dim \mathcal{T}_x = n-1, \quad \dim \mathcal{T}_x^\perp = 1 \tag{35.1}$$

where $\mathcal{T}_x$ is the tangent space at x.

We say that a surface $\mathcal{S}$ is **orientable** if there exists a continuous mapping $\mathbf{n} : \mathcal{S} \to \mathcal{V}$ with $|\mathbf{n}| = 1$ and $\mathbf{n}(x) \in \mathcal{T}_x^\perp$ and hence

$$\mathcal{T}_x^\perp = \mathbf{R}\,\mathbf{n}(x) \quad \text{for all} \quad x \in \mathcal{S}. \tag{35.2}$$

Such a mapping $\mathbf{n}$ is called a **surface normal on** $\mathcal{S}$, and we say that $\mathbf{n}$ endows $\mathcal{S}$ with an **orientation**. Although it is evidently always possible to find a mapping of type (35.2), there may be no mapping of type (35.2) that is also continuous. In this case, $\mathcal{S}$ is **non-orientable**. A *Mobius-strip* is an example of a non-orientable surface.

We now assume that a surface of class C^r is given.

Proposition 1: *Every point z on $\mathcal{S}$ has an open neighborhood $\mathcal{R}$ in $\mathcal{S}$ that has a surface normal $\mathbf{n} : \mathcal{R} \to \mathcal{V}$ of class C^{r-1} and hence (the neighborhood $\mathcal{R}$) is orientable.*

Proof: Let $z \in \mathcal{S}$ be given. We choose $\mathbf{u} \in \mathcal{T}_z^\perp \setminus \{\mathbf{0}\}$. Let $\mathbf{E} : \mathcal{S} \to \operatorname{Lin} \mathcal{V}$ be the idempotent mapping of $\mathcal{S}$ as defined by Def.2 in Sect.33. Noting that $\mathbf{E}$ is continuous, we see that $\mathcal{R} := \{x \in \mathcal{S} \mid |(\mathbf{1}_\mathcal{V} - \mathbf{E}_x)\mathbf{u}| > 0\}$ is an open neighborhood of z in $\mathcal{S}$. We define $\mathbf{n} : \mathcal{R} \to \mathcal{V}$ by

$$\mathbf{n}(x) := \frac{(\mathbf{1}_\mathcal{V} - \mathbf{E}_x)\mathbf{u}}{|(\mathbf{1}_\mathcal{V} - \mathbf{E}_x)\mathbf{u}|} \in \mathcal{T}_x^\perp \quad \text{for all} \quad x \in \mathcal{R}.$$

Since $\mathbf{E}$ is of class C^{r-1} by the Idempotent Mapping Theorem of Sect.33, it is evident that $\mathbf{n}$ is a surface normal of class C^{r-1} on $\mathcal{R}$. ∎

The Prop.1 shows that locally every surface $\mathcal{S}$ is orientable.

Proposition 2: *An orientable connected surface $\mathcal{S}$ has exactly two orientations. If $\mathbf{n}$ is the surface normal of the first, then $-\mathbf{n}$ is the surface normal of the second, if $\mathcal{S}$ is of class C^r then $\mathbf{n}$ is of class C^{r-1}.*

Proof: Let $\mathbf{n}$ and $\mathbf{n}'$ be surface normals of $\mathcal{S}$. Since, for each $x \in \mathcal{S}$, $\dim \mathcal{T}_x^\perp = 1$ and hence $\mathcal{T}_x^\perp$ contains exactly two unit vectors such that one is the opposite of the other. We must have $\mathbf{n}'(x) = \xi(x)\mathbf{n}(x)$ with $\xi(x) = 1$ or $\xi(x) = -1$. Since $\mathbf{n}$ and $\mathbf{n}'$ are both continuous, so is $\xi = \mathbf{n} \cdot \mathbf{n}'$ Hence, since $\mathcal{S}$ is connected, ξ must be either the constant 1 or the constant -1, i.e. $\mathbf{n} = \mathbf{n}'$ or $\mathbf{n} = -\mathbf{n}'$.

If $\mathbf{n}$ is a surface normal on $\mathcal{S}$ and $\mathcal{R}$ an open subset of $\mathcal{S}$ then $\mathbf{n}|_{\mathcal{R}}$ is a surface normal on $\mathcal{R}$. Thus, it follows from Prop.1 that every point in $\mathcal{S}$ has an open neighborhood such that the restriction of $\mathbf{n}$ to that neighborhood is of class C^{r-1}. Hence $\mathbf{n}$ itself is of class C^{r-1}. ∎

Proposition 3: *Assume that $\mathcal{S}$ is a surface of class C^2 and that $\mathbf{n} : \mathcal{S} \to \mathcal{V}$ a surface normal. Let $x \in \mathcal{S}$ be given. Let $\mathcal{T}_x$ be the tangent space and $\mathbf{\Lambda}_x \in \mathrm{Sym}_2(\mathcal{T}_x^2, \mathcal{V})$ the curving tensor at x, as defined by Def.1 of Sect.33. Then:*

$$\mathrm{Rng}\,\nabla_x\mathbf{n} \subset \mathcal{T}_x, \tag{35.3}$$

$$\mathbf{L}_x := -\nabla_x\mathbf{n}|^{\mathcal{T}_x} \in \mathrm{Sym}\,\mathcal{T}_x, \tag{35.4}$$

$$\mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) = \mathbf{n}(x)(\mathbf{u}_2 \cdot \mathbf{L}_x\mathbf{u}_1) \quad \text{for all} \quad \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{T}_x, \tag{35.5}$$

$$\mathbf{u}_2 \cdot \mathbf{L}_x\mathbf{u}_1 = \mathbf{n}(x) \cdot \mathbf{\Lambda}_x({}_1, \mathbf{u}_2) \quad \text{for all} \quad \mathbf{u}_1, \mathbf{u}_2 \in \mathcal{T}_x\ . \tag{35.6}$$

Proof: Taking the gradient of $\mathbf{n} \cdot \mathbf{n} = 1$ at x, we get $(\nabla_x\mathbf{n})^\top\mathbf{n}(x) = 0$. Since $\mathcal{T}_x^\perp = \mathbf{R}\,\mathbf{n}(x)$, it follows that $\mathcal{T}_x^\perp \subset (\mathrm{Null}\,(\nabla_x\mathbf{n})^\top$. Using the Theorem on Annihilators and Transposes (see Sect.21 of Vol.I), we conclude that $\mathcal{T}_x^\perp \subset (\mathrm{Rng}\,(\nabla_x\mathbf{n}))^\perp$, which implies (35.3). The idempotent mapping $\mathbf{E}$ of $\mathcal{S}$ is obviously related to $\mathbf{n}$ by

$$\mathbf{E}(y) = \mathbf{1}_{\mathcal{V}} - \mathbf{n}(y) \otimes \mathbf{n}(y) \quad \text{for all} \quad y \in \mathcal{S}. \tag{35.7}$$

Taking the gradient of (35.7) at $y := x$ gives, with the help of the Product Rule,

$$(\nabla_x\mathbf{E})\mathbf{u}_1 = -\big((\nabla_x\mathbf{n})\mathbf{u}_1 \otimes \mathbf{n}(x) + \mathbf{n}(x) \otimes (\nabla_x\mathbf{n})\mathbf{u}_1\big) \quad \text{for all} \quad \mathbf{u}_1 \in \mathcal{T}_x.$$

Using the Idempotent Mapping Theorem of Sect.33 we obtain

$$\mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) = -\mathbf{n}(x)(\mathbf{u}_2 \cdot (\nabla_x\mathbf{n})\mathbf{u}_1) \quad \text{for all} \quad (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{T}_x^2,$$

which is (35.5). Taking the inner product of (35.5) with $\mathbf{n}(x)$ yield (35.6). The symmetry of $\mathbf{L}_x$ follows from (35.6) and the symmetry of $\mathbf{\Lambda}_x$ (see (33.2)). ∎

Definition 1. *The symmetric lineon $\mathbf{L}_x \in \mathrm{Lin}\,\mathcal{T}_x$ defined in (35.4) will be called the* **curving lineon** *at x.*

$\mathbf{L}_x$ depends not only on x and $\mathcal{S}$, but also on the orientation of $\mathcal{S}$. A change of orientation transforms $\mathbf{L}_x$ into $-\mathbf{L}_x$. It follows from (35.5) that the curving lineon $\mathbf{L}_x$ determines the curving tensor $\mathbf{\Lambda}_x$ and from (35.6) that $\mathbf{\Lambda}_x$ determines $\mathbf{L}_x$ within sign.

If $\varphi : \mathcal{P} \to \mathcal{E}$ satisfies the conditions of Prop.2, Sect.33, we have, with $x := \varphi(p) \in \mathcal{S}$ and $\mathbf{F} := \nabla_p\varphi$,

$$\mathbf{n}(x) \cdot \nabla_p^{(2)}\varphi(\mathbf{z}_1, \mathbf{z}_2) = \mathbf{z}_2 \cdot (\mathbf{F}^\top\mathbf{L}_x\mathbf{F})\mathbf{z}_1 \quad \text{for all} \quad \mathbf{z}_1, \mathbf{z}_2 \in \mathcal{Z}. \tag{35.8}$$

This result is obtained by taking the inner product of (35.7) with $\mathbf{n}(x)$ and then using (35.6) and (35.7).

Remark 1: If $\varphi : \mathcal{P} \to \mathcal{E}$, with $\mathcal{P} \subset \mathbf{R}^{n-1}$, is a coordinate representation of $\mathcal{S} := \mathrm{Rng}\,\varphi$ as explained in Remark 2 of Sect. 32 and Remark 1 of Sect.33, then (35.8) gives

$$\mathbf{n}(x) \cdot \varphi_{,j,k}(p) = \varphi_{,j}(p) \cdot \mathbf{L}_x\varphi_{,k}(p) \quad \text{for all} \quad j, k \in (k-1)^], \tag{35.9}$$

and from (35.4) we get

$$(\mathbf{n} \circ \varphi)_{,j}(p) = -\mathbf{L}_x \varphi_{,j}(p) \qquad \text{for all} \quad j \in (k-1)^] \ . \blacksquare \tag{35.10}$$

Remark 2: One can prove that if a surface $\mathcal{S}$ has a parametric representation, it is orientable. For example, let $\varphi : \mathcal{P} \to \mathcal{E}$, where $\mathcal{P}$ is an open subset of $\mathbf{R}^2$, be a coordinate representation of a surface $\mathcal{S} := \operatorname{Rng}\varphi$ in a 3-dimensional Euclidean space $\mathcal{E}$. Then $(\varphi_{,1}(p), \varphi_{,2}(p))$ is a basis of $\mathcal{T}_{\varphi(p)}$ and, if $\times : \mathcal{V}^2 \to \mathcal{V}$ is a cross product on $\mathcal{V}$, then $\varphi_{,1}(p) \times \varphi_{,2}(p)$ is non-zero and normal to $\mathcal{T}_{\varphi(p)}$. Hence

$$\mathbf{n} \circ \varphi := \frac{\varphi_{,1} \times \varphi_{,2}}{|\varphi_{,1} \times \varphi_{,2}|} \tag{35.11}$$

defines a surface normal $\mathbf{n}$ on $\mathcal{S}$. $\blacksquare$

The spectral values of the curving lineon $\mathbf{L}_x$ are called the **principal curvatures** of the oriented surface at x. The spectral spaces of $\mathbf{L}_x$ are said to define the **principal directions** in $\mathcal{T}_x$. The **mean curvature** H_x at x is defined by

$$H_x := \frac{1}{n-1} \operatorname{tr} \mathbf{L}_x \tag{35.12}$$

and the **Gauss curvature** K_x by

$$K_x := \det \mathbf{L}_x. \tag{35.13}$$

Of course, H_x is the mean value of the principal curvatures and K_x is the product of the principal curvatures, counted with appropriate multiplicity.

It is useful to consider, with $\mathbf{L}_x \in \operatorname{Sym}(\mathcal{T}_x)$, also the corresponding quadratic form $\overline{\mathbf{L}}_x := \mathbf{L}_x \circ (\mathbf{1}_{\mathcal{T}_x}, \mathbf{1}_{\mathcal{T}_x})$ (see Def.1 of Sect.27 in Vol.I). A point x on a surface $\mathcal{S}$ of class C^2 is called:

(i) an **elliptic point** if $\overline{\mathbf{L}}_x$ is single-signed and non-degenerate (for $n = 3$ this is equivalent to $\mathbf{K}_x > 0$),

(ii) a **hyperbolic point** if $\overline{\mathbf{L}}_x$ is non-degenerate but not single-signed (for $n = 3$ this is equivalent to $\mathbf{K}_x < 0$),

(iii) a **parabolic point** if $\overline{\mathbf{L}}_x$ is degenerate but not zero (for $n = 3$ this is equivalent to $\mathbf{K}_x = 0, \mathbf{H}_x \neq 0$),

(iv) an **umbilic point** if $\mathbf{L}_x = \kappa\, \mathbf{1}_{\mathcal{T}_x}$, for some $\kappa \in \mathbf{R}$,

(v) a **flat point** if $\mathbf{L}_x = 0$.

The principal curvatures and the mean curvature change sign when the orientation of $\mathcal{S}$ is changed. If n is even, the Gauss curvature is independent of orientation. The concepts (i)–(v) above do not depend on the choice of orientation.

Let $p : J \to \mathcal{E}$ be an arclength-parametrization of a regular path of class C^2 on a surface $\mathcal{S}$ of class C^2. If we apply (35.8) to p instead of φ, we obtain

$$\mathbf{n}(p(s)) \cdot p^{\bullet\bullet}(s) = p^{\bullet}(s) \cdot \mathbf{L}_{p(s)} p^{\bullet}(s) \quad \text{for all} \quad s \in J. \tag{35.14}$$

If the curvature $\kappa(s) = |p^{\bullet\bullet}(s)|$ is not zero at a given $s \in J$, we have $p^{\bullet\bullet}(s) = \kappa(s)\mathbf{v}(s)$, where $\mathbf{v}(s)$ is the principal normal. Using (35.5) and (34.7), we see that (35.14) is equivalent to

$$\kappa(s)|\mathbf{n}(p(s)) \cdot \mathbf{v}| = |p^{\bullet}(s) \cdot \mathbf{L}_{p(s)} p^{\bullet}(s)| = \kappa_{N}(s), \tag{35.15}$$

where $\kappa_{N}(s)$ is the normal curvature at s. This result has the following geometric interpretation (Theorem of Meusnier).

Proposition 4: *Let $\mathcal{S}$ be a surface of class C^2 and $\mathbf{u}$ a unit vector tangent to $\mathcal{S}$ at $x \in \mathcal{S}$ such that $\mathbf{u} \cdot \mathbf{L}_x \mathbf{u} \neq 0$. Then the circles of curvature of all paths on $\mathcal{S}$ through x with unit tangent $\mathbf{u}$ lies on a sphere. This sphere is tangent to the surface at x and has radius $\rho_{_N} := \dfrac{1}{\mathbf{u} \cdot \mathbf{L}_x \mathbf{u}}$. If a path has pricipal normal $\mathbf{v}$ at x, with $\mathbf{v} \notin \mathcal{T}_x$, then its circle of curvature is the intersection of the sphere with plane through x with direction space $\mathrm{Lsp}\{\mathbf{u}, \mathbf{v}\}$. Its radius is $\rho := |\cos\theta|\rho_{_N}$, where θ is the angle between $\mathbf{v}$ and the surface normal $\mathbf{n}(x)$.*

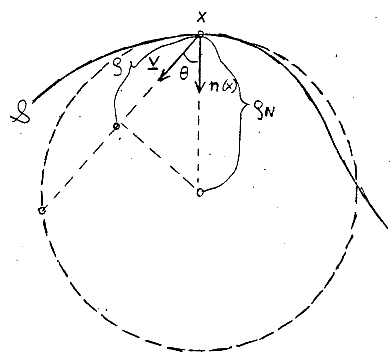

Figure 2: circles of curvature

We note that, using (35.4) and the Chain Rule, we obtain

$$(n \circ p)^{\bullet}(s) = -\mathbf{L}_{p(s)} p^{\bullet}(s) \qquad \text{for all} \quad s \in J \tag{35.16}$$

The path is called a **line of curvature** if $p^{\bullet}(s)$ is a spectral vector of $\mathbf{L}_{p(s)}$ for all $s \in J$, i.e. if one can determine a function $\gamma \in \mathrm{Map}(J, \mathbf{R})$ such that

$$\mathbf{L}_{p(s)} p^{\bullet}(s) = \gamma(s) p^{\bullet}(s) \quad \text{for all} \tag{35.17}$$

By (35.16), this condition is equivalent to

$$(\mathbf{n} \circ p)^{\bullet} + \gamma p^{\bullet} = 0 \quad \text{valuewise} \tag{35.18}$$

(Formula of Olinde Rodringues).

By (35.14) and (35,17), we have $(\mathbf{n} \circ p) \cdot p^{\bullet\bullet} = \gamma = \pm\kappa_{_N}$.

The path is called an **asymptotic line** if

$$p^{\bullet}(s) \cdot \mathbf{L}_{p(s)} p^{\bullet}(s) = 0 \quad \text{for all} \quad s \in J. \tag{35.19}$$

By (35.14) and (35.16), this condition is equivalent to each of the following:

$$(\mathbf{n} \circ p) \cdot p^{\bullet\bullet} = 0, \quad (\mathbf{n} \circ p)^{\bullet} \cdot p^{\bullet} = 0, \quad \kappa_{_N} = 0. \tag{35.20}$$

Of course, for p to define an asymptotic line it is necessary that $\mathrm{Rng}\, p$ contains no elliptic points.

The concepts of line of curvature and asymptotic line do not depend on the orientation of $\mathcal{S}$ and are invariant under reversal of path.

Note: What we call the curving Lineon $\mathbf{L}_x$ is often called the *Weingarten map*, and the associated symmetric bilinear form on $\mathcal{T}_x$ the *second fundamental form*. The *first fundamental form* is merely the inner product on

$\mathcal{T}_x$. ∎

36. Implicitly defined Manifolds.

Many, if not most, imbedded manifolds that occur in applications can be defined implicitly by an equation $\mathbf{h}(x) = \mathbf{0}$. To make this precise, assume that $\mathcal{Z}$ is a linear space of dimension m, $\mathcal{D}$ an open subset of $\mathcal{E}$ and

$$\mathbf{h} : \mathcal{D} \to \mathcal{Z}$$

a mapping of class C^r $(r \in \mathbb{N}^\times)$. To say that the subset $\mathcal{S}$ of $\mathcal{E}$ is defined implicitly by the equation

$$?\ x \in \mathcal{E},\ \ \mathbf{h}(x) = \mathbf{0}$$

means that $\mathcal{S}$ is the set of all solutions, i.e.

$$\mathcal{S} := \mathbf{h}^<(\{\mathbf{0}\}) = \{x \in \mathcal{D} \mid \mathbf{h}(x) = \mathbf{0}\}. \tag{36.1}$$

Theorem On Implicily Defined Manifolds: *The set $\mathcal{S}$ defined by (36.1) is a manifold of class C^r if $\nabla_x\mathbf{h} \in \mathrm{Lin}\,(\mathcal{V}, \mathcal{Z})$ is surjective for all $x \in \mathcal{S}$. In this case $\mathcal{S}$ has dimension $n - m$ and its tangent spaces are given by*

$$\mathcal{T}_x = \mathrm{Null}\,\nabla_x\mathbf{h} \quad \text{for all} \quad x \in \mathcal{S}. \tag{36.2}$$

Proof: We must construct, for each $x \in \mathcal{S}$, a mapping π_x that satisfies the axioms (S_1)–(S_4) for a local curving. Let $x \in \mathcal{S}$ be given. We define $\mathcal{T}_x$ by (36.2) and denote the symmetric idempotent with range $\mathcal{T}_x$ by $\mathbf{E}_x \in \mathrm{Lin}\,\mathcal{V}$. We then define $\mathbf{H}_x : \mathcal{D} \to \mathcal{T}_x \times \mathcal{Z}$ by

$$\mathbf{H}_x(y) := (\mathbf{E}_x(y - x), \mathbf{h}(y)) \quad \text{for all} \quad y \in \mathcal{D}.$$

It is clear that $\mathbf{H}_x$ is of class C^r and that $\nabla_x\mathbf{H}_x \in \mathrm{Lin}\,(\mathcal{V}, \mathcal{T}_x \times \mathcal{Z})$ is given by

$$(\nabla_x\mathbf{H}_x)\mathbf{v} = (\mathbf{E}_x\mathbf{v}, (\nabla_x\mathbf{h})\mathbf{v}) \quad \text{for all} \quad \mathbf{v} \in \mathcal{V}.$$

Hence we have

$$\mathrm{Null}\,\nabla_x\mathbf{H}_x = \{\mathbf{v} \in \mathcal{V} \mid \mathbf{E}_x\mathbf{v} = \mathbf{0}, \mathbf{v} \in \mathrm{Null}\,\nabla_x\mathbf{h}\} = \mathcal{T}_x^\perp \cap \mathcal{T}_x = \{\mathbf{0}\}.$$

Therefore, $\nabla_x\mathbf{H}_x$ is injective. On the other hand, since $\nabla_x\mathbf{h}$ is surjective, we have $\dim\mathcal{T}_x = \dim\mathcal{V} - \dim\mathcal{Z} = n - m$ and hence $\dim\mathcal{V} = \dim(\mathcal{T}_x \times \mathcal{Z}) = n$. It follows that $\nabla_x\mathbf{H}_x$ is invertible, and hence, by the Local Inversion Theorem (see Sect.68 in Vo.I), that $\mathbf{H}_x$ is locally invertible near $x \in \mathcal{S}$. Since $\mathbf{H}_x(x) = (\mathbf{0}, \mathbf{0})$, this means that there is an open neighborhood $\mathcal{N}_x$ of $\mathbf{0}$ in $\mathcal{T}_x$, an open neighborhood $\mathcal{M}_x$ of $\underset{\sim}{0}$ in $\mathcal{Z}$, and a mapping

$$\widetilde{\pi}_x : \mathcal{N}_x \times \mathcal{M}_x \to \mathcal{E}$$

of class C^r such that $\widetilde{\pi}_x(\mathbf{0}, \mathbf{0}) = x$ and $\mathbf{H}_x(\widetilde{\pi}_x(\mathbf{u}, \mathbf{z})) = (\mathbf{u}, \mathbf{z})$, ie.

$$\mathbf{E}_x(\widetilde{\pi}_x(\mathbf{u}, \mathbf{z}) - x) = \mathbf{u} \quad , \quad \mathbf{h}(\widetilde{\pi}_x(\mathbf{u}, \mathbf{z})) = \mathbf{z} \tag{36.3}$$

for all $\mathbf{u} \in \mathcal{N}_x$ and all $\mathbf{z} \in \mathcal{M}_x$. The range $\mathcal{G}_x := \mathrm{Rng}\,\widetilde{\pi}_x$ is an open neighborhood of $x \in \mathcal{E}$.

We now define $\pi_x : \mathcal{N}_x \to \mathcal{E}$ by $\pi_x := \widetilde{\pi}_x(\cdot, \mathbf{0})$. It is clear that π_x is of class C^r and that $\pi_x(\mathbf{0}) = x$. Moreover, by $(36.3)_2$ we have $\mathbf{h}(\pi_x(\mathbf{u})) = \mathbf{0}$ for all $\mathbf{u} \in \mathcal{N}_x$ and hence $\nabla_x\mathbf{h}(\nabla_{\mathbf{u}}\pi_x) = \mathbf{0}$. It follows that $\mathrm{Rng}\,\nabla_{\mathbf{0}}\pi_x \subset \mathrm{Null}\,\nabla_x\mathbf{h} = \mathcal{T}_x$. Therefore π_x satisfies (S_1)–(S_2). Using $(36.3)_1$ with $\mathbf{z} := \mathbf{0}$, we see that π_x also satisfies (S_3). To prove that (S_4) is valid, we note that $y \in \mathcal{G}_x \cap \mathcal{S}$ if and only if $y = \widetilde{\pi}_x(\mathbf{u}, \mathbf{z})$ for some $\mathbf{u} \in \mathcal{N}_x$, $\mathbf{z} \in \mathcal{M}_x$ and $\mathbf{h}(y) = \mathbf{0}$. For this to be the case, we must have $\mathbf{0} = \mathbf{h}(y) = \mathbf{h}(\widetilde{\pi}_x(\mathbf{u}, \mathbf{z})) = \mathbf{z}$ and hence

$\mathbf{z} = \mathbf{0}$ and $y = \widetilde{\pi}_x(\mathbf{u}, \mathbf{0}) = \pi_x(\mathbf{u})$. Therefore $y \in \mathcal{G}_x \cap \mathcal{S}$ if and only if $y = \pi_x(\mathbf{u})$ for some $\mathbf{u} \in \mathcal{N}_x$, which means that $\operatorname{Rng} \pi_x = \mathcal{G}_x \cap \mathcal{S}$. ∎

The special case when $\mathcal{Z} := \mathbf{R}$ is of particular interest and described by the following result.

Proposition 1: *Let $h : \mathcal{D} \to \mathbb{R}$, where $\mathcal{D}$ is open in $\mathcal{E}$, be of class C^r ($r \in \mathbb{N}^\times$) and assume that $\nabla_x h \neq \mathbf{0}$ for all $x \in \mathcal{S}$. Then*

$$\mathcal{S} := h^{<}(\{0\}) = \{x \in \mathcal{D} \mid h(x) = 0 \ )$$

is an orientable surface of class C^r and

$$\mathbf{n}(x) := \frac{\nabla_x h}{|\nabla_x h|} \quad \text{for all} \quad x \in \mathcal{S} \tag{36.4}$$

defines a surface normal of class C^{r-1}.

Proof: We have $\nabla_x h \in \mathcal{V}^* \cong \mathcal{V}$. The range of $\nabla_x h$ is either $\{0\}$ or all of $\mathbf{R}$. Hence $\nabla_x h$ is surjective. By the Theorem on Implicitly Defined Manifolds, $\mathcal{S}$ is a surface of class C^r. The tangent spaces are given by

$$\mathcal{T}_x := \operatorname{Null} \nabla_x h = \{\mathbf{u} \in \mathcal{V} \mid \mathbf{u} \cdot \nabla_x h = 0\} = \{\nabla_x h\}^\perp \ . \tag{36.5}$$

It follows that $\mathcal{T}_x^\perp = \mathbf{R}\, \nabla_x h$ and hence that (36.4) defines a surface normal, which is of class C^{r-1}. ∎

Using the formulas (P6.3) and (P6.4) in Vol.I, it follows from (36.4) that

$$\nabla_x \mathbf{n} = \frac{1}{|\nabla_x h|} \big(\nabla_x^{(2)} h - \mathbf{n}(x) \otimes \nabla_x |\nabla h|\big)\big|_{\mathcal{T}_x}. \tag{36.6}$$

Now, by the Theorem on Symmetry of Second Gradients (see Sect.611 of Vol.I) $\nabla_x^{(2)} h$ is symmetric. Also, by (35.4) of Prop.3 in Sect.35, $\mathbf{L}_x = -\nabla_x \mathbf{n}\big|^{\mathcal{T}_x}$ is symmetric. Using these facts and $\mathcal{T}_x \cap \mathbf{R}\,\mathbf{n}(x) = \{\mathbf{0}\}$, it easily follows from (36.6) that

$$\big(\mathbf{n}(x) \otimes \nabla_x |\nabla h|\big)\big|_{\mathcal{T}_x} = \big(\nabla_x |\nabla h| \otimes \mathbf{n}(x)\big)\big|_{\mathcal{T}_x} = \mathbf{0} \ .$$

Hence we conclude from (36.6) that the curving lineon is given by

$$\mathbf{L}_x = -\frac{1}{|\nabla_x h|} \nabla_x^{(2)} h\big|_{\mathcal{T}_x}^{\mathcal{T}_x}. \tag{36.7}$$

Examples:

(A) Spheres : The sphere $\mathcal{S}$ of radius ρ centered at a point $c \in \mathcal{E}$ is defined by

$$\mathcal{S} := \{x \in \mathcal{E} \ \big| \ |x - c| = \rho\}.$$

If we define $h : \mathcal{E} \to \mathbf{R}$ by

$$h(x) := \rho^2 - (x - c)^{\cdot 2} \quad \text{for all} \quad x \in \mathcal{E} \tag{36.8}$$

then $\mathcal{S} = h^{<}(\{0\})$, and we can apply Prop.1. Since $\nabla_x h = -2(x - c)$ and hence $|\nabla h| = 2\rho \neq 0$ for all $x \in \mathcal{S}$, the sphere $\mathcal{S}$ is an orientable surface of class C^ω and

$$\mathbf{n}(x) := -\frac{1}{\rho}(x - c) \quad \text{for all} \quad x \in \mathcal{S} \tag{36.9}$$

defines a surface normal. The tangent spaces are given by

$$\mathcal{T}_x = \{\mathbf{u} \in \mathcal{V} \mid \mathbf{u} \cdot (x - c) = 0\}. \tag{36.10}$$

It follows from (36.9) that

$$\nabla_x \mathbf{n} = -\frac{1}{\rho}\, \mathbf{1}_{\mathcal{T}_x \subset \mathcal{V}}.$$

Hence, by (35.4) the curving lineon is $\mathbf{L}_x = \frac{1}{\rho}\mathbf{1}_{\mathcal{T}_x}$ and by (35.5) the curving tensor is given by

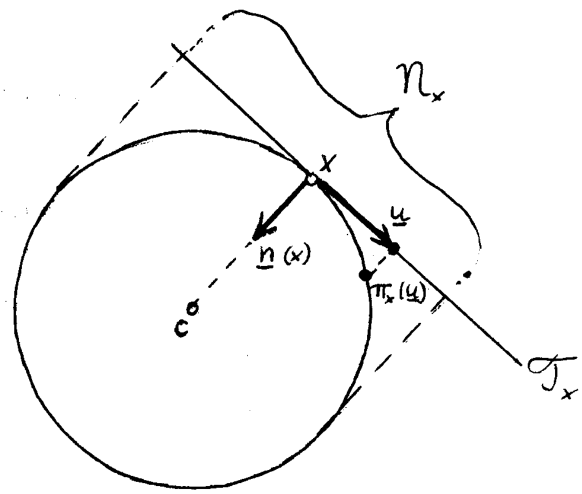

Figure 3: local curving for a sphere

$$\mathbf{\Lambda}_x(\mathbf{u}_1, \mathbf{u}_2) = -\frac{1}{\rho^2}(\mathbf{u}_1 \cdot \mathbf{u}_2)(x - c) \quad \text{for all} \quad (\mathbf{u}_1, \mathbf{u}_2) \in \mathcal{T}_x^2.$$

The only principal curvature is $\frac{1}{\rho}$, which is also the mean curvature. The Gauss curvature is $\frac{1}{\rho^{n-1}}$. Every point on $\mathcal{S}$ is umbilic. All paths on $\mathcal{S}$ are lines of curvature.

To determine the geodesics on $\mathcal{S}$ we use (34.11) to find that

$$p^{\bullet\bullet} = -\frac{1}{\rho^2}(p^\bullet \cdot p^\bullet)(p - c) = -\frac{1}{\rho^2}(p - c)$$

must hold for the arclength parametrizations p of the geodesics. Thus, the geodesics are those solutions of the differential equation

$$\rho^2 p^{\bullet\bullet} + (p - c) = \mathbf{0}$$

for which $\operatorname{Rng} p \subset \mathcal{S}$. It is easily seen that these solutions are of the form

$$p(s) = c + \rho\left(\cos(s/\rho)\,\mathbf{e}_1 + \sin(s/\rho)\,\mathbf{e}_2\right) \quad \text{for all} \quad s \in J,$$

where $(\mathbf{e}_1, \mathbf{e}_2)$ is an orthonormal pair in $\mathcal{V}$. These solutions represent the great circles on $\mathcal{S}$.

A local curving π_x of $\mathcal{S}$ at x must satisfy $(\pi_x(\mathbf{u}) - c)^{\cdot 2} = \rho^2$ and $\pi_x(\mathbf{u}) - (x + \mathbf{u}) \in \mathcal{T}_x^\perp$ for all $\mathbf{u}$ in the domain $\mathcal{N}_x$ of π_x. Since $\mathcal{T}_x^\perp$ is spanned by $(x - c)$, we conclude that

$$\pi_x(\mathbf{u}) = x + \mathbf{u} + \lambda(\mathbf{u})(x - c)$$

for a suitable $\lambda : \mathcal{N}_x \to \mathbf{R}$. Since $\mathbf{u} \cdot (x-c) = 0$, we obtain

$$\rho^2 = (\pi_x(\mathbf{u}) - c)^{\cdot 2} = (1+\lambda(\mathbf{u}))^2 (x-c)^{\cdot 2} + \mathbf{u}^{\cdot 2} = \rho^2(1+\lambda(\mathbf{u}))^2 + \mathbf{u}^{\cdot 2}$$

and hence that $1+\lambda(\mathbf{u}) = \frac{1}{\rho}\sqrt{\rho^2 - \mathbf{u}^{\cdot 2}}$. Therefore, we have

$$\pi_x(\mathbf{u}) = c + \frac{1}{\rho}\sqrt{\rho^2 - \mathbf{u}^{\cdot 2}}(x-c) + \mathbf{u}. \tag{36.11}$$

For the domain $\mathcal{N}_x$ of π_x we can take $\mathcal{N}_x := \{\mathbf{u} \in \mathcal{T}_x \mid \ |\mathbf{u}| < \rho\}$, i.e. the open ball of radius ρ in $\mathcal{T}_x$ (see Fig.1). We could have obtained the curving tensor $\mathbf{\Lambda}_x$ by differentiation form (36.8), but the method used above is much easier.

(B) Simple Ellipsoids : Let $c_1, c_2 \in \mathcal{E}$ and $\rho \in \mathbb{P}^\times$ be given such that $|c_1 - c_2| < 2\rho$ holds. Let $\mathcal{S}$ be the simple ellipsoid, with foci c_1 and c_2, given by

$$\mathcal{S} := \left\{ x \in \mathcal{E} \;\middle|\; |x - c_1| + |x - c_2| = 2\rho \right\} .$$

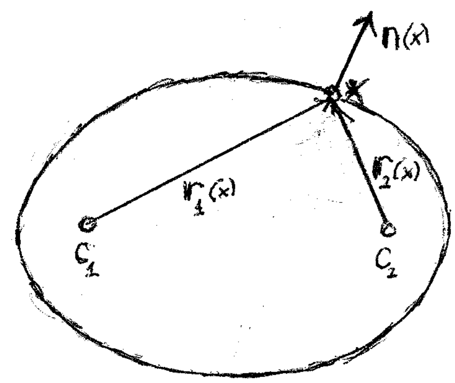

Figure 4: simple ellipsoid

If we define $h : \mathcal{E} \to \mathbf{R}$ by

$$h(x) := 2\rho - (|x-c_1| + |x-c_2|) \qquad \text{for all} \qquad x \in \mathcal{E} \ ,$$

we have $\mathcal{S} = h^{<}(\{0\})$ and we can apply Prop.1 above. We define $\mathbf{r}_1, \mathbf{r}_2 : \mathcal{E} \longrightarrow \mathcal{V}$ and $r_1,\ r_2 : \mathcal{E} \longrightarrow \mathbf{R}$ by

$$\mathbf{r}_1(x) := x - c_1 \ , \ \mathbf{r}_2(x) := x - c_2 \quad \text{ for all } \quad x \in \mathcal{E} \text{ and } r_1 := |\mathbf{r}_1| \ , \ r_2 := |\mathbf{r}_2| \tag{36.12}$$

so that

$$h = 2\rho - (r_1 + r_2) \ . \tag{36,13}$$

Using the differentiation rules of Problems 4 and 5 of Chap.6 of Vol.I, it follows from (36.13) that

$$\nabla h = -\left(\frac{\mathbf{r}_1}{r_1} + \frac{\mathbf{r}_2}{r_2}\right) \tag{36.14}$$

and

$$\nabla^{(2)} h = \frac{1}{r_1^3}\left(r_1^2 \mathbf{1}_{\mathcal{V}} - \mathbf{r}_1 \otimes \mathbf{r}_1\right) + \frac{1}{r_2^3}\left(r_2^2 \mathbf{1}_{\mathcal{V}} - \mathbf{r}_2 \otimes \mathbf{r}_2\right) \ , \tag{36.15}$$

where the domain of ∇h and $\nabla^{(2)} h$ is taken to be $\mathcal{E}\backslash\{c_1, c_2\}$ because h fails to be differentiable at c_1 and c_2. It follows from (36.14) that

$$|\nabla_x h| = \Big|\frac{\mathbf{r}_1(x)}{r_1(x)} + \frac{\mathbf{r}_2(x)}{r_2(x)}\Big| = \left(2 + 2\frac{\mathbf{r}_1(x)\cdot\mathbf{r}_2(x)}{r_1(x)r_2(x)}\right)^{1/2} \quad \text{for all} \quad x \in \mathcal{S}\ . \tag{36.16}$$

Since this is not zero, it follows from Prop.1 above that the ellipsoid $\mathcal{S}$ is an orientable surface of class C^2. By (36.4) and (36.14) and (36.16), a surface normal is given by

$$\mathbf{n}(x) = \Big|\frac{\mathbf{r}_1(x)}{r_1(x)} + \frac{\mathbf{r}_2(x)}{r_2(x)}\Big|^{-1/2} \left(\frac{\mathbf{r}_1(x)}{r_1(x)} + \frac{\mathbf{r}_2(x)}{r_2(x)}\right) \qquad \text{for all} \quad x \in \mathcal{S} \tag{36.17}$$

This result shows that the direction of the normal $\mathbf{n}(x)$ bisects the directions of the vectors $\mathbf{r}_1(x)$ and $\mathbf{r}_2(x)$ (see Figure 4).

In view of (36.15) and (36.16) we find that the curving lineon (36.7) for the ellipsoid is given by the complicated formula

$$\mathbf{L}_x = \left(\Big|\frac{\mathbf{r}_1}{r_1} + \frac{\mathbf{r}_2}{r_2}\Big|^{-1/2}\Big(\frac{1}{r_1^3}\left(r_1^2\mathbf{1}_{\mathcal{V}} - \mathbf{r}_1\otimes\mathbf{r}_1\right) + \frac{1}{r_2^3}\left(r_2^2\mathbf{1}_{\mathcal{V}} - \mathbf{r}_2\otimes\mathbf{r}_2\right)\Big)\right)(x)\Big|_{\mathcal{T}_x}^{\mathcal{T}_x} \tag{36.18}$$

Remark: When $c_1 = c_2$, the example "Simple Ellipsoids" reduces to the case of the example "Spheres" above. These simple ellipsoids are actually surfaces of revolution and can also be analized with the method described in the next section. ∎

(C) Orthogonal groups : Let a genuine inner product space $\mathcal{U}$ be given. Its orthogonal group is given by

$$\mathrm{Orth}\,\mathcal{U} := \{\mathbf{R} \in \mathrm{Lin}\,\mathcal{U} \mid \mathbf{R}^\top\mathbf{R} = \mathbf{1}_{\mathcal{U}}\}. \tag{36.19}$$

(See Sect.43 of Vol.I.)

Now, Lin $\mathcal{U}$ has a natural inner product, defined by

$$\mathbf{L}\cdot\mathbf{M} := \mathrm{tr}\,(\mathbf{L}^\top\mathbf{M}) \quad \text{for all} \quad \mathbf{L}, \mathbf{M} \in \mathrm{Lin}\ \mathcal{U},$$

and hence can be regarded as a genuine Euclidean space. (See Sect.44 of Vol.I.) Also, Sym $\mathcal{U}$ is a subspace of Lin $\mathcal{U}$. If we define $\mathbf{h} : \mathrm{Lin}\ \mathcal{U} \to \mathrm{Sym}\ \mathcal{U}$ by

$$\mathbf{h}(\mathbf{L}) := \mathbf{L}^\top\mathbf{L} - \mathbf{1}_{\mathcal{U}} \quad \text{for all} \quad \mathbf{L} \in \mathrm{Lin}\ \mathcal{U}, \tag{36.20}$$

then $\mathrm{Orth}\,\mathcal{U} = \mathbf{h}^{<}\{\mathbf{0}\}$. We will apply the Theorem on Implicitly Defined Manifolds with $\mathcal{E} = \mathcal{V} := \mathrm{Lin}\ \mathcal{U}$, $\mathcal{Z} := \mathrm{Sym}\ \mathcal{U}$, and $\mathcal{S} := \mathrm{Orth}\,\mathcal{U}$. It follows from (36.20) that

$$(\nabla_{\mathbf{L}}\mathbf{h})\mathbf{M} = \mathbf{L}^\top\mathbf{M} + \mathbf{M}^\top\mathbf{L} = \mathbf{L}^\top\mathbf{M} + (\mathbf{L}^\top\mathbf{M})^\top \in \mathrm{Sym}\ \mathcal{U} \tag{36.21}$$

for all $\mathbf{L}, \mathbf{M} \in \mathrm{Lin}\ \mathcal{U}$. Let $\mathbf{R} \in \mathrm{Orth}\ \mathcal{U}$ be given. It follows from (36.21) that $\nabla_{\mathbf{R}}\mathbf{h}(\frac{1}{2}\mathbf{RS}) = \mathbf{S}$ for all $\mathbf{S} \in \mathrm{Sym}\ \mathcal{U}$, which prove that $\nabla_{\mathbf{R}}\mathbf{h}$ is surjective. By the Theorem on Implicitly Defined Manifolds, Orth $\mathcal{U}$ is a manifold of class C^ω imbedded in Lin $\mathcal{U}$. The tangent space $\mathcal{T}_{\mathbf{R}}$ is given by

$$\mathcal{T}_{\mathbf{R}} = \mathrm{Null}\,\nabla_{\mathbf{R}}\mathbf{h} = \{\mathbf{L} \in \mathrm{Lin}\ \mathcal{U} \mid \mathbf{R}^\top\mathbf{L} \in \mathrm{Skew}\ \mathcal{U}\}.$$

Since $\mathbf{R}^\top = \mathbf{R}^{-1}$, we have $\mathbf{R}^\top\mathbf{L} \in \mathrm{Skew}\,\mathcal{U}$ if and only if $\mathbf{L} = \mathbf{RW}$ for some $\mathbf{W} \in \mathrm{Skew}\,\mathcal{U}$. It follows that

$$\mathcal{T}_{\mathbf{R}} = \mathbf{R}\,\mathrm{Skew}\ \mathcal{U} = \{\mathbf{RW} \mid \mathbf{W} \in \mathrm{Skew}\ \mathcal{U}\}. \tag{36.22}$$

The orthogonal supplement of $\mathcal{T}_{\mathbf{R}}$ is $\mathcal{T}_{\mathbf{R}}^{\perp} = \mathbf{R}\,\mathrm{Sym}\;\mathcal{U}$. In particular, we have $\mathcal{T}_{\mathbf{1}_{\mathcal{T}_x}} = \mathrm{Skew}\;\mathcal{U}$, $\mathcal{T}_{\mathbf{1}_{\mathcal{T}_x}}^{\perp} = \mathrm{Sym}\;\mathcal{U}$. If $m := \dim \mathcal{U}$, then the dimension of $\mathrm{Orth}\,\mathcal{U}$ is the same as that of Skew $\mathcal{U}$, namely $\frac{m(m-1)}{2}$ (see (27.8) in Vol.I).

The idempotent mapping $\mathbf{E} : \mathrm{Orth}\,\mathcal{U} \to \mathrm{Lin}\;(\mathrm{Lin}\;\mathcal{U})$, as defined by Def.2 of Sect.33 is easily seen to be given by

$$\mathbf{E}_{\mathbf{R}}\mathbf{L} = \mathbf{R}\frac{1}{2}(\mathbf{R}^{\top}\mathbf{L} - (\mathbf{R}^{\top}\mathbf{L})^{\top}) \in \mathbf{R}\,\mathrm{Skew}\;\mathcal{U} = \mathcal{T}_{\mathbf{R}}$$

for all $\mathbf{R} \in \mathrm{Orth}\,\mathcal{U}$ and all $\mathbf{L} \in \mathrm{Lin}\;\mathcal{U}$. Differentiating with respect to $\mathbf{R}$, we find

$$(\nabla_{\mathbf{R}}\,\mathbf{E})(\mathbf{T},\mathbf{L}) = -\frac{1}{2}(\mathbf{T}\,\mathbf{L}^{\top}\mathbf{R} + \mathbf{R}\,\mathbf{L}^{\top}\mathbf{T}) \quad \text{for all} \quad \mathbf{T} \in \mathcal{T}_{\mathbf{R}}.$$

Using the Idempotent Mapping Theorem of Sect.33 and (36.22), we see that the curving tensor $\mathbf{\Lambda}_{\mathbf{R}}$ is described by

$$\mathbf{\Lambda}_{\mathbf{R}}(\mathbf{R}\mathbf{W}_1, \mathbf{R}\mathbf{W}_2) = \mathbf{R}\frac{1}{2}(\mathbf{W}_1\mathbf{W}_2 + \mathbf{W}_2\mathbf{W}_1) \in \mathbf{R}\,\mathrm{Sym}\;\mathcal{U} = \mathcal{T}_{\mathbf{R}}^{\perp}, \tag{36.23}$$

for all $\mathbf{W}_1, \mathbf{W}_2 \in \mathrm{Skew}\;\mathcal{U}$.

By (34.11), the arclength parameterization $\mathbf{P} : J \to \mathrm{Orth}\,\mathcal{U}$ of a geodesic must satisfy the differential equation $\mathbf{P}^{\bullet\bullet}(s) = \mathbf{\Lambda}_{\mathbf{P}(s)}(\mathbf{P}^{\bullet}(s), \mathbf{P}^{\bullet}(s))$. Differentiating $\mathbf{P}^{\top}\mathbf{P} = \mathbf{1}_{\mathcal{U}}$, we get $\mathbf{P}^{\bullet\top}\mathbf{P} + \mathbf{P}^{\top}\mathbf{P}^{\bullet} = \mathbf{0}$, i.e. $\mathbf{P}^{\top}\mathbf{P}^{\bullet} \in \mathrm{Skew}\;\mathcal{U}$. Since $\mathbf{P}^{\top}(s) = \mathbf{P}^{-1}(s)$ for all $s \in J$, there is a process $\mathbf{W} : J \to \mathrm{Skew}\;\mathcal{U}$ such that $\mathbf{P}^{\bullet} = \mathbf{P}\mathbf{W}$. In view of (36.23) together with $\mathbf{P}^{\bullet\bullet}(s) = \mathbf{\Lambda}_{\mathbf{P}(s)}(\mathbf{P}^{\bullet}(s), \mathbf{P}^{\bullet}(s))$, the differential equation for $\mathbf{P}$ becomes $\mathbf{P}^{\bullet\bullet} = \mathbf{P}\mathbf{W}^2$. On the other hand, differentiation of $\mathbf{P}^{\bullet} = \mathbf{P}\mathbf{W}$ gives $\mathbf{P}^{\bullet\bullet} = \mathbf{P}^{\bullet}\mathbf{W} + \mathbf{P}\mathbf{W}^{\bullet} = \mathbf{P}\mathbf{W}^2 + \mathbf{P}\mathbf{W}^{\bullet}$. We conclude that $\mathbf{P}\mathbf{W}^{\bullet} = \mathbf{0}$ and hence, since $\mathbf{P}$ has invertible values, that $\mathbf{W}$ is constant. We denote the value of $\mathbf{W}$ by $\mathbf{W}_0 \in \mathrm{Skew}\;\mathcal{U}$. Then $\mathbf{P}$ is a solution of the differential equation $\mathbf{P}^{\bullet} = \mathbf{P}\mathbf{W}_0$. The general solution is

$$\mathbf{P} = \mathbf{R}_0 \exp_{\mathcal{U}} \circ (\iota\mathbf{W}_0), \quad \text{where} \quad \mathbf{R}_0 \in \mathrm{Orth}\,\mathcal{U}\;. \tag{36.24}$$

(See Sect.612 of Vol.I.) Since $\mathbf{P}$ is an arclength parameterization, the condition that $1 = |\mathbf{P}^{\bullet}| = |\mathbf{P}\mathbf{W}_0|$ requires

$$1 = |\mathbf{W}_0|^2 = \mathrm{tr}(\mathbf{W}_0^{\top}\mathbf{W}_0) = -\mathrm{tr}(\mathbf{W}_0^2)\;. \tag{36.25}$$

Now let $\mathbf{R} \in\in \mathrm{Orth}$ be given. By Def.1 of Sect.31, a local curving $\pi_{\mathbf{R}}$ of $\mathrm{Orth}\,\mathcal{U}$ must satisfy $\pi_{\mathbf{R}}(\mathbf{U}) \in \mathrm{Orth}\,\mathcal{U}$ and $\pi_{\mathbf{R}}(\mathbf{U}) - (\mathbf{R} - \mathbf{U}) \in \mathcal{T}_{\mathbf{R}}^{\perp} = \mathbf{R}\,\mathrm{Sym}\;\mathcal{U}$ for all $\mathbf{U}$ in the domain $\mathcal{N}_{\mathbf{R}} \subset \mathbf{R}\,\mathrm{Skew}\mathcal{U}$ of $\pi_{\mathbf{R}}$. Let a local curving $\pi_{\mathbf{1}_{\mathcal{U}}} : \mathcal{N}_{\mathbf{1}_{\mathcal{U}}} \to \mathrm{Lin}\;\mathcal{U}$ at $\mathbf{1}_{\mathcal{U}} \in \mathrm{Orth}\;\mathcal{U}$ be given. It it is easily seen that a local curving at $\mathbf{R} \in \mathrm{Orth}\,\mathcal{U}$ can then be defined by the formula

$$\pi_{\mathbf{R}}(\mathbf{R}\mathbf{W}) := \mathbf{R}\,\pi_{\mathbf{1}_{\mathcal{U}}}(\mathbf{W}) \qquad \text{for all} \quad \mathbf{W} \in \mathcal{N}_{\mathbf{1}_{\mathcal{U}}} \subset \mathrm{Skew}\mathcal{U} \tag{36.26}$$

Hence it is suffices to find a formula for $\pi_{\mathbf{1}_{\mathcal{U}}}$.

Let $\mathbf{W} \in \mathcal{N}_{\mathbf{1}_{\mathcal{U}}}$ be given and put $\mathbf{Q} := \pi_{\mathbf{1}_{\mathcal{U}}}(\mathbf{W})$. then

$$\mathbf{Q} - (\mathbf{1}_{\mathcal{U}} + \mathbf{W}) =: \mathbf{S} \in \mathrm{Sym}\;\mathcal{U}.$$

Hence we have

$$\mathbf{0} = \mathbf{S} - \mathbf{S}^{\top} = \mathbf{Q} - \mathbf{Q}^{\top} - (\mathbf{W} - \mathbf{W}^{\top}) = \mathbf{Q} - \mathbf{Q}^{\top} - 2\mathbf{W}.$$

Multiplying this equation by $\mathbf{Q}$ and using $\mathbf{Q}^{\top}\mathbf{Q} = \mathbf{1}_{\mathcal{U}} = \mathbf{Q}\mathbf{Q}^{\top}$, we get

$$\mathbf{Q}^2 - 2\mathbf{Q}\mathbf{W} - \mathbf{1}_{\mathcal{U}} = \mathbf{0}\;. \tag{36.27}$$

We now define the domain of of $\pi_{\mathbf{1}_{\mathcal{U}}}$ by

$$\mathcal{N}_{\mathbf{1}_{\mathcal{U}}} := \{\mathbf{W} \in \mathrm{Skew}\;\mathcal{U} \mid \mathbf{1}_{\mathcal{U}} + \mathbf{W}^2 \in \mathrm{Pos}^{+}\mathcal{U}\}\;. \tag{36.28}$$

In the case when $\mathbf{W} \in \mathcal{N}_{\mathbf{1}_\mathcal{U}}$, the quadratic equation (36.27) can be solved for $\mathbf{Q}$ using the lineonic square root defined in Sect.85 of Vol.I. The result is $\pi_{\mathbf{1}_\mathcal{U}}(\mathbf{W}) = \mathbf{Q} = \mathbf{W} \pm \sqrt{\mathbf{W}^2 + \mathbf{1}_\mathcal{U}}$. Since $\pi_{\mathbf{1}_\mathcal{U}}(\mathbf{0}) = \mathbf{1}_\mathcal{U}$, we must use the + sign. Thus we have

$$\pi_{\mathbf{1}_\mathcal{U}}(\mathbf{W}) = \mathbf{W} + \sqrt{\mathbf{W}^2 + \mathbf{1}_\mathcal{U}} \qquad \text{for all} \qquad \mathbf{W} \in \mathcal{N}_{\mathbf{1}_\mathcal{U}}. \tag{36.29}$$

37. Graphs and Surfaces of Revolution

In this section, we continue to assume that is a genuine Euclidean space $\mathcal{E}$ is given. We also assume that a flat $\mathcal{F}$ in $\mathcal{E}$ and an open subset $\mathcal{P}$ of $\mathcal{F}$ are given. We denote the direction space of $\mathcal{F}$ by $\mathcal{U}$. We also assume that a function $f : \mathcal{P} \to \mathbf{R}$ is given.

In the case when $\dim \mathcal{F} = \dim \mathcal{U} = n-1$, i.e. when $\mathcal{F}$ is a hyperplane, we seclect $\mathbf{w} \in \mathcal{U}^\perp$ such that $|\mathbf{w}| = 1$. Then

$$\mathcal{S} := \{p + f(p)\mathbf{w} \mid p \in \mathcal{P}\} \tag{37.1}$$

is called the **graph** in $\mathcal{E}$ of the function f.

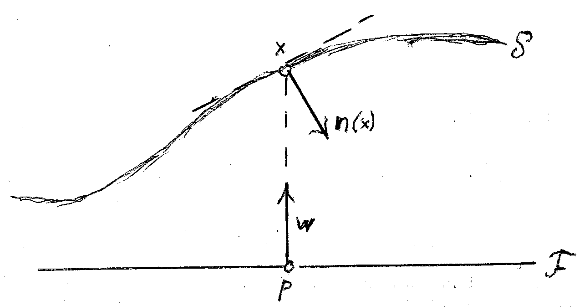

Figure 5: graph

In the case when $\dim \mathcal{F} = \dim \mathcal{U} < n-1$ and $\operatorname{Rng} f \subset \mathbb{P}^\times$, we put

$$\mathcal{S} := \{p + f(p)\mathbf{w} \mid p \in \mathcal{P},\ \mathbf{w} \in \operatorname{Usph} \mathcal{U}^\perp\} \tag{37.2}$$

and call it the **surface of revolution** about $\mathcal{P}$ generated by f.

Proposition 1: *If f is of class C^r ($r \in \mathbb{N}^\times$) then the graph (37.1) of f, or the surface of revolution (37.2) generated by f, is an orientable surface of class C^r and*

$$\mathbf{n}(x) := \frac{\nabla_p f - \mathbf{w}}{\sqrt{1 + |\nabla_p f|^2}} \quad \text{with} \quad x := p + f(p)\mathbf{w} \in \mathcal{S} \tag{37.3}$$

for all $p \in \mathcal{P}$ (and all $\mathbf{w} \in \operatorname{Usph} \mathcal{U}^\perp$ if $\mathcal{S}$ is a surface of revolution) defines a surface normal of class C^{r-1}.

Proof: Consider the mapping $\varepsilon : \mathcal{E} \to \mathcal{E}$ characterised by

$$\varepsilon(p + \mathbf{v}) := p \quad \text{for all} \quad p \in \mathcal{F},\ \mathbf{v} \in \mathcal{U}^\perp . \tag{37.4}$$

We have $\operatorname{Rng}\varepsilon = \mathcal{F}$ and it is easily seen that $\mathbf{E} := \nabla\varepsilon \in \operatorname{Lin}\,\mathcal{V}$ is a symmetric idempotent with $\operatorname{Rng}\mathbf{E} = \mathcal{U}$. (See Problem 3 for Chapt.3 in Vol.I.)

We now define a function $h : \mathcal{P} + \mathcal{U}^{\perp} \to \mathbf{R}$ as follows: When $\mathcal{S}$ is the graph of f as given by (37.1), we put

$$h(x) := f(\varepsilon(x)) - (x - \varepsilon(x)) \cdot \mathbf{w} \qquad \text{for all} \quad x \in \mathcal{P} + \mathcal{U}^{\perp}\ . \tag{37.5}$$

When $\mathcal{S}$ is the surface of revolution generated by f as given by (37.2), we put

$$h(x) := f(\varepsilon(x))^2 - (x - \varepsilon(x))^{\cdot 2} \qquad \text{for all} \quad x \in \mathcal{P} + \mathcal{U}^{\perp} \tag{37.6}$$

In either case, it is easily seen that $\mathcal{S} = h^{<}(\{0\})$. Let $x \in \mathcal{S}$ be given and put $p := \varepsilon(x)$. In the case when $\mathcal{S}$ is the graph of f, differentiation of (37.5) gives

$$\nabla_x h = \mathbf{E}^{\top}\nabla_p f - (\mathbf{1}_{\mathcal{V}} - \mathbf{E})^{\top}\mathbf{w}$$

and hence, since $\mathbf{E}^{\top} = \mathbf{E}$, $\mathbf{E}|_{\mathcal{U}} = \mathbf{1}_{\mathcal{U}\subset\mathcal{V}}$ and $(\mathbf{1}_{\mathcal{V}} - \mathbf{E})|_{\mathcal{U}^{\perp}} = \mathbf{1}_{\mathcal{U}^{\perp}\subset\mathcal{V}}$, we have

$$\nabla_x h = \nabla_p f - \mathbf{w} \ \text{ with } \ x = p + f(p)\mathbf{w}\ . \tag{37.7}$$

When $\mathcal{S}$ is the surface of revolution generated by f, differentiation of (37.6) gives

$$\nabla_x h = 2f(p)\,\mathbf{E}^{\top}\nabla_p f - 2(\mathbf{1}_{\mathcal{V}} - \mathbf{E})^{\top}(x - \varepsilon(x))$$

and hence, since $x - \varepsilon(x) \in \mathcal{U}^{\perp}$,

$$\nabla_x h = 2f(p)(\nabla_p f - \mathbf{w}), \quad \text{with} \quad x = p + f(x)\mathbf{w}, \ \ \mathbf{w} \in \mathcal{U}^{\perp}\ . \tag{37.8}$$

In either case, we have $\nabla_x h \neq \mathbf{0}$ for all $x \in \mathcal{S}$. Hence we can apply Prop.1 of Sect.36 to conclude that $\mathcal{S}$ is an orientable surface of class C^r with a surface normal given by (37.3). ▮

The curving lineon of a graph or a surface of revolution is described by the following result:

Proposition 2: *Assume that f is of class C^2 and that $x = p + f(p)\mathbf{w}$, with $p \in \mathcal{P}$ and $\mathbf{w} \in \mathcal{U}^{\perp}$, is a point of the surface $\mathcal{S}$ defined by (37.1) or (37.2). Consider the linear mapping $\mathbf{J}_x \in \operatorname{Lin}(\mathcal{U}, \mathcal{T}_x)$ defined by*

$$\mathbf{J}_x := \mathbf{1}_{\mathcal{U}\subset\mathcal{V}} + \mathbf{w} \otimes \nabla_p f \tag{37.9}$$

and the lineon $\mathbf{B}_x \in \operatorname{Lin}\,\mathcal{U}$ defined by

$$\mathbf{B}_x := \frac{1}{\sqrt{1 + |\nabla_p f|^2}^{\,3}} \left(\nabla_p f \otimes (\nabla_p^{(2)} f)\nabla_p f - (1 + |\nabla_p f|^2)\nabla_p^{(2)} f\right). \tag{37.10}$$

The curving lineon $\mathbf{L}_x \in \operatorname{Lin}\,\mathcal{T}_x$ then satisfies

$$\mathbf{L}_x\mathbf{J}_x = \mathbf{J}_x\mathbf{B}_x. \tag{37.11}$$

If $\mathcal{S}$ is a graph defined by (37.1), then $\mathbf{J}_x$ is invertible. the principal curvatures are the spectral values of $\mathbf{B}_x$ and the principal direction spaces are $\mathbf{J}_{x>}(\mathcal{W})$, where $\mathcal{W}$ is a spectral space of $\mathbf{B}_x$.

If $\mathcal{S}$ is a surface of revolution defined by (37.2), then $(\,\mathcal{U}^{\perp} \cap \mathbf{w}^{\perp}, \operatorname{Rng}\mathbf{J}_x)$ is a orthogonal decomposition of $\mathcal{T}_x$ with $\mathbf{L}_x$-invariant terms. A principal curvature of multiplicity at least $(n-1) - \dim\mathcal{U}$ is $\frac{1}{f(p)\sqrt{1+|\nabla_p f|^2}}$ and $\mathcal{U}^{\perp} \cap \mathbf{w}^{\perp}$ is a subspace of the correponding principal directional space. The remaining principal curvatures are the spectral values of $\mathbf{B}_x$, and the corresponding principal direction spaces are $\mathbf{J}_{x>}(\mathcal{W})$, where $\mathcal{W}$ is a spectral space of $\mathbf{B}_x$.

Proof: It is clear that $\mathbf{J}_x$, as defined by (37.9), is the gradient at p of the mapping $q \mapsto q + f(q)\mathbf{w}$ from $\mathcal{P}$ into $\mathcal{S}$ and hence is indeed a linear mapping from $\mathcal{U}$ to $\mathcal{T}_x$. We also note that $\mathbf{J}_x$ is injective.

We now consider the mapping $\mathbf{k} : \mathcal{P} \to \mathcal{V}^\times$ defined by

$$\mathbf{k}(q) := \nabla_q f - \mathbf{w} \quad \text{for all} \quad q \in \mathcal{P}, \tag{37.12}$$

so that

$$\gamma(q) := |\mathbf{k}(q)| = \sqrt{1 + |\nabla_q f|^2} \quad \text{for all} \quad q \in \mathcal{P}. \tag{37.13}$$

By Prop. 1, we have

$$\mathbf{n}(q + f(q)\mathbf{w}) = \frac{\mathbf{k}(q)}{\gamma(q)} = \frac{\mathbf{k}}{|\mathbf{k}|}(q) \quad \text{for all} \quad q \in \mathcal{P}$$

Differentiation with respect to q at the point $q := p$, using the Chain Rule, yields

$$(\nabla_x \mathbf{n})\mathbf{J}_x = \nabla_p \left(\frac{\mathbf{k}}{|\mathbf{k}|} \right)$$

By (35.4), (37.13), and by (P6.4) of Vol.I, we conclude that

$$\mathbf{L}_x \mathbf{J}_x = \frac{1}{\gamma(p)^3} \Big(\mathbf{k}(p) \otimes (\nabla_p \mathbf{k})^\top \mathbf{k}(p) - \gamma(p)^2 \nabla_p \mathbf{k} \Big)\Big|^{\mathcal{T}_x}. \tag{37.14}$$

Now, it follows from (37.12) that $\nabla_p \mathbf{k} = \nabla_p^{(2)} f|^{\mathcal{V}}$ and hence that $(\nabla_p \mathbf{k})^\top = \nabla_p^{(2)} \mathbf{P}$, where $\mathbf{P} \in \operatorname{Lin}(\mathcal{V}, \mathcal{U})$ is the projection on $\mathcal{U}$ that satisfies $\operatorname{Null} \mathbf{P} = \mathcal{U}^\perp$, so that $\mathbf{P}\mathbf{w} = \mathbf{0}$. This observation and (37.12) show that (37.14) reduces to

$$\mathbf{L}_x \mathbf{J}_x = \frac{1}{\gamma(p)^3} \Big((\nabla_p f - \mathbf{w}) \otimes (\nabla_p^{(2)} f)(\nabla_p f) - \gamma(p)^2 \nabla_p^{(2)} f \Big)\Big|^{\mathcal{T}_x}. \tag{37.15}$$

On the other hand, an easy calculation from (37.10) and (37.13) shows that

$$\nabla_p f \cdot \mathbf{B}_x \mathbf{u} = \frac{1}{\gamma(p)^3} \Big((\nabla_p^2 f) \nabla_p f \Big) \cdot \mathbf{u} \quad \text{for all} \quad \mathbf{u} \in \mathcal{U}.$$

From this result, (37.9) and (37.15) the desired conclusion (37.11) follows immediately.

Assume now that $\mathcal{S}$ is a graph. Then $\dim \mathcal{U} = \dim \mathcal{T}_x = n - 1$ and, since $\mathbf{J}_x$ is injective, it must be invertible. Therefore, by (37.6), we have $\mathbf{L}_x = \mathbf{J}_x \mathbf{B}_x \mathbf{J}_x^{-1}$. It follows that $\mathbf{L}_x$ has the same spectrum as $\mathbf{B}_x$ and the spectral space of $\mathbf{L}_x$ are of the form $\mathbf{J}_{x>}(\mathcal{W})$, where $\mathcal{W}$ is a spectral space of $\mathbf{B}_x$.

Finally, assume that $\mathcal{S}$ is a surface of revolution, in which case $\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp$ is not the zero-space. It follows from (37.3) that $\mathbf{n}_x(x) \in \mathcal{U} + \mathbb{R}\,\mathbf{w}_x$ and hence that $\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp = (\mathcal{U} + \mathbb{R}\,\mathbf{w})^\perp \subset \{\mathbf{n}_x\}^\perp = \mathcal{T}_x$. By Prop. 1, we have

$$\mathbf{n}_x(p + f(p)\mathbf{z}) = \frac{1}{\gamma(p)} (\nabla_p f - \mathbf{z}) \quad \text{for all} \quad \mathbf{z} \in \operatorname{Usph} \mathcal{U}^\perp \ . \tag{37.16}$$

Differentiating (37.16) with respect to $\mathbf{z}$ at $\mathbf{z} := \mathbf{w}$ and using the Chain Rule, it easily follows that

$$f(p) \nabla_x \mathbf{n}\big|_{\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp} = -\frac{1}{\gamma(p)}\ \mathbf{1}_{\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp \subset \mathcal{U}}$$

and hence, by (35.4),

$$\mathbf{L}_x\big|_{\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp} = \frac{1}{\gamma(p) f(p)}\ \mathbf{1}_{\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp \subset \mathcal{T}_x}\ , \tag{37.17}$$

which shows that $\frac{1}{\gamma(p) f(p)}$ is the spectral value of $\mathbf{L}_x$ corresponding to the spectral space that includes $\mathcal{U}^\perp \cap \{\mathbf{w}\}^\perp$. The remaining spectral values and spectral spaces are obtained in the same way as in the case when $\mathcal{S}$ is a graph. ∎

The mean curvature H_x of a graph is

$$\mathrm{H}_x = \frac{1}{n-1}\,\mathrm{tr}\,\mathbf{L}_x = \frac{1}{n-1}\,\mathrm{tr}\,\mathbf{B}_x.$$

Hence, by (37.10), we have

$$(n-1)\sqrt{1+|\nabla_p f|^2}^{\,3}\,\mathrm{H}_x = \nabla_p^{(2)} f\,(\nabla_p f, \nabla_p f) - (1+|\nabla_p f|^2)\Delta_p f. \tag{37.18}$$

If $n = 3$ and $\dim \mathcal{T}_x = 1$, (37.2) defines a surface of revolution about a portion $\mathcal{P}$ of the line $\mathcal{F}$. In this case, $\mathcal{P}$ can be represented by an open subset I of $\mathbf{R}$ and f can be replaced by a function $f : I \to \mathbb{P}^{\times}$. The principal curvatures are

$$\kappa_1 = \frac{1}{f(p)\sqrt{1+f^{\bullet}(p)^2}} \quad , \quad \kappa_2 = -\frac{f^{\bullet\bullet}(p)}{\sqrt{1+f^{\bullet}(p)^2}^{\,3}}. \tag{37.19}$$

A **minimal surface** is a surface that has zero mean curvature everywhere. It follows from (37.18) that the graph of f is a minimal surface if and only if f satisfies the non-linear partial differential equation

$$(1+|\nabla_p f|^2)\Delta f = \nabla_p^{(2)} f\,(\nabla_p f, \nabla_p f)\ . \tag{37.20}$$

The theory of the solutions of this equation is very difficult. It follows from (37.18) that a surface of revolution, generated by f about a line in a 3-dimensional Euclidean space, is a minimal surface if and only if $\kappa_1 + \kappa_2 = 0$, i.e.

$$1 + (f^{\bullet})^2 = f f^{\bullet\bullet}.$$

This is an ordinary differential equation for f. Its general solution is given by

$$f = a\cosh\circ\left(\frac{\iota - p_0}{a}\right) \quad , \quad a \in \mathbf{R}^{\times}\ ,\ p_0 \in \mathbf{R}\,.$$

It describes a **catenoid of revolution**.

38. Curvatures of Paths.

Let $\mathcal{E}$ be a genuine Euclidean space, with translation space $\mathcal{V}$, such that $\dim \mathcal{E} = \dim \mathcal{V} \geq 3$.

Let $\mathfrak{p}$ be a regular path of class C^3 (see Sect.34). By Prop.1 of Sect.34. we can then choose an arclength parametrization $p : J \to \mathcal{E}$ of $\mathfrak{p}$ that is of class C^3. We assume that the triple $(p^{\bullet}(s),\ p^{\bullet\bullet}(s),\ p^{\bullet\bullet\bullet}(s))$ is linearly independent for all $s \in J$. Since $|p^{\bullet}| = 1$, we have $p^{\bullet\bullet}\cdot p^{\bullet} = 0$.

By applying the Gram-Schmidt orthonormalization process (see Problem (1) of Chap.4 in Vol.I), we obtain, after an easy calculation, a valuewise orthonormal list $(\mathbf{t}, \mathbf{n}, \mathbf{b}) : J \to \mathcal{V}^3$ of mappings as follows:

$$\mathbf{t} = p^{\bullet} \qquad , \qquad \mathbf{n} = \frac{p^{\bullet\bullet}}{|p^{\bullet\bullet}|} \qquad , \qquad \mathbf{b} = \frac{\mathbf{n}^{\bullet} - (\mathbf{n}^{\bullet}\cdot\mathbf{t})\mathbf{t}}{|\mathbf{n}^{\bullet} - (\mathbf{n}^{\bullet}\cdot\mathbf{t})\mathbf{t}|}. \tag{38.1}$$

The unit-vector functions $\mathbf{t}, \mathbf{n}, \mathbf{b}$ from J to $\mathcal{V}$ are called the **tangent, normal**, and **binormal**, respectively, to the path $\mathfrak{p}$.

For a more general case, let $\mathfrak{p}$ be a regular path of class C^{k+1}, where $k \leq \dim \mathcal{V}$. Assume that $\mathfrak{p}$ has an arclength parametrization $p : J \to \mathcal{E}$ such that the list $(p^{(i)}(s) \mid i \in k^]\,)$ is linearly independent for all $s \in J$. By applying the Gram-Schmidt Orthonormalization process (see Problem 1 (b) of Sect.48 in vol.I), we obtain a list $(\mathbf{v}_i \mid i \in k^]\,) : J \to \mathcal{V}^k$ of orthonormal mappings by the following recursive definition:

$$\mathbf{v}_i := \frac{p^{(i)} - \sum\limits_{j \in (i-1)^]} (p^{(i)} \cdot \mathbf{v}_{i-j})\mathbf{v}_{i-j}}{\left| p^{(i)} - \sum\limits_{j \in (i-1)^]} (p^{(i)} \cdot \mathbf{v}_{i-j})\mathbf{v}_{i-j} \right|} . \tag{38.2}$$

We assume that the list $(p^{(i)}(s) \mid i \in (k+1)^]\,)$ is linearly dependent for all $s \in J$, so the recursion (38.6) will break off after $i := k$ because the denominater will become zero. This is always the case when $k := n$.

Note that for $i := 1$, the sums on the right side of (38.2) are taken over the empty list and hence are zero, so that

$$\mathbf{v}_1 = \frac{p^\bullet}{|p^\bullet|} = p^\bullet = \mathbf{t} \ .$$

For $i := 2$ we observe, as before, that $p^{\bullet\bullet} \cdot p^\bullet = p^{\bullet\bullet} \cdot \mathbf{t} = p^{\bullet\bullet} \cdot \mathbf{v}_1 = 0$. Hence (38.2), observing $(38.1)_2$, gives

$$\mathbf{v}_2 := \frac{p^{\bullet\bullet}}{|p^{\bullet\bullet}|} = \frac{\mathbf{v}_1^\bullet}{|\mathbf{v}_1^\bullet|} = \mathbf{n} \ . \tag{38.3}$$

It is easily seen that, for $i = 3$, observing $(38.1)_3$, the formula (38.2) gives $\mathbf{v}_3 = \mathbf{b}$.

Since the list $(\mathbf{v}_i \mid i \in k^]\,) : J \to \mathcal{V}^k$ is orthonormal, we have $\mathbf{v}_i \cdot \mathbf{v}_j = \delta_{i,j}$ for all $i, j, \in k^]$ (see (41.21) in Vol.I). By the Product Rule, it follows that

$$\mathbf{v}_i^\bullet \cdot \mathbf{v}_j = -\mathbf{v}_j^\bullet \cdot \mathbf{v}_i \qquad \text{for all} \quad i, j, \in k^] \ , \tag{38.4}$$

which means that the matrix

$$M := \left(\mathbf{v}_i^\bullet \cdot \mathbf{v}_j \;\middle|\; i, j \in k^] \right)$$

is skew.

Now, differentiation of (38.2) shows that, for every $i \in k^]$, $\mathbf{v}_i^\bullet$ is a linear combination of the list $(\mathbf{v}_j \mid j \in (i+1)^]\,)$, which implies that the terms $M_{i,j}$ are zero when $j > i+1$. Since M is skew, there is a list $(\kappa_i \mid i \in (k-1)^]\,)$, with terms in $^\times$, such that $M_{i,i+1} = \kappa_i$ and $M_{i+1,i} = -\kappa_i$ for all $i \in (k-1)^[$. All other terms of M are zero. Thus we can describe the formulas for the list $(\mathbf{v}_i^\bullet \mid i \in k^]\,)$ of the derivatives of the $\mathbf{v}_i$ in terms of the orthonormal list by $(\mathbf{v}_i \mid i \in k^]\,)$ by the scheme

$$\begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \vdots \\ \mathbf{v}_{k-1} \\ \mathbf{v}_k \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & \kappa_1 & 0 & \dots & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & \dots & 0 & 0 \\ 0 & -\kappa_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & \kappa_{k-1} \\ 0 & 0 & 0 & \dots & -\kappa_{k-1} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ \vdots \\ \mathbf{v}_{k-1} \\ \mathbf{v}_k \end{pmatrix} . \tag{38.5}$$

The functions in the list $(\kappa_i \mid i \in (k-1)^]\,)$ are called the **curvature functions** of the path $\mathfrak{p}$.

As noted above, we have $\mathbf{v}_1 = \mathbf{t}$, the tangent, $\mathbf{v}_2 = \mathbf{n}$, the normal, and $\mathbf{v}_3 = \mathbf{b}$, the binormal. Hence, if $k := 3$, (38.5) reduces to

$$\mathbf{t}^\bullet = \qquad \kappa\mathbf{n}$$

$$
\begin{aligned}
\mathbf{n}^{\bullet} &= -\kappa\mathbf{t} \qquad + \tau\mathbf{b} \\
\mathbf{b}^{\bullet} &= \qquad -\tau\mathbf{n}
\end{aligned}
\tag{38.6}
$$

and $\kappa := \kappa_1$ is called the **curvature** and $\tau := \kappa_2$ the **torsion** of the path.

Note. The fomulas described by (38.6) are often called the “Formulas of Frenet”. One can prove that two paths of the type considered here are congruent (see Def.2 in Sect.46 of Vol.I) if and only if they have have the same lists of curvature functions. ∎