Ax-Schanuel Type Inequalities in Differentially Closed Fields

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To my parents for their endless love and support.

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Abstract

In this thesis we study Ax-Schanuel type inequalities for abstract differential equations. A motivating example is the exponential differential equation. The Ax-Schanuel theorem states positivity of a predimension defined on its solutions. The notion of a predimension was introduced by Hrushovski in his work from the 1990s where he uses an amalgamation-with-predimension technique to refute Zilber's Trichotomy Conjecture. In the differential setting one can carry out a similar construction with the predimension given by Ax-Schanuel. In this way one constructs a limit structure whose theory turns out to be precisely the first-order theory of the exponential differential equation (this analysis is due to Kirby (for semiabelian varieties) and Crampin, and it is based on Zilber's work on pseudo-exponentiation). One says in this case that the inequality is adequate.

Thus, by an Ax-Schanuel type inequality we mean a predimension inequality for a differential equation. Our main question is to understand for which differential equations one can find an adequate predimension inequality. We show that this can be done for linear differential equations with constant coefficients by generalising the Ax-Schanuel theorem.

Further, the question turns out to be closely related to the problem of recovering the differential structure in reducts of differentially closed fields where we keep the field structure (which is quite an interesting problem in its own right). So we explore that question and establish some criteria for recovering the derivation of the field. We also show (under some assumptions) that when the derivation is definable in a reduct then the latter cannot satisfy a non-trivial adequate predimension inequality.

Another example of a predimension inequality is the analogue of Ax-Schanuel for the differential equation of the modular j-function due to Pila and Tsimerman. We carry out a Hrushovski construction with that predimension and give an axiomatisation of the first-order theory of the strong Fraïssé limit. It will be the theory of the differential equation of j under the assumption of adequacy of the predimension. We also show that if a similar predimension inequality (not necessarily adequate) is known for a differential equation then the fibres of the latter have interesting model theoretic properties such as strong minimality and geometric triviality. This, in particular, gives a new proof for a theorem of Freitag and Scanlon stating that the differential equation of j defines a trivial strongly minimal set.

Contents

Chapter 1

Introduction

1.1 The main question

In [Lan66] Serge Lang mentions that Stephen Schanuel conjectured that for any $\mathbb{Q}$-linearly independent complex numbers $z_1, \ldots, z_n$ one has

$$\operatorname{td}_{\mathbb{Q}} \mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n. \tag{1.1}$$

This is now known as Schanuel's conjecture. It generalises many results (e.g. the Lindemann-Weierstrass theorem) and conjectures in transcendental number theory and is widely open. For example, a simple consequence of Schanuel's conjecture is algebraic independence of e and π which is a long standing open problem.

Schanuel's conjecture (and its real version) is closely related to the model theory of the complex (real) exponential field $\mathbb{C}_{\exp} = (\mathbb{C}; +, \cdot, \exp)$ (respectively, $\mathbb{R}_{\exp} = (\mathbb{R}; +, \cdot, \exp)$, see [MW96]). Most notably, Boris Zilber noticed that the inequality (1.1) states the positivity of a predimension. The notion of a predimension was defined by Ehud Hrushovski in [Hru93] where he uses an amalgamation-with-predimension technique (which is a variation of Fraïssé's amalgamation construction) to refute Zilber's Trichotomy Conjecture. More precisely, Schanuel's conjecture is equivalent to the following statement: for any $z_1, \ldots, z_n \in \mathbb{C}$ the inequality

$$\delta(\bar{z}) = \operatorname{td}_{\mathbb{Q}} \mathbb{Q}(\bar{z}, \exp(\bar{z})) - \operatorname{ldim}_{\mathbb{Q}}(\bar{z}) \geq 0 \tag{1.2}$$

holds, where td and ldim stand for transcendence degree and linear dimension respectively. Here δ satisfies the submodularity law[1] which allows one to carry out a Hrushovski construction. In this way Zilber constructed pseudo-exponentiation on algebraically closed fields of characteristic zero. He proved that there is a unique model of that (non first-order) theory in each uncountable cardinality and conjectured that the model of cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}_{\exp}$. Since (1.2) holds for pseudo-exponentiation (it is included in the axiomatisation given by Zilber), Zilber's conjecture implies Schanuel's conjecture. For details on pseudo-exponentiation see [Zil04b, Zil05, Zil02, Zil16, KZ14, Kir13].

[1] This is not quite right as we will see in Chapter 3. It will be explained there in what sense δ is submodular.

Zilber's work also gave rise to a diophantine conjecture (Conjecture on Intersection with Tori) which was later generalised by Pink and is now known as the Zilber-Pink conjecture ([Zil02, KZ14, Pin05]). It generalises many diophantine conjectures and theorems such as Mordell-Lang, Manin-Mumford, and André-Oort, and is being actively studied by model theorists and number theorists.

Though Schanuel's conjecture seems to be out of reach, James Ax proved its differential analogue in 1971 ([Ax71]). It is now known as the Ax-Schanuel theorem or inequality.

Theorem 1.1.1 (Ax-Schanuel)**.** *Let $\mathcal{K} = (K; +, \cdot, \mathrm{D}, 0, 1)$ be a differential field with field of constants C. If $(x_1, y_1), \ldots, (x_n, y_n)$ are non-constant solutions to the exponential differential equation $\mathrm{D}\, y = y \,\mathrm{D}\, x$ then*

$$\mathrm{td}_C\, C(\bar{x}, \bar{y}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/C) \geq 1, \tag{1.3}$$

where $\mathrm{ldim}_{\mathbb{Q}}(\bar{x}/C)$ is the dimension of the $\mathbb{Q}$-span of $x_1, \ldots, x_n$ in the quotient vector space K/C.

Here again we have a predimension inequality, which will be part of the first order theory of the reduct $\mathcal{K}_{\mathrm{Exp}} = (K; +, \cdot, \mathrm{Exp}, 0, 1)$ of $\mathcal{K}$ where $\mathrm{Exp}(x, y)$ is a binary predicate for the set of solutions of the exponential differential equation. Therefore a natural question arises: if one carries out a Hrushovski construction with this predimension and class of reducts, will one end up with a similar reduct of a (saturated) differentially closed field? In other words, we can ask whether a Hrushovski construction will yield the theory $T_{\mathrm{Exp}} = \mathrm{Th}(\mathcal{F}_{\mathrm{Exp}})$, where $\mathcal{F}$ is a differentially closed field. Zilber calls predimensions with this property *adequate*. Thus the question is whether the Ax-Schanuel inequality is adequate.

Cecily Crampin studied the exponential differential equation in her DPhil thesis [Cra06] and gave a criterion for a system of exponential differential equations to have a solution (analogous to pseudo-exponentiation), known as *existential* or *exponential closedness* (in fact, it is a special case of the full existential closedness property proved by Kirby). She also considered the predimension function and proved some form of strong existential closedness for reducts of differentially closed fields. Her results can be used to show that Ax-Schanuel is adequate, though it is not explicit in her work as she does not construct the strong Fraïssé limit.

Jonathan Kirby considered this problem in a much more general context. He studied exponential differential equations of semiabelian varieties, observed that Ax-Schanuel holds in that setting too and, using the amalgamation-with-predimension construction, proved, in our terminology, that it is adequate, along with giving an axiomatisation of the complete theory of the corresponding reducts (see [Kir06, Kir09]). The axiomatisation is again very similar to pseudo-exponentiation (and adaptations of many arguments and concepts from Zilber's work are used in the analysis of the exponential differential equations). An important property that shows adequacy of Ax-Schanuel is strong existential closedness which means that saturated models of T_{Exp} are existentially closed in strong extensions. This can be given an equivalent algebraic formulation stating that certain varieties have generic exponential points. In other words, we can think of this property as an "exponential Nullstellensatz". More details on this, in particular an axiomatisation of T_{Exp}, will be presented in Section 5.1.

Once this is done, one naturally asks the question of whether something similar can be done for other differential equations. In other words, one wants to find adequate predimension inequalities for differential equations. Thus, by an Ax-Schanuel type inequality we mean a predimension inequality. Adequacy gives us a good understanding of the model theoretic and geometric properties of the differential equation under consideration. In particular, considering reducts of differentially closed fields with the field structure and a relation for solutions of our equation (and possibly their derivatives) one normally gets some criteria for a system of equations in the reduct to have a solution. These criteria are dictated by the "strong existential closedness" property. Then one obtains an axiomatisation of the (first-order) theory of the equation, i.e. of the corresponding reduct. Understanding which systems have a solution is equivalent to asking which algebraic varieties contain a point that is a solution (coordinate-wise) of our differential equation. In this regard the nature of the reduct and its axiomatisation is geometric. These ideas will be illustrated on the example of the exponential differential equation in Section 5.1. More details will be given in Chapter 6 where we study the differential equation of the j-invariant, carry out a Hrushovski construction with the predimension given by the Ax-Schanuel theorem for j, and give an axiomatisation of the amalgam.

Thus, the main question of our interest is the following.

Question 1.1.2. *Which differential equations satisfy an adequate predimension inequality?*

This question is also important from a number theoretic point of view since Ax-Schanuel type statements (often combined with o-minimality, a branch of model theory) have interesting applications in number theory and in particular contribute to our understanding of the corresponding number theoretic conjectures like Schanuel's conjecture ([BKW10, Pil15, Kir10]). In particular, the Ax-Schanuel theorem was used by Zilber to establish a weak form of the CIT conjecture ([Zil02]) and by Kirby to prove a weak version of Schanuel's conjecture (in exponential fields) and to deduce from this that there are at most countably many "essential" counterexamples to Schanuel's conjecture ([Kir10]).

Apart from this, Ax-Schanuel type statements give information about algebraic relations between the complex meromorphic solutions of a given differential equation. This is useful from the point of view of the theory of differential equations, complex functions and functional transcendence.

This thesis is constructed around the above question. We give some partial answers which contribute to our understanding of the general picture.

1.2 Summary of the thesis

Now we give a summary of the thesis and indicate the main results of each chapter.

In Chapter 2 we present basic definitions and results on differential fields and strongly minimal sets that will be used throughout the thesis.

Chapter 3 gives an axiomatic approach to predimensions and Hrushovski style amalgamation-with-predimension constructions. In particular, we define adequacy of a predimension inequality. We also consider some examples and show how they fit with the presented approach.

Question 1.1.2 turns out to be closely related to another interesting problem, namely, recovering the differential structure in reducts of differentially closed fields. The idea is that definability of a derivation would imply that there is no non-trivial adequate predimension inequality for the given differential equation. So we study the question of definability of a derivation in reducts of differentially closed fields in Chapter 4. This is a natural problem to explore even outside the context of our main question. Indeed, it is the differential analogue of the problem of recovering the field structure from reducts of algebraically closed fields and from strongly minimal sets in general, which is a well-studied problem in model theory of fields (see Section 4.1 for more details and references). Note however that we will not study arbitrary reducts but only those that have the field structure of the original differential field.

We obtain several criteria for definability of a derivation which we present below. But first let us fix the setting. Let $\mathcal{F} = (F; +, \cdot, \mathrm{D}, 0, 1)$ be a sufficiently saturated model of DCF_0 (differentially closed fields of characteristic zero). Pick an element $t \in F$ with $\mathrm{D}\, t = 1$ and add it to our language as a constant symbol. Then by definable we mean definable without parameters in this new language. Fix some collection R of definable sets in $\mathcal{F}$ and consider the reduct $\mathcal{F}_R := (F; +, \cdot, 0, 1, t, P)_{P \in R}$. In order to distinguish between the same notions in the differentially closed field $\mathcal{F}$ and its reduct $\mathcal{F}_R$ we will add a subscript D or R respectively to their notations. For instance, MR_{D} stands for Morley rank in $\mathcal{F}$ while MR_R means Morley rank in $\mathcal{F}_R$.

Fix a generic (over the empty set) element a in $\mathcal{F}$. This means that $\mathrm{MR}_{\mathrm{D}}(a) = \omega$ or, equivalently, a is differentially transcendental over $\mathbb{Q}$. Denote $T_R := \mathrm{Th}\, \mathcal{F}_R$ and $T_R^+ := \mathrm{Th}(F; +, \cdot, 0, 1, t, a, P)_{P \in R}$. Let also $k_0 := \mathbb{Q}(t)$. In Chapter 4 we will prove the following theorem.

Theorem 4.6.11. *For a generic point $a \in F$ the following are equivalent:*

1. D *is definable in the reduct* $\mathcal{F}_R$*,*
2. $\mathrm{MR}_R(\mathrm{D}\, a/a) < \omega$*,*
3. $\mathrm{MR}_R(\mathrm{D}\, a/a) = 0$*,*
4. $\mathrm{tp}_R(\mathrm{D}\, a/a)$ *forks over the empty set,*
5. *The set* $\{\mathrm{D}^n\, a : n \geq 0\}$ *is not (forking) independent in* $\mathcal{F}_R$*,*
6. $\mathrm{dcl}_R(a) = k_0\langle a \rangle (= \mathrm{dcl}_{\mathrm{D}}(a))$*,*
7. $\mathrm{dcl}_R(a) \supsetneq k_0(a)$*,*
8. $\mathrm{acl}_R(a) = (k_0\langle a \rangle)^{\mathrm{alg}} (= \mathrm{acl}_{\mathrm{D}}(a))$*,*

9. $\mathrm{acl}_R(a) \supsetneq (k_0(a))^{\mathrm{alg}}$*,*

10. Every model of T_R^+ is the R-reduct (with canonical interpretation) of a differentially closed field,

11. Every automorphism of $\mathcal{F}_R$ fixes $\mathbb{D}$ (the graph of D*) setwise.*

Further, using these criteria (in fact, only the equivalence of 1 and 2 above suffices) we establish the following result.

Theorem 4.8.2. *If T_R is inductive (i.e. $\forall\exists$-axiomatisable) and defines* D *then T_R is model complete.*

The contrapositive of this theorem states that if D is definable in a reduct of a differentially closed field which is not model complete then it cannot be $\forall\exists$-axiomatisable. Model completeness of reducts is expected to be rare, so for reducts where D is definable we should not expect an $\forall\exists$-axiomatisation.

We can also consider another formulation of Theorem 4.8.2: if T_R is not model complete and is inductive, then it does not define D. As we said above, T_R is not expected to be model complete, so in inductive reducts definability of a derivation is expected to be rare. This implies in particular that the exponential differential equation does not define D since its first order theory is inductive and not model complete (see Sections 4.8 and 5.1).

Theorem 4.8.2 will be used in Section 3.4 to justify our point that definability of a derivation in a reduct of a differentially closed field implies that the reduct in consideration cannot have a (strongly) adequate predimension inequality. We have one more result in this direction.

Theorem 4.7.1. *Assume the underlying fields of finitely generated structures from our strong amalgamation class are algebraically closed of finite transcendence degree over $\mathbb{Q}$. Assume further that generic* 1*-types (in the sense of the pregeometry associated to the predimension) are not algebraic. If* D *is definable in $\mathcal{F}_R$ and δ is strongly adequate, then the reduct is model complete and hence δ is trivial.*

Next, we study linear differential equations with constant coefficients (the exponential differential equation being a special case of it) in Chapter 5. We generalise Ax-Schanuel and thus establish predimension inequalities for those equations. We also prove they are adequate and axiomatise the first-order theories of those equations. Our results (of Chapter 5) rely heavily on the aforementioned analysis of the exponential differential equation by Kirby.

We formulate the main results of Chapter 5 below. For a differential field $\mathcal{K}$ and a non-constant element $x \in K$ define a derivation $\partial_x : K \to K$ by $\partial_x = (\mathrm{D}\, x)^{-1} \cdot \mathrm{D}$. Then consider the differential equation

$$\partial_x^n y + c_{n-1}\partial_x^{n-1} y + \ldots + c_1 \partial_x y + c_0 y = 0, \tag{2.1}$$

where the coefficients are constants with $c_0 \neq 0$. Let $\lambda_1, \ldots, \lambda_n$ be the roots of the characteristic polynomial $p(\lambda) = \lambda^n + \sum_{0 \leq i < n} c_i \lambda^i$. Then the Ax-Schanuel theorem for (2.1) is as follows.

Theorem 5.2.4. *Let (x_i, y_i), $i = 1, \ldots, m$, be non-constant solutions to the equation (2.1) in a differential field $\mathcal{K}$ such that $y_i, \partial_{x_i} y_i, \ldots, \partial_{x_i}^{n-1} y_i$ are linearly independent over C for every i. Then*

$$\operatorname{td}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \ldots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq \operatorname{ldim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \ldots, \lambda_n \bar{x}/C) + 1, \tag{2.2}$$

where $\partial_{\bar{x}}^j \bar{y} := (\partial_{x_1}^j y_1, \ldots, \partial_{x_m}^j y_m)$.

We work with reducts of differential fields with a relation $\mathrm{E}_n(x, y_0, y_1, \ldots, y_{n-1})$ which consists of all tuples $(x, y, \partial_x y, \ldots, \partial_x^{n-1} y)$ where (x, y) is a solution to (2.1). Then (2.2) can be axiomatised in a first-order way in the language of those reducts. For a complete axiomatisation of these reducts we will need an important axiom scheme called Existential Closedness. For a structure F in our language it can be formulated as follows. (For the definition of E_n-Exp-rotundity see section 5.3).

EC' For each irreducible E_n-Exp-rotund variety $V \subseteq F^{m(n+1)}$ the intersection $V(F) \cap \mathrm{E}_n^m(F)$ is non-empty.

We will see that this axiom scheme along with the inequality (2.2) and some basic axioms (which reveal the relationship between E_n and Exp) axiomatise the first order theory T_{E_n} of reducts of differentially closed fields in the corresponding language.

In Theorem 5.5.2 we give a reformulation of Theorem 5.2.4 where we deal with arbitrary solutions without assuming linear independence of $y_i, \partial_{x_i} y_i, \ldots, \partial_{x_i}^{n-1} y_i$. This allows us to rewrite (2.2) as a predimension inequality. Finally, we prove that it is adequate (Theorem 5.5.4) using the adequacy of the Ax-Schanuel inequality for the exponential differential equation.

Chapter 6 is devoted to the differential equation of the j-function. Starting with the Ax-Schanuel inequality for j (established by Pila and Tsimerman in [PT16]), we show that the class of models of a certain theory (which is essentially the universal theory of reducts of differential fields with a relation for the equation of j) has the strong amalgamation property. Then we construct the strong Fraïssé limit and give an axiomatisation of its first-order theory. Thus, the given axiomatisation will be a candidate for the theory of the differential equation of the j-function assuming adequacy of the Pila-Tsimerman inequality. Note however that adequacy is still open and we do not have an answer to that question.

Chapter 7 establishes some connections between predimension inequalities of a certain type (similar to Ax-Schanuel for j) and strongly minimal sets in DCF_0. Assuming that a differential equation $E(x, y)$ satisfies a predimension inequality of the form "td − dim" where dim is a certain dimension of trivial type we deduce that the fibre $U := \{y : E(t, y) \wedge \mathrm{D}\, y \neq 0\}$ (where t is an element with $\mathrm{D}\, t = 1$) is strongly minimal and geometrically trivial (we work in a differentially closed field $\mathcal{K}$). Thus we get a necessary condition for E to satisfy an Ax-Schanuel inequality of the given form. This is a step towards the solution of our main problem. In particular it gives rise to an inverse problem: given a one-variable differential equation which is strongly minimal and geometrically trivial, can we say anything about the Ax-Schanuel properties of its two-variable analogue? See Section 7.4 for more details.

Our main result of Chapter 7 is as follows.

Theorem 7.1.7. *Let $\mathcal{K}$ be a differentially closed field with field of constants C and $\mathcal{P}$ be a non-empty collection of algebraic polynomials $P(X,Y) \in C[X,Y]$. Let also $E(x,y)$ be given by a differential equation $f(x,y) = 0$ with $m := \operatorname{ord}_Y f(X,Y)$. Assume E satisfies the following Ax-Schanuel condition.*

> *Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be non-constant elements of K with $f(x_i, y_i) = 0$. If $P(y_i, y_j) \neq 0$ for all $P \in \mathcal{P}$ and $i \neq j$ then*
>
> $$\operatorname{td}_C C(x_1, y_1, \partial_{x_1} y_1, \ldots, \partial_{x_1}^{m-1} y_1, \ldots, x_n, y_n, \partial_{x_n} y_n, \ldots, \partial_{x_n}^{m-1} y_n) \geq mn + 1.$$

Assume finally that the differential polynomial $g(Y) := f(t,Y)$ is absolutely irreducible. Then

- *$U := \{y : f(t,y) = 0 \wedge \mathrm{D}\, y \neq 0\}$ is strongly minimal with trivial geometry.*
- *If, in addition, $\mathcal{P} = \{X - Y\}$ then U is strictly disintegrated and hence it has $\aleph_0$-categorical induced structure.*

We also obtain a characterisation of the induced structure on Cartesian powers of U in terms of *special* subvarieties. Further, we prove some generalisations of the above theorem. See Section 7.1 for more details.

A motivating example for this is the Ax-Schanuel inequality for the differential equation of the j-function. It is a predimension inequality of the above form. Given this, one can easily see that the theorem of Freitag and Scanlon [FS15], stating that the differential equation of j defines a strongly minimal set with trivial geometry, is a special case of Theorem 7.1.7.

1.3 Notations

In this section we fix some notations that will be used throughout the thesis.

- We use upper-case letters $X, Y, \ldots$, possibly with subscripts, for indeterminates of polynomials (whether differential or algebraic). Lower-case letters (with subscripts and superscripts) will be used for elements of a set and for variables in formulas (it will be clear from the context which one we mean). In particular if $f(X)$ is a (differential) polynomial then $f(X) = 0$ means that f is identically zero, $f(a) = 0$ means that f vanishes at a and $f(x) = 0$ is a formula with a free variable x.
- Upper-case letters are also used to denote sets or structures. Often structures with domains $A, B, M, N, K, F, \ldots$ will be denoted by $\mathcal{A}, \mathcal{B}, \mathcal{M}, \mathcal{N}, \mathcal{K}, \mathcal{F}, \ldots$ respectively.
- The length of a tuple $\bar{a}$ will be denoted by $|\bar{a}|$. For a set A and a tuple $\bar{a}$ we will sometimes write $\bar{a} \in A$ or $\bar{a} \subseteq A$ and mean that all coordinates of $\bar{a}$ are in A, i.e. $\bar{a} \in A^{|\bar{a}|}$.

- For two sets X, Y the notation $X \subseteq_{fin} Y$ means X is a finite subset of Y. The union $X \cup Y$ will sometimes be written as XY. The power set of X is denoted by $\mathfrak{P}(X)$.

- All fields considered in this work will be of characteristic zero. The algebraic closure of a field is denoted by K^{alg}, and we use K^{dif} for the differential closure of a differential field. The derivation of a differential field will be denoted by $'$ or D. The field of constants of a differential field will normally be denoted by C.

- If F is a differential field and $A \subseteq F$ then we denote by $\langle A \rangle$ the differential subfield generated by A. If $K \subseteq F$ are differential fields and $A \subseteq F$ then $K\langle A \rangle$ is the differential subfield generated by K and A. The algebraic subfield generated by K and A is denoted by $K(A)$. The ring of differential polynomials of n variables over a differential field F is denoted by $F\{X_1, \ldots, X_n\}$. For a non-constant point $x \in F$ the *differentiation with respect to* x is a derivation $\partial_x : F \to F$ defined by $y \mapsto \frac{y'}{x'}$, where $'$ is the derivation of F.

- We write $\operatorname{span}_K(A)$ for the linear span of a subset $A \subseteq V$ of a K-vector space V. For the linear dimension of V over K we use the shorthand $\operatorname{ldim}_K V$.

- If $K \subseteq F$ are fields, the transcendence degree of F over K will be denoted by $\operatorname{td}_K F$ or $\operatorname{td}(F/K)$. When we work in an ambient algebraically closed field F and V is a variety defined over F, we will normally identify V with the set of its F-points $V(F)$. The algebraic locus (Zariski closure) of a tuple $\bar{a} \in F$ over K will be denoted by $\operatorname{Loc}_K(\bar{a})$ or $\operatorname{Loc}(\bar{a}/K)$ (and identified with the set of its F-points). By an irreducible variety we always mean absolutely irreducible.

- Given an algebraically closed field K and a variety $V \subseteq K^{n+m}$ defined over $\mathbb{Q}$, let P be the projection of V onto the last m coordinates, and for $p \in P$ let $V(p)$ (also denoted V_p) be the fibre of the projection above $p \in P$. Then we have a *parametric family of varieties* $(V(p))_{p \in P}$. For a subfield $C \subseteq K$ the family $(V(p))_{p \in P(C)}$ will be called a parametric family over C.

- Morley rank, U-rank and differential rank will be denoted by MR, U and DR respectively.

- If $\mathcal{M}$ is a structure and $\bar{a} \in M^n$ is a finite tuple, then the complete type of $\bar{a}$ in $\mathcal{M}$ over a parameter set $A \subseteq M$ will be denoted by $\operatorname{tp}^{\mathcal{M}}(\bar{a}/A)$ while $\operatorname{qftp}^{\mathcal{M}}(\bar{a}/A)$ stands for the quantifier-free type. We often omit the superscript $\mathcal{M}$ if the ambient model is clear.

- We use the symbol $\mathop{\smile\!\!\!\!\!\!|}$ for forking independence.

For convenience, we will recall some of the above notations throughout the thesis when we use them.

Chapter 2

Preliminaries

In this chapter we present some preliminary definitions and results that we will use in the thesis. We assume the reader is familiar with basics of model theory, referring to to [Mar02] and [TZ12] for a general introduction to the subject. For background on algebra, Lang's book [Lan02] is a very good reference, while for basics of algebraic geometry that we will need the reader is referred to [Sha13].

2.1 Differentially closed fields

In this section we present basic definitions and facts about differential fields. For more details and proofs of the results stated here we refer the reader to [Mar05b, Poi00, Kap57, Pil01, Pil03, vdD07].

Throughout the thesis we assume all rings that we deal with are commutative with identity and have characteristic zero.

The language of differential rings is $\mathfrak{L}_{\mathrm{D}} = \{+, \cdot, 0, 1, \mathrm{D}\}$ where D is a unary function symbol for derivation of the ring[1]. In this language we can axiomatise the theory of differential (rings) fields with the axioms of (rings) fields with two extra axioms stating that D is additive and satisfies Leibniz's rule, i.e. $\forall x, y \ \mathrm{D}(x+y) = \mathrm{D}\,x + \mathrm{D}\,y$ and $\forall x, y \ \mathrm{D}(xy) = x\,\mathrm{D}\,y + y\,\mathrm{D}\,x$. The theory of differential fields of characteristic zero is denoted by DF_0.

The *field of constants* of a differential field $(F; +, \cdot, 0, 1, \mathrm{D})$ is defined as the kernel of the derivation, i.e. $C_F = \{x \in F : \mathrm{D}\,x = 0\}$. This is always a relatively algebraically closed subfield of F.

If F is a differential field then the ring of *differential polynomials* over F is a differential ring extension defined as $F\{X\} = F[X, \mathrm{D}(X), \mathrm{D}^2(X), \ldots]$ with $\mathrm{D}(\mathrm{D}^n(X)) = \mathrm{D}^{n+1}(X)$. Thus, differential polynomials are of the form $p(X, \mathrm{D}\,X, \ldots, \mathrm{D}^n\,X)$ where $p(X_0, \ldots, X_n) \in F[X_0, \ldots, X_n]$ is an algebraic polynomial over F. A *differential rational function* over F is the quotient of two differential polynomials over F. The field of all differential rational functions of X over F will be denoted by $F\langle X\rangle$. We can also consider differential polynomials and rational functions in several variables,

[1] We will sometimes use the symbol $'$ for derivation.

which are defined analogously. If $f(X_1, X_2, \ldots, X_n)$ is a differential polynomial, then $f(x_1, \ldots, x_n) = 0$ is a *differential equation* over F.

Further, for F a differential field and $A \subseteq F$ a subset we denote by $\langle A\rangle$ or $\mathbb{Q}\langle A\rangle$ the differential subfield generated by A. If $K \subseteq F$ are differential fields and $A \subseteq F$ then $K\langle A\rangle$ is the differential subfield generated by K and A. The algebraic subdfield generated by K and A is denoted by $K(A)$. One can easily verify that $K\langle A\rangle = K(\{\mathrm{D}^n a : a \in A,\ n \in \mathbb{N}\})$.

The *order* of $f(X)$, denoted $\mathrm{ord}(f)$, is the biggest n for which $\mathrm{D}^n X$ occurs in f. For $n = \mathrm{ord}(f)$ the greatest m for which $(\mathrm{D}^n X)^m$ occurs in f is the *degree* of f. In the case of polynomials of several variables we will write $\mathrm{ord}_{X_i}(f)$ for the order of f with respect to X_i.

The theory DF_0 has a model completion. It is called the theory of *differentially closed fields* of characteristic zero. To axiomatise this theory we add the existential closedness axiom scheme: a differential field $(F; +, \cdot, \mathrm{D}, 0, 1)$ is *differentially closed* if for any non-constant differential polynomials $f(X)$ and $g(X)$ over F with $\mathrm{ord}(g) < \mathrm{ord}(f)$ there exists $x \in F$ such that $f(x) = 0$ and $g(x) \neq 0$. We let DCF_0 denote the theory of differentially closed fields of characteristic 0. In [PP98] D. Pierce and A. Pillay give a geometric axiomatisation of DCF_0. It immediately follows from the definition that differentially closed fields are algebraically closed (in the field theoretic sense). Hence, the field of constants is algebraically closed as well.

Suppose $K \subseteq F$ are two models of DF_0. For an element $a \in F$ one defines the *differential rank* (or *dimension* or *order*) of a over K, denoted $\mathrm{DR}(a/K)$ (or $\dim(a/K)$ or $\mathrm{ord}(a/K)$), as the transcendence degree of $K\langle a\rangle$ over K. If it is finite, say n, then there is a differential polynomial $f(X) \in K\{X\}$ of order n with $f(a) = 0$. If f is the simplest (i.e. of lowest order and degree) among such polynomials, then it is called *the minimal polynomial* of a over K. This polynomial must be irreducible. The elements $a, \mathrm{D}\,a, \ldots, \mathrm{D}^{n-1} a$ are algebraically independent, while $a, \mathrm{D}\,a, \ldots, \mathrm{D}^n a$ are algebraically dependent over K. In this case a is called *differentially algebraic* over K, otherwise it is called *differentially transcendental* over K. In the latter case $\mathrm{DR}(a/K)$ is defined to be ω.

In analogy with pure fields one can define the *differential transcendence degree* of a differential field F over a differential subfield K. Elements $x_1, \ldots, x_n \in F$ are called *differentially independent* over K if there is no non-zero differential polynomial $f(X_1, \ldots, X_n) \in K\{X_1, \ldots, X_n\}$ with $f(x_1, \ldots, x_n) = 0$. The cardinality of a maximal differentially independent set is the *differential transcendence degree* of F over K, denoted $\mathrm{dif.\,tr.\,deg.}(F/K)$. When we omit K, we mean the differential transcendence degree over $\mathbb{Q}$. This is well defined. In fact, differential independence defines a pregeometry on models of DCF_0.

The theory of differentially closed fields is model theoretically very nice. Namely, it admits elimination of quantifiers, elimination of imaginaries, it is complete and model complete. Further, DCF_0 is ω-stable with Morley rank ω.

Suppose $K \models \mathrm{DF}_0$ and $K \subseteq F$ is a differentially closed extension of K. Then for any element $a \in F$ the following inequality holds

$$\mathrm{U}(a/K) \leq \mathrm{MR}(a/K) \leq \mathrm{DR}(a/K),$$

where $\mathrm{U}(a/K)$ and $\mathrm{MR}(a/K)$ stand respectively for Morley rank and U-rank of a over K. Moreover, a is differentially transcendental over K if and only if $\mathrm{U}(a/K) = \mathrm{MR}(a/K) = \mathrm{DR}(a/K) = \omega$. In this case a is called *generic* over K (if we omit K then it means a is generic over the empty set or, equivalently, over the prime differential subfield).

There is a unique type of a differentially transcendental element (over a subfield K) which is determined by formulas $\{f(x) \neq 0 : f(X) \in K\{X\}\}$. This is called the *generic type* over K and has rank ω. The generic n-type is defined analogously and has rank $\omega \cdot n$.

Every differential field K has a *differential closure* which is defined as the prime model of DCF_0 over K. It always exists and is unique up to isomorphism (over K) in ω-stable theories. We will denote *the* differential closure of K by K^{dif}. Note that differential closures may not be minimal nevertheless.

Let us also describe the model theoretic algebraic and definable closures of a set in differentially closed fields. We will use these results later in Chapter 4.

Proposition 2.1.1. *Suppose $F \models \mathrm{DCF}_0$ and $A \subseteq F$ is a subset. Then*

- *The definable closure of A coincides with the differential subfield generated by A, that is,* $\mathrm{dcl}(A) = \mathbb{Q}\langle A \rangle$.
- *The model theoretic algebraic closure of A coincides with the field theoretic algebraic closure of the differential subfield generated by A, i.e.* $\mathrm{acl}(A) = (\mathbb{Q}\langle A \rangle)^{\mathrm{alg}}$.

At the end we present Seidenberg's embedding theorem, which will be mentioned and used several times in the thesis. For a proof, see [Mar05b], Appendix A.

Theorem 2.1.2 (Seidenberg's embedding theorem)**.** *Every countable differential field can be embedded into the field of germs of meromorphic functions at the origin.*

2.2 Differential algebraic curves

Now we define differential curves and make some easy observations about them that will be used in Chapter 4.

Definition 2.2.1. A *differential algebraic curve* E in a differential field K is a set in K^2 defined by a differential equation of two variables, i.e. $E = \{(x, y) \in K^2 : f(x, y) = 0\}$ for some $f(X, Y) \in K\{X, Y\}$. For brevity we will sometimes say *differential curve* instead of differential algebraic curve.

It makes sense to consider curves not only in K^2 (those are in fact *plane* curves) but also in K^n for any integer $n \geq 1$ (those can be defined as Kolchin closed sets of "dimension" 1, i.e. with Morley rank at least ω and less than $\omega \cdot 2$). However, we do not need that generality in this work.

Note also that by an algebraic curve we mean a set defined by an algebraic equation of two variables. Let $\mathbb{D} := \{(x, \mathrm{D}\, x) : x \in K\}$ be the graph of D in K. This is an example of a differential curve.

Further, we note that DCF_0 has minimality properties. By this we mean that $\mathrm{MD}(x = x) = 1$, where MD stands for Morley degree (as we have already mentioned $\mathrm{MR}(x = x) = \omega$). As in algebraically closed fields "small" means finite, in differentially closed fields small means of finite rank (any of the ranks mentioned above). Thus any definable set is either small or co-small, i.e. its complement is small.

Definition 2.2.2. A *(differential) curve in general sense* or an *almost curve* in K is a definable subset of K^2 the generic fibres of which are of finite Morley rank. In contrast to this we will sometimes use the nomenclature *proper (differential) curve* for a differential algebraic curve.

Thus a definable set $E \subseteq K^2$ is a curve in general sense if for any generic points $a, b \in K$ the fibres $E_a = \{y \in K : (a, y) \in E\}$ and $E_b = \{x \in K : (x, b) \in E\}$ are small. Clearly any proper differential curve is a curve in general sense. On the other hand it is easy to notice that any curve in general sense must be contained in a proper differential curve. This means it must be defined by a formula of the form $\varphi(x, y) = [f(x, y) = 0 \wedge \psi(x, y)]$ where f is a differential polynomial and ψ is any formula. In this case the Morley rank of generic fibres of E is uniformly bounded (by the number $\max(\mathrm{ord}_X(f), \mathrm{ord}_Y(f))$).

We could alternatively define curves in general sense to be definable sets of Morley rank less than $\omega \cdot 2$. One can also require $\mathrm{MR}(E)$ to be at least ω in order to avoid any degeneracies like $\mathrm{D}\, x = 0 \wedge \mathrm{D}\, y = 0$ (which correspond to finite sets in ACF_0).

2.3 Two theorems on dimension of algebraic varieties

Here we formulate two classical results from algebraic geometry which will be used several times throughout the thesis. We refer the reader to [Sha13] for proofs.

Theorem 2.3.1 (Dimension of intersection)**.** *Let U be a smooth irreducible algebraic variety and $V, W \subseteq U$ be subvarieties. Then any non-empty component X of the intersection $V \cap W$ satisfies*

$$\dim X \geq \dim V + \dim W - \dim U.$$

The inequality is equivalent to $\operatorname{codim} X \leq \operatorname{codim} V + \operatorname{codim} W$ (codimensions in U).

Definition 2.3.2. Let U, V, W be as above. A non-empty component X of $V \cap W$ is said to be *typical* if $\dim X = \dim V + \dim W - \dim U$ and *atypical* otherwise.

The following fact is the "additive formula for fibres".

Theorem 2.3.3. *Let $f : V \to W$ be a surjective regular map between irreducible varieties. If $n = \dim V$, $m = \dim W$ then $n \geq m$, and*

(i) $\dim X \geq n - m$ *for any $w \in W$ and any component X of the fibre $f^{-1}(w)$,*

(ii) *there is a non-empty open subset $U \subseteq W$ such that* $\dim f^{-1}(w) = n - m$ *for any $w \in U$.*

2.4 Pregeometries

Definition 2.4.1. A *pregeometry* is a pair (X, cl) where X is a non-empty set and $\mathrm{cl} : \mathfrak{P}(X) \to \mathfrak{P}(X)$ is a map satisfying the following conditions for all $A \subseteq X$ and $a, b \in X$:

(i) (Reflexivity) $A \subseteq \mathrm{cl}(A)$,

(ii) (Finite character) $\mathrm{cl}(A) = \bigcup_{A_0 \subseteq_{fin} A} \mathrm{cl}(A)$,

(iii) (Transitivity) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$,

(iv) (Exchange) $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A) \Rightarrow b \in \mathrm{cl}(Aa)$.

If cl satisfies the first three conditions then it is called a *closure operator*. A vector space with linear closure and an algebraically closed field with algebraic closure are basic examples of pregeometries.

In this section (X, cl) will always be a pregeometry.

Definition 2.4.2. A subset $A \subseteq X$ is said to be

(i) *independent*, if $a \notin \mathrm{cl}(A \setminus a)$ for all $a \in A$,

(ii) a *generating* set if $X = \mathrm{cl}(A)$,

(iii) a *basis* if it is an independent generating set.

For a subset $Y \subseteq X$ we can define a pregeometry on Y, called the *restriction* (to Y) and denoted cl^Y, by $\mathrm{cl}^Y(A) = \mathrm{cl}(A) \cap Y$ for $A \subseteq Y$. The *relativisation* is a pregeometry (X, cl_Y) defined by $\mathrm{cl}_Y(A) = \mathrm{cl}(A \cup Y)$.

All bases of a pregeometry have the same cardinality called the *dimension* of X and denoted by $d(X)$ or $\dim(X)$. The dimension of a subset Y is the dimension of the pregeometry (Y, cl^Y). The relative dimension $d(X/Y)$ is defined as the dimension of (X, cl_Y).

The dimension function of a pregeometry is *submodular*, that is, for any $A, B \subseteq X$ we have

$$d(AB) + d(A \cap B) \leq d(A) + d(B).$$

If equality holds for all closed A, B then the pregeometry is called *modular*, while if equality holds for all closed A, B with $d(A \cap B) > 0$ then it is said to be *locally modular*.

A pregeometry is called *trivial* if $\mathrm{cl}(A) = \bigcup_{a \in A} \mathrm{cl}(a)$ for all $A \subseteq X$.

Example 2.4.3.

- An infinite set with no structure is a trivial pregeometry ($\mathrm{cl}(\{x\}) = \{x\}$ for all $x \in X$). The dimension of a set is equal to its cardinality.

- A vector space with linear closure is a modular, non-trivial pregeometry. The dimension is the linear dimension.
- An algebraically closed field with algebraic closure is a non-locally modular pregeometry. Its dimension is equal to the transcendence degree over the prime subfield.

The above examples are three standard examples of pregeometries. More generally, a strongly minimal set with (model theoretic) algebraic closure is a pregeometry. Zilber's famous Trichotomy Conjecture states that all strongly minimal sets must be similar to one of those, i.e. they must be either trivial, or "vector space-like", or "field-like". The conjecture was refuted by Hrushovski in 1993 (see [Hru93]).

2.5 Trivial strongly minimal sets

In this section we define some standard properties of strongly minimal sets that will be used throughout Chapter 7. We also prove that geometric triviality of a strongly minimal set does not depend on the set of parameters over which it is defined. It is of course a well known classical result, but we sketch a proof here since we use it in the proof of Corollary 7.1.8. For a detailed account of strongly minimal sets and geometric stability theory in general we refer the reader to [Pil96].

As it was said in the previous section, algebraic closure defines a pregeometry on a strongly minimal set. More precisely, if X is a strongly minimal set in a structure $\mathcal{M}$ defined over $A \subseteq M$ then the operator

$$\mathrm{cl} : Y \mapsto \mathrm{acl}(AY) \cap X, \text{ for } Y \subseteq X,$$

is a pregeometry.

Definition 2.5.1. Let $\mathcal{M}$ be a structure and $X \subseteq M$ be a strongly minimal set defined over a finite set $A \subseteq M$.

- We say X is *geometrically trivial* (over A) if whenever $Y \subseteq X$ and $z \in \mathrm{acl}(AY) \cap X$ then $z \in \mathrm{acl}(Ay)$ for some $y \in Y$. In other words, geometric triviality means that the closure of a set is equal to the union of closures of singletons.
- X is called *strictly disintegrated* (over A) if any distinct elements $x_1, \ldots, x_n \in X$ are independent (over A).
- X is called $\aleph_0$-*categorical* (over A) if it realises only finitely many 1-types over AY for any finite $Y \subseteq X$. This is equivalent to saying that $\mathrm{acl}(AY) \cap X$ is finite for any finite $Y \subseteq X$.

Note that strict disintegratedness implies $\aleph_0$-categoricity and geometric triviality.

Theorem 2.5.2. *Let $\mathcal{M}$ be a model of an ω-stable theory and $X \subseteq M$ be as above. If X is geometrically trivial over A then it is geometrically trivial over any superset $B \supseteq A$.*

Proof. By expanding the language with constant symbols for elements of A we can assume that X is $\emptyset$-definable. Also we can assume $B = \{b_1, \dots, b_n\}$ is finite. Let $z \in \mathrm{acl}(BY)$ for some finite $Y \subseteq X$. By stability $\mathrm{tp}(\bar{b}/X)$ is definable over a finite $C \subseteq X$ and we may assume that $C \subseteq \mathrm{acl}(B) \cap X$. Therefore $z \in \mathrm{acl}(CY)$. By geometric triviality of X (over $\emptyset$) we have $z \in \mathrm{acl}(c)$ for some $c \in C$ or $z \in \mathrm{acl}(y)$ for some $y \in Y$. This shows geometric triviality of X over B. $\square$

As we saw in the proof, all definable subsets of X^m over B are definable over $\mathrm{acl}(B) \cap X$ (which is the *stable embedding* property). The same argument shows that $\aleph_0$-categoricity does not depend on parameters (see also [NP16]). Of course this is not true for strict disintegratedness but a weaker property is preserved. Namely if X is strictly disintegrated over A then any distinct non-algebraic elements over B are independent over B.

Chapter 3

Predimensions and Hrushovski Constructions

In this chapter we define predimensions and strong embeddings and observe several standard facts about them. Then we give a brief account of Hrushovski's amalgamation-with-predimension construction. It is the uncollapsed version of a full Hrushovski construction [Hru93, Hru92]. This will be used to define *adequacy* of a predimension inequality. The Ax-Schanuel inequality for the exponential differential equation (Ax [Ax71]) and its analogue for the differential equation of the modular j-function (Pila-Tsimerman [PT16]) are our main examples.

We will observe the close relationship between triviality of an adequate predimension inequality and model completeness of the corresponding strong Fraïssé limit. This will be used to prove that if a derivation is definable from a differential equation then the latter cannot satisfy any non-trivial adequate predimension inequality (assuming $\forall\exists$-axiomatisability). We will also see that adequate predimensions of certain kind do not allow definability of a derivation without any assumptions on the theory of the reduct.

We mainly follow Wagner [Wag94] and Baldwin [Bal02] in defining predimensions and related notions. They give an axiomatic approach to Hrushovski constructions. Wagner works in a relational language, while Baldwin's setting does not have this restriction. We need that generality since we always have a field structure in our examples. Note that Baldwin imposes stronger definability conditions for the predimension than we do. The reason is that the Ax-Schanuel predimension does not satisfy his definability axioms. Our approach is motivated by Kirby's analysis of the exponential differential equations [Kir09] and Zilber's approach to complex exponentiation and Schanuel's conjecture.

Hrushovski invented the aforementioned constructions in order to produce structures with "exotic" geometry and refute some conjectures on categorical theories and answer some questions. Most notably, he refuted Zilber's Trichotomy Conjecture [Zil84a, Zil84b] stating that any uncountably categorical and non-locally modular theory is bi-interpretable with an algebraically closed field, and Lachlan's conjecture [Lac74] stating that any stable $\aleph_0$-categorical theory is totally transcendental. Later on Hrushovski's techniques were adapted and used in various settings to construct interesting structures. The reader is referred to [Hru92, Hru93, Wag94, Wag09, Bal02, BH00, Zil04b] for details on Hrushovski constructions and examples of "exotic" structures (theories) that can be obtained by those constructions.

3.1 Predimensions

Let $\mathfrak{L}$ be a countable language and $\mathfrak{C}$ be a collection of $\mathfrak{L}$-structures closed under isomorphism and intersections. The latter can be understood in a category theoretic sense, but for us it will be enough to assume that if $A_i \in \mathfrak{C},\ i \in I$, are substructures of some $A \in \mathfrak{C}$ then $\bigcap_{i\in I} A_i \in \mathfrak{C}$. We will also assume that $\mathfrak{C}$ has the joint embedding property, i.e. for any $A, B \in \mathfrak{C}$ there is $C \in \mathfrak{C}$ such that A and B can be embedded into C. Assume further that $\mathfrak{C}$ contains a smallest structure $S \in \mathfrak{C}$, that is, S can be embedded into all structures of $\mathfrak{C}$.

Definition 3.1.1. For $B \in \mathfrak{C}$ and $X \subseteq B$ the $\mathfrak{C}$-*closure* of X inside B (or the $\mathfrak{C}$-substructure of B generated by X) is the structure[1]

$$\langle X\rangle_B := \bigcap_{A\in\mathfrak{C}: X\subseteq A\subseteq B} A.$$

A structure $A \in \mathfrak{C}$ is *finitely generated* if $A = \langle X\rangle_A$ for some finite $X \subseteq A$. The collection of all finitely generated structures from $\mathfrak{C}$ will be denoted by $\mathfrak{C}_{f.g.}$.

Note that in general finitely generated in this sense is different from being finitely generated as a structure. We will assume however that finitely generated structures are countable.

Since S is the smallest structure in $\mathfrak{C}$, it is in fact generated by the empty set, i.e. $S = \langle\emptyset\rangle$. So, by abuse of notation, we will normally write $\emptyset$ instead of S.

For $A, B \in \mathfrak{C}$ by $A \subseteq_{f.g.} B$ we mean A is a finitely generated substructure of B. When we have two structures $A, B \in \mathfrak{C}$ we would like to have a notion of a structure generated by A and B. However, this cannot be well-defined without embedding A and B into a bigger C. Given such a common extension C, we will denote $AB_C := \langle A \cup B\rangle_C$. Often we will drop the subscript C meaning that our statement holds for every common extension C (or it is obvious in which common extension we work). This remark is valid also when we write $A \cap B$ which should be understood as the intersection of A and B after identifying them with their images in a common extension.

In general a substructure of a finitely generated structure may not be finitely generated. However, in our examples this does not happen. So we make the following assumption.

Assumption 3.1.2. Assume $\mathfrak{C}$ satisfies the following condition.

[1] In the differential setting the notation $\langle A\rangle$ is used to denote the differential subfield generated by a set A. The meaning of this notation will be clear from the context. In particular it is used only for $\mathfrak{C}$-closure in this chapter.

FG If $A \in \mathfrak{C}, B \in \mathfrak{C}_{f.g.}$ with $A \subseteq B$ then $A \in \mathfrak{C}_{f.g.}$.

Definition 3.1.3. A *predimension* on $\mathfrak{C}_{f.g.}$ is a function $\delta : \mathfrak{C}_{f.g.} \to \mathbb{Z}$ with the following properties:

P1 $\delta(\emptyset) = 0$,

P2 If $A, B \in \mathfrak{C}_{f.g.}$ with $A \cong B$ then $\delta(A) = \delta(B)$,

P3 (Submodularity) For all $A, B \in \mathfrak{C}_{f.g.}$ and $C \in \mathfrak{C}$ with $A, B \subseteq C$ we have

$$\delta(AB) + \delta(A \cap B) \leq \delta(A) + \delta(B). \tag{1.1}$$

If, in addition, such a function is monotonic, i.e. $A \subseteq B \Rightarrow \delta(A) \leq \delta(B)$, and hence takes only non-negative values, then δ is called a *dimension*.

Definition 3.1.4. Given a predimension δ, for a finite subset $X \subseteq_{fin} A \in \mathfrak{C}$ one defines

$$\delta_A(X) := \delta(\langle X \rangle_A).$$

The following is Hrushovski's *ab initio* example from [Hru93].

Example 3.1.5. Let $\mathfrak{C}$ be the class of all structures $(M; R)$ in a language $\mathfrak{L} = \{R\}$ consisting of one ternary relation R. Then $\mathfrak{C}_{f.g.}$ is the collection of all finite $\mathfrak{L}$-structures. For $A \in \mathfrak{C}_{f.g.}$ define

$$\delta(A) := |A| - |R(A)|.$$

Then δ is a predimension.

Other examples of predimensions, which are more relevant to our work, will be given in Section 3.3.

Now we define the relative predimension of two structures, which depends on a common extension of those structures (so we work in such a common extension without explicitly mentioning it).

Definition 3.1.6. The *relative predimension* is defined as follows.

- For $A, B \in \mathfrak{C}_{f.g.}$ define $\delta(A/B) := \delta(AB) - \delta(B)$.
- For $X \in \mathfrak{C}_{f.g.}$ and $A \in \mathfrak{C}$ define $\delta(X/A) \geq k$ for an integer k if for all $Y \subseteq_{f.g.} A$ there is $Y \subseteq Y' \subseteq_{f.g.} A$ such that $\delta(X/Y') \geq k$. Also, $\delta(X/A) = k$ if $\delta(X/A) \geq k$ and $\delta(X/A) \ngeq k + 1$.

In the next definition B is the ambient structure that we work in.

Definition 3.1.7. Let $A \subseteq B \in \mathfrak{C}$. We say A is *strong* (or *self-sufficient*) in B, denoted $A \leq B$, if for all $X \subseteq_{f.g.} B$ we have $\delta(X/A) \geq 0$. One also says B is a *strong extension* of A. An embedding $A \hookrightarrow B$ is strong if the image of A is strong in B.

It is easy to notice that the above definition will not change if we take a finite set X instead of a finitely generated structure X.

Lemma 3.1.8. *Let $A, B \in \mathfrak{C}$. Then $A \leq B$ if and only if for all $X \subseteq_{f.g.} B$ we have $\delta(X \cap A) \leq \delta(X)$.*

Proof. Let $A \leq B$ and $X \subseteq_{f.g.} B$. Choose $Y = X \cap A \subseteq_{f.g.} A$. Then by definition there is $Y \subseteq Y' \subseteq_{f.g.} A$ such that $\delta(XY') \geq \delta(Y')$. Now by submodularity

$$\delta(X \cap A) = \delta(Y) = \delta(X \cap Y') \leq \delta(X) + \delta(Y') - \delta(XY') \leq \delta(X).$$

Conversely, assume the condition given in the lemma holds. We need to prove that $A \leq B$. Let $X \subseteq_{f.g.} B$ and $Y \subseteq_{f.g.} A$. Choose $Y' = XY \cap A \supseteq Y$. Then

$$\delta(XY') = \delta(XY) \geq \delta(XY \cap A) = \delta(Y')$$

as $XY \subseteq_{f.g.} B$. □

Definition 3.1.9. For $B \in \mathfrak{C}$ and $X \subseteq B$ we define the *self-sufficient closure* of X in B by

$$\lceil X \rceil_B := \bigcap_{A \in \mathfrak{C}: X \subseteq A \leq B} A.$$

It is easy to see that the intersection of finitely many strong substructures is strong as well. This can be used to show that an arbitrary intersection of strong substructures is strong. It follows from this that $\lceil X \rceil_B \leq B$. Note also that $\leq$ is transitive.

Lemma 3.1.10. *Let $M \in \mathfrak{C}$ be saturated. If $X, Y \subseteq_{fin} M$ (with some enumeration) have the same type in M then $\langle X \rangle_M \cong \langle Y \rangle_M$ and $\lceil X \rceil_M \cong \lceil Y \rceil_M$ and hence $\delta_M(X) = \delta_M(Y)$.*

Proof. Since M is saturated and $\mathrm{tp}(X) = \mathrm{tp}(Y)$, there is an automorphism that sends X to Y. Now the lemma follows from P2. □

From now on we assume $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$. In other words $\emptyset$ is strong in all structures of $\mathfrak{C}$. Instead of assuming this we could work with the subclass $\mathfrak{C}^0$ of all structures with non-negative predimension. However, we find it more convenient to assume δ is non-negative on $\mathfrak{C}_{f.g.}$ since anyway this will be the case in our examples.

Lemma 3.1.11. *If $B \in \mathfrak{C}$ and $X \subseteq_{f.g.} B$ then*

- *$\lceil X \rceil_B$ is finitely generated, and*
- *$\delta(\lceil X \rceil_B) = \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\}$.*

Proof. Let $A \subseteq_{f.g.} B$ be such that $\delta(A) = \min\{\delta(A') : X \subseteq A' \subseteq_{f.g.} B\}$. We claim that $A \leq B$. Indeed, for any $Y \subseteq_{f.g.} B$ we have

$$\delta(A \cap Y) \leq \delta(A) - \delta(AY) + \delta(Y) \leq \delta(Y).$$

Thus $A \leq B$ and hence $\lceil X \rceil_B$ is contained in finitely generated A and so is finitely generated itself.

Further, $\lceil X \rceil_B \leq A$ so $\delta(\lceil X \rceil_B) \leq \delta(A)$. Now by minimality of $\delta(A)$ we conclude that $\delta(\lceil X \rceil_B) = \delta(A)$. □

A predimension gives rise to a dimension in the following way.

Definition 3.1.12. For $X \subseteq_{f.g.} B$ define

$$d_B(X) := \min\{\delta(Y) : X \subseteq Y \subseteq_{f.g.} B\} = \delta(\lceil X \rceil_B).$$

For $X \subseteq_{fin} B$ set $d_B(X) := d_B(\langle X \rangle_B)$.

It is easy to verify that d is a dimension function and therefore we have a natural pregeometry associated with δ. More precisely, we define $\mathrm{cl}_B : \mathfrak{P}(B) \to \mathfrak{P}(B)$ by

$$\mathrm{cl}_B(X) = \{b \in B : d_B(b/X) = 0\}.$$

Then (B, cl_B) is a pregeometry and d_B is its dimension function.

Self-sufficient embeddings can be defined in terms of d. Indeed, if $A \subseteq B$ then $A \leq B$ if and only if for any $X \subseteq_{fin} A$ one has $d_A(X) = d_B(X)$.

Definition 3.1.13. A predimension δ is *trivial* if all embeddings are strong. Equivalently, δ is trivial if it is monotonic and hence equal to the dimension associated with it.

Proposition 3.1.14. *Let $A, B \in \mathfrak{C}$ be saturated and $A \preceq B$. Then $A \leq B$.*

Proof. If $A \nleq B$ then for some $X \subseteq_{f.g.} B$ one has $\delta(X/A) < 0$. This means that there is $Y \subseteq_{f.g.} A$ such that for all $Y \subseteq Y' \subseteq_{f.g.} A$ we have $\delta(X/Y') < 0$. Choose $Y' = \lceil Y \rceil_A$. The latter is finitely generated by Lemma 3.1.11. Suppose $X = \langle \bar{x} \rangle_B$ and $Y' = \langle \bar{y} \rangle_A$ for some finite tuples $\bar{x}$ and $\bar{y}$. Let $\bar{z}$ be a realisation of the type $\mathrm{tp}^B(\bar{x}/\bar{y})$ in A. If $Z = \langle \bar{z} \rangle_A$ then $\delta(Z/Y') < 0$ (by Lemma 3.1.10) which means $\delta(Y'Z) < \delta(Y')$ contradicting Lemma 3.1.11. $\square$

3.2 Amalgamation with predimension

Now we formulate conditions under which one can carry out an amalgamation-with-predimension construction. Let $\mathfrak{C}$ be as above and let δ be a non-negative predimension on $\mathfrak{C}_{f.g.}$.

Definition 3.2.1. The class $\mathfrak{C}$ is called a *strong amalgamation class* if the following conditions hold.

- C1 Every $A \in \mathfrak{C}_{f.g.}$ has at most countably many finitely generated strong extensions up to isomorphism.
- C2 $\mathfrak{C}$ is closed under unions of countable strong chains $A_0 \leq A_1 \leq \ldots$.
- SAP $\mathfrak{C}_{f.g.}$ has the *strong amalgamation property*, that is, for all $A_0, A_1, A_2 \in \mathfrak{C}_{f.g.}$ with $A_0 \leq A_i,\ i = 1, 2$, there is $B \in \mathfrak{C}_{f.g.}$ such that A_1 and A_2 are strongly embedded into B and the corresponding diagram commutes.

Remark 3.2.2. Since $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$, it follows that $\emptyset$ is strong in all finitely generated structures and hence the strong amalgamation property implies the *strong joint embedding* property.

The following is a standard theorem that follows in particular from the category theoretic version of Fraïssé's amalgamation construction due to Droste and Göbel [DG92] (see [Kir09] for a nice exposition, without a proof though).

Theorem 3.2.3 (Amalgamation theorem)**.** *If $\mathfrak{C}$ is a strong amalgamation class then there is a unique (up to isomorphism) countable structure $U \in \mathfrak{C}$ with the following properties.*

U1 *U is* universal *with respect to strong embeddings, i.e. every countable $A \in \mathfrak{C}$ can be strongly embedded into U.*

U2 *U is* saturated *with respect to strong embeddings, i.e. for every $A, B \in \mathfrak{C}_{f.g.}$ with strong embeddings $A \hookrightarrow U$ and $A \hookrightarrow B$ there is a strong embedding of B into U over A.*

Furthermore, any isomorphism between finitely generated strong substructures of U can be extended to an automorphism of U.

This U is called the *generic model*, *strong amalgam*, *strong Fraïssé limit* or *Fraïssé-Hrushovski limit* of $\mathfrak{C}_{f.g.}$. It has a natural pregeometry associated with the predimension function as described in the previous section. Note that U2 is normally known as the *richness* property in literature (we used the terminology of [DG92] above).

Remark 3.2.4. Since we have assumed $\emptyset$ is strong in all structures from $\mathfrak{C}$, the property U2 implies U1. Indeed, for $A \in \mathfrak{C}_{f.g.}$ we have $\emptyset \leq A$ and $\emptyset \leq U$. Hence by U2 there is a strong embedding $A \hookrightarrow U$. Now since every countable structure in $\mathfrak{C}$ is the union of a strong chain of finitely generated structures, every such structure can be strongly embedded into U. Thus, U2 determines the Fraïssé limit uniquely.

Now we consider a stronger amalgamation property known as the *asymmetric amalgamation property*. However, in our examples the class $\mathfrak{C}_{f.g.}$ does not have this property, so we need to assume a subclass has that property.

Assumption 3.2.5. Assume there is a subclass $\hat{\mathfrak{C}} \subseteq \mathfrak{C}$ with the following properties[2].

C3 Every structure $A \in \mathfrak{C}$ has a unique (up to isomorphism over A) extension $\hat{A} \in \hat{\mathfrak{C}}$ which is $\hat{\mathfrak{C}}$-generated by A. If $A \in \mathfrak{C}_{f.g.}$ then $\hat{A} \in \hat{\mathfrak{C}}_{f.g.}$.

C4 If $A, B \in \mathfrak{C}$ with a strong embedding $A \hookrightarrow B$ then it can be extended to a strong embedding $\hat{A} \hookrightarrow \hat{B}$.

C5 $\hat{\mathfrak{C}}$ is closed under unions of countable strong chains.

[2] $\hat{\mathfrak{C}}_{f.g.}$ denotes the collection of structures from $\hat{\mathfrak{C}}$ that are finitely $\hat{\mathfrak{C}}$-generated.

AAP (Asymmetric Amalgamation Property) If $A_0, A_1, A_2 \in \hat{\mathfrak{C}}_{f.g.}$ with a strong embedding $A_0 \leq A_1$ and an embedding $A_0 \hookrightarrow A_2$ (not necessarily strong), then there is $B \in \hat{\mathfrak{C}}_{f.g.}$ with an embedding $A_1 \hookrightarrow B$ and a strong embedding $A_2 \leq B$ such that the corresponding diagram commutes. Moreover, if A_0 is strong in A_2 then A_1 is strong in B.

Proposition 3.2.6. *If $\hat{\mathfrak{C}}$ satisfies* AAP *then $\mathfrak{C}_{f.g.}$ has the strong amalgamation property.*

Proof. Let $A, B_1, B_2 \in \mathfrak{C}_{f.g.}$ with strong embeddings $A \hookrightarrow B_1,\ A \hookrightarrow B_2$. By our assumptions we have strong extensions $A \leq \hat{A},\ B_1 \leq \hat{B}_1,\ B_2 \leq \hat{B}_2$ with $\hat{A}, \hat{B}_1, \hat{B}_2 \in \hat{\mathfrak{C}}$. Moreover, there are strong embeddings of $\hat{A}$ into $\hat{B}_1$ and $\hat{B}_2$. Now we can use the AAP property of $\hat{\mathfrak{C}}$ to construct a strong amalgam B' of $\hat{B}_1$ and $\hat{B}_2$ over $\hat{A}$. Let B be the substructure of B' $\mathfrak{C}$-generated by B_1 and B_2. Clearly $B \in \mathfrak{C}_{f.g.}$ and it is a strong amalgam of B_1 and B_2 over A. □

Notation. For $A \in \hat{\mathfrak{C}}$ and a subset $X \subseteq A$, the substructure of A $\mathfrak{C}$-generated by X will be denoted by $\langle X\rangle_A^{\mathfrak{C}}$ while $\langle X\rangle_A^{\hat{\mathfrak{C}}}$ stands for the substructure of A $\hat{\mathfrak{C}}$-generated by X. The same pertains to strong substructures generated by X in the two classes. When no confusion can arise, we will drop the superscript.

Proposition 3.2.7. *Under the assumptions* C1-5, AAP*, the classes $\mathfrak{C}_{f.g.}$ and $\hat{\mathfrak{C}}_{f.g.}$ are strong amalgamation classes and have the same strong Fraïssé limit.*

Proof. Firstly, we show that $\hat{\mathfrak{C}}$ is a strong amalgamation class. For this we need to prove that every countable $A \in \hat{\mathfrak{C}}$ has at most countably many strong finitely generated extensions in $\hat{\mathfrak{C}}$, up to isomorphism.

Let $B \in \hat{\mathfrak{C}}_{f.g.}$ be generated by $\bar{b}$ over A as a $\hat{\mathfrak{C}}$-structure. Denote $B_0 := \lceil A\bar{b}\rceil_B^{\mathfrak{C}}$. Then $B_0 \leq B$ and $B = \langle B_0\rangle_B^{\hat{\mathfrak{C}}}$ which shows that $B = \hat{B}_0$. Since $A \leq B$, we have $A \leq B_0$ and so there are countably many choices for B_0 and hence countably many choices for B.

Let U be the strong Fraïssé limit of $\hat{\mathfrak{C}}$. We will show that it satisfies U2 for $\mathfrak{C}_{f.g.}$. Let $A, B \in \mathfrak{C}_{f.g.}$ with strong embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow U$. We can extend f and g to strong embeddings $\hat{A} \hookrightarrow \hat{B}$ and $\hat{A} \hookrightarrow U$ over A. Therefore $\hat{B}$ can be strongly embedded into U over $\hat{A}$. The restriction of this embedding to B will be a strong embedding of B into U over A.

Thus U is also strongly saturated for $\mathfrak{C}$, hence U is isomorphic to the Fraïssé limit of $\mathfrak{C}$. □

Proposition 3.2.8. *Under the above assumption U has the following* Asymmetric Richness Property.

ARP *If $A \leq B \in \hat{\mathfrak{C}}_{f.g.}$ then any embedding $A \hookrightarrow U$ extends to an embedding $B \hookrightarrow U$. Moreover, if the former embedding is strong then so is the latter.*

Proof. Let $\lceil A\rceil \in \hat{\mathfrak{C}}_{f.g.}$ be the self-sufficient closure of A in U (in the sense of $\hat{\mathfrak{C}}$). By AAP there is $B' \in \hat{\mathfrak{C}}_{f.g.}$ with embeddings $\lceil A\rceil \leq B'$ and $B \hookrightarrow B'$ over A. Now richness of U implies the desired result. □

The ARP property says that the amalgam U is existentially closed in strong extensions, which is normally used to give a first-order axiomatisation of the amalgam.

In general U1 and U2 are not first-order axiomatisable, nor is ARP. Normally they are $\mathfrak{L}_{\omega_1,\omega}$-axiomatisable provided the predimension has some definability properties (which we specify below). In order to extract a first-order axiomatisation from this $\mathfrak{L}_{\omega_1,\omega}$-axiomatisation, one normally approximates U1 and U2 by finitary axioms which are first-order. Wagner considers this problem in [Wag94] and gives the appropriate conditions under which it can be done, working in a relational language though. In particular, if the language is finite and relational and $\mathfrak{C}_{f.g.}$ consists of finite structures then one can find a first-order axiomatisation of the amalgam. In general it is possible to give a similar first-order axiomatisation of $\mathrm{Th}(U)$ imposing quite strong definability conditions on δ. However it seems those conditions would fail for the Ax-Schanuel predimension (see Section 3.3) and so we consider weaker definability conditions.

Let $M \in \mathfrak{C}$ be an arbitrary structure.

Definition 3.2.9. We say δ is *(infinitely) definable* in M if for any $n, m \in \mathbb{N}$ the set $\{\bar{a} \in M^n : \delta(\bar{a}) \geq m\}$ is definable by a possibly infinite Boolean combination of first-order formulas, i.e. an $\mathfrak{L}_{\omega_1,\omega}$-formula of the form

$$\bigwedge_{i<\omega} \bigvee_{j<\omega} \varphi_{i,j}^{m,n}(\bar{x}), \tag{2.1}$$

where $\varphi_{i,j}^{m,n}(\bar{x})$ are first-order formulae. We say δ is *universally definable* if the formulas $\varphi_{i,j}^{m,n}$ can be chosen to be equivalent to universal formulas in M.

Recall that we assumed δ is non-negative. This means, in particular, that

$$\delta(\bar{x}) \geq 0 \text{ for all finite tuples } \bar{x} \subseteq M. \tag{2.2}$$

Lemma 3.2.10. *If $M \in \mathfrak{C}$ is saturated and δ is definable then the inequality* (2.2) *is first-order axiomatisable.*

Proof. By (2.2) we know that for each i we have

$$M \models \forall \bar{x} \bigvee_{j<\omega} \varphi_{i,j}^{0,n}(\bar{x}).$$

Since M is saturated, there is a positive integer N_i such that

$$M \models \forall \bar{x} \bigvee_{j<N_i} \varphi_{i,j}^{0,n}(\bar{x}).$$

Then (2.2) is axiomatised by the following collection of axioms:

$$\forall \bar{x} \bigvee_{j<N_i} \varphi_{i,j}^{0,n}(\bar{x}),\ i < \omega, n < \omega.$$

□

For a finite set $\bar{a} \subseteq M$ we say $\bar{a}$ is strong in M if $\langle \bar{a} \rangle \leq M$. Definability of δ implies that for a finite set being strong in M is $\mathfrak{L}_{\omega_1,\omega}$-definable.

Lemma 3.2.11. *Assume U is saturated and δ is universally definable in U. Then* $\mathrm{Th}(U)$ *is nearly model complete, that is, every formula is equivalent to a Boolean combination of existential formulas in U.*

Proof. For a finite tuple $\bar{a} \subseteq U$ its type (in U) is determined by the isomorphism type of $\lceil \bar{a} \rceil_U$ which is determined by finitely generated non-strong extensions of $\langle \bar{a} \rangle$ in U. If $\bar{a}$ and $\bar{b}$ satisfy exactly the same existential formulae (and hence exactly the same universal formulae), then for any non-strong extension of $\langle \bar{a} \rangle$ there is an isomorphic non-strong extension of $\langle \bar{b} \rangle$. Hence $\lceil \bar{a} \rceil_U \cong \lceil \bar{b} \rceil_U$. Thus, $\mathrm{tp}(\bar{a})$ is determined by existential formulae and their negations that are true of $\bar{a}$. Therefore $\mathrm{Th}(U)$ is nearly model complete. $\square$

When one knows the first-order theory of U, one can normally understand whether U is saturated or not. It is saturated in our main examples, i.e. the exponential differential equation and the equation of the j-function (see Chapter 6). However, in general, it is possible to have a non-saturated Fraïssé limit. Baldwin and Holland [BH00] give a criterion (called *separation of quantifiers*) for saturatedness of U (working under stronger definability conditions for δ though).

Definition 3.2.12. We say δ is trivial on $\hat{\mathfrak{C}}$ if all embeddings of structures from $\hat{\mathfrak{C}}$ are strong.

Note that in general δ is not defined on $\hat{\mathfrak{C}}$ (nor on $\hat{\mathfrak{C}}_{f.g.}$), so to be more precise we could say that strong embeddings induced by δ are trivial on $\hat{\mathfrak{C}}$. ***From now on, triviality of δ should be understood in this sense.***

Proposition 3.2.13. *Assume U is saturated. If δ is non-trivial on $\hat{\mathfrak{C}}$ then* $\mathrm{Th}(U)$ *is not model complete.*

Proof. Non-triviality of the predimension means there are finitely generated $A \subseteq B \in \hat{\mathfrak{C}}_{f.g.}$ with $A \not\leq B$. By universality of U we know that there is a strong embedding of A into U. Using the asymmetric amalgamation property we find a structure $U' \in \mathfrak{C}$ which extends U and extends B strongly such that the corresponding diagram commutes. This can be done since the amalgam U is the union of a countable strong chain of finitely generated structures. So we can inductively use the asymmetric amalgamation for each of these structures and take the union of amalgams obtained in each step (these amalgams form a strong increasing chain). Then it is easy to see that $U \not\leq U'$. On the other hand, U' is countable and hence it can be embedded into U. Thus we have embeddings $U \hookrightarrow U' \hookrightarrow U$ and the first one is non-strong. Therefore we have a non-strong embedding of U into itself. By Proposition 3.1.14 this embedding is not elementary which means $\mathrm{Th}(U)$ is not model complete. $\square$

Now we define what it means for the inequality (2.2) to be adequate.

Definition 3.2.14. Let $\hat{\mathfrak{C}} \subseteq \mathfrak{C}$ be classes of structures closed under isomorphism and intersections and such that $\emptyset \in \mathfrak{C}$. Assume they satisfy FG, C1-5, AAP and δ is a non-negative universally definable predimension on $\mathfrak{C}_{f.g.}$. Let $M \in \mathfrak{C}$ be a countable structure.

- We say that δ (or the inequality (2.2)) is *adequate* for M if $U \equiv M$.
- We say δ is *strongly adequate* for M if $M \cong U$.

In other words, adequacy of a predimension inequality means that $\mathrm{Th}(M)$ can be obtained by a Hrushovski construction and strong adequacy means that the structure M itself can be obtained by a Hrushovski construction. These notions will make more sense in differential setting where M is always taken to be a reduct of a differentially closed field. Note also that when M and U are saturated, adequacy of δ implies its strong adequacy.

Note that we do not need definability of δ or AAP for some subclass $\hat{\mathfrak{C}}$ in order to construct the strong Fraïssé limit U and define adequacy. However, these are natural assumptions since in most cases (in differential setting) the properties FG, C1-5 and definability of δ will be evident while strong amalgamation of $\mathfrak{C}_{f.g.}$ will be deduced from strong amalgamation of $\hat{\mathfrak{C}}_{f.g.}$, and in fact $\hat{\mathfrak{C}}_{f.g.}$ will have the asymmetric amalgamation property. That is the reason that we included all those conditions in the definition of adequacy. This will be illustrated in Chapter 6.

3.3 Examples

In this section we give examples of predimensions that are the main motivating factor for this work.

3.3.1 Complex exponentiation

Let $\mathbb{C}_{\exp} := (\mathbb{C}; +, \cdot, 0, 1, \exp)$ be the complex exponential field. Let $E(x, y)$ be the graph of the exponential function and consider the structure $\mathbb{C}_E := (\mathbb{C}; +, \cdot, 0, 1, E)$. Note that it is not saturated and its first-order theory is not stable since $\mathbb{Z}$ is definable.

For complex numbers $x_1, \ldots, x_n$ and their exponentials $y_1, \ldots, y_n$ define

$$\delta(\bar{x}, \bar{y}) := \mathrm{td}_{\mathbb{Q}}\, \mathbb{Q}(\bar{x}, \bar{y}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}).$$

Schanuel's conjecture states non-negativity of this function.

Consider the class $\mathfrak{C}$ of all (field-theoretically) algebraically closed substructures of $\mathbb{C}_E$. For a finitely generated (i.e. of finite transcendence degree over $\mathbb{Q}$) substructure A define

$$\begin{aligned}\sigma(A) := \max\{n : &\text{ there are } a_i, b_i \in A,\ i = 1, \ldots, n, \text{ with } a_i\text{'s} \\ &\text{linearly independent over } \mathbb{Q} \text{ and } A \models E(a_i, b_i)\}\end{aligned}$$

and

$$\delta(A) := \mathrm{td}_{\mathbb{Q}}(A) - \sigma(A).$$

Then σ is finite provided Schanuel's conjecture holds and δ is a well-defined non-negative predimension. However the inequality $\delta \geq 0$ is not first-order axiomatisable even assuming the conjecture holds.

Schanuel's conjecture is widely open and so we cannot say much about this example. It is quite complicated from a model theoretic point of view. In particular, $\mathbb{Z}$ is definable in $\mathbb{C}_E$. So its first order theory is quite difficult to study. In spite of this Zilber discovered a nice way of treating the complex exponential field using infinitary logic. He considered algebraically closed fields with a relation which has some of the properties of complex exponentiation. Then he took all those structure where the analogue of Schanuel's conjecture holds. By a Hrushovski style construction he obtained a theory called *pseudo-exponentiation*. It is axiomatised in the language $\mathfrak{L}_{\omega_1,\omega}(Q)$ where Q is a quantifier for "there exist uncountably many". This theory (and its first-order part) is a natural candidate for the $\mathfrak{L}_{\omega_1,\omega}(Q)$-theory (respectively, first-order theory) of $\mathbb{C}_E$. Nevertheless, all these questions seem to be out of reach at the moment. We refer the reader to [Zil04b, KZ14, Zil02, Zil16, Zil15, Kir13] for details. Note also that many ideas in the analysis of the exponential differential equation (see below) originate in Zilber's work on pseudo-exponentiation.

Remark 3.3.1. Submodularity does not hold for finite sets.[3] Indeed, let $a, b \in \mathbb{C}$ with $\delta(a) = \delta(b) = 1,\ \delta(a, b) = 0$. Then taking $A = \{a, b\}, B = \{2a, b\}$ we get

$$\delta(A \cup B) + \delta(A \cap B) = 0 + 1 > 0 + 0 = \delta(A) + \delta(B).$$

3.3.2 Exponential differential equation

Let $\mathcal{K} := (K; +, \cdot, ', 0, 1)$ be a countable saturated differentially closed field with field of constants C. Let $\mathrm{Exp}(x, y)$ be defined by the exponential differential equation $y' = yx'$ and denote $\mathcal{K}_{\mathrm{Exp}} := (K; +, \cdot, \mathrm{Exp}, 0, 1)$. Fix the language $\mathfrak{L}_{\mathrm{Exp}} := \{+, \cdot, \mathrm{Exp}, 0, 1\}$. Consider the following axioms for an $\mathfrak{L}_{\mathrm{Exp}}$-structure F ($\mathbb{G}_a$ and $\mathbb{G}_m$ denote the additive and multiplicative groups of a field and $G_n := \mathbb{G}_a^n \times \mathbb{G}_m^n$).

- A1 F is an algebraically closed field of characteristic 0.
- A2 $C_F := \{c \in F : F \models \mathrm{Exp}(c, 1)\}$ is a algebraically closed subfield of F.
- A3 $\mathrm{Exp}(F) = \{(x, y) \in F^2 : \mathrm{Exp}(x, y)\}$ is a subgroup of $G_1(F)$ containing $G_1(C_F)$.
- A4 The fibres of Exp in $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$ are cosets of the subgroups $\mathbb{G}_a(C_F)$ and $\mathbb{G}_m(C_F)$ respectively.
- AS For any $x_i, y_i \in F,\ i = 1, \ldots, n$, if $F \models \bigwedge_{i=1}^n \mathrm{Exp}(x_i, y_i)$ and $\mathrm{td}_{C_F}(\bar{x}, \bar{y}/C_F) \leq n$ then there are integers $m_1, \ldots, m_n$, not all of them zero, such that $m_1x_1 + \ldots + m_nx_n \in C_F$.

[3] Thanks to Felix Weitkamper for pointing this out.

Note that AS can be given by a first-order axiom scheme (see Section 5.1).

Let T^0_{Exp} be the theory axiomatised by A1-A4, AS. The class $\mathfrak{C}$ consists of all countable models of T^0_{Exp} with a fixed field of constants C (which is a countable algebraically closed field with transcendence degree $\aleph_0$). For $F \in \mathfrak{C}$ and $X \subseteq F$ we have $\langle X \rangle = C(X)^{\mathrm{alg}}$ with the induced structure from F. A structure $A \in \mathfrak{C}$ is finitely generated if and only if it has finite transcendence degree over C.

For finite tuples $\bar{x}, \bar{y} \in K^n$ with $\mathrm{Exp}(x_i, y_i)$ define

$$\delta(\bar{x}, \bar{y}) := \mathrm{td}_C\, C(\bar{x}, \bar{y}) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/C).$$

The Ax-Schanuel theorem states positivity of this function (for non-constant tuples). It is easy to see that δ is universally definable. We want to extend δ to $\mathfrak{C}_{f.g.}$. Following [Kir09] for $A \in \mathfrak{C}_{f.g.}$ define

$$\begin{aligned}\sigma(A) := \max\{n : &\text{ there are } a_i, b_i \in A,\ i = 1, \ldots, n, \text{ with } a_i\text{'s}\\ &\text{linearly independent over } \mathbb{Q} \text{ mod } C \text{ and } A \models \mathrm{Exp}(a_i, b_i)\}\end{aligned}$$

and

$$\delta(A) := \mathrm{td}_C(A) - \sigma(A).$$

Firstly note that σ is well defined and finite since the Ax-Schanuel inequality bounds the number n in consideration by $\mathrm{td}_C\, C(\bar{a}, \bar{b})$ which, in its turn, is bounded by $\mathrm{td}_C\, A$.

Secondly, it is quite easy to prove that for any $A, B \in \mathfrak{C}_{f.g.}$

$$\sigma(A \cup B) \geq \sigma(A) + \sigma(B) - \sigma(A \cap B).$$

This implies that δ is submodular. Invariance of δ under isomorphism is clear too. Hence it is a predimension.

The Ax-Schanuel inequality is equivalent to saying that $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$ where equality holds if and only if $A = C$.

The class $\mathfrak{C}$ satisfies the strong amalgamation property but not the asymmetric amalgamation property. So we let $\hat{\mathfrak{C}}$ be the subclass of $\mathfrak{C}$ consisting of *full* structures. A structure $A \in \mathfrak{C}$ is *full* if for every $a \in A$ there are $b_1, b_2 \in A$ with $A \models \mathrm{Exp}(a, b_1) \wedge \mathrm{Exp}(b_2, a)$. Then $\hat{\mathfrak{C}}$ has the AAP property and satisfies all the assumptions made in previous sections.

Theorem 3.3.2 ([Kir09])**.** *The Ax-Schanuel inequality is strongly adequate for $\mathcal{K}_{\mathrm{Exp}}$.*

See Section 5.1 for a complete axiomatisation of $\mathrm{Th}(\mathcal{K}_{\mathrm{Exp}})$. In Chapter 6 we study the predimension given by the Ax-Schanuel inequality for the j-function and give full details of construction and axiomatisation of the Fraïssé limit.

3.4 Predimensions in the differential setting

Let $\mathcal{K} := (K; +, \cdot, ', 0, 1)$ be a countable saturated differentially closed field with field of constants C. Suppose $f(X, Y) \in \mathbb{Q}\{X, Y\}$ is a differential polynomial with $\mathrm{ord}_Y(f) = m + 1$. Consider the differential equation

$$f(x, y) = 0. \tag{4.1}$$

Let $E(x, y_0, \ldots, y_m)$ be an $(m+2)$-ary relation defined by

$$f(x, y_0) = 0 \wedge \bigwedge_{i=0}^{m-1} y_i' = y_{i+1} x'.$$

We fix the language $\mathfrak{L}_E := \{+, \cdot, E, 0, 1\}$. Let $\mathfrak{C}$ be a class of $\mathfrak{L}_E$-structures satisfying all requirements set in Sections 3.1 and 3.2 (in particular, the existence of $\hat{\mathfrak{C}}$ with the appropriate properties is assumed). Assume δ is a non-negative predimension on $\mathfrak{C}_{f.g.}$. Normally $\mathfrak{C}$ will consist of algebraically closed fields with a relation E satisfying some basic universal axioms of E-reducts of differential fields. These axioms will depend on functional equations satisfied by E. Most importantly, we should have an axiom scheme for the inequality $\delta \geq 0$.

Definition 3.4.1. We say δ is (strongly) adequate (for the differential equation E) if it is (strongly) adequate for the reduct $\mathcal{K}_E := (K; +, \cdot, E, 0, 1)$.

Remark 3.4.2. It makes sense to consider just a binary relation for the set of solutions of our differential equation, without including derivatives, and study predimensions in that setting. More generally, we can do the same for an arbitrary reduct of a differentially closed field and define adequacy as above.

Now we consider a special kind of predimension motivated by the Ax-Schanuel inequality for the exponential differential equation and its analogue for the j-function. Assume d is a modular dimension function on $\mathcal{K}$. Suppose whenever (x_i, y_i) are solutions of equation (4.1), the following inequality holds:

$$\operatorname{td}_C C(\{x_i, \partial_{x_i}^j y_i : i = 1, \ldots, n,\ j = 0, \ldots, m\}) - (m+1) d(\bar{x}, \bar{y}) \geq 0. \tag{4.2}$$

The inequality (4.2) is first-order axiomatisable provided that d is type-definable in the algebraically closed field K, i.e. for each m and n the set $\{\bar{x} \in K^n : d(\bar{x}) \geq m\}$ is type definable (in the language of rings).

For $A \in \mathfrak{C}_{f.g.}$ define

$$\begin{gathered}\sigma(A) := \max\{d(\bar{a}, \bar{b}) : a_i, b_i \in A \text{ and there are } b_i^1, \ldots, b_i^m \in A, \\ \text{with } A \models E(a_i, b_i, b_i^1, \ldots, b_i^m)\}\end{gathered}$$

and

$$\delta(A) := \operatorname{td}(A/C) - (m+1) \cdot \sigma(A).$$

It is easy to see that σ is finite and hence δ is well defined. On the other hand for $A, B \in \mathfrak{C}_{f.g.}$ one can easily prove (using modularity of d) that

$$\sigma(A \cup B) \geq \sigma(A) + \sigma(B) - \sigma(A \cap B).$$

Thus, δ is submodular. In this manner we obtain a predimension on $\mathfrak{C}_{f.g.}$ and it makes sense to ask whether it is adequate or not.

The main question of our thesis is the following.

Question 3.4.3. *Which differential equations do satisfy a (strongly) adequate predimension inequality?*

As we have already mentioned adequacy means that the reduct $\mathcal{K}_E$ is "geometric" and the predimension governs its geometry. In our setting this intuitive idea can be clarified a bit, based on the analysis of pseudo-exponentiation and the exponential differential equation (and the differential equation of the j-function in Chapter 6).

In order to understand the structure of our differential equation, one has to understand which systems of equations in the language of the reduct $\mathcal{K}_E$ do have a solution. Then a predimension inequality (like (4.2)) implies that "overdetermined" systems cannot have solutions. Adequacy means that this is the only obstacle: if having a solution does not contradict our inequality then there is a solution. It is not difficult to see that this question is equivalent to understanding which varieties contain (generic enough) points that are solutions to our differential equation[4] (we call them E-points). This is in fact how one axiomatises the first-order theory of a differential equation (i.e. the theory of the corresponding reduct) with an adequate predimension inequality.

Indeed, as we noted in Section 3.2, one normally approximates the richness property (which determines the strong Fraïssé limit uniquely up to isomorphism) by first-order axioms in order to give an axiomatisation of $\mathrm{Th}(U)$. Richness of the strong Fraïssé limit U implies that it is existentially closed in strong extensions. So if a variety contains an E-point in a strong extension of U then such a point exists already in U. When one tries to axiomatise this property, one normally proves that varieties with certain properties always contain an E-point. However, according to the richness property, we need also make sure that when we work over a strong substructure as a set of parameters then there exists an E-point in our variety which is strongly embedded into U. So, our axioms should state that varieties with the appropriate properties contain an E-point which cannot be extended to another point with lower predimension. In this case the axiomatisation is $\forall\exists\forall$.

However, in our main examples, that is, the exponential differential equation and the equation of the j-function, we end up with simpler axioms which are in fact $\forall\exists$. Let us explain how one obtains those axioms. Suppose we work over a strong substructure $A \leq U$ and V is a variety defined over A. If we know that V contains an E-point $\bar{b}$ and $\delta(\bar{b}/A) > 0$ then it is possible that $\bar{b}$ is not strong in U. This can happen if V has high dimension. In such a situation one uses the tool of intersecting varieties with generic hyperplanes (see Lemma 6.6.4) and decreases the dimension of V, more precisely, one replaces V with a subvariety V' defined over some A' with $A \leq A' \leq U$. Now if $\dim V'$ is small enough then an E-point $\bar{b}$ in V' satisfies $\delta(\bar{b}/A') = 0$ which shows that $\bar{b}$ is strong in U (since $A' \leq U$). Thus, the existence of E-points in certain varieties is enough to deduce the existence of E-points which are strong in U. Hence one axiomatises the existential closedness property by saying that certain varieties contain E-points. Then one normally ends up with an $\forall\exists$ axiom scheme which, along

[4]We do not state precisely what we mean by this because we will see it in the case of the exponential differential equation and the equation of j which will be enough to understand the question in general.

with the basic universal axioms (including an axiom scheme stating non-negativity of δ), is expected to give a complete axiomatisation of the theory of the strong Fraïssé limit. So, in this case the axiomatisation is expected to be $\forall\exists$.

This observation justifies the condition of $\forall\exists$-axiomatisability in Theorem 4.8.2. Nevertheless, we recall once more that those speculations are based on the aforementioned examples, and in general we expect an $\forall\exists\forall$-axiomatisation rather than just $\forall\exists$. On the other hand, the procedure described above and in particular the method of intersecting varieties with generic hyperplanes is quite general and can be carried out for various differential equations with a predimension inequality. So in "nice" examples we hope to get an $\forall\exists$ theory. In Chapter 6 we illustrate those ideas on the example of the differential equation of the j-function. Finally, let us remark that getting a first order axiomatisation for the Fraïssé limit is by no means "automatic" since some technical issues may arise depending on the setting as we will see in Section 6.8.

Now we can show that if the reduct $\mathcal{K}_E$ of $\mathcal{K}$ is not proper (in the sense that the derivation of $\mathcal{K}$ is definable in $\mathcal{K}_E$) and has an $\forall\exists$-axiomatisation then E cannot have a non-trivial strongly adequate predimension inequality. This follows immediately from Theorem 4.8.2 and Proposition 3.2.13.

In fact, we have one more result in this direction.

Theorem 4.7.1. *Assume the underlying fields of finitely generated structures from our strong amalgamation class are algebraically closed of finite transcendence degree over $\mathbb{Q}$. Assume further that generic 1-types (in the sense of pregeometry associated to the predimension) are not algebraic. If D is definable in $\mathcal{F}_R$ and δ is strongly adequate, then the reduct is model complete and hence δ is trivial.*

This cannot be applied to the exponential differential equation since finitely generated structures in $\mathfrak{C}$ have infinite transcendence degree over $\mathbb{Q}$. Instead, the result mentioned above (Theorem 4.8.2) helps in that situation.

Thus, we know that under a natural assumption if we can recover the differential structure (i.e. define the derivation) in the reduct $\mathcal{K}_E$ then we do not have a non-trivial strongly adequate predimension inequality (in fact this holds for all reducts, not only for differential equations). Therefore we need to understand when we can recover the differential structure. This is one of the motivating factors for the next chapter. It would be nice if we could prove a converse of this, i.e. if the derivation is not definable in a reduct and the latter is $\forall\exists$-axiomatisable (we can also add an assumption about near model completeness) then we can find a non-trivial predimension inequality. This problem, however, seems to be quite difficult at the moment to tackle.

Finally, we remark that Question 3.4.3 makes sense for any reduct and not only for differential equations. If we want to restrict the question to differential equations of two variables then it will make more sense to ask when there is an adequate predimension of the form (4.2).

Chapter 4

Definability of Derivations in the Reducts of Differentially Closed Fields

4.1 Setup and the main question

For a differentially closed field $\mathcal{F} = (F; +, \cdot, 0, 1, \mathrm{D})$ we consider its reducts of the form $\mathcal{F}_R = (F; +, \cdot, 0, 1, P)_{P \in R}$ where R is some collection of definable sets in $\mathcal{F}$. Our main problem in this chapter is to understand when the derivation D is definable in $\mathcal{F}_R$. Ideally, we would like to find a dividing line for definability of D like local modularity in the problem of recovering the field structure in the reducts of algebraically closed fields (see the discussion below).

Question 4.1.1. *When is* D *definable in the reduct* $\mathcal{F}_R$*?*

As we will see, when D is definable, it is in fact definable with using just one parameter, namely an element $t \in F$ with $\mathrm{D}\, t = 1$. So it is more convenient to add t to our language as a constant symbol and work in the reducts of $\mathcal{F} = (F; +, \cdot, 0, 1, t, \mathrm{D})$ (we do this starting from Section 4.4). We will assume for simplicity that the sets from R are 0-definable in this language and also we will be interested in 0-definability of D.

Note that we could ask a more general question: whether there is *some* derivation definable in the reduct. But in that case such a derivation will also be definable in the differentially closed field $\mathcal{F}$. Since it is known that any such derivation is of the form $a \cdot \mathrm{D}$ for some $a \in F$, i.e. it coincides with D up to a constant multiple (and coincides absolutely with D if we add t to our language and require that a derivation takes the value 1 at t), it is no loss of generality if we restrict our attention to definability of D only. Another point is that we can assume without loss of generality that R is finite since any possible definition of D can contain only finitely many relations from R.

This is by nature a classification problem. We do not have a comprehensive solution yet, but we give some partial answers to our question, and draw some conclusions based on our analysis. We will not pose any explicit conjectures, but one may nevertheless expect intuitively that definability of D is very rare, i.e. in most cases it is not definable. In other words, our general expectation is that for "generic" (in some sense) reducts D is not definable.

The motivation to consider this kind of problem comes from two independent sources. Firstly, the analogous problem for pure fields, that is, recovering the field structure from reducts of algebraically closed fields or from non-locally modular strongly minimal sets in general, is very important in model theory of fields and Zariski geometries. It was initiated by Zilber's famous "Trichotomy conjecture" and is still not entirely resolved. It has been (and still is) a topic of active research during the past few decades and proved to be very useful and important. Zariski geometries, introduced by B. Zilber and E. Hrushovski, are structures where that theory works ideally. For more details on this we refer the reader to [Zil09, Rab93, HS15, Mar05a].

Secondly, as we already mentioned in previous chapters, this problem turns out to be related to the existence of a predimension inequality for a given differential equation $E(x, y)$ which is the main question considered in this work (in this case we will work in the reduct $\mathcal{F}_E = (F; +, \cdot, 0, 1, E)$ with $R = \{E\}$). As we observed in Section 3.4 we can use Theorem 4.8.2 (which is one of the main results of this chapter) to show that definability of a derivation would imply that there is no non-trivial strongly adequate predimension inequality for the given differential equation (assuming the reduct is $\forall\exists$-axiomatisable). Theorem 4.7.1 supports this viewpoint too.

Now let us briefly outline the chapter. In Section 4.2 we show that the definable derivations in models of DCF_0 are the trivial ones. We study the reducts of differentially closed fields from a general model theoretic point of view and establish some properties of them in Section 4.3. In particular we will see that the reducts always have rank ω and do not admit quantifier elimination unless R consists only of algebraic relations only (assuming R is finite).

Section 4.4 will be devoted to an important example of a reduct that allows a definition of D. Namely, we will see that if E is a differential curve containing the graph of D then D is quantifier-free definable in $\mathcal{F}_E$. This example will be crucial for the main results of this chapter.

Further, we show in Section 4.5 that only the behaviour of D at generic (differentially transcendental) points is important for definability of D, that is, if we can define $\mathrm{D}\, a$ from a for a generic element a then the whole of D is definable. We will develop this idea further and prove that if for a generic element a the Morley rank (in the reduct) of $\mathrm{D}\, a$ over a is finite then D is definable. Thus $\mathrm{D}\, a$ can be either generic or algebraic (in fact, definable) over a in the reduct. This can trivially be given a stability-theoretic reformulation (in terms of forking) which will be generalised later.

We use those criteria to give further examples of differential equations that define D (Section 4.6). In particular, we will show that from an algebraic function of x and its derivatives one can define $\mathrm{D}\, x$. This will be used to obtain a characterisation of definable and algebraic closures in $\mathcal{F}_R$. Note, however, that those examples will not be used in later sections. Theorem 4.6.11 sums up most of our results obtained up to that point giving a list of conditions equivalent to definability of D in the reducts.

Using the results on generic points we will show that when D is definable, the reduct cannot satisfy an adequate non-trivial predimension inequality of a certain form (Section 4.7).

Section 4.8 will be devoted to a result which shows that definability of D in reducts is "rare" and partially justifies the above ideas about the relation of definability of D and existence of an adequate predimension inequality. Namely, we will prove that if D is definable and $\mathrm{Th}(\mathcal{F}_R)$ is inductive then this theory must in fact be model complete. We have already discussed how this is related to adequate predimension inequalities. In particular, it immediately implies that one cannot define D from the equation $\mathrm{D}\, y = y\, \mathrm{D}\, x$.

Finally, in the last section we define the standard pregeometry (obtained by forking the generic type) on reducts and see how it is related to the definability of D.

The results of this chapter constitute the material of the preprint [Asl16c].

4.2 Definable derivations

If D is a derivation on a field $(F; +, \cdot, 0, 1)$ then for any element $a \in F$ the map $a \cdot \mathrm{D}$ will be a derivation as well. If our field is differentially closed then it is differentially closed with respect to this new derivation too. We show in this section that in a differentially closed field all definable derivations are of that form.

This fact, though proved independently here, is actually well known. A proof can be found for example in [Sue07] (in a general form for definable derivations in differentially closed fields with several commuting derivations). We present our proof here for completeness.

The following well-known result is a characterisation of definable functions in a differentially closed field (see, for example, [Pil01] or [TZ12], Exercise 6.1.14).

Lemma 4.2.1. *Let $\mathcal{F}$ be a differentially closed field and $f : F^k \to F$ be a definable (possibly with parameters) function in $\mathcal{F}$. Then there is a partition of F^k into a finite number of definable subsets U_i such that f is given by a differential rational function on each of them (this means, in particular, that each of these rational functions is determined on the corresponding set).*

Proof. Suppose $\phi(\bar{v}, w)$ defines f. Consider the following set of formulae:

$$\Delta := \{\neg(h(\bar{v}) \cdot f(\bar{v}) = g(\bar{v}) \ \wedge\ h(\bar{v}) \neq 0) : g, h \in F\{X\}\} \cup \mathrm{EDiag}(\mathcal{F}),$$

where $\mathrm{EDiag}(\mathcal{F})$ is the set of all sentences with parameters from F that are true in $\mathcal{F}$.

Claim. Δ is not satisfiable.

Proof. Suppose otherwise. Let $\mathcal{L} = (L; +, \cdot, 0, 1, \mathrm{D})$ be a differentially closed field satisfying Δ, which implies $\mathcal{F} \preceq \mathcal{L}$. Then the formula ϕ defines a function $\tilde{f} : L^k \to L$ which extends f. For any tuple $\bar{a} \in L^k$ we must have $\tilde{f}(\bar{a}) \in \mathrm{dcl}(F, \bar{a}) = F\langle \bar{a} \rangle$ due to Proposition 2.1.1. This means that $\tilde{f}(\bar{a})$ is the value of a differential rational function at $\bar{a}$. Hence, there are differential polynomials g and h over F such that $h(\bar{a}) \neq 0$ and $\tilde{f}(\bar{a}) = \frac{g(\bar{a})}{h(\bar{a})}$. This is a contradiction. □

Thus, Δ is not satisfiable. By compactness, a finite subset of Δ is not satisfiable. Therefore there is a finite number of differential polynomials $g_1, \ldots, g_n, h_1, \ldots, h_n$ such that

$$\mathcal{F} \models \forall \bar{v} \bigvee_i (h_i(\bar{v}) \cdot f(\bar{v}) = g_i(\bar{v}) \wedge h_i(\bar{v}) \neq 0).$$

Now define $U_i = \{\bar{x} \in F^k : h_i(\bar{x}) \cdot f(\bar{x}) = g_i(\bar{x}) \wedge h_i(\bar{x}) \neq 0\}$. These sets are definable and f is given by a differential rational function on each of them. $\square$

Lemma 4.2.2. *Suppose* D *and* D_1 *are derivations on a field* $(F;+,\cdot,0,1)$ *such that there is* $t \in F$ *with* $\mathrm{D}\,t = 1$. *Let* $P(X_0, \ldots, X_n, Y)$ *be a non-zero polynomial over* F *such that*

$$P(X, \mathrm{D}\,X, \ldots, \mathrm{D}^n\,X, \mathrm{D}_1\,X) = 0. \tag{2.1}$$

Then $\mathrm{D}_1 = a \cdot \mathrm{D}$, *where* $a = \mathrm{D}_1\,t$.

Proof. For an element $x \in F$ and an arbitrary rational number r one has $P(x + r, \mathrm{D}\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x) = 0$, hence

$$P(X, \mathrm{D}\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x) = 0$$

(as a polynomial of X). Therefore all coefficients of this polynomial are zeros. Since $P(X_0, \ldots, X_n, Y)$ is non-zero, if we consider it as a polynomial of X_0, it will have a non-zero coefficient that is a polynomial of $X_1, \ldots, X_n, Y$. It must vanish at $(\mathrm{D}\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x)$. This is true for all $x \in F$.

Thus for a non-zero polynomial P_1 we have

$$P_1(\mathrm{D}\,X, \ldots, \mathrm{D}^n\,X, \mathrm{D}_1\,X) = 0.$$

Again, fixing an element $x \in F$ we see that for any rational r one has $P_1(\mathrm{D}\,x + r, \mathrm{D}^2\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x + ar) = 0$ (we substitute $X = x + rt$). This implies

$$P_1(X, \mathrm{D}^2\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x - a\,\mathrm{D}\,x + aX) = 0.$$

Replacing X by a fixed element $y \in F$ and taking $x + rt^2$ instead of x we get

$$P_1(y, \mathrm{D}^2\,x + 2r, \mathrm{D}^3\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x - a\,\mathrm{D}\,x + ay) = 0.$$

Therefore

$$P_1(y, X, \mathrm{D}^3\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x - a\,\mathrm{D}\,x + ay) = 0.$$

Arguing as above we show that for some non-zero polynomial P_2 we have

$$P_2(y, \mathrm{D}^3\,x, \ldots, \mathrm{D}^n\,x, \mathrm{D}_1\,x - a\,\mathrm{D}\,x + ay) = 0$$

for all $x, y \in F$.

Proceeding this way one can prove that there is a non-zero polynomial $Q(Z_1, Z_2) \in F[Z_1, Z_2]$ such that

$$Q(Y, \mathrm{D}_1\,X - a\,\mathrm{D}\,X + aY) = 0.$$

Now suppose for some $u \in F$ we have $\mathrm{D}_1\,u \neq a\,\mathrm{D}\,u$. Then for any natural number n one has $\mathrm{D}_1(nu) \neq a\,\mathrm{D}(nu)$. This means that for any $y \in F$ the polynomial $Q(y, ay + Z)$ equals zero for infinitely many values of Z, hence, it is identically zero. This yields $Q(Y, Z) = 0$. We arrived at a contradiction, therefore $\mathrm{D} = a\,\mathrm{D}_1$. $\square$

Theorem 4.2.3. *Let $\mathcal{F} = (F; +, \cdot, 0, 1, \mathrm{D})$ be a differentially closed field and $\tilde{\mathrm{D}}$ be a definable (possibly with parameters) derivation. Then there exists an element $a \in F$ such that $\tilde{\mathrm{D}} = a\,\mathrm{D}$.*

Proof. From Lemma 4.2.1 it follows that there are definable sets $U_i \subseteq F$ such that $\tilde{\mathrm{D}}$ is given by a differential rational function on each U_i. Therefore there are differential polynomials $f_i(X), g_i(X) \in F\{X\}$ such that $f_i(x) \cdot \tilde{D}(x) = g_i(x)$ and $f_i(x) \neq 0$ for all $x \in U_i$. We know that $f_i(X) = P_i(X, \mathrm{D}\,X, \ldots, \mathrm{D}^m\,X)$, $g_i(X) = Q_i(X, \mathrm{D}\,X, \ldots, \mathrm{D}^m\,X)$ for some polynomials P_i and Q_i over F. Form the polynomial

$$P(X_0, \ldots, X_m, Y) = \prod_i (P_i(X_0, \ldots, X_m) \cdot Y - Q_i(X_0, \ldots, X_m)).$$

This is a non-zero polynomial and

$$P(X, \mathrm{D}\,X, \ldots, \mathrm{D}^m\,X, \tilde{\mathrm{D}}X) = 0.$$

As $\mathcal{F}$ is differentially closed, there exists $t \in F$ with $\mathrm{D}\,t = 1$. Now Lemma 4.2.2 yields the desired result. □

4.3 Model theoretic properties of the reducts

From now on we will work in a differentially closed field $\mathcal{F} = (F; +, \cdot, 0, 1, \mathrm{D})$ which we will assume to be sufficiently saturated. Thus, it will serve as a monster model for us.

For a collection R of definable sets in (Cartesian powers of) $\mathcal{F}$ we define the *R-reduct* $\mathcal{F}_R$ of $\mathcal{F}$ to be the structure $(F; +, \cdot, 0, 1, P)_{P \in R}$ in the language $\mathfrak{L}_R = \{+, \cdot, 0, 1\} \cup R$ (the elements of R are relation symbols in the language $\mathfrak{L}_R$). We will omit R and just say "reduct" whenever no confusion can arise. We will say that R (or the reduct $\mathcal{F}_R$) is *algebraic* if all relations of R can be defined in the pure field $(F; +, \cdot, 0, 1)$. If R consists of just one relation E then we will write $\mathcal{F}_E$ for the corresponding E-reduct.

In this section we examine basic model theoretic properties of the reducts $\mathcal{F}_R$. Though we will sometimes assume R is finite, most of our results will be valid for an arbitrary R. From the point of view of Question 4.1.1 the assumption of finiteness of R is no loss of generality as a possible definition of D would anyway contain only finitely many occurrences of relation symbols from R.

We start by introducing a piece of notation. In order to distinguish between the same concepts in the differentially closed field $\mathcal{F}$ and in the reduct $\mathcal{F}_R$, we will add a subscript D or R respectively to their notations. Thus MR_{D}, MD_{D}, tp_{D}, $\mathrm{dcl}_{\mathrm{D}}$, $\mathrm{acl}_{\mathrm{D}}$ stand for Morley rank, Morley degree, type, definable closure and algebraic closure respectively in $\mathcal{F}$ while MR_R, MD_R, tp_R, dcl_R, acl_R stand for the same notions in $\mathcal{F}_R$.

Also we will need to consider generic elements and types. **By generic we will always mean generic in the differentially closed field $\mathcal{F}$ (rather than in $\mathcal{F}_R$) unless explicitly stated otherwise.** If we do not specify over which set an element is generic then we mean over the empty set.

Finally, we turn to model theoretic properties of the reducts. Clearly $\mathcal{F}_R$ is an ω-stable structure. We show that its Morley rank is ω.

Proposition 4.3.1. *$\mathcal{F}_R$ has Morley rank ω unless R is algebraic.*

Proof. First of all, since $\mathcal{F}_R$ is a reduct of $\mathcal{F}$, and the latter has Morley rank ω, we have $\mathrm{MR}(\mathcal{F}_R) \leq \omega$. So we need to prove $\mathrm{MR}(\mathcal{F}_R) \geq \omega$.

It obviously suffices to prove this for $R = \{P\}$ where P is a non-algebraic unary relation which has finite Morley rank in the differentially closed field $\mathcal{F}$. The case $P = C$ (the field of constants) is a well known example. In this case the reduct is just an algebraically closed field with a unary predicate for an algebraically closed subfield. Our proof below is an adaptation of a known proof for this special case (see, for example, [Mar02], exercise 6.6.17, d).

As P is non-algebraic, it must be infinite and hence $\mathrm{MR}_R(P) \geq 1$. Also P has finite Morley rank in $\mathcal{F}$, so $(\mathbb{Q}(P))^{\mathrm{alg}} \neq F$. Now for an element $x \in F \setminus (\mathbb{Q}(P))^{\mathrm{alg}}$ define

$$X_n = \left\{ y \in F : \exists a_0, \ldots, a_{n-1} \in P \ \left(y = \sum a_i x^i \right) \right\}.$$

The map $\pi : P^{n+1} \to X_{n+1}$ given by $(a_0, \ldots, a_n) \mapsto a_0 + a_1 x + \ldots + a_n x^n$ is a definable bijection. Hence $\mathrm{MR}_R(X_n) = \mathrm{MR}_R(P^n) \geq n$. Therefore $\mathrm{MR}_R(F) = \omega$. $\square$

We will assume throughout this chapter that R is not algebraic and so $\mathcal{F}_R$ has Morley rank ω.

Remark 4.3.2. As we saw in the proof, if $a \in F$ is a differentially transcendental element then for each $n < \omega$ there is a definable (in $\mathcal{F}_R$) set $X_n \subseteq F$, defined over a, such that $n \leq \mathrm{MR}_R(X_n) < \omega$.

Further, observe that $\mathcal{F}_R$ has Morley degree 1. If $\varphi(x)$ is a formula (of one variable) in the language $\mathfrak{L}_R = \{+, \cdot, 0, 1\} \cup R$ then in the language $\mathfrak{L}_{\mathrm{D}}$ it is equivalent to a quantifier-free formula. If it is an equation in conjunction with something else, then $\mathrm{MR}_R(\varphi) < \omega$ otherwise $\mathrm{MR}_R(\varphi) = \omega$. Also, $\mathrm{MR}_R(\varphi) \leq \mathrm{MR}_{\mathrm{D}}(\varphi)$ and these ranks are finite or infinite simultaneously. Indeed, if $\mathrm{MR}_{\mathrm{D}}(\varphi) = \omega$ then $\mathrm{MR}_{\mathrm{D}}(\neg\varphi) < \omega$, and so $\mathrm{MR}_R(\neg\varphi) < \omega$. Therefore, $\mathrm{MR}_R(\varphi) = \omega$ since $\mathrm{MR}_R(x = x) = \omega$ as proven above.

There is a unique generic 1-type in $\mathcal{F}_R$ given by

$$\{\varphi(x) : \varphi \in \mathfrak{L}_R, \ \mathrm{MR}_R(\varphi) = \omega\} = \{\neg\varphi(x) : \varphi \in \mathfrak{L}_R, \ \mathrm{MR}_R(\varphi) < \omega\}.$$

Similarly, the unique generic n-type is given by formulas of Morley rank $\omega \cdot n$.

Now let us see whether $\mathcal{F}_R$ can have quantifier elimination or not. First notice that even when $R = \{\mathbb{D}\}$ (the graph of D), $\mathcal{F}_R$ does not admit quantifier elimination for $y = \mathrm{D}^2 x$ is existentially definable but not quantifier-free definable. It turns out that this is a general phenomenon.

Corollary 4.3.3. *If R is non-algebraic and finite then the reduct $\mathcal{F}_R$ does not admit elimination of quantifiers.*

Proof. Suppose R is not algebraic but $\mathcal{F}_R$ has quantifier elimination. Then any formula with one free variable must be equivalent to a Boolean combination of algebraic polynomial equations (in the language of rings) and formulas of the form

$$Q(p_1(x), \dots, p_n(x))$$

where $Q \in R$ is an n-ary predicate and p_i's are algebraic polynomials. But clearly if such a formula has finite Morley rank then the latter is uniformly bounded (remember that R is finite). This contradicts Proposition 4.3.1. $\square$

One sees that although in the case $R = \{\mathbb{D}\}$ the reduct does not have quantifier elimination, it is nevertheless model complete. In general it is true if D is existentially definable. We show this below.

Lemma 4.3.4. *Let $\mathcal{M}$ be a structure. If a function $f : M^n \to M$ is existentially definable in $\mathcal{M}$ then it is also universally definable.*

Proof. If $\phi(\bar{x}, y)$ defines f then so does $\forall z(z = y \vee \neg\phi(\bar{x}, z))$. $\square$

Proposition 4.3.5. *If* D *is existentially definable in $\mathcal{F}_R$ then $T_R := \mathrm{Th}(\mathcal{F}_R)$ is model complete.*

Proof 1. Suppose that D is existentially definable. Take an arbitrary formula $\varphi \in \mathfrak{L}_R$. In the language of differential rings it is equivalent to a quantifier-free formula, i.e. to a Boolean combination of differential equations. Each differential equation is existentially definable in the reduct and, by Lemma 4.3.4, it is also universally definable. Substituting existential definitions in positive parts (i.e. equations) and universal definitions in negative parts (inequations), we get an existential formula in the language $\mathfrak{L}_R$. Thus any formula in the language of the reduct is equivalent to an existential formula. This is equivalent to model completeness. $\square$

Proof 2. Suppose $\varphi(x, y)$ is an existential formula defining D. Let $\mathcal{K}_R \subseteq \mathcal{L}_R$ be two models of T_R. Since D is definable, there are derivations D_K and D_L on K and L respectively which are *compatible* with R (see the beginning of the next section). Take any $a, b \in K$. Then $\mathcal{K}_R \models \varphi(a, b)$ if and only if $\mathcal{L}_R \models \varphi(a, b)$. This shows that $\mathrm{D}_K \subseteq \mathrm{D}_L$ which, with model completeness of DCF_0, implies model completeness of T_R. $\square$

Thus, model completeness is the deepest possible level of quantifier elimination that we can have for T_R. As we will see in the last section, under a natural assumption, definability of D will imply that T_R is model complete.

4.4 An example

In this section we show that in a certain class of reducts D is definable. It will be used later to establish some criteria for definability of D.

Choose an element $t \in F$ with $\mathrm{D}\, t = 1$ (it exists because our field is differentially closed) and add it as a constant symbol to our language. Thus from now on we work in the language $\{+, \cdot, \mathrm{D}, 0, 1, t\}$ for differential fields, which by abuse of notation we will again denote by $\mathfrak{L}_{\mathrm{D}}$. Correspondingly all reducts will be considered in the language $\mathfrak{L}_R = \{+, \cdot, 0, 1, t\} \cup R$. Again abusing the nomenclatures we will call $\mathfrak{L}_{\mathrm{D}}$ the language of differential rings and $\mathfrak{L}_R$ the language of the reducts. This means that we do not count t as a parameter in our formulas, i.e. we are free to use t in formulas and declare that something is definable without parameters. Note that this does not affect any of the results proved in the previous section. Let us also mention that after adding t to our language (and requiring that a derivation takes the value 1 at t) the only candidate for a definable derivation can be D (see Theorem 4.2.3).

For a formula $\varphi(\bar{x})$ in the language $\mathfrak{L}_{R,\mathrm{D}} = \mathfrak{L}_R \cup \mathfrak{L}_{\mathrm{D}}$ and a tuple $\bar{a} \in F$ we will sometimes write $\mathcal{F} \models \varphi(\bar{a})$. This is an abuse since in general φ is not in the language of differential rings, but clearly $\mathcal{F}$ can be canonically made into an $\mathfrak{L}_{R,\mathrm{D}}$-structure.

In general, if the relations in R are defined with parameters and D is definable then it will be definable with parameters as well. But in many cases we do not use any extra parameters to define D. So for simplicity we will assume that R consists of 0-definable relations in $\mathcal{F}$, i.e. relations defined over $k_0 = \mathbb{Q}(t) = \mathrm{dcl}(\emptyset)$. Thus from now on by definable we will mean definable without parameters unless explicitly stated otherwise.

We denote the theory of the reduct by $T_R := \mathrm{Th}(\mathcal{F}_R)$. We will sometimes say that there is a derivation D_K on a model $\mathcal{K}_R \models T_R$ which is *compatible* with R. This means that $(K; +, \cdot, \mathrm{D}_K, 0, 1, t, P)_{P \in R} \equiv (F; +, \cdot, \mathrm{D}, 0, 1, t, P)_{P \in R}$, i.e. the differential field $\mathcal{K} = (K; +, \cdot, \mathrm{D}_K, 0, 1, t)$ is differentially closed with $\mathrm{D}_K\, t = 1$ and the sets from R are defined by the same formulas as in $\mathcal{F}$.

Throughout this chapter we let E be a differential curve (possibly in general sense); as we noted above the corresponding reduct will be denoted $\mathcal{F}_E$. Recall also that $\mathbb{D} = \{(x, \mathrm{D}\, x) : x \in F\}$ is the graph of D.

Now we prove an auxiliary result which will be used several times throughout this chapter. It states that $(\mathbb{Q}(t))^n$ is Kolchin-dense in F^n for each n.

Lemma 4.4.1 (cf. [Mar05b], Lemma A.4)**.** *For any non-zero differential polynomial $f(X_1, \ldots, X_n)$ over $\mathbb{Q}(t)$ there are elements $t_1, \ldots, t_n \in \mathbb{Q}[t]$ such that $f(t_1, \ldots, t_n) \neq 0$.*

Proof. First assume f is a polynomial of one variable X. Let $\mathrm{ord}(f) = n$. Since $\mathcal{F}$ is differentially closed, we can find an element $u \in F$ with $\mathrm{D}^{n+1} u = 0 \wedge f(u) \neq 0$. Then clearly

$$u = c_n t^n + \ldots + c_1 t + c_0$$

for some constants $c_0, \ldots, c_n \in C$.

Now for constants $\lambda_0, \ldots, \lambda_n$ denote

$$p(t, \bar{\lambda}) = \lambda_n t^n + \ldots + \lambda_1 t + \lambda_0.$$

Since t is transcendental over C, there are algebraic polynomials $q_i(X_0, \ldots, X_n) \in \mathbb{Q}[X_0, \ldots, X_n]$, $i = 1, \ldots, m$, such that for all $\bar{\lambda} \in C^{n+1}$

$$f(p(t, \bar{\lambda})) = 0 \text{ iff } \bigwedge_{i=1}^{m} q_i(\bar{\lambda}) = 0.$$

Let $V \subseteq C^{n+1}$ be the algebraic variety over $\mathbb{Q}$ defined by $\bigwedge_{i=1}^{m} q_i(\bar{\lambda}) = 0$. Then as we saw above $V(C) \neq C^{n+1}$, and hence $V(\mathbb{Q}) \subsetneq \mathbb{Q}^{n+1}$. So there is a tuple $\bar{r} \in \mathbb{Q}^{n+1}$ with $\bar{r} \notin V(\mathbb{Q})$. Therefore $f(p(t, \bar{r})) \neq 0$ and $p(t, \bar{r}) \in \mathbb{Q}[t]$.

Now we prove the general case (when f has more than one variables) by induction on n. If $f = f(X_1, \ldots, X_n)$ with $n > 1$ then consider it as a differential polynomial $g(X_1, \ldots, X_{n-1})$ of $n - 1$ variables over the differential ring $\mathbb{Q}(t)\{X_n\}$. Choose a non-zero coefficient of g which will be a non-zero differential polynomial $h(X_n) \in \mathbb{Q}(t)\{X_n\}$. As we proved above there is $t_n \in \mathbb{Q}[t]$ such that $h(t_n) \neq 0$. Now the polynomial $f(X_1, \ldots, X_{n-1}, t_n)$ is a non-zero polynomial of $n - 1$ variables over $\mathbb{Q}(t)$ and we are done by the induction hypothesis. $\square$

Remark 4.4.2. The proof shows that we can choose $t_1, \ldots, t_n$ from $\mathbb{Z}[t]$ (and even from $\mathbb{N}[t]$).

Definition 4.4.3. Introduce the reverse lexicographical order on $(n+1)$-tuples of integers, that is, $(\alpha_0, \ldots, \alpha_n) < (\beta_0, \ldots, \beta_n)$ if and only if for some j, $\alpha_i = \beta_i$ for $i > j$ and $\alpha_j < \beta_j$. The *multi-degree* of an algebraic polynomial $Q(X_0, \ldots, X_n)$ is the greatest (with respect to this order) $(n+1)$-tuple $(\alpha_0, \ldots, \alpha_n)$ for which $X_0^{\alpha_0} \cdot \ldots \cdot X_n^{\alpha_n}$ appears in Q with a non-zero coefficient. The multi-degree of a differential polynomial $f(X) = P(X, \mathrm{D}\, X, \ldots, \mathrm{D}^n\, X)$ is defined as that of P.

Theorem 4.4.4. *If E (a differential algebraic curve) contains the graph of* D *then* D *is quantifier-free definable in $\mathcal{F}_E$.*

Proof. [1] Let E be given by a differential equation $f(x, y) = 0$. We know that $f(X, \mathrm{D}\, X)$ identically vanishes. Denote $U := Y - \mathrm{D}\, X$ and consider the differential polynomial $g(X, U) := f(X, U + \mathrm{D}\, X)$. Clearly $g(X, 0) = 0$.

First we intersect additive translates to "eliminate" x and define a differential equation $h(u) = 0$ for some differential polynomial $h(U)$. If $g(X, U)$ depends on X (i.e. $g(X, U) \in k_0\{X, U\} \setminus k_0\{U\}$) then we can find (see Lemma 4.4.1) $p(t) \in \mathbb{Q}[t]$ such that $g(X + p(t), U) \neq g(X, U)$. Clearly, U is invariant under the transformation $X \mapsto X + p(t),\ Y \mapsto Y + p'(t)$ where $p'(Z) = \frac{\partial p}{\partial Z}$. So consider the formula $E(x, y) \wedge E(x + p(t), y + p'(t))$. It is equivalent to $g(x, u) = 0 \wedge g(x + p(t), u) = 0$ which implies $g_1(x, u) := g(x, u) - g(x + p(t), u) = 0$. The leading terms of the differential polynomials $g(X, U)$ and $g(X + p(t), U)$ in variable X (i.e. the sums of monomials in these polynomials that have highest multi-degree in X) are the same and hence they cancel out in the difference $g_1(X, U) := g(X, U) - g(X + p(t), U)$. On the other hand $g_1(X, U) \neq 0$ by our choice of p and the multi-degree of g_1 in X is strictly less than that of g. In other words, if the multi-degree of g in X is bigger than $(0, \ldots, 0)$ then we can reduce it. Now if $g_1(X, U)$ depends on X then we do the same for g_1. We keep repeating this process and reduce the multi-degree of our differential polynomial step by step until it becomes $(0, \ldots, 0)$. This means we get a curve $h(u) = 0$ for a non-zero differential polynomial h, which contains a quantifier-free definable set in our reduct. It is also clear that the latter contains the curve $u = 0$ (the graph of D).

[1] I am grateful to Ehud Hrushovski for detecting a gap in the initial version of the proof and helping me to fix it.

Now we use multiplicative translates to define the curve $u = 0$ (which is actually $y = \mathrm{D}\, x$). Let $p(t) \in \mathbb{Q}[t]$. When we substitute $X \mapsto p(t)X,\ Y \mapsto p'(t)X + p(t)Y$ then U is replaced by $p(t)U$. Then $h(u) = 0 \wedge h(p(t)u) = 0$ is implied by a quantifier-free formula in the language of the reduct and implies $h_1(u) := p(t)^\alpha h(u) - h(p(t)u) = 0$ for any positive integer α. If $(\alpha_0, \ldots, \alpha_n)$ is the multi-degree of h then taking $\alpha := \alpha_0 + \ldots + \alpha_n$ the leading terms of the differential polynomials $p(t)^\alpha h(U)$ and $h(p(t)U)$ will coincide and will cancel out in the difference $h_1(U) := p(t)^\alpha h(U) - h(p(t)U)$. By an appropriate choice of p we can also guarantee that $h_1(U)$ is non-zero unless $h(U) = h(1) \cdot U^\alpha$. Indeed, if $h(U) \neq h(1) \cdot U^\alpha$ then the polynomial $h(V \cdot U) - V^\alpha \cdot h(U)$ is non-zero and hence there is $p(t) \in \mathbb{Q}[t]$ such that $h(p(t) \cdot U) \neq p(t)^\alpha \cdot h(U)$, therefore $h_1(U) \neq 0$. Thus, if $h(U)$ is not a homogeneous algebraic polynomial then h_1 is non-zero and its multi-degree is strictly less than that of h. Now if $h_1(U)$ is not algebraic homogeneous then we repeat the above procedure for h_1. Iterating this process we will eventually obtain an equation $u^\alpha = 0$ for some positive integer α which is equivalent to $u = 0$. Taking into account that all the sets defined this way contain $u = 0$ we see that at the last step we have defined $u = 0$ which, in terms of x and y, is the curve $y = \mathrm{D}\, x$.

Finally note that we only take conjunctions of atomic formulas here, hence the definition is quantifier-free. □

Remark 4.4.5. Strictly speaking, for the "quantifier-free" part of the theorem to be true we need to pick $p(t) \in \mathbb{N}[t]$ each time. Alternatively, we could add unary functions for multiplicative and additive inverses to our language.

Example 4.4.6. Suppose E is given by $(\mathrm{D}\, y - \mathrm{D}^2\, x) \cdot \mathrm{D}\, x = 0$. Then $E(x+t, y+1) = [(\mathrm{D}\, y - \mathrm{D}^2\, x) \cdot (\mathrm{D}\, x + 1) = 0]$. The conjunction $E(x, y) \wedge E(x + t, y + 1)$ implies $\mathrm{D}\, y - \mathrm{D}^2\, x = 0$. Now we substitute $x \mapsto tx,\ y \mapsto x + ty$ and get $t(\mathrm{D}\, y - \mathrm{D}^2\, x) + y - \mathrm{D}\, x = 0$. Subtracting the previous equation multiplied by t we get $y - \mathrm{D}\, x = 0$. Thus the formula $E(x, y) \wedge E(x + t, y + 1) \wedge E(tx, x + ty) \wedge E(tx + t, x + ty + 1)$ defines D.

Corollary 4.4.7. *If E is a curve in general sense that contains $\mathbb{D}$ then* D *is quantifier-free definable.*

Proof. Being a curve in general sense, E is defined by a formula of the form $f(x, y) = 0 \wedge \psi(x, y)$ for ψ a quantifier free formula in the language of differential fields. Now for the curve E' given by the equation $f(x, y) = 0$ we have a definition of D. Suppose it is given by the formula $\varphi(x, y)$ in the reduct $\mathcal{F}_{E'}$. We claim that the same formula defines D in $\mathcal{F}_E$. Indeed, as we take only conjunctions to define D from E', the set defined by $\varphi(x, y)$ in $\mathcal{F}_E$ will be contained in $\mathbb{D}$. On the other hand it clearly contains $\mathbb{D}$. Therefore it defines $\mathbb{D}$. □

We will give further examples and non-examples (of differential equations defining D) in Section 4.6, but first we need to establish some facts on generic points which we do in the next section.

4.5 Generic points

Recall that we work in a saturated differentially closed field $\mathcal{F}$. From now on we fix a generic (in the sense of DCF_0, that is, differentially transcendental) point $a \in F$. We first prove that if $\mathrm{D}\, a$ can be defined from a then we can recover the whole of D.

Proposition 4.5.1. *Suppose a formula $\varphi(x, y) \in \mathfrak{L}_R$ defines $\mathrm{D}\, a$ from a, that is,*

$$\mathcal{F} \models \forall y(\varphi(a, y) \leftrightarrow y = \mathrm{D}\, a).$$

Then D *is definable (without parameters). Moreover, if φ is existential then* D *is existentially definable.*

First proof. First of all observe that since the generic type is unique, for any differentially transcendental element $b \in F$ we have

$$\mathcal{F} \models \forall y(\varphi(b, y) \leftrightarrow y = \mathrm{D}\, b).$$

Let A be the set defined by $\varphi(x, y)$ and define

$$B := \{(b, \mathrm{D}\, b) : b \text{ generic in } \mathcal{F}\} \subseteq A.$$

At generic points b the formula φ defines $\mathrm{D}\, b$ but we do not have any information about non-generic points. So we need shrink the set A to a subset of $\mathbb{D}$ in order to avoid any possible problems at non-generic points. The set A being a curve in general sense must be defined by a formula $f(x, y) = 0 \wedge \psi(x, y)$ (in the language of differential rings). Then $f(a, \mathrm{D}\, a) = 0$ and hence $f(X, \mathrm{D}\, X) = 0$. Therefore D can be defined from the differential curve $f(x, y) = 0$ by Theorem 4.4.4. Taking into account that for a generic element b the elements $b + p(t)$ and $p(t)b$ are generic as well for any $p(t) \in \mathbb{Q}[t] \setminus \{0\}$, we see that the sets $\varphi(x, y) \wedge \varphi(x + p(t), y + p'(t))$ and $\varphi(x, y) \wedge \varphi(p(t)x, p(t)y + p'(t)x)$ contain B. Arguing as in the proofs of Theorem 4.4.4 and Corollary 4.4.7, after taking sufficiently many conjunctions of such formulas we will eventually define a set B' such that it contains B and is contained in the graph $\mathbb{D}$ of D. Note that B' is 0-definable.

Treating $\mathbb{D}$ as an additive group we prove the following.

Claim. $\mathbb{D} = B' + B'$.

Clearly $B' + B' \subseteq \mathbb{D}$. Let us show that the converse inclusion holds. Any element $d \in F$ has a representation $d = b_1 + b_2$ with b_1 and b_2 generic. Indeed, take b_1 to be generic over d and choose $b_2 = d - b_1$. Hence $(d, \mathrm{D}\, d) = (b_1, \mathrm{D}\, b_1) + (b_2, \mathrm{D}\, b_2) \in B + B \subseteq B' + B'$.

This gives a definition of D without parameters. Moreover, if φ is existential then we get an existential definition. $\square$

Remark 4.5.2. The group $\mathbb{D}$ is in fact a connected ω-stable group (its Morley degree is one). Therefore the equality $\mathbb{D} = G + G$ holds for any definable subset G of $\mathbb{D}$ with $\mathrm{MR}(G) = \mathrm{MR}(\mathbb{D})$ (see, for example, [Mar02], Chapter 7, Corollary 7.2.7). We could use this to show that $\mathbb{D} = B' + B'$ since $\mathrm{MR}(B') = \mathrm{MR}(\mathbb{D}) = \omega$. In fact, the idea is the same as in the above claim; one just passes to a saturated extension and uses the above argument there.

We will shortly give another proof to Proposition 4.5.1. For this we first observe that if D is definable with independent parameters then it is also definable without parameters.

Lemma 4.5.3. *Suppose $\psi(x, y, u_1, \ldots, u_n) \in \mathfrak{L}_R$ and $b_1, \ldots, b_n$ are differentially independent elements in $\mathcal{F}$. If the formula $\psi(x, y, \bar{b})$ defines $y = \mathrm{D}\, x$ then there are 0-definable elements $t_1, \ldots, t_n \in k_0 = \mathbb{Q}(t)$ such that $\psi(x, y, \bar{t})$ defines D (and so D is 0-definable).*

Proof. We have

$$\mathcal{F} \models \psi(x, y, \bar{b}) \longleftrightarrow y = \mathrm{D}\, x.$$

Therefore

$$q(\bar{z}) := \mathrm{tp}_{\mathrm{D}}(\bar{b}) \models \psi(x, y, \bar{z}) \longleftrightarrow y = \mathrm{D}\, x.$$

Since $q(\bar{z})$ is the generic m-type in DCF_0, it consists only of differential inequations. Applying compactness and taking into account that conjunction of finitely many inequations is an inequation as well, we conclude that there is a differential polynomial $f(Z_1, \ldots, Z_m)$ over k_0 such that

$$\mathcal{F} \models \forall \bar{z}(f(\bar{z}) \neq 0 \longrightarrow \forall x, y(\psi(x, y, \bar{z}) \leftrightarrow y = \mathrm{D}\, x)).$$

By Lemma 4.4.1 we can find elements $t_1, \ldots, t_m \in k_0$ such that $f(t_1, \ldots, t_m)$ is non-zero. Now we see that

$$\mathcal{F} \models \psi(x, y, \bar{t}) \longleftrightarrow y = \mathrm{D}\, x$$

and we are done. $\square$

Second proof of Proposition 4.5.1. Let $(b_1, b_2) \in F^2$ be a differentially independent tuple. Then for every $d \in F$ the differential transcendence degree of $d, d + b_1, d + b_2$ is at least 2. It is easy to deduce from this that the following formula defines D:

$$\exists u_1, u_2(\varphi(b_1, u_1) \wedge \varphi(b_2, u_2) \wedge [(\varphi(x, y) \wedge \varphi(x + b_1, y + u_1))$$
$$\vee(\varphi(x, y) \wedge \varphi(x + b_2, y + u_2)) \vee (\varphi(x + b_2, y + u_2) \wedge \varphi(x + b_1, y + u_1))]).$$

Now Lemma 4.5.3 concludes the proof. $\square$

The idea that the behaviour of D at generic (differentially transcendental) points determines its global behaviour as a function can be developed further. We proceed towards this goal in the rest of this section.

Next we show that if $\mathrm{D}\, a$ is not generic over a (in the reduct) then it is in fact definable and hence D is definable. Let $p(y) := \mathrm{tp}_R(\mathrm{D}\, a/a)$ be the type of $\mathrm{D}\, a$ over a in $\mathcal{F}_R$.

Theorem 4.5.4. *The derivation D is definable in $\mathcal{F}_R$ if and only if p has finite Morley rank (in $\mathcal{F}_R$).*

Proof. Obviously, if D is definable then p is algebraic and hence has Morley rank 0. Let us prove the other direction.

Let $\varphi(a, y) \in p$ be a formula of finite Morley rank. Trivially $\mathcal{F} \models \varphi(a, \mathrm{D}\, a)$ and $\varphi(x, y)$ defines a curve in general sense. As in the proof of Proposition 4.5.1 we can define a big subset $\psi(x, y)$ of $\mathbb{D}$, that is, a subset of Morley rank ω. This set certainly contains the point $(a, \mathrm{D}\, a)$ and $\psi(a, y)$ defines $\mathrm{D}\, a$. Thus $\mathrm{D}\, a$ is definable over a and Proposition 4.5.1 finishes the proof. □

Remark 4.5.5. The proof shows that if $\varphi(x, y)$ is an existential formula of rank $< \omega \cdot 2$ which is true of $(a, \mathrm{D}\, a)$ then D is existentially definable.

Corollary 4.5.6. *In the reduct,* $\mathrm{D}\, a$ *is either generic or algebraic (in fact, definable) over* a*.*

Lemma 4.5.7. *If* p *is isolated then it has finite Morley rank (in the reduct).*

Proof. The argument here is an adaptation of the proof of the fact that in differentially closed fields the generic type is not isolated.

Suppose p is isolated but has rank ω, i.e. it is the generic type over a (in the reduct). Then

$$p(y) = \{\neg\varphi(a, y) : \varphi \in \mathfrak{L}_R,\ \mathcal{F} \models \varphi(a, \mathrm{D}\, a) \text{ and } \mathrm{MR}_R(\varphi(a, y)) < \omega\}.$$

Suppose $\neg\psi(a, y)$ isolates p. By Remark 4.3.2 there is a formula $\varphi(a, y)$ for which $\mathrm{MR}_R(\psi(a, y)) < \mathrm{MR}_R(\varphi(a, y)) < \omega$. Then $\varphi(a, y) \wedge \neg\psi(a, y)$ is consistent. A realisation of this formula cannot be generic, for φ has finite Morley rank. This is a contradiction. □

As an immediate consequence one gets the following result.

Corollary 4.5.8. *The derivation* D *is definable in* $\mathcal{F}_R$ *if and only if* p *is isolated.*

Remark 4.5.9. We can consider the quantifier-free type $q(y) := \mathrm{qftp}(\mathrm{D}\, a/a)$. Then D is quantifier-free definable if and only if this type is isolated, if and only if it has finite Morley rank.

Notice that in stability-theoretic language we have proved that D is definable if and only if $\mathrm{tp}_R(\mathrm{D}\, a/a)$ forks over the empty set. Indeed, $\mathrm{MR}_R(\mathrm{D}\, a) = \omega$ (since it is generic in the differentially closed field) and forking in ω-stable theories means that Morley rank decreases, hence $\mathrm{tp}_R(\mathrm{D}\, a/a)$ forks over $\emptyset$ if and only if $\mathrm{MR}_R(\mathrm{D}\, a/a) < \omega$.

In terms of forking independence we have the following formulation: D is definable if and only if $a \not\perp\!\!\!\perp \mathrm{D}\, a$ in $\mathcal{F}_R$. This will be generalised in the next section.

Note that all the above results will remain true if we replace Morley rank everywhere with U-rank.

Now add a differentially transcendental element a to our language and consider the reducts in this new language. Denote the theory of $\mathcal{F}_R$ in this language by T_R^+. Assume that each model of T_R^+ comes from a differentially closed field, that is, each model $\mathcal{K}_R$ is the reduct of a differentially closed field $\mathcal{K} = (K; +, \cdot, \mathrm{D}_K, 0, 1, t, a)$ in which a is generic (differentially transcendental) and relations from R are interpreted canonically (i.e. they are defined in $\mathcal{K}$ by the same formulas as in $\mathcal{F}$). Then the type $p(y)$ will be realised by $\mathrm{D}_K a$ in $\mathcal{K}_R$. The omitting types theorem now yields that p must be isolated. Thus, we have established the following result.

Theorem 4.5.10. *If each model of T_R^+ is the R-reduct (with canonical interpretation[2]) of a model of DCF_0, then D is definable.*

In other words, this means that if each model of T_R^+ is equipped with a derivation which is compatible with R then D is definable. The converse of this holds as well trivially.

This is similar to Beth's definability theorem in spirit (see [Poi00]). Beth's theorem in this setting means that if each model of T_R^+ has at most one derivation compatible with R then D is definable. We showed that if each model has at least one derivation then D is definable. Also it is worth mentioning that unlike Beth's definability theorem, this statement is not true in general for arbitrary theories.

4.6 Further examples

In this section we will see that there is another class of differential equations defining D. It will be used to characterise definable and algebraic closures of generic elements in the reducts. At the end of the section we will give two non-examples. Note that the results of this section will not be used in further sections.

We will show first that differential rational functions define the derivation.

Proposition 4.6.1. *If $E(x,y)$ is given by $g(x)\cdot y = f(x)$ where $\frac{f(X)}{g(X)}$ is a differential rational function which is not an algebraic rational function, then* D *is definable in* $\mathcal{F}_E$.

Lemma 4.6.2. *Let D_1 and D_2 be derivations on a field K and $t \in K$ be such that $\mathrm{D}_1 t = \mathrm{D}_2 t = 1$. If there is a non-zero algebraic polynomial $P(X_0,\ldots,X_n,Y_1,\ldots,Y_m)$ over K such that*

$$P(X, \mathrm{D}_1 X, \ldots, \mathrm{D}_1^n X, \mathrm{D}_2 X, \ldots, \mathrm{D}_2^m X) = 0$$

then $\mathrm{D}_1 = \mathrm{D}_2$.

Proof. We can assume without loss of generality that $n = m$. As in the proof of Lemma 4.2.2 we can show there is a non-zero polynomial $P_1(\bar{X},\bar{Y})$ such that

$$P_1(X_1,\ldots,X_n, \mathrm{D}_2 Y - \mathrm{D}_1 Y + X_1, \ldots, \mathrm{D}_2^n Y - \mathrm{D}_1^n Y + X_n) = 0.$$

Clearly $\mathrm{D} := \mathrm{D}_2 - \mathrm{D}_1$ is a derivation of K. The above identity implies that for some non-zero polynomial Q we have

$$Q(\mathrm{D} X, \mathrm{D}^2 X, \ldots, \mathrm{D}^n X) = 0.$$

[2]Recall that this means that every relation from R is interpreted in the given model by the appropriate differential equation (or formula in the language $\mathfrak{L}_{\mathrm{D}}$).

If $\mathrm{D}_1 \neq \mathrm{D}_2$ then $\mathrm{D} \neq 0$ and there is an element $b \in K$ with $\mathrm{D}\, b \neq 0$. Dividing D by $\mathrm{D}\, b$ we can assume that $\mathrm{D}\, b = 1$. But then substituting $X \mapsto X + rb^j$ for $r \in \mathbb{Q}$ and $j = 1, \ldots, n$, we see that $Q = 0$, which is a contradiction. $\square$

Proof of Proposition 4.6.1. Suppose

$$f(X) = P(X, \mathrm{D}\, X, \ldots, \mathrm{D}^n\, X), \;\; g(X) = Q(X, \mathrm{D}\, X, \ldots, \mathrm{D}^m\, X).$$

We will use Beth's definability theorem to show that D is definable in $T_E := \mathrm{Th}(\mathcal{F}_E)$. Indeed, if we have two derivations D_1 and D_2 on a model $\mathcal{K}_E \models T_E$ that are compatible with E (and K is differentially closed with either of these derivations and $\mathrm{D}_1\, t = \mathrm{D}_2\, t = 1$), then

$$\begin{gathered} P(X, \mathrm{D}_1\, X, \ldots, \mathrm{D}_1^n\, X) \cdot Q(X, \mathrm{D}_2\, X, \ldots, \mathrm{D}_2^m\, X) = \\ P(X, \mathrm{D}_2\, X, \ldots, \mathrm{D}_2^n\, X) \cdot Q(X, \mathrm{D}_1\, X, \ldots, \mathrm{D}_1^m\, X). \end{gathered}$$

Since $f(X)/g(X)$ is not an algebraic rational function, the above identity shows that the conditions of Lemma 4.6.2 are satisfied. Therefore $\mathrm{D}_1 = \mathrm{D}_2$. $\square$

Remark 4.6.3. Note that even in the simple cases $y = \mathrm{D}^2\, x$ and $y = (\mathrm{D}\, x)^2$ the differentiation is not definable without using t since we can not distinguish between D and $-\mathrm{D}$.

Remark 4.6.4. A special case of Proposition 4.6.1, when $g(X) = 1$, i.e. E is given by a differential polynomial, can be proven straightforwardly. Indeed, let E be given by $y = f(x)$ where f is a differential polynomial of order at least one. Then, as in the proof of Theorem 4.4.4, we consider polynomials of the form $f(p(t)X) - (p(t))^{\alpha} f(X)$ for an integer α. All these polynomials are definable and by an appropriate choice of α we can reduce the multi-degree of f. Iterating this process, at some point we are going to get an algebraic polynomial. One can see that in the previous step we must have a differential polynomial of the form $R_1(X) \cdot \mathrm{D}\, X + R_2(X)$ where R_1, R_2 are algebraic polynomials and $R_1 \neq 0$. From this we can easily define $\mathrm{D}\, X$. Note that in fact it is enough to take $p(t) = t$.

Now we prove that from an algebraic function of $x, \mathrm{D}\, x, \ldots, \mathrm{D}^n\, x$ one can define $\mathrm{D}\, x$. But we need to exclude some trivial counterexamples like $y \cdot \mathrm{D}\, x = 0$.

Definition 4.6.5. A differential polynomial $f(X, Y)$ is said to be *non-degenerate* if it cannot be decomposed into a product $g(X)h(X, Y)$ where g is a differential polynomial and h is an algebraic polynomial. An irreducible non-algebraic polynomial which depends on both variables is obviously non-degenerate.

Proposition 4.6.6. *Suppose $E(x, y)$ is defined by a non-degenerate equation $f(x, y) = 0$ where $\mathrm{ord}_X(f) > 0$ and $\mathrm{ord}_Y(f) = 0$. Then D is definable in $\mathcal{F}_E$.*

Proof. Pick a differentially transcendental element $a \in F$ and let

$$f(a, Y) = \prod_{i=1}^{k} f_i(a, Y)^{e_i}$$

be the irreducible factorisation of $f(a,Y)$ over $k_0\langle a\rangle$. Denote

$$g(a,Y) := \prod_{i=1}^{k} f_i(a,Y) = \sum_{i=0}^{m} g_i(a)\cdot Y^i,$$

where $g_i(X) \in k_0\langle X\rangle$ and $g_m \neq 0$.

Consider the formula

$$\psi(x, z_0, \ldots, z_m) = \exists y_1, \ldots, y_m \left(\bigwedge_{i\neq j} y_i \neq y_j \wedge \bigwedge_{i=1}^{m} E(x, y_i) \wedge \bigwedge_{i=1}^{m} \sum_{j=0}^{m} z_j \cdot y_i^j = 0 \right).$$

Clearly, $E(a,y)$ holds if and only if $g(a,y)=0$. The polynomial $g(a,Y)$ has m different roots. Therefore $\varphi(a, z_0, \ldots, z_m)$ holds if and only if the roots of $\sum_{i=0}^{m} z_i \cdot Y^i$ are exactly the same as those of $g(a,Y)$ (as these two polynomials have the same degree in Y). This can happen if and only if $\sum_{i=0}^{m} z_i \cdot Y^i$ is equal to $g(a,Y)$ up to a constant which depends on a. This means that

$$\frac{z_i}{z_m} = \frac{g_i(a)}{g_m(a)},$$

for all i. At least one of $\frac{g_i(X)}{g_m(X)}$ is not an algebraic rational function since otherwise f would be degenerate. But then we can define $\mathrm{D}\, a$ from that differential rational function by Proposition 4.6.1 and we are done. □

Next, we will apply Proposition 4.6.6 to work out definable and algebraic closures of generic points in the reducts. As before, let $a \in F$ be a generic point. We will show that the definable closure of a in $\mathcal{F}_R$ coincides either with the definable closure in the differentially closed field or with that in the pure algebraically closed field.

It is well known what the definable and algebraic closures of arbitrary sets in differentially closed fields look like (Proposition 2.1.1). Taking into account the fact that we have added t as a constant symbol to the language, we see that for a set $A \subseteq F$ the definable and algebraic closures in $\mathcal{F}$ are given by $\mathrm{dcl}_{\mathrm{D}}(A) = k_0\langle A\rangle$ and $\mathrm{acl}_{\mathrm{D}}(A) = (k_0\langle A\rangle)^{\mathrm{alg}}$, where $k_0 = \mathbb{Q}(t)$ and $k_0\langle A\rangle$ is the differential subfield generated by (k_0 and) A. This immediately implies that in the reduct we have $k_0(A) \subseteq \mathrm{dcl}_R(A) \subseteq k_0\langle A\rangle$ and $(k_0(A))^{\mathrm{alg}} \subseteq \mathrm{acl}_R(A) \subseteq (k_0\langle A\rangle)^{\mathrm{alg}}$.

We show that for generic elements one of these two extremal cases must happen.

Theorem 4.6.7. *For $a \in F$ a generic point exactly one of the following statements holds:*

- $\mathrm{dcl}_R(a) = k_0(a)$*; this holds if and only if* $\mathrm{acl}_R(a) = (k_0(a))^{\mathrm{alg}}$ *if and only if* D *is not definable;*
- $\mathrm{dcl}_R(a) = k_0\langle a\rangle$*; this holds if and only if* $\mathrm{acl}_R(a) = (k_0\langle a\rangle)^{\mathrm{alg}}$ *if and only if* D *is definable.*

Proof. It will be enough to show that if $\mathrm{acl}_R(a) \supsetneq (k_0(a))^{\mathrm{alg}}$ then D is definable. Thus, let $\mathrm{acl}_R(a) \supsetneq (k_0(a))^{\mathrm{alg}}$. Choose $b \in (k_0\langle a\rangle)^{\mathrm{alg}} \setminus (k_0(a))^{\mathrm{alg}}$ which is algebraic (in the model theoretic sense) over a in $\mathcal{F}_R$. There is a formula $\varphi(x, y) \in \mathfrak{L}_R$ such that $\varphi(a, b)$ holds and $\varphi(a, y)$ has finitely many realisations. Because $\varphi(a, y)$ defines a finite set in the differentially closed field $\mathcal{F}$, it is equivalent to an algebraic polynomial equation over $k_0\langle a\rangle$. The latter is clearly non-degenerate and is not defined over $k_0(a)$ since b is its root. Applying Proposition 4.6.6 we define $\mathrm{D}\, a$ (over a). Hence D is definable. □

Now using Proposition 4.6.1 we generalise Theorem 4.5.4.

Proposition 4.6.8. *Let $a \in F$ be a differentially transcendental element. If*

$$\mathrm{MR}_R(a, \mathrm{D}\, a, \dots, \mathrm{D}^n a) < \omega \cdot (n + 1)$$

then D *is definable.*

Proof. We proceed to the proof by induction on n. The case $n = 1$ is done in Theorem 4.4.4. Assuming the theorem is true for all numbers less than n, we prove it for n.

There is a formula $\varphi(x_0, x_1, \dots, x_n) \in \mathfrak{L}_R$ with $\mathrm{MR}_R(\varphi) < \omega \cdot (n + 1)$ and

$$\mathcal{F}_R \models \varphi(a, \mathrm{D}\, a, \dots, \mathrm{D}^n a).$$

Since φ does not have "full" rank, we can assume without loss of generality it is given by a differential equation $f(x_0, x_1, \dots, x_n) = 0$ in the language of differential rings. Since $f(a, \mathrm{D}\, a, \dots, \mathrm{D}^n a) = 0$ and a is generic, f must be equal to $g(X, U_1, \dots, U_n)$ for some differential polynomial g with $g(X, 0, \dots, 0) = 0$ where $X := X_0$, $U_i := X_i - \mathrm{D}^i X$. Further, applying the method of additive translates as in the proof of Theorem 4.4.4, we can assume that g does not depend on the first variable, so we write $g(U_1, \dots, U_n)$. However, we cannot proceed as in Theorem 4.4.4 and use multiplicative translates as there exist non-algebraic "homogeneous" differential polynomials of several variables.

Claim. $\mathrm{D}^n a \in \mathrm{dcl}_R(a, \mathrm{D}\, a, \dots, \mathrm{D}^{n-1} a)$.

Proof. The set defined by $\varphi(a, \mathrm{D}\, a, \dots, \mathrm{D}^{n-1} a, y)$ contains $\mathrm{D}^n a$. Moreover, that formula is given by $h(y - \mathrm{D}^n a) = 0$ where $h(U) = g(0, \dots, 0, U)$. Consider the formula

$$\varphi\left[p(t)a, \mathrm{D}(p(t)a), \dots, \mathrm{D}^{n-1}(p(t)a), \mathrm{D}^n(p(t)a) - p(t)\,\mathrm{D}^n a + p(t)y\right], \tag{6.1}$$

for a non-zero polynomial $p(t) \in \mathbb{Q}[t]$.

It is easy to see that this is a formula in the language of reducts with parameters $a, \mathrm{D}\, a, \dots, \mathrm{D}^{n-1} a$ and it is true of $y = \mathrm{D}^n a$. The formula (6.1) is equivalent to $h(p(t)(y - \mathrm{D}^n a)) = 0$. Taking the conjunction of $\varphi(a, \mathrm{D}\, a, \dots, \mathrm{D}^{n-1} a, y)$ and the formula (6.1) we get a formula in the language $\mathfrak{L}_R$ equivalent to[3]

$$h[y - \mathrm{D}^n a] = 0 \wedge h[p(t)(y - \mathrm{D}^n a)] = 0.$$

[3]Here we use square brackets for ease of reading. They do not have any special meaning.

This contains the point $\mathrm{D}^n a$ since $h(0) = 0$ and is contained in sets defined by $h[p(t)(y - \mathrm{D}^n a)] - (p(t))^\alpha h[y - \mathrm{D}^n a] = 0$ for α a positive integer. By an appropriate choice of α we reduce the multi-degree of h and by a choice of p we make sure the difference is not identically zero (see the proof of Theorem 4.4.4). This can be done unless h is an algebraic homogeneous polynomial. Iterating this process we will eventually reach a situation where h has been replaced by an algebraic homogeneous polynomial in which case our formula defines $\mathrm{D}^n a$. $\square$

We proved that there is a formula $\psi(\bar{x}) \in \mathfrak{L}_R$ such that $\psi(a, \mathrm{D}\, a, \ldots, \mathrm{D}^{n-1} a, y)$ has a unique solution which is $\mathrm{D}^n a$. We can assume $\mathrm{MR}_R(a, \mathrm{D}\, a, \ldots, \mathrm{D}^{n-1} a) = \omega \cdot n$ since otherwise D is definable by the induction hypothesis. Let $b_1, \ldots, b_{n-1}$ be differentially independent elements over a. Then

$$\mathrm{tp}_R(a, b_1, \ldots, b_{n-1}) = \mathrm{tp}_R(a, \mathrm{D}\, a, \ldots, \mathrm{D}^{n-1} a) =: p(x_0, \ldots, x_{n-1}).$$

Evidently $\exists! y \psi(x_0, \ldots, x_{n-1}, y) \in p$ where "$\exists!$" stands for "there is a unique" (it is obviously first-order expressible). Therefore

$$\mathcal{F}_R \models \exists! y \psi(a, b_1 \ldots, b_{n-1}, y),$$

and the unique solution of $\psi(a, b_1 \ldots, b_{n-1}, y)$ is a differential rational function of $a, b_1, \ldots, b_{n-1}$. Denote it by $r(a, b_1, \ldots, b_{n-1})$. If r is an algebraic rational function then

$$\xi(\bar{x}) := \forall y (\psi(x_0, \ldots, x_{n-1}, y) \leftrightarrow y = r(x_0, \ldots, x_{n-1}))$$

is a formula in the language of reducts and is true of $(a, b_1, \ldots, b_{n-1})$. Hence it must be true of $(a, \mathrm{D}\, a, \ldots, \mathrm{D}^{n-1} a)$ too, which means $\mathrm{D}^n a = r(a, \mathrm{D}\, a, \ldots, \mathrm{D}^{n-1} a)$ which is impossible since a is differentially transcendental.

Thus, r is not algebraic. By a compactness argument (as in Lemma 4.5.3) we can choose $t_1, \ldots, t_{n-1} \in \mathbb{Q}(t, a)$ such that $\mathcal{F} \models \xi(a, t_1 \ldots, t_{n-1})$ and $r(a, t_1, \ldots, t_{n-1}) \in \mathbb{Q}(t)\langle a \rangle \setminus \mathbb{Q}(t, a)$. This guarantees that the formula $\psi(a, t_1 \ldots, t_{n-1}, y)$ (which is in the language of reducts) defines a non-algebraic (in the field theoretic sense) element over a and so $\mathrm{dcl}_R(a) \supsetneq k_0(a)$. So D is definable due to Theorem 4.6.7. $\square$

Recall that in a stable theory a set A (in the monster model) is called *independent* (over B) if for any $a \in A$ we have $a \mathop{\smile\!\!\!\!\!\!\;|}_B A \setminus \{a\}$.

Corollary 4.6.9. D *is definable in* $\mathcal{F}_R$ *if and only if the sequence* $a, \mathrm{D}\, a, \mathrm{D}^2 a, \ldots$ *is not independent (over the empty set) in* $\mathcal{F}_R$.

Proof. If the sequence $a, \mathrm{D}\, a, \mathrm{D}^2 a, \ldots$ is not independent then for some n the set $\{a, \mathrm{D}\, a, \ldots, \mathrm{D}^n a\}$ is not independent. Therefore $\mathrm{MR}_R(a, \mathrm{D}\, a, \ldots, \mathrm{D}^n a) < \omega \cdot (n + 1)$. $\square$

As a common generalisation of Theorem 4.4.4 and Proposition 4.6.6 we prove the following result.

Proposition 4.6.10. *Suppose* E *(a curve in general sense) contains a differential curve defined by a non-degenerate equation* $f(x, y) = 0$ *where* $\mathrm{ord}_X(f) > 0$ *and* $\mathrm{ord}_Y(f) = 0$. *Then* D *is definable in* $\mathcal{F}_E$.

Proof. Let $g(X,Y) = p(X, \mathrm{D}\,X, \ldots, \mathrm{D}^n\,X, Y)$ be an irreducible non-degenerate factor of $f(X,Y)$. Furthermore, as $\mathrm{ord}_X(f) > 0$ we can assume that $\mathrm{ord}_X(g) > 0$. Consider the formula

$$\varphi(x, y_1, \ldots, y_n) := \exists z (E(x,z) \wedge p(x, y_1, \ldots, y_n, z) = 0).$$

Clearly $\mathcal{F}_E \models \varphi(a, \mathrm{D}\,a, \ldots, \mathrm{D}^n\,a)$. Further, if $\varphi(a, b_1, \ldots, b_n)$ holds then for some c we have

$$p(a, \bar{b}, c) = 0 \wedge E(a,c).$$

Since p is irreducible, $a, b_1, \ldots, b_n$ are algebraically dependent over c. Moreover, $\mathrm{ord}_X(g) > 0$ implies that $b_1, \ldots, b_n$ are algebraically dependent over $\{a, c\}$. On the other hand, c is differentially algebraic over a. Therefore $a, \bar{b}$ are differentially dependent and hence $\mathrm{MR}_{\mathrm{D}}(\varphi) < \omega \cdot (n+1)$. Now Proposition 4.6.8 finishes the proof. □

One will certainly notice at this point that we found a number of conditions on $\mathcal{F}_R$ which are all equivalent to definability of D. We sum up all these conditions in the following theorem.

Theorem 4.6.11. *For a generic point $a \in F$ the following are equivalent:*

1. D *is definable in the reduct $\mathcal{F}_R$ without parameters,*
2. $\mathrm{MR}_R(\mathrm{D}\,a/a) < \omega$,
3. $\mathrm{MR}_R(\mathrm{D}\,a/a) = 0$,
4. $\mathrm{tp}_R(\mathrm{D}\,a/a)$ *forks over the empty set,*
5. *The sequence $(\mathrm{D}^n\,a)_{n \geq 0}$ is not (forking) independent,*
6. $\mathrm{dcl}_R(a) \supsetneq k_0(a)$,
7. $\mathrm{acl}_R(a) \supsetneq (k_0(a))^{\mathrm{alg}}$,
8. *Every model of T_R^+ is the $\mathfrak{L}_R$-reduct (with canonical interpretation) of a differentially closed field,*
9. *Every automorphism of $\mathcal{F}_R$ fixes $\mathbb{D}$ setwise.*

Proof. We need only show $9 \Rightarrow 1$. Take any automorphism σ of $\mathcal{F}_R$ which fixes a. It fixes $\mathbb{D}$ setwise, hence $(\sigma(a), \sigma(\mathrm{D}\,a)) \in \mathbb{D}$. This means $\sigma(\mathrm{D}\,a) = \mathrm{D}(\sigma a) = \mathrm{D}\,a$. Thus any automorphism of $\mathcal{F}_R$ fixing a fixes $\mathrm{D}\,a$. Since $\mathcal{F}_R$ is saturated, $\mathrm{D}\,a$ is definable over a. Therefore D is definable. □

We conclude this section by giving examples of differential equations that do not define D.

Example 4.6.12. We will show that unary relations cannot define D.

Let R consist of unary relations, i.e. definable subsets of F (by quantifier elimination of DCF_0 we may assume R consists of sets of solutions of one-variable equations). Then D is not definable in $\mathcal{F}_R$.

Consider the differential closure of k_0 inside $\mathcal{F}$, that is,

$$K = \{d \in F : \mathrm{DR}(d) < \omega\}.$$

This is obviously a differentially closed field. Take a generic element $a \in F$, i.e. an element outside K. Let $L \supseteq K$ be the differential closure of $K\langle a \rangle$ inside $\mathcal{F}$. Further, denote $a_i = \mathrm{D}^i a,\ i \geq 0$ and let A be a transcendence basis of L over K containing these elements (not differential transcendence basis, which would consist only of a).

Define a new derivation D_1 on L as follows. Set $\mathrm{D}_1 = \mathrm{D}$ on $K \cup A \setminus \{a_0, a_1\}$ and $\mathrm{D}_1 a_0 = a_2,\ \mathrm{D}_1 a_1 = a_0$. This can be uniquely extended to a derivation of L. The field automorphism $\sigma \in \mathrm{Aut}(L/K)$ which fixes $A \setminus \{a_0, a_1\}$ and swaps a_0 and a_1 is in fact an isomorphism of differential fields $\mathcal{L} = (L; +, \cdot, \mathrm{D})$ and $\mathcal{L}_1 = (L; +, \cdot, \mathrm{D}_1)$. Therefore the latter is differentially closed.

Thus we have a field L equipped with two different derivations D and D_1 and L is a differentially closed field with respect to each of them. Further, $K \subset L$ consists of all differentially algebraic elements in $\mathcal{L}$. Since $\mathcal{L}$ and $\mathcal{L}_1$ are isomorphic over K, the differential closure of k_0 in $\mathcal{L}_1$ is equal to K as well. Therefore the interpretations of relation symbols for one-variable differential equations in $\mathcal{L}$ and $\mathcal{L}_1$ are contained in K. But D and D_1 agree on K and therefore those interpretations agree in $\mathcal{L}$ and $\mathcal{L}_1$. This shows that D is not definable in the structure $\mathcal{F}_R$.

Example 4.6.13. Now we give a more interesting example.

Proposition 4.6.14. *The exponential differential equation* $\mathrm{D}\, y = y \,\mathrm{D}\, x$ *does not define* D.

We show first that for a differential equation E if D is definable in T_E then E is uniquely determined by T_E.

Lemma 4.6.15. *If* D *is definable in* T_E *then for any differential equation* $E'(x, y)$

$$T_E = T_{E'} \Rightarrow E = E'.$$

Proof. Let E be given by the equation $f(x, y) = 0$. Since D is definable, the formula $\forall x, y (E(x, y) \leftrightarrow f(x, y) = 0)$ (more precisely, its translation into the language of the reducts) is in T_E. In other words, the fact that E is defined by the equation $f(x, y) = 0$ is captured by T_E. Therefore if E' has the same theory as E it must be defined by the same equation $f(x, y) = 0$. $\square$

Proof of Proposition 4.6.14. An axiomatisation of the complete theory of the exponential differential equation is given in [Kir09] (see Section 5.1). One can deduce from the axioms that the equation $\mathrm{D}\, y = 2y \,\mathrm{D}\, x$ is elementarily equivalent to the exponential equation. But clearly those two equations define different sets in differentially closed fields. Hence the previous lemma shows that D is not definable if E is given by $\mathrm{D}\, y = y \,\mathrm{D}\, x$. $\square$

We will give another proof to Proposition 4.6.14 in Section 4.8.

4.7 Connection to predimensions

As we have already mentioned, definability of D in a reduct should show that the latter does not have any adequate predimension inequality. Now we prove a precise result in this direction. The main result of the next section contributes to that idea as well.

Let $\mathcal{F}$ be a countable saturated differentially closed field. Assume $\mathfrak{C}$ is a collection of structures in the language of reducts $\mathfrak{L}_R$ and δ is a predimension on $\mathfrak{C}_{f.g.}$ satisfying all necessary conditions given in Chapter 3. Let d be the dimension associated to δ. Below by a d-generic type (over some parameter set A) we mean the type of an element a with $d(a/A) = 1$.

Theorem 4.7.1. *Assume the underlying fields of structures from $\mathfrak{C}_{f.g.}$ are algebraically closed of finite transcendence degree over $\mathbb{Q}$. Assume further that d-generic 1-types (over finite sets) are not algebraic. If D is definable in $\mathcal{F}_R$ and δ is strongly adequate, then the reduct is model complete and hence δ is trivial.*

In general, it is possible that d-generic 1-type is not unique. Moreover, in some trivial examples such a type may be algebraic. So our assumption excludes such degenerate cases. In particular, if the free amalgamation property holds for $\mathfrak{C}_{f.g.}$ then d-generic types cannot be algebraic. Actually, it will suffice to assume that generic 1-types have more than one realisation. In fact, we expect d-generic types to be generic in the sense of the reduct of a differentially closed field. As we know those are unique and have maximal rank.

Proof. Strong adequacy means that $\mathcal{F}_R$ is the Fraïssé limit of $\mathfrak{C}_{f.g.}$. Let $a \in F$ be differentially transcendental. Denote $A := \lceil a \rceil$ (the strong closure of a in $\mathcal{F}_R$) and $A' := \lceil a, \mathrm{D}\, a \rceil$.

If $d(\mathrm{D}\, a/a) = 1$ then by our assumption $\mathrm{tp}_R(\mathrm{D}\, a/a)$ (which is a d-generic type) has more than one realisation which contradicts definability of $\mathrm{D}\, a$ over a. Thus, $d(\mathrm{D}\, a/a) = 0$ and so $\delta(A') = d(a, \mathrm{D}\, a) = d(a) = \delta(A)$. Since $A \subseteq A'$, we have $\delta(A'/A) = \delta(A') - \delta(A) = 0$. Let A' be $\mathfrak{C}$-generated by $(a, \mathrm{D}\, a, \bar{u})$. Extending $\bar{u}$ if necessary we can assume that $A' = \mathbb{Q}(a, \mathrm{D}\, a, \bar{u})^{\mathrm{alg}}$ (here we use the fact that $\mathrm{td}(A'/\mathbb{Q}) < \aleph_0$).

If $(v, \bar{w})$ is a realisation of the existential type $\mathrm{etp}_R(\mathrm{D}\, a, \bar{u}/a)$ then we claim that $v = \mathrm{D}\, a$. Indeed, in a differentially closed field the type of a (field-theoretically) algebraic element is isolated by its minimal algebraic equation (and so all algebraic conjugates of that element have the same type), hence $B := \mathbb{Q}(a, v, \bar{w})^{\mathrm{alg}} \cong \mathbb{Q}(a, \mathrm{D}\, a, \bar{u})^{\mathrm{alg}} = A'$ where the isomorphism is in the sense of $\mathfrak{L}_R$-structures (induced from $\mathcal{F}_R$). Therefore $B \in \mathfrak{C}$ and $\delta(B) = \delta(A')$. If $d(a) = 0$ then $\delta(A') = 0$ and so $\delta(B) = 0$. If $d(a) = 1$ then $A = \langle a \rangle$ is the structure $\mathfrak{C}$-generated by a. Since $a \in B$, we must have $A \subseteq B$ and $\delta(B/A) = \delta(B) - \delta(A) = \delta(A') - \delta(A) = 0$. In both cases $B \leq \mathcal{F}_R$. Now by homogeneity of the Fraïssé limit for strong substructures, the above isomorphism between B and A' extends to an automorphism of $\mathcal{F}_R$. This implies $\mathrm{tp}_R(\mathrm{D}\, a, \bar{u}/a) = \mathrm{tp}_R(v, \bar{w}/a)$ and so $\mathrm{tp}_R(\mathrm{D}\, a/a) = \mathrm{tp}_R(v/a)$. On the other hand, $\mathrm{D}\, a$ is definable over a, hence we must have $v = \mathrm{D}\, a$.

Thus, for $p(x, y, \bar{z}) := \mathrm{etp}_R(a, \mathrm{D}\, a, \bar{u})$ we have

$$\mathcal{F}_R \models \exists \bar{z} \bigwedge p(a, y, \bar{z}) \longleftrightarrow y = \mathrm{D}\, a.$$

A standard compactness argument shows that there is an (existential) $\varphi(x, y, \bar{z}) \in p$ so that

$$\mathcal{F}_R \models \exists \bar{z} \varphi(a, y, \bar{z}) \longleftrightarrow y = \mathrm{D}\, a.$$

By Remark 4.5.5, D is existentially definable in $\mathcal{F}_R$. $\square$

The result will still hold if instead of assuming that finitely generated structures have finite transcendence degree we assume δ is quantifier-free (infinitely) definable. Nonetheless, we cannot apply these results to the exponential differential equation since there finitely generated means of finite transcendence degree over C. However, Theorem 4.8.2 helps in that situation.

4.8 Model completeness

In Section 4.5 we showed that if a formula $\varphi(x, y)$ defines a small set which contains the point $(a, \mathrm{D}\, a)$ for a differentially transcendental element a then D is definable. Moreover, if φ is existential then D is existentially definable. Smallness of a set can be verified as follows. If b is a generic (differentially transcendental) element over a, that is, (a, b) is a generic pair (differentially independent), then $\varphi(x, y)$ defines a small set if and only if $\neg\varphi(a, b)$. Thus, instead of working with formulas defining D we can work with formulas $\varphi(x, y)$ with $\varphi(a, \mathrm{D}\, a) \wedge \neg\varphi(a, b)$.

Definition 4.8.1. A formula $\varphi(x, y) \in \mathfrak{L}_R$ is a D-*formula* if $\mathcal{F} \models \varphi(a, \mathrm{D}\, a) \wedge \neg\varphi(a, b)$, where (a, b) is a differentially independent pair.

Here we worked over the empty set. In particular, a is differentially transcendental over the empty set and the definitions that we consider are again over the empty set, i.e. without parameters. However, it is clear that we could in fact work over any set $A \subseteq F$. In this case we should let a be differentially transcendental over A. If $\varphi(x, y)$ is a formula over A such that $\varphi(a, \mathrm{D}\, a) \wedge \neg\varphi(a, b)$ holds where b is differentially transcendental over Aa (in this case we will say φ is a D-formula over A), then certainly D is definable over A. Moreover, if $\varphi(x, y)$ is existential then D is existentially definable over A. In this section we use this fact to prove that under a natural assumption, if D is definable then it is existentially definable.

As above $a \in F$ is a differentially transcendental element and $k_0 = \mathbb{Q}(t) = \mathrm{dcl}_R(\emptyset)$ (recall that t is an element with $\mathrm{D}\, t = 1$).

Theorem 4.8.2. *If T_R is inductive (i.e. $\forall\exists$-axiomatisable) and defines* D *then it defines* D *existentially and, therefore, T_R is model complete.*

As we saw in Section 3.4, this result can be used to show that if a derivation is definable in a reduct which is $\forall\exists$-axiomatisable then the latter cannot have a non-trivial strongly adequate predimension. Recall also that as we observed in Lemma

3.2.11, under reasonable definability assumptions on the predimension a Hrushovski construction yields a nearly model complete theory. Theorem 4.8.2 shows that if we also assume D is definable then we have a much stronger quantifier elimination result, namely, model completeness.

We now establish an auxiliary result which will be used in the proof of Theorem 4.8.2.

Lemma 4.8.3. *Let $\varphi(x, \bar{u}) \in \mathfrak{L}_R$ be a quantifier-free formula and $p(X, Y, \bar{U}) \in k_0[X, Y, \bar{U}]$ be an algebraic polynomial which is monic in the Y variable. Denote*

$$\chi(x, y) := \forall \bar{u}(\varphi(x, \bar{u}) \to p(x, y, \bar{u}) = 0).$$

If $\exists \bar{u} \varphi(a, \bar{u})$ and $\chi(a, \mathrm{D}\, a)$ hold then D *is existentially definable.*

Proof. Let the tuple $(b_1, \ldots, b_m, e_1, \ldots, e_s)$ be of maximal differential transcendence degree m over a such that $\mathcal{F}_R \models \varphi(a, b_1, \ldots, b_m, e_1, \ldots, e_s)$ and assume that $b_1, \ldots, b_m$ are differentially independent over a.

Consider the formula

$$\psi(x, y, \bar{z}) = \exists v_1, \ldots, v_s(\varphi(x, \bar{z}, \bar{v}) \wedge p(x, y, \bar{z}, \bar{v}) = 0).$$

Clearly $\psi(a, \mathrm{D}\, a, \bar{b})$ holds. Moreover, if $\psi(a, d, \bar{b})$ holds then for some $d_1, \ldots, d_s$ we have

$$\mathcal{F}_R \models \varphi(a, \bar{b}, \bar{d}),$$

which implies that $d_1, \ldots, d_s$ have finite rank over $\{a, b_1, \ldots, b_m\}$. Since p is monic as a polynomial of Y and $p(a, d, \bar{b}, \bar{d}) = 0$, we conclude that $d \in (k_0(a, \bar{b}, \bar{d}))^{\mathrm{alg}}$ and hence d is not generic over $\{a, b_1, \ldots, b_m\}$.

Thus working over the parameter set $B = \{b_1, \ldots, b_m\}$ we see that a is generic over B and $\psi(x, y, \bar{b})$ is a D-formula over B. Hence we can make it into a proper definition of D with parameters from B. Thus, we get an existential definition of D with differentially independent parameters $b_1, \ldots, b_m$. By Lemma 4.5.3 we have an existential definition without parameters. □

If we assume D has a universal definition then the proof of Theorem 4.8.2 can be somewhat simplified. Though we will not work under this assumption, the following lemma (combined with Lemma 3.2.11) shows that it would be a natural condition from the point of view of adequate predimension inequalities.

Lemma 4.8.4. *If there is a* D*-formula which is a Boolean combination of existential formulas then there is a universal* D*-formula.*

Proof. Assume D is not existentially definable. Let

$$\delta(x, y) = \bigvee_i (\varphi_i(x, y) \wedge \psi_i(x, y))$$

be a D-formula where φ_i is universal and ψ_i is existential. Since $\delta(a, \mathrm{D}\, a)$ holds, there is some i_0 for which $\varphi_{i_0}(a, \mathrm{D}\, a) \wedge \psi_{i_0}(a, \mathrm{D}\, a)$ holds. If ψ_{i_0} is small then D is existentially definable. Therefore φ_{i_0} is small and it is a universal D-formula. □

Now we are ready to prove our main theorem.

Proof of Theorem 4.8.2. Let $\delta(x,y)$ be a formula defining D. We assume that D is not existentially definable, hence δ is not existential. The main idea of the proof is that unless one says explicitly that $\forall x \exists y \delta(x,y)$, one cannot guarantee that δ defines a function. In other words we will prove that $\forall x \exists y \delta(x,y)$ (which is not an $\forall\exists$-sentence) is not implied by the $\forall\exists$-part of T_R, as otherwise we will be able to find an existential definition of D. This will contradict our assumption of inductiveness.

Let T be the $\forall\exists$-part of T_R, i.e. the subset of T_R consisting of $\forall\exists$-sentences. In other words

$$T = \{\forall \bar{x} \exists \bar{y} \varphi(\bar{x},\bar{y}) : \varphi \text{ is a quantifier-free formula in } \mathfrak{L}_R,\ \mathcal{F}_R \models \forall \bar{x} \exists \bar{y} \varphi(\bar{x},\bar{y})\}.$$

Denote $\Phi := \{\varphi(\bar{x},\bar{y}) : \forall \bar{x} \exists \bar{y} \varphi(\bar{x},\bar{y}) \in T\}$.

By our assumption T is an axiomatisation of T_R. However, we will get a contradiction to this by showing that T has a model in which $\forall x \exists y \delta(x,y)$ does not hold. The construction of that model will go as follows. We start with the field $k = \mathbb{Q}(t,a) = k_0(a)$ and add solutions of the formulas $\varphi \in \Phi$ step by step (for $\varphi(\bar{x},\bar{y}) \in \Phi$ we think of $\bar{x}$ as coefficients and of $\bar{y}$ as solutions). We also make sure that we do not add $\mathrm{D}\, a$ in any step. If the latter is not possible then we show that D is existentially definable.

In order to implement this idea, we expand the language by adding constant[4] symbols for solutions of all $\varphi \in \Phi$. First, take $C_0 = \{a\}$. We will inductively add new constant symbols to C_0 countably many times.

If C_l is constructed then C_{l+1} is the expansion of C_l by new constant symbols as follows. For each $\varphi(\bar{x},\bar{y}) \in \Phi$ with $|\bar{x}| = m,\ |\bar{y}| = n$ and for all $\bar{c} \in C_l^m$ add new constant symbols $d^1_{\varphi,\bar{c}}, \ldots, d^n_{\varphi,\bar{c}}$. After adding these new constants for all $\varphi \in \Phi$ we get C_{l+1}. Finally set $C = \bigcup_l C_l$. This is a countable set.

Now consider the following sets of sentences in the expanded language $\mathfrak{L}_R \cup C$. First, denote

$$\Gamma(C) := \{\varphi(c_1,\ldots,c_m,d^1_{\varphi,\bar{c}},\ldots,d^n_{\varphi,\bar{c}}) : \varphi(\bar{x},\bar{y}) \in \Phi,\ |\bar{x}| = m,\ |\bar{y}| = n,\ \bar{c} \in C^m\}.$$

Further, let

$$\Delta(C) := \{\neg\delta(a,c) : c \in C\}.$$

Finally we set

$$\Sigma(C) := T_R \cup \mathrm{tp}_R(a) \cup \Gamma(C) \cup \Delta(C).$$

Claim. $\Sigma := \Sigma(C)$ is satisfiable.

Proof. If it is not satisfiable, then a finite subset $\Sigma_0 \subseteq \Sigma$ is not satisfiable. Denote the set of constants from C that occur in sentences from Σ_0 by $\{a, e_1, \ldots, e_n\}$ (if necessary, we can assume a occurs in Σ_0 inessentially). We are going to give a its

[4] In this proof we use the word "constant" for constant symbols only and not for constants in the sense of differential algebra. In particular, the interpretations of those constant symbols may not be constants in the differential sense.

canonical interpretation in $\mathcal{F}$ and this is the reason that we separated it from the other constant symbols. Let $\psi(a, e_1, \ldots, e_n) := \bigwedge(\Sigma_0 \cap \Gamma)$. The formula $\psi(x, u_1, \ldots, u_n)$ is clearly quantifier-free and without parameters.

Thus

$$T_R \cup \mathrm{tp}_R(a) \cup \{\psi(a, e_1, \ldots, e_n)\} \cup \{\neg\delta(a, e_i) : i = 1, \ldots, n\}$$

is inconsistent. This means that in particular we cannot find interpretations for $e_1, \ldots, e_n$ in $\mathcal{F}_R$ which will make the latter into a model of Σ_0. As already mentioned above, a is interpreted canonically in $\mathcal{F}$, i.e. its interpretation is the element $a \in F$.

Therefore

$$\mathcal{F}_R \not\models \exists u_1, \ldots, u_n \left[\psi(a, \bar{u}) \wedge \bigwedge_i \neg\delta(a, u_i)\right].$$

This means

$$\mathcal{F}_R \models \forall \bar{u} \left[\psi(a, \bar{u}) \longrightarrow \bigvee_i u_i = \mathrm{D}\, a\right].$$

Note that evidently $\mathcal{F}_R \models \exists \bar{u}\psi(a, \bar{u})$, i.e. the implication above does not hold vacuously. So the formula

$$\chi(x, y) := \forall \bar{u} \left[\psi(x, \bar{u}) \longrightarrow \prod_{i=1}^{n}(y - u_i) = 0\right]$$

satisfies the conditions of Lemma 4.8.3. Hence, D is existentially definable. This contradiction proves the claim. □

Thus Σ is satisfiable. Take a model $\mathcal{M}$ of Σ and inside this model consider the subset K consisting of interpretations of the constant symbols from C. We claim that K is closed under addition and multiplication and contains $0, 1, t$. This is because the sentence $\forall x, y \exists z, w(x + y = z \wedge x \cdot y = w)$, being $\forall\exists$, belongs to T. So, by our construction of C, for each $c_1, c_2 \in C$ we have elements $d_1, d_2 \in C$ such that the sentences $c_1 + c_2 = d_1$, $c_1 \cdot c_2 = d_2$ are in Σ. Similarly $0, 1, t \in K$ since the sentences $\exists x(x = 0)$, $\exists x(x = 1)$, and $\exists x(x = t)$ are in T. Therefore K is a structure in the language of rings. In fact it is an algebraically closed field (containing k) since ACF_0 is $\forall\exists$-axiomatisable. Hence K is a structure $\mathcal{K}_R = (K; +, \cdot, 0, 1, t, P)_{P \in R}$ in the language of the reducts (with the induced structure from $\mathcal{M}$). By the choice of Σ we know that $\mathcal{K}_R$ is a model of T.

If we chose $\mathcal{M}$ to be saturated of cardinality $|F|$ (such a model exists due to stability) then we can identify it with $\mathcal{F}_R$.[5] In that case $\mathcal{K}_R$ is a substructure obtained by starting with $k_0(a)$ and inductively adding solutions to formulas from Φ.

Suppose for a moment that δ is universal in order to illustrate what we are going to do next. Let

$$\delta(x, y) = \forall \bar{v}\rho(x, y, \bar{v})$$

with ρ quantifier-free. Since $\mathcal{F}_R \models \neg\delta(a, s)$ for any $s \in K$, there is a witness $\bar{l}_s \in F$ such that $\mathcal{F}_R \models \neg\rho(a, s, \bar{l}_s)$. However this witness may not be in K. So we add all

[5]Note that this is not essential, but it helps to understand how the proof works.

those witnesses to K and then repeat the above procedure to make it a model of T. We also make sure we never add $\mathrm{D}\, a$, which is possible as above (otherwise D would be existentially definable). Iterating this process countably many times and taking the union of all the constructed substructures we end up with a structure $\mathcal{N}_R$ in the language of reducts which is a model of T and contains witnesses for each of the formulas $\exists \bar{v} \neg \rho(a, s, \bar{v})$ where $s \in N$. Thus, $\mathcal{N}_R \models \neg \exists y \delta(a, y)$ which means that T is not an axiomatisation of T_R. This contradiction proves the theorem.

Now we consider the general case. Let δ be of the form

$$\delta(x, y) = \forall \bar{v}_1 \exists \bar{w}_1 \forall \bar{v}_2 \ldots \forall \bar{v}_n \exists \bar{w}_n \rho(x, y, \bar{v}_1, \ldots, \bar{v}_n),$$

where ρ is quantifier-free (the tuples $\bar{v}_1$ and $\bar{w}_n$ can be empty). Then

$$\neg\delta(x, y) = \exists \bar{v}_1 \forall \bar{w}_1 \exists \bar{v}_2 \ldots \exists \bar{v}_n \forall \bar{w}_n \neg\rho(x, y, \bar{v}_1, \ldots, \bar{v}_n).$$

We add new constant symbols as follows. Firstly, for each $s \in C$ we add a tuple of constants $\bar{l}^1_s$ of the same length as $\bar{v}_1$. Then for each i and each tuple $\bar{c} \in C^{|\bar{w}_i|}$ we add new constants $\bar{l}^{i+1}_{\bar{c}}$ with $|\bar{l}^{i+1}_{\bar{c}}| = |\bar{v}_{i+1}|$. Denote this extension of C by C'. Then we add new constant symbols to C' for solutions of all formulas $\varphi \in \Phi$ as above. We denote this set by C^1. Then we iterate this procedure by adding new constants to witness $\neg\delta(a, s)$ (for each s from the set of constants already constructed) and then adding new constants for solutions of $\varphi \in \Phi$. Thus, we get a chain $C \subseteq C^1 \subseteq C^2 \subseteq \ldots$. Let $\tilde{C}$ be their union.[6]

For $A \subseteq \tilde{C}$ denote

$$\Xi(A) := \{\neg\rho(a, s, \bar{l}^1_s, \bar{c}_1, \bar{l}^2_{\bar{c}_1}, \ldots, \bar{l}^n_{\bar{c}_{n-1}}, \bar{c}_n) : s \in A,\ \bar{c}_i \in A^{|\bar{w}_i|}\}.$$

For any $s \in K$ we know that $\mathcal{F}_R \models \neg\delta(a, s)$, therefore $\Sigma(C) \cup \Xi(C)$ is satisfiable (note that this collection of sentences contains parameters from C'). The proof of the above claim shows that $\Sigma(C^1) \cup \Xi(C)$ is satisfiable and so $\Sigma(C^1) \cup \Xi(C^1)$ is satisfiable too. Proceeding inductively we see that $\Sigma(C^i) \cup \Xi(C^i)$ is satisfiable for each $i < \omega$. Hence, by compactness, $\Sigma(\tilde{C}) \cup \Xi(\tilde{C})$ is satisfiable.

The interpretation of $\tilde{C}$ in a model of $\Sigma(\tilde{C}) \cup \Xi(\tilde{C})$ gives a structure $\mathcal{N}_R$ in the language of reducts which is a model of T and contains witnesses for each of the formulas $\exists \bar{v}_1 \forall \bar{w}_1 \exists \bar{v}_2 \ldots \exists \bar{v}_n \forall \bar{w}_n \neg\rho(a, s, \bar{v}_1, \ldots, \bar{v}_n)$ where $s \in N$. Hence $\mathcal{N}_R \models \neg\exists y \delta(a, y)$ which means that T is not an axiomatisation of T_R, which is a contradiction. $\square$

As an immediate application of Theorem 4.8.2 we give another proof to Proposition 4.6.14 which states that if E is the exponential differential equation, i.e. it is given by $\mathrm{D}\, y = y \,\mathrm{D}\, x$, then D is not definable in $\mathcal{F}_E$.[7] Indeed, we will see in Section 5.1 that the first-order theory of the exponential differential equation has an

[6]In this construction for an element $s \in C$ (and for tuples $\bar{c}_i$) we add one $\bar{l}^1_s$ in each step. Though this does not cause any problems, at each step we can add the corresponding sets of constants only for new constant symbols. In particular, after adding one $\bar{l}^1_s$ we do not add any such tuple for the same s any more. Alternatively, we could just require all those different tuples for the same element s to be equal by adding the appropriate formulas stating their equality.

[7]Note that Kirby [Kir09] gives yet another proof of this fact by considering a lattice of reducts for exponential differential equations of some collections of semiabelian varieties, and showing that each of these can be expanded properly inside DCF_0.

$\forall\exists$-axiomatisation ([Kir09]). It is not model complete however, hence D cannot be definable due to Theorem 4.8.2. Of course it is the Ax-Schanuel inequality that is responsible for this. As Kirby proved it is an adequate predimension inequality.

4.9 Pregeometry on differentially closed fields and reducts

Working in a model $\mathcal{F}$ of DCF_0 we can define a closure operator $\mathrm{cl}_{\mathrm{D}} : \mathfrak{P}(F) \to \mathfrak{P}(F)$ as follows: for a subset $A \subseteq F$ set

$$\mathrm{cl}_{\mathrm{D}}(A) := \{a \in F : \mathrm{MR}_{\mathrm{D}}(a/A) < \omega\}.$$

Thus, $\mathrm{cl}_{\mathrm{D}}(A)$ is just the set of all differentially algebraic elements over A. It is obviously a differentially closed field.

It is quite easy to see that cl_{D} defines a pregeometry on F. We observe that the same is valid for reducts $\mathcal{F}_R$. Indeed, define the R-closure on F by

$$\mathrm{cl}_R(A) := \{a \in F : \mathrm{MR}_R(a/A) < \omega\}.$$

We show below that this is a pregeometry. In fact it is a classical result and is true in a much more general setting, when one associates a pregeometry with a regular type (see [Pil83], Proposition 9.34, or [PT11], Theorem 1).

Proposition 4.9.1. *The operator* cl_R *defines a pregeometry.*

Proof. Reflexivity and finite character are evident. Exchange follows from symmetry of forking independence. Indeed, if $b \in \mathrm{cl}_R(Aa) \setminus \mathrm{cl}_R(A)$ then $b \not\perp\!\!\!\perp_A a$. Hence $a \not\perp\!\!\!\perp_A b$ and so $a \in \mathrm{cl}_R(Ab)$.

Transitivity is a bit harder to prove. We need to show that $\mathrm{cl}_R(\mathrm{cl}_R(A)) = \mathrm{cl}_R(A)$. We are going to work with U-rank rather than Morley rank.[8]

Suppose $b \in \mathrm{cl}_R(\mathrm{cl}_R(A))$. Then there are $a_1, \ldots, a_n \in \mathrm{cl}_R(A)$ such that $b \in \mathrm{cl}_R(\bar{a})$. By Lascar's inequality we have

$$\mathrm{U}_R(b, \bar{a}/A) = \mathrm{U}_R(b/A\bar{a}) + \mathrm{U}_R(\bar{a}/A) < \omega.$$

Note that the equality above holds since both ranks on the right hand side are finite. Now we deduce that $\mathrm{U}_R(b/A) < \omega$ and hence $b \in \mathrm{cl}_R(A)$. $\square$

The following result is a direct consequence of Theorem 4.5.4.

Theorem 4.9.2. *The derivation* D *is definable in the reduct* $\mathcal{F}_R$ *if and only if* $\mathrm{cl}_{\mathrm{D}} = \mathrm{cl}_R$.

[8]It is not difficult to see that in reducts of differentially closed fields finiteness of Morley rank is equivalent to finiteness of U-rank

It will be more interesting to prove that isomorphism of those two pregeometries implies definability of D. If that can be proven then one can hope that an intrinsic property of cl_R can characterise definability of D. Accomplishing this would lead to a deeper problem, namely, interpreting a differential field in a topological structure (the topology of which is similar to the Kolchin topology) with a pregeometry which should be understood as the abstract version of cl_R. This is the differential analogue of the well known problem of recovering the field structure in Zariski geometries solved by Hrushovski and Zilber [HZ96, Zil09]. It seems to be quite a difficult problem and we do not study it here.

Chapter 5

Ax-Schanuel for Linear Differential Equations

This chapter is devoted to the analysis of linear differential equations with constant coefficients. We show that the Ax-Schanuel theorem can be generalised to such differential equations of any order. Using results on the exponential differential equation by Kirby [Kir06, Kir09] and Crampin [Cra06] we give a complete axiomatisation of the first order theories of linear differential equations and show that the generalised Ax-Schanuel inequalities are adequate for them.

The preprint [Asl16a] consists of the material of this chapter.

5.1 The exponential differential equation

In this section we give an axiomatisation of the theory of the exponential differential equation. We will work in the language $\mathfrak{L}_{\mathrm{Exp}} := \{+, \cdot, 0, 1, \mathrm{Exp}\}$ where Exp is a binary predicate which will be interpreted in a differential field $\mathcal{K} = (K; +, \cdot, 0, 1, \mathrm{D})$ as the set $\{(x, y) \in K^2 : \mathrm{D}\, y = y \,\mathrm{D}\, x\}$. In this case the reduct of $\mathcal{K}$ to the language $\mathfrak{L}_{\mathrm{Exp}}$ will be denoted by $\mathcal{K}_{\mathrm{Exp}}$. For a differentially closed field $\mathcal{K}$ we denote the complete first-order theory of $\mathcal{K}_{\mathrm{Exp}}$ by T_{Exp}. For an $\mathfrak{L}_{\mathrm{Exp}}$-structure $\mathcal{F}_{\mathrm{Exp}}$[1] and for a natural number n we let

$$\mathrm{Exp}^n(F) := \{(\bar{x}, \bar{y}) \in F^{2n} : \mathcal{F}_{\mathrm{Exp}} \models \mathrm{Exp}(x_i, y_i) \text{ for each } i\}.$$

As we already mentioned, an axiomatisation of T_{Exp} has been given by Kirby [Kir06, Kir09] and partially by Crampin [Cra06] (Kirby's work is much more general, he studies exponential differential equations of semiabelian varieties). The original idea of such an axiomatisation is due to Zilber in the context of pseudo-exponentiation [Zil04b]. We refer the reader to [Kir09, BK16, Zil04b, Cra06] for details and proofs of the results presented in this section.

Throughout the chapter $\mathcal{K} = (K; +, \cdot, \mathrm{D}, 0, 1)$ will be a differential field.

[1]We will normally add a subscript Exp in the notations of $\mathfrak{L}_{\mathrm{Exp}}$-structures to emphasise the fact that they are $\mathfrak{L}_{\mathrm{Exp}}$-structures and to distinguish them from $\mathfrak{L}_{\mathrm{E}_n}$-structures considered later. It does not mean that they are reducts of some differential fields unless we explicitly state that they are.

Theorem 5.1.1 ([Ax71], Theorem 3). *For any $x_i, y_i \in K$, $i = 1, \ldots, n$, if $\mathcal{K} \models \bigwedge_{i=1}^{n} \operatorname{Exp}(x_i, y_i)$ and $\operatorname{td}_C C(\bar{x}, \bar{y}) \leq n$ then there are integers $m_1, \ldots, m_n$, not all of them zero, such that $m_1 x_1 + \ldots + m_n x_n \in C$ or, equivalently, $y_1^{m_1} \cdot \ldots \cdot y_n^{m_n} \in C$.*

This can be given a geometric formulation. For a field we let $\mathbb{G}_a$ be its additive group and $\mathbb{G}_m$ be the multiplicative group. Also for a natural number n we denote $G_n := \mathbb{G}_a^n \times \mathbb{G}_m^n$. Thus, as varieties, $G_n(F) = F^n \times (F^\times)^n$ for a field F. Observe that for a differential field $\mathcal{K}$ the set $\operatorname{Exp}(K) \subseteq K^2$ is a subgroup of $G_n(K)$. Notice that $\prod y_i^{m_i} = c \in C$ means that $(y_1, \ldots, y_n)$ lies in a C-coset of the subgroup of $\mathbb{G}_m^n(K)$ defined by $\prod y_i^{m_i} = 1$. The analogous fact holds for x_i's and the additive group $\mathbb{G}_a^n$.

The tangent space of $\mathbb{G}_m^n$ at the identity can be identified with $\mathbb{G}_a^n$. For an algebraic subgroup H of $\mathbb{G}_m^n$ its tangent space at the identity, denoted $T_e H$, is an algebraic subgroup of $\mathbb{G}_a^n$. Following [Kir06] we denote it by $\operatorname{Log} H$. The tangent bundle of H will be denoted by TH. Also, for an integer n we let $\operatorname{Exp}^n(K) := \{(\bar{x}, \bar{y}) \in K^{2n} : \mathcal{K} \models \bigwedge_{i=1}^{n} \operatorname{Exp}(x_i, y_i)\}$.

These observations allow one to reformulate Theorem 5.1.1 in a geometric language.

Theorem 5.1.2 (Ax-Schanuel - version 2). *Let $V \subseteq G_n(K)$ be an algebraic variety defined over C with $\dim(V) \leq n$. If $(\bar{x}, \bar{y}) \in V(K) \cap \operatorname{Exp}^n(K)$ then there is a proper algebraic subgroup H of $\mathbb{G}_m^n$ such that $(\bar{x}, \bar{y})$ lies in a C-coset of TH, that is, $\bar{y} \in \gamma H$ and $\bar{x} \in \gamma' + \operatorname{Log} H$ for some constant points $\gamma \in \mathbb{G}_m^n(C)$ and $\gamma' \in \mathbb{G}_a^n(C)$.*

If V is a variety as above and $V(K) \cap \operatorname{Exp}^n(K) \neq \emptyset$ then we say V has an exponential point. The Ax-Schanuel theorem can be thought of as a necessary condition for a variety to have an exponential point. We will shortly present the existential closedness statement, which is a sufficient condition for this. But for now we consider some basic axioms for an $\mathfrak{L}_{\mathrm{Exp}}$-structure $\mathcal{F}_{\mathrm{Exp}}$.

- A1 F is an algebraically closed field of characteristic 0.
- A2 $C := C_F = \{c \in F : \mathcal{F}_{\mathrm{Exp}} \models \operatorname{Exp}(c, 1)\}$ is an algebraically closed subfield of F.
- A3 $\operatorname{Exp}(F) = \{(x, y) \in F^2 : \operatorname{Exp}(x, y)\}$ is a subgroup of $G_1(F)$ containing $G_1(C)$.
- A4 The fibres of Exp in $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$ are cosets of the subgroups $\mathbb{G}_a(C)$ and $\mathbb{G}_m(C)$ respectively.
- AS For any $x_i, y_i \in F$, $i = 1, \ldots, n$, if $\mathcal{F}_{\mathrm{Exp}} \models \bigwedge_{i=1}^{n} \operatorname{Exp}(x_i, y_i)$ and $\operatorname{td}_C C(\bar{x}, \bar{y}) \leq n$ then there are integers $m_1, \ldots, m_n$, not all of them zero, such that $m_1 x_1 + \ldots + m_n x_n \in C$.
- NT $F \supsetneq C$.

Note that AS can be given by an axiom scheme. A compactness argument gives a uniform version of AS. That is, given a parametric family of varieties $V(\bar{c})$ over C, there is a finite number N, such that if for some $\bar{c}$ we have $(\bar{x}, \bar{y}) \in V(\bar{c})$ and $\dim V(\bar{c}) \leq n$ then $m_1 x_1 + \ldots + m_n x_n \in C$ for some integers m_i with $|m_i| \leq N$.

These axioms basically constitute the universal part of T_{Exp} with the exception that A1 is $\forall\exists$ and NT is existential. Models of the theory A1-A4,AS will be called Exp-*fields*.

Now we turn to existential closedness. For a $k \times n$ matrix M of integers we define $[M] : G_n(F) \to G_k(F)$ to be the map given by $[M] : (\bar{x}, \bar{y}) \mapsto (u_1, \ldots, u_k, v_1, \ldots, v_k)$ where

$$u_i = \sum_{j=1}^{n} m_{ij} x_j \text{ and } v_i = \prod_{j=1}^{n} y_j^{m_{ij}}.$$

Definition 5.1.3. An irreducible variety $V \subseteq G_n(F)$ is Exp-*rotund* if for any $1 \leq k \leq n$ and any $k \times n$ matrix M of integers $\dim[M](V) \geq \operatorname{rank} M$. If for any non-zero M the stronger inequality $\dim[M](V) \geq \operatorname{rank} M + 1$ holds then we say V is *strongly* Exp-*rotund*.

The definition of (Exp-)rotundity is originally due to Zilber though he initially used the word *normal* for these varieties [Zil04b]. The term *rotund* was coined by Kirby in [Kir09].

Strong Exp-rotundity fits with the Ax-Schanuel inequality in the sense that it is a sufficient condition for a variety defined over C to contain a non-constant exponential point. More precisely, if $\mathcal{F}$ is differentially closed and $V \subseteq G_n(F)$ is a strongly Exp-rotund variety defined over the constants, then the intersection $V(F) \cap \mathrm{Exp}^n(F)$ contains a non-constant point.

Nevertheless, the existential closedness axiom we will use for the axiomatisation of T_{Exp} is slightly different. One needs to consider varieties that are not necessarily defined over C.

The existential closedness property for an Exp-field $\mathcal{F}_{\mathrm{Exp}}$ is as follows.

EC For each irreducible Exp-rotund variety $V \subseteq G_n(F)$ the intersection $V(F) \cap \mathrm{Exp}^n(F)$ is non-empty.

As noted above, V is not necessarily defined over C and the point in the intersection may be constant.

Exp-rotundity of a variety is a definable property. This allows one to axiomatise the above statement by a first-order axiom scheme. Reducts of differentially closed fields satisfy EC and it gives a complete theory together with the axioms mentioned above.

Theorem 5.1.4 ([Kir09])**.** *The theory T_{Exp} is axiomatised by the following axioms and axiom schemes:* A1-A4, AS, EC, NT.

We also define free varieties ([Kir09, Zil04b]) and present a result from [Kir09] below. Although this is not essential for our main results, we will use it to establish a similar fact for linear differential equations of higher order (Section 5.4) which gives us a better understanding of the general picture.

Definition 5.1.5. An irreducible variety $V \subseteq G_n(K)$ (defined over C) is Exp-*free* if it does not have a generic (over C) point $(\bar{a}, \bar{b})$ for which

$$\sum m_i a_i \in C \text{ or } \prod y_i^{k_i} \in C$$

for some integers m_i and k_i (not all of them zero).

Note that this notion corresponds to *absolute freeness* in [Kir06, Kir09].

Proposition 5.1.6 ([Kir06, Kir09])**.** *Let V be an* Exp*-free variety defined over C. If V has a generic (over C) exponential point then it is strongly* Exp*-rotund.*

Finally let us make an easy observation which will be useful later.

Lemma 5.1.7. *Let $\mathcal{K}$ be a differentially closed field. If $V \subseteq G_n(K)$ is* Exp*-rotund then for any constant $c \in C^{\times}$ there is a point $(\bar{a}, \bar{b}) \in V(K)$ such that $\mathcal{K}_{\mathrm{Exp}} \models$ $\mathrm{Exp}(ca_i, b_i)$ for all i.*

Proof. Let $L : K^{2n} \to K^{2n}$ be the map $(\bar{x}, \bar{y}) \mapsto (c\bar{x}, \bar{y})$. It is easy to check that $V' := L(V)$ is Zariski closed and Exp-rotund. Therefore there is a point $(\bar{u}, \bar{v}) \in V'(K) \cap \mathrm{Exp}^n(K)$. If $a_i = c^{-1}u_i$, $b_i = v_i$, then $(\bar{a}, \bar{b}) \in V(K)$ and $\mathrm{Exp}(ca_i, b_i)$ holds. $\square$

5.2 Higher order linear differential equations

In this section we will use some facts and notions from the theory of abstract linear differential equations in differential fields (see [Mar05b], Section 4).

Let us start with a motivating example which will make it clear which differential equations we should consider. If $x(t)$ and $y(t)$ are complex analytic functions with $y(t) = \exp(x(t))$ then they satisfy the differential equation $\frac{d}{dt}y(t) = y(t) \cdot \frac{d}{dt}x(t)$. Since we are interested in non-constant solutions, this equation can be written as $\frac{dy}{dx} = \frac{\frac{d}{dt}y}{\frac{d}{dt}x} = y$. Now if we replace $\frac{d}{dt}$ with D, we will obtain the abstract exponential differential equation $\frac{\mathrm{D}\, y}{\mathrm{D}\, x} = y$. Here we could also argue as follows. In the differential equation $\frac{dy}{dx} = y$ replace differentiation with respect to x, that is, $\frac{d}{dx}$ with $\frac{1}{\mathrm{D}\, x} \cdot \mathrm{D}$ to get $\frac{\mathrm{D}\, y}{\mathrm{D}\, x} = y$. If $x \in K$ is a non-constant element then $\partial_x = \frac{1}{\mathrm{D}\, x} \cdot \mathrm{D}$ is a derivation of K and the exponential differential equation can be written as $\partial_x(y) = y$. Here ∂_x can be thought of as abstract differentiation with respect to x.

Now we want to generalise this to higher order linear differential equations with constant coefficients. Consider the equation

$$\frac{d^n y}{dx^n} + c_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + c_1\frac{dy}{dx} + c_0 y = 0. \tag{2.1}$$

Its solutions are linear combinations of exponential functions. We want to form the corresponding abstract differential equations whose solutions will be analogues of those combinations. As above we replace $\frac{d}{dx}$ by ∂_x to obtain the equation

$$\partial_x^n y + c_{n-1}\partial_x^{n-1} y + \ldots + c_1 \partial_x y + c_0 y = 0.$$

The left hand side of this equation is a differential rational function with denominator $(\mathrm{D}\,x)^{2n-1}$. We multiply through by this factor to make the left hand side into a polynomial. It will also allow us to define the field of constants. Thus we consider the abstract differential equation

$$\Delta(x,y) := (\mathrm{D}\,x)^{2n-1}\left[\partial_x^n y + c_{n-1}\partial_x^{n-1} y + \ldots + c_1\partial_x y + c_0 y\right] = 0 \tag{2.2}$$

in a differential field $\mathcal{K}$. The coefficients are supposed to be constants.

Note that ∂_x^i above is the i-th iterate of the map $\partial_x : K \to K$. The notation ∂_x may misleadingly suggest that x is fixed in the equation (2.2) which is not the case. It should be considered as a two-variable equation. We prefer this way of writing our equation since otherwise it would be cumbersome. Note however that $\Delta(x,y)$ is not linear as a two-variable differential polynomial, it is linear with respect to y only. We will assume that $c_0 \neq 0$ in order to avoid any possible degeneracies (like $\mathrm{D}\,y = 0$).

Observe that by introducing new variables $z_0, \ldots, z_n$ we can write (2.2) as the following system of equations

$$\begin{cases} z_n + c_{n-1}z_{n-1} + \ldots + c_1 z_1 + c_0 z_0 = 0, \\ z_0 = y, \\ \mathrm{D}\,z_i = z_{i+1}\,\mathrm{D}\,x,\ i = 0,\ldots,n-1. \end{cases} \tag{2.3}$$

Let $p(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \ldots + c_0$ be the characteristic polynomial of (2.2). Let $\lambda_1, \ldots, \lambda_n$ be its roots and let $\mu_1, \ldots, \mu_k$ be its different roots with multiplicities $n_1, \ldots, n_k$ respectively. Since we have assumed c_0 is non-zero, λ_i's are also non-zero.

Now we establish some auxiliary results which will be used in the proof of the Ax-Schanuel theorem for the equation (2.2). Since it is a universal statement, we can assume without loss of generality that $\mathcal{K}$ is differentially closed. This is not very important but makes our arguments easier as we do not have to worry about the existence of solutions of differential equations.

Lemma 5.2.1. *Let x be a non-constant element of K and let $y_i \in K \setminus \{0\}$ be such that $\partial_x y_i = \mu_i y_i$ for $i = 1, \ldots, k$. Then $\bigcup_{i=1}^k \{y_i, xy_i, \ldots, x^{n_i-1}y_i\}$ forms a fundamental system of solutions*[2] *to $\Delta(x,y) = 0$.*

Though the proof is very similar to that in the complex setting (see, for example, [BR78]), we nevertheless present it here for completeness.

Proof. Since x is non-constant, the equation (2.2) can be written as $p(\partial_x)y = 0$. The operator $p(\partial_x)$ can be factored as

$$p(\partial_x) = \prod_{i=1}^k (\partial_x - \mu_i)^{n_i}.$$

It is easy to see that for any $0 \leq l < n_i$

$$(\partial_x - \mu_i)^{n_i}(x^l y_i) = 0.$$

[2] This means that those solutions form a C-linear basis for the space of all solutions.

Hence we have $p(\partial_x)(x^l y_i) = 0$ and thus we have n solutions to $\Delta(x, y) = 0$. Now we prove they are linearly independent.

Assume

$$\sum_{i=1}^{k} \sum_{j=0}^{n_i - 1} a_{ij} x^j y_i = 0 \tag{2.4}$$

for some constants a_{ij}. Let i be such that there is a non-zero coefficient a_{ij}. Let t be the biggest number with $a_{it} \neq 0$. Consider the operator

$$q(\partial_x) = (\partial_x - \mu_i)^t \prod_{s \neq i} (\partial_x - \mu_s)^{n_s}.$$

Clearly

$$q(\partial_x)(x^j y_r) = \begin{cases} 0, & \text{if } r \neq i \text{ or } j < t, \\ t! \cdot \prod_{s \neq i} (\mu_i - \mu_s)^{n_s} \cdot y_i \neq 0, & \text{if } r = i, \ j = t. \end{cases}$$

Now applying $q(\partial_x)$ to (2.4) we get $a_{it} = 0$, a contradiction.

Thus, we found n linearly independent solutions. Since the order of the equation is n, the set of solutions is an n-dimensional C-vector space, therefore the above solutions form a basis for that vector space. □

If $y_1, \ldots, y_k$ are as in Lemma 5.2.1, then for any non-zero constants a_{ij} the set $\bigcup_{i=1}^{k} \{a_{i0} y_i, a_{i1} x y_i, \ldots, a_{i,n_i-1} x^{n_i - 1} y_i\}$ is a fundamental system of solutions to our equation. This kind of fundamental systems will be called *canonical*. There is a unique such system up to multiplication by constants. Note also that we will treat canonical fundamental systems as ordered tuples, rather than as sets. Thus if we say $v_1, \ldots, v_n$ is a canonical fundamental system, then we mean that the first n_1 elements coincide (up to constants) with $y_1, x y_1, \ldots, x^{n_1 - 1} y_1$ respectively, and so on. Of course we assume a certain ordering $\mu_1, \ldots, \mu_k$ of different eigenvalues is fixed.

Definition 5.2.2. Given a non-constant $x \in K$, let $v_1, \ldots, v_n$ be a canonical fundamental system and let $y \in K$ be such that $\Delta(x, y) = 0$. Then y (or the pair (x, y)) is said to be a *proper* solution if $y = \sum a_i v_i$ with $a_i \in C^{\times}$, that is, if y is not in the linear span of a proper subset of $\{v_1, \ldots, v_n\}$.

A solution is proper if and only if it does not satisfy a linear differential equation of lower order.

Lemma 5.2.3. *A pair $(x, y) \in K^2$ is a proper solution to (2.2) if and only if $y, \partial_x y, \ldots, \partial_x^{n-1} y$ are C-linearly independent.*

Proof. Let $v_1, \ldots, v_n$ be as above and $y = \sum a_i v_i$. Since $v_1, \ldots, v_n$ are C-linearly independent, the Wronskian $W(\bar{v}) = \det(\partial_x^l v_i)$ is non-zero. It is easy to check that $\partial_x^l(v_i) = f_{li}(x) v_i$ where f_{li} is a rational function over $\mathbb{Q}(\mu_1, \ldots, \mu_k)$. Furthermore, none of the $f_{li}(x)$ is zero (as x is non-constant). Let H_x be the $n \times n$ matrix with entries $f_{li}(x)$. Then $W(\bar{v}) = \det(H_x) \cdot \prod_{i=1}^m v_i$. Consider the following system of equations with respect to v's:

$$\partial_x^l(y) = \sum_{i=1}^m a_i f_{li}(x) v_i, \; l = 0, \ldots, n-1.$$

Its determinant is $\det(H_x) \cdot \prod_{i=1}^m a_i$ which is non-zero if and only if none of the a_i's is zero. This finishes the proof. $\square$

Let (x, y) be a proper solution. Then we can assume $y = v_1 + \ldots + v_n$. Let H_x be as in the proof and denote its rows by H_x^l. It is an invertible linear transformation of K^n. Let L_x be its inverse with coordinate functions (rows) $L_x^i : K^n \to K$. Thus

$$\partial_x^l(y) = H_x^l(v_1, \ldots, v_n) \text{ and } v_i = L_x^i(y, \partial_x y, \ldots, \partial_x^{n-1} y).$$

It is also worth mentioning that when $p(\lambda)$ does not have multiple roots, H_x and L_x do not depend on x, they depend only on λ_i's. Note also that if $\Delta(x, y) = 0$ and x is non-constant then $\Delta(x, \partial_x y) = 0$. In particular, if (x, y) is a proper solution then $y, \partial_x y, \ldots, \partial_x^{n-1} y$ form a fundamental system of solutions. These considerations will be useful in Section 5.4.

Now we are ready to prove the Ax-Schanuel inequality for (2.2).

Theorem 5.2.4. *Let* (x_i, y_i), $i = 1, \ldots, m$, *be proper solutions to the equation* (2.2) *in* $\mathcal{K}$. *Then*

$$\operatorname{td}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \ldots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq \operatorname{ldim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \ldots, \lambda_n \bar{x}/C) + 1, \tag{2.5}$$

where $\partial_{\bar{x}}^j \bar{y} = (\partial_{x_1}^j y_1, \ldots, \partial_{x_m}^j y_m)$

In particular, if we assume $\lambda_1 \bar{x}, \ldots, \lambda_n \bar{x}$ are $\mathbb{Q}$-linearly independent modulo C then $\operatorname{td}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \ldots, \partial_{\bar{x}}^{n-1} \bar{y}) \geq mn + 1$. This is possible only if $\lambda_1, \ldots, \lambda_n$ are linearly independent over $\mathbb{Q}$. In fact we can always assume it is the case; otherwise both the transcendence degree and the linear dimension will decrease and we will be reduced to the same inequality for a smaller n. Note also that the case $n = 1$ is exactly Ax's theorem for the exponential differential equation.

Proof of Theorem 5.2.4. For each i let $v_{ij} \in K^{\times}$, $j = 1, \ldots, n$, be a canonical fundamental system of solutions to $\Delta(x_i, y) = 0$. Then for every i the C-linear span of $v_{i1}, \ldots, v_{in}$ is the same as that of $y_i, \partial_{x_i} y_i, \ldots, \partial_{x_i}^{n-1} y_i$, for (x_i, y_i) is a proper solution. In particular, the field $C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \ldots, \partial_{\bar{x}}^{n-1} \bar{y})$ is equal to the field extension of C generated by $\bar{x}$ and all the v_{ij}. Therefore

$$\begin{aligned} \operatorname{td}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}, \ldots, \partial_{\bar{x}}^{n-1} \bar{y}) &= \operatorname{td}_C C(\mu_1 \bar{x}, \ldots, \mu_k \bar{x}, \bar{v}_1, \ldots, \bar{v}_n) \\ &\geq \operatorname{ldim}_{\mathbb{Q}}(\mu_1 \bar{x}, \ldots, \mu_k \bar{x}/C) + 1 \\ &= \operatorname{ldim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \ldots, \lambda_n \bar{x}/C) + 1 \end{aligned}$$

where $\bar{v}_j$ is the tuple $(v_{1j}, v_{2j}, \ldots, v_{mj})$. The inequality follows from Ax's theorem applied to the tuple $(\mu_1 \bar{x}, \ldots, \mu_k \bar{x})$ taking into account that $(\bar{v}_1, \ldots, \bar{v}_n)$ contains a solution y_{ij} for each of the equations $\operatorname{Exp}(\mu_i x_j, y_{ij})$. $\square$

Let us also note that one can prove (with a similar argument) an analogue of Theorem 5.2.4 for fields with several commuting derivations (using the corresponding version of Ax's theorem). Nevertheless, we prefer working in differential fields with a single derivation and do not consider a multi-derivative version of the above theorem.

5.3 The complete theory

Having established a predimension inequality (see Section 5.5) for higher order linear differential equations, we want to find an appropriate existential closedness property and thus give an axiomatisation of the complete theory of the corresponding reducts.

First, let us see which language we should work in. An obvious option would be simply taking a binary predicate for the solutions of the equation (2.2). But the inequality (2.5) cannot be written as a first order statement (axiom scheme) in this language. This is because derivatives of y_i's are involved in (2.5). Therefore we need to take a predicate of higher arity which will have variables for the derivatives of y's as well. Thus we will work in the language $\mathfrak{L}_{\mathrm{E}_n} = \{+, \cdot, \mathrm{E}_n, 0, 1, \lambda_1, \ldots, \lambda_n\}$ where $\lambda_1, \ldots, \lambda_n$ are constant symbols for the eigenvalues and $\mathrm{E}_n(x, z_0, z_1, \ldots, z_{n-1})$ is an $(n+1)$-ary predicate. It will be interpreted in a differential field $\mathcal{K}$ as the set

$$\left\{(x, \bar{z}) \in K^{n+1} : \exists z_n \left[z_n + \sum_{i=0}^{n-1} c_i z_i = 0 \wedge \bigwedge_{i=0}^{n-1} \mathrm{D}\, z_i = z_{i+1} \mathrm{D}\, x\right]\right\}.$$

Note that since $\lambda_1, \ldots, \lambda_n$ are in the language, the coefficients $c_0, \ldots, c_{n-1}$ are $\emptyset$-definable. The theory of reducts of differentially closed fields to the language $\mathfrak{L}_{\mathrm{E}_n}$ will be denoted by T_{E_n}. Also the field of constants can be defined as $C = \{c : \mathrm{E}_n(c, 0, 1, 0, \ldots, 0)\}$.

Observe that Exp can be defined in an E_n-reduct of a differential field. Namely,

$$\mathcal{K} \models \mathrm{Exp}(\lambda_i x, y) \leftrightarrow \mathrm{E}_n(x, y, \lambda_i y, \ldots, \lambda_i^{n-1} y)$$

for any $i \in \{1, \ldots, n\}$. Indeed, if $\mathrm{Exp}(\lambda_i x, y)$ holds then by Lemma 5.2.1 $\Delta(x, y) = 0$ and $\partial_x^j y = \lambda_i^j y$ for each j and so $(x, y, \lambda_i y, \ldots, \lambda_i^{n-1} y) \in \mathrm{E}_n$. Conversely, if $(x, y, \lambda_i y, \ldots, \lambda_i^{n-1} y) \in \mathrm{E}_n$ then y can be written as a C-linear combination of the fundamental system of solutions. Moreover, we must have $\partial_x^j y = \lambda_i^j y$ for $j = 0, \ldots, n-1$. This system of equations implies that $\partial_x y = \lambda_i y$ and so $\mathrm{Exp}(\lambda_i x, y)$ holds.

In fact Exp and E_n are interdefinable. So we can just translate the axiomatisation for the exponential differential equation to the language $\mathfrak{L}_{\mathrm{E}_n}$ and get an axiomatisation of T_{E_n}. However we want an axiomatisation based on the Ax-Schanuel inequality proved in Section 5.2. In other words, we want to understand which systems of equations in $\mathfrak{L}_{\mathrm{E}_n}$-reducts of differentially closed fields have solutions, and prove that (2.5) is an adequate predimension inequality.

Notation. If $\partial_x y_i = \mu_i y_i$ then let $g_{ijl}(X)$ be the algebraic polynomial (over $\mathbb{Q}(\mu_i)$) for which $\partial_x^l(x^j y_i) = g_{ijl}(x) y_i$. In particular $g_{i0l} = \mu_i^l$. Also denote $N_i := 1 + \sum_{j<i} n_j$.

Now we formulate a number of axioms and axiom schemes for an $\mathfrak{L}_{\mathrm{E}_n}$-structure $\mathcal{F}_{\mathrm{E}_n}$. In such a structure we let $\mathrm{Exp}(x,y)$ be the relation defined by the formula $\mathrm{E}_n(\lambda_1^{-1}x, y, \lambda_1 y, \ldots, \lambda_1^{n-1}y)$.

A1' F is an algebraically closed field.

A2' $C := \{c \in F : \mathcal{F}_{\mathrm{E}_n} \models \mathrm{E}_n(c,1,0,\ldots,0)\}$ is an algebraically closed subfield of F and $\lambda_1,\ldots,\lambda_n$ are non-zero elements of C satisfying the appropriate algebraic relations. In particular $\lambda_{N_i} = \lambda_{N_i+1} = \ldots = \lambda_{N_i+n_i-1} =: \mu_i$ for every i.

A3' $\mathrm{E}_n(x, z_0, \ldots, z_{n-1})$ holds if and only if there are $y_1,\ldots,y_k \in F^\times$ with $\mathrm{Exp}(\mu_i x, y_i)$ and elements $a_{ij} \in C$ such that

$$z_l = \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} g_{ijl}(x) y_i,$$

for $l = 0,\ldots,n-1$.

A4' $\mathrm{Exp}(F) = \{(x,y) \in F^2 : \mathrm{Exp}(x,y)\}$ is a subgroup of $G_1(F)$ containing $G_1(C)$.

A5' The fibres of Exp in $\mathbb{G}_a(F)$ and $\mathbb{G}_m(F)$ are cosets of the subgroups $\mathbb{G}_a(C)$ and $\mathbb{G}_m(C)$ respectively.

AS' Let $x_i, z_{ij} \in F \setminus C,\ 1 \leq i \leq m, 0 \leq j < n$, be such that $z_{i0},\ldots,z_{i,n-1}$ are C-linearly independent and

$$\mathcal{F}_{\mathrm{E}_n} \models \bigwedge_i \mathrm{E}_n(x_i, z_{i0}, \ldots, z_{i,n-1}).$$

Then for each $1 \leq d \leq mn$ if $\mathrm{ldim}_{\mathbb{Q}}(\lambda_1 \bar{x}, \ldots, \lambda_n \bar{x}/C) \geq d$ then

$$\mathrm{td}_C\, C(\bar{x}, \bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{n-1}) \geq d+1.$$

NT' $F \supsetneq C$.

As in the case of AS, a compactness argument can be used here as well to show that AS' can be expressed as a first-order axiom scheme.

Definition 5.3.1. An E_n-field is a model of A1'-A5',AS'.

Lemma 5.3.2. *Let $\mathcal{F}_{\mathrm{E}_n}$ be a model of* A1'-A5'. *Then it satisfies* AS' *iff the relation* $\mathrm{Exp}(x,y)$ *satisfies* AS.

Proof. Let $x_1,\ldots,x_m \in F$ be $\mathbb{Q}$-linearly independent modulo C. Then $\mu_1 x_1,\ldots,\mu_1 x_m$ are such as well. Denote $\mu_s x_i =: u_{m(s-1)+i}$ for $i = 1,\ldots,m,\ s = 1,\ldots,k$. If $\mathrm{ldim}_{\mathbb{Q}}(u_1,\ldots,u_{mk}/C) = d \geq m$ then assume without loss of generality that $u_1,\ldots,u_d$ are linearly independent over the rationals modulo C. Let $v_i \in F$ be such that $\mathcal{F}_{\mathrm{E}_n} \models \mathrm{Exp}(u_i, v_i)$. Then AS' implies that

$$\mathrm{td}_C\, C(x_1,\ldots,x_m,v_1,\ldots,v_{mk}) \geq d+1.$$

For each $i > d$ there are integers $m_i, m_{i1}, \ldots, m_{id}$ such that $m_i u_i + m_{i1}u_1 + \ldots + m_{id}u_d = c \in C$.

Denote $v = v_i^{m_i} \prod_{j=1}^{d} v_j^{m_{ij}}$. By A4' we have $\mathrm{Exp}(c, v)$. But also $\mathrm{Exp}(c, 1)$ holds and using A5' we deduce that $v \in C$. Hence $v_1, \ldots, v_d, v_i$ are algebraically dependent over C. Therefore

$$\mathrm{td}_C\, C(\bar{x}, v_1, \ldots, v_d) \geq d + 1.$$

Now we can easily deduce that $\mathrm{td}_C\, C(\bar{x}, v_1, \ldots, v_m) \geq m + 1$ and we are done.

The converse follows from the proof of Theorem 5.2.4. $\square$

Notation. Let $\mathrm{pr}_j : K^{m(n+1)} \to K^{2m}$ be defined as

$$\mathrm{pr}_j : (\bar{x}, \bar{v}_0, \ldots, \bar{v}_{n-1}) \mapsto (\bar{x}, \bar{v}_j),$$

where $\bar{v}_j = (v_{1j}, \ldots, v_{mj})$.

Also we will denote the set $\{(\bar{x}, \bar{z}_0, \ldots, \bar{z}_{n-1}) \in F^{m(n+1)} : \mathcal{F}_{\mathrm{E}_n} \models \bigwedge_{i=1}^{m} \mathrm{E}_n(x_i, \bar{z}^i)\}$ by $\mathrm{E}_n^m(F)$ where $\bar{z}^i = (z_{i0}, \ldots, z_{i,n-1})$.

Definition 5.3.3. An irreducible variety $V \subseteq K^{m(n+1)}$ is called E_n-Exp-*rotund* if $V_1 := \mathrm{pr}_1(V) \subseteq G_m(K)$ is Exp-rotund and

$$(\bar{x}, \bar{y}) \in V_1 \Longrightarrow (\bar{x}, \bar{y}, \mu_1\bar{y}, \ldots, \mu_1^{n-1}\bar{y}) \in V. \tag{3.1}$$

We could of course replace μ_1 in (3.1) by any μ_i. As Exp-rotundity is a definable property, so is E_n-Exp-rotundity.

Now we formulate the existential closedness property for an E_n-field $\mathcal{F}_{\mathrm{E}_n}$.

EC' For each irreducible E_n-Exp-rotund variety $V \subseteq F^{m(n+1)}$ the intersection $V(F) \cap \mathrm{E}_n^m(F)$ is non-empty.

This statement can be given by a first-order axiom scheme, for E_n-Exp-rotundity is a first-order property.

Lemma 5.3.4. *If $\mathcal{K}$ is a differentially closed field then $\mathcal{K}_{\mathrm{E}_n}$ satisfies* EC'.

Proof. Let $V \subseteq K^{m(n+1)}$ be an E_n-Exp-rotund variety. Then $V_1 = \mathrm{pr}_1(V)$ is an Exp-rotund variety. So by Theorem 5.1.4 and Lemma 5.1.7 there is a point $(\bar{x}, \bar{y}) \in V_1$ such that $\mathcal{K}_{\mathrm{E}_n} \models \mathrm{Exp}(\mu_1 x_i, y_i)$ for each $i = 1, \ldots, m$. By definition we have

$$(\bar{x}, \bar{y}, \mu_1\bar{y}, \ldots, \mu_1^{n-1}\bar{y}) \in V.$$

It is also clear that

$$\mathcal{K}_{\mathrm{E}_n} \models \mathrm{E}_n(\bar{x}, \bar{y}, \mu_1\bar{y}, \ldots, \mu_1^{n-1}\bar{y})$$

and we are done. $\square$

Lemma 5.3.5. *If $\mathcal{F}_{\mathrm{E}_n}$ satisfies* A1'-A5', AS', EC' *then* $\mathrm{Exp}(x, y)$ *satisfies* EC.

Proof. Suppose $W \subseteq G_m(F)$ is an Exp-rotund variety defined over a set $A \subseteq F$. Let $\mathbb{F} \supseteq F$ be a saturated algebraically closed field and pick $(\bar{a}, \bar{b}) \in \mathbb{F}^{2n}$ a generic point of W. Let $V \subseteq F^{m(n+1)}$ be the algebraic locus of $(\mu_1^{-1}\bar{a}, \bar{b}, \mu_1\bar{b}, \dots, \mu_1^{n-1}\bar{b})$ over $A\mu_1$. Then V is E_n-Exp-rotund and hence $V(F) \cap \mathrm{E}_n^m(F) \neq \emptyset$. By our construction of V we also know that a point in that intersection must be of the form $(\mu_1^{-1}\bar{x}, \bar{y}, \mu_1\bar{y}, \dots, \mu_1^{n-1}\bar{y})$. Then $(\bar{x}, \bar{y}) \in W$ and by A3' $\mathcal{F}_{\mathrm{E}_n} \models \mathrm{Exp}(\bar{x}, \bar{y})$. So $W(F) \cap \mathrm{Exp}^n(F) \neq \emptyset$. □

Finally, we can deduce that the given axioms form a complete theory.

Theorem 5.3.6. *The axioms and axiom schemes* A1'-A5', AS', EC', NT' *axiomatise the complete theory* T_{E_n}.

Proof. Indeed, Lemmas 5.3.2, 5.3.4 and 5.3.5 show that an $\mathfrak{L}_{\mathrm{E}_n}$-structure $\mathcal{F}_{\mathrm{E}_n}$ satisfies A1'-A5', AS', and EC' if and only if the relation $\mathrm{Exp}(x, y)$ satisfies the axioms A1-A4, AS, and EC. The latter collection of axioms axiomatises the theory T_{Exp} by Theorem 5.1.4. Now the desired result follows as the relations Exp and E_n are interdefinable due to A3'. □

5.4 Rotundity and freeness

Though EC' is an appropriate existential closedness property for E_n-fields, our definition of E_n-Exp-rotundity is not that natural. Indeed, the inequality given by AS' is not reflected in it and also the notion of E_n-Exp-rotundity is far from being a necessary condition for a variety to intersect E_n. As we saw, E_n-Exp-rotund varieties have a very special E_n-point, which is essentially (made of) an exponential point. For these reasons we define another notion of rotundity (and strong rotundity) which will be more intuitive and natural. (That definition will not be as simple as Definition 5.3.3 though.) We will see in particular that strongly rotund varieties will contain proper E_n-points.

Recall that $\mu_1, \dots, \mu_k$ are the different eigenvalues of our differential equation. As before, for $z_{ij}, i = 1, \dots, m, j = 0, \dots, n-1$, denote $\bar{z}^i = (z_{i0}, \dots, z_{i,n-1})$ and $\bar{z}_j = (z_{1j}, \dots, z_{mj})$. Define the map

$$\tilde{L} : K^{m(n+1)} \to K^{m(n+1)}$$

by

$$\tilde{L} : (\bar{x}, \bar{z}_0, \dots, \bar{z}_{n-1}) \mapsto (\bar{x}, L^1_{x_1}(\bar{z}^1), \dots, L^1_{x_m}(\bar{z}^m), \dots, L^n_{x_1}(\bar{z}^1), \dots, L^n_{x_m}(\bar{z}^m)),$$

where $L^j_{x_i}$ is as in Section 5.2. Let $\tilde{H}$ be its inverse map. Recall that for $1 \leq i \leq k$ we denoted $N_i = 1 + \sum_{j<i} n_j$. Define maps $R : F^{m(n+1)} \to F^{m(k+1)}$ and $\tilde{R} : F^{m(n+1)} \to F^{2km}$ as follows:

$$R : (\bar{x}, \bar{v}_1, \dots, \bar{v}_n) \mapsto (\bar{x}, \bar{v}_{N_1}, \dots, \bar{v}_{N_k}),$$

$$\tilde{R} : (\bar{x}, \bar{v}_1, \dots, \bar{v}_n) \mapsto (\mu_1\bar{x}, \dots, \mu_k\bar{x}, \bar{v}_{N_1}, \dots, \bar{v}_{N_k}).$$

Definition 5.4.1. An irreducible variety $V \subseteq F^{m(n+1)}$ is called (*strongly*) E_n*-rotund* if $V' := \tilde{R} \circ \tilde{L}(V) \subseteq G_{km}(F)$ is (strongly) Exp-rotund and $V'' := R(\tilde{L}(V))$ satisfies the following property:

$$(\bar{x}, \bar{y}_1, \ldots, \bar{y}_k) \in V'' \Rightarrow \tilde{H}(\bar{x}, \bar{y}_1, x\bar{y}_1, \ldots, x^{n_1-1}\bar{y}_1, \ldots, \bar{y}_k, x\bar{y}_k, \ldots, x^{n_k-1}\bar{y}_k) \in V.$$

One can use this notion of rotundity to formulate an appropriate existential closedness statement (that is, the above EC' but for E_n-rotund varieties instead of E_n-Exp-rotund ones) which, with A1'-A5' and AS', axiomatises T_{E_n}. The following result shows that this notion of rotundity fits better with our differential equation.

Proposition 5.4.2. *Let $\mathcal{K}$ be a differentially closed field. If $V \subseteq K^{m(n+1)}$ is a strongly* E_n*-rotund variety defined over C then $V(K)$ has a proper* E_n*-point.*

Proof. Indeed strong Exp-rotundity of V' implies that it has a non-constant Exp-point. This point obviously gives rise to a proper E_n-point on V. □

In sufficiently saturated models of T_{E_n} every (strongly) E_n-rotund variety contains a generic (proper) E_n-point. The converse holds for "free" varieties: (strong) E_n-rotundity is a necessary condition for a free variety to have a generic (proper) E_n-point. We give precise definitions below.

For simplicity we assume in the rest of this section that $\mu_1, \ldots, \mu_k$ are linearly independent over $\mathbb{Q}$. Otherwise we would have to take a basis and thus introduce new notations which is not desirable.

Definition 5.4.3. An irreducible variety $V \subseteq F^{m(n+1)}$ is called E_n*-free* if $V' := \tilde{R} \circ \tilde{L}(V) \subseteq G_{km}(F)$ is Exp-free.

Note that if we do not require $\mu_1, \ldots, \mu_k$ to be linearly independent over $\mathbb{Q}$ then the above definition would not make sense. Of course in that case we could just change the definition of the map R appropriately and get the same notion of freeness. The following result follows from Proposition 5.1.6 and some obvious observations on generic points. It can also be proven using Theorem 5.2.4.

Proposition 5.4.4. *Suppose $V \subseteq F^{m(n+1)}$ is an irreducible and free variety defined over C. If V has a proper generic (over C)* E_n*-point then it must be strongly* E_n*-rotund.*

5.5 Adequacy

We denote by T^0_{Exp} the $\mathfrak{L}_{\mathrm{Exp}}$-theory given by the axioms A1-A4, AS. Similarly, $T^0_{\mathrm{E}_n}$ is the $\mathfrak{L}_{\mathrm{E}_n}$-theory consisting of the axioms A1'-A5', AS'. The results of Section 5.3 show that T^0_{Exp} and $T^0_{\mathrm{E}_n}$ (as well as T_{Exp} and T_{E_n}) are essentially the same theory given in two different languages. In particular, every model $\mathcal{F}_{\mathrm{E}_n}$ of $T^0_{\mathrm{E}_n}$ (or T_{E_n}) can be canonically made into a model $\mathcal{F}_{\mathrm{Exp}}$ of T^0_{Exp} (respectively T_{Exp}) and vice versa. This relationship allows us to deduce that the "predimension" inequality (2.5) is adequate. We proceed towards this goal in this section.

We will first prove that an embedding of E_n-fields is the same as an embedding of Exp-fields and thus establish the isomorphism of categories of E_n-fields and Exp-fields. Then we will define the predimension function on the class of E_n-fields and see that it is equal to the predimension on corresponding Exp-fields. It will allow us to deduce strong adequacy of that predimension from Theorem 3.3.2 proven by Kirby in [Kir06, Kir09].

We saw that Exp is quantifier-free definable in an E_n-field and that E_n is existentially definable in an Exp-field. The following lemma implies immediately that E_n is also universally definable in an Exp-field. For an E_n-field (or Exp-field) $\mathcal{F}_{\mathrm{E}_n}$ we let C_F be its field of constants.

Lemma 5.5.1. *Let $\mathcal{K}_{\mathrm{E}_n}$ and $\mathcal{F}_{\mathrm{E}_n}$ be two E_n-fields with an embedding of fields $f : K \hookrightarrow F$. Then $f : \mathcal{K}_{\mathrm{E}_n} \hookrightarrow \mathcal{F}_{\mathrm{E}_n}$ is an embedding of E_n-fields if and only if it is an embedding of* Exp*-fields $f : \mathcal{K}_{\mathrm{Exp}} \hookrightarrow \mathcal{F}_{\mathrm{Exp}}$.*

Proof. Since Exp is quantifier-free definable in an E_n-field, we only need to show that an embedding of Exp-fields is also an embedding of the corresponding E_n-fields. Identifying K with $f(K)$ we can assume $\mathcal{K}_{\mathrm{Exp}} \subseteq \mathcal{F}_{\mathrm{Exp}}$. Let $a, b_0, \ldots, b_{n-1} \in K$ be such that

$$\mathcal{F}_{\mathrm{Exp}} \models \mathrm{E}_n(a, b_0, \ldots, b_{n-1}).$$

We shall show that

$$\mathcal{K}_{\mathrm{Exp}} \models \mathrm{E}_n(a, b_0, \ldots, b_{n-1}).$$

We can assume that a is non-constant. By A3' we know that there are $e_1, \ldots, e_k \in F^{\times}$ with $\mathrm{Exp}(\mu_i a, e_i)$ and elements $a_{ij} \in C_F$ such that

$$b_l = \sum_{i=1}^{k} \sum_{j=0}^{n_i-1} a_{ij} g_{ijl}(a) e_i,$$

for $l = 0, \ldots, n-1$ (here g_{ijl} is as in Section 5.3). If $\mathrm{Exp}(u, v)$ holds for some u, v then $\mathrm{Exp}(u, cv)$ holds as well for any constant c. Hence we can assume without loss of generality that a_{ij} is either 0 or 1. As $g_{ijl}(X) \in C_K[X]$, we can express all e_i's with $a_{i0} = 1$ in terms of $g_{ijl}(a)$ and b_l (this is because the corresponding determinant does not vanish). Hence $e_i \in K$ and we are done by A3' again. $\square$

This lemma shows that the category of E_n-fields with morphisms being embeddings is isomorphic to the category of Exp-fields again with embeddings as morphisms.

In order to define the predimension we first observe that Theorem 5.2.4 can be reformulated to give a lower bound for transcendence degree not only for proper solutions but for arbitrary ones. Recall that $\mu_1, \ldots, \mu_k$ are all the different eigenvalues of our equation with multiplicities $n_1, \ldots, n_k$ respectively. Let $v_1, \ldots, v_n$ be a canonical fundamental system of solutions of $\Delta(x, y) = 0$. For a solution (x, y) (with x non-constant) we have a unique representation $y = c_1 v_1 + \ldots + c_n v_n$ with $c_i \in C$. For $1 \leq i \leq k$ we define

$$\epsilon_i(y) := \begin{cases} 1, & \text{if for some } j \text{ with } N_i \leq j < N_{i+1} \text{ we have } c_j \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then set $\epsilon(y) := (\epsilon_1(y), \dots, \epsilon_k(y))$ and denote $\epsilon(y)x := (\epsilon_1(y)\mu_1 x, \dots, \epsilon_k(y)\mu_k x)$.

Now it is easy to see that Theorem 5.2.4 is equivalent to the following.

Theorem 5.5.2. *Let (x_i, y_i), $i = 1, \dots, m$, be solutions to the equation* (2.2) *in $\mathcal{K}$ with $x_i \in K \setminus C$. Then*

$$\operatorname{td}_C C(\bar{x}, \bar{y}, \partial_{\bar{x}}\bar{y}, \dots, \partial_{\bar{x}}^{n-1}\bar{y}) - \operatorname{ldim}_{\mathbb{Q}}(\epsilon(y_1)x_1, \dots, \epsilon(y_m)x_m/C) \geq 1. \tag{5.1}$$

Now we define the predimension. Fix a countable algebraically closed field C with $\operatorname{td}(C/\mathbb{Q}) = \aleph_0$ and let $\mathfrak{C}$ be the collection of all E_n-fields with field of constants C.

For $\mathcal{K}_{\mathrm{E}_n} \in \mathfrak{C}_{f.g.}$ (with domain K) define

$$\begin{aligned} \sigma(\mathcal{K}_{\mathrm{E}_n}) := \max\{ \operatorname{ldim}_{\mathbb{Q}}(\epsilon^1 a_1, \dots, \epsilon^m a_m/C) : \text{ where } \epsilon^i \in \{0,1\}^k,\ a_i \in K \text{ such that} \\ \text{there are } b_i^0, \dots, b_i^{n-1} \in K \text{ with } \epsilon(b_i^0) = \epsilon^i \text{ and} \\ \mathcal{K}_{\mathrm{E}_n} \models \mathrm{E}_n(a_i, b_i^0, \dots, b_i^{n-1}),\ i = 1, \dots, m\} \end{aligned}$$

and

$$\delta(\mathcal{K}_{\mathrm{E}_n}) := \operatorname{td}_C(\mathcal{K}_{\mathrm{E}_n}) - \sigma(\mathcal{K}_{\mathrm{E}_n}).$$

Then the inequality (5.1) states precisely that $\delta(\mathcal{K}_{\mathrm{E}_n}) \geq 0$ for all $\mathcal{K}_{\mathrm{E}_n} \in \mathfrak{C}_{f.g.}$ and equality holds if and only if $\mathcal{K}_{\mathrm{E}_n} = C$.

Lemma 5.5.3. *For an E_n-field $\mathcal{K}_{\mathrm{E}_n} \in \mathfrak{C}_{f.g.}$ we have*

$$\sigma(\mathcal{K}_{\mathrm{E}_n}) = \sigma(\mathcal{K}_{\mathrm{Exp}}),\ \delta(\mathcal{K}_{\mathrm{E}_n}) = \delta(\mathcal{K}_{\mathrm{Exp}}).$$

Hence an embedding of E_n-fields $\mathcal{K}_{\mathrm{E}_n} \hookrightarrow \mathcal{F}_{\mathrm{E}_n}$ is strong if and only if it is strong as an embedding of the corresponding Exp-fields $\mathcal{K}_{\mathrm{Exp}} \hookrightarrow \mathcal{F}_{\mathrm{Exp}}$. Furthermore, the category of E_n-fields with morphisms being strong embeddings is isomorphic to the category of Exp-fields with strong embeddings.

Proof. We only need to show that $\sigma(\mathcal{K}_{\mathrm{E}_n}) = \sigma(\mathcal{K}_{\mathrm{Exp}})$. It is quite easy to see. Let $m = \sigma(\mathcal{K}_{\mathrm{Exp}})$ and $(a_1, \dots, a_m) \in K^m$ be linearly independent over $\mathbb{Q}$ mod C such that for some $b_1, \dots, b_m \in K^\times$ we have $\mathcal{K}_{\mathrm{Exp}} \models \mathrm{Exp}(\mu_1 a_i, b_i)$ for each i. Then $\mathcal{K}_{\mathrm{E}_n} \models \mathrm{E}_n(a_i, b_i, \mu_1 b_i, \dots, \mu_1^{n-1} b_i)$. Clearly, $\epsilon(b_i) = 1$ only for $i = 1$. Hence $\operatorname{ldim}_{\mathbb{Q}}(\epsilon^1 a_1, \dots, \epsilon^m a_m/C) = m$ and so $\sigma(\mathcal{K}_{\mathrm{E}_n}) \geq m = \sigma(\mathcal{K}_{\mathrm{Exp}})$. A similar argument shows that $\sigma(\mathcal{K}_{\mathrm{Exp}}) \geq \sigma(\mathcal{K}_{\mathrm{E}_n})$. ∎

Now Theorem 3.3.2 implies adequacy of (5.1).

Theorem 5.5.4. *The inequality* (5.1) *is strongly adequate for E_n.*

Chapter 6

The j-function

In this chapter we will study the Ax-Schanuel inequality for the j-function established by Pila and Tsimerman. Adequacy of that inequality is still open and we do not answer that question here. However, we show that the models of a theory (which is essentially the universal theory of appropriate reducts of differential fields) have the strong amalgamation property (along with all other necessary properties), construct the strong Fraïssé limit U and give an axiomatisation of its first-order theory. Thus, the given axiomatisation will be a candidate for the theory of the differential equation of the j-function if we believe the predimension inequality is adequate. We will also see that U is saturated and hence adequacy of the predimension inequality implies strong adequacy.

The definitions and results of this chapter are analogous to their exponential counterparts. Many proofs are adapted from [Kir09] and [BK16]. However, we should note that some things are simpler for j while others are subtler and more complicated.

6.1 Background on the j-function

We do not need to know much about the j-function itself, nor need we know its precise definition. Being familiar with some basic properties of j will be enough for this chapter. We summarise those properties below referring the reader to [Lan73, Ser73, Mas03, Sil09] for details.

Let $\mathrm{GL}_2(\mathbb{C})$ be the group of 2×2 matrices with non-zero determinant. This group acts on the complex plane (more precisely, Riemann sphere) by linear fractional transformations. Namely, for a matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$ we define

$$gz = \frac{az+b}{cz+d}.$$

This action is obviously the same as the action of the subgroup $\mathrm{SL}_2(\mathbb{C})$ consisting of matrices with determinant 1 (to be more precise, the action of $\mathrm{GL}_2(\mathbb{C})$ factors through $\mathrm{SL}_2(\mathbb{C})$).

The function j is a modular function of weight 0 for the *modular group* $\mathrm{SL}_2(\mathbb{Z})$, which is defined and analytic on the upper half-plane $\mathbb{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. It is

$\mathrm{SL}_2(\mathbb{Z})$-invariant. Moreover, by means of j the quotient $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$ is identified with $\mathbb{C}$ (thus, j is a bijection from the fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$ to $\mathbb{C}$).

The j-function is often called the j-invariant as the j-invariant of an elliptic curve determines its isomorphism class. Given a point $\tau \in \mathbb{H}$ we let $\Lambda(\tau)$ be the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Then $E_\tau := \mathbb{C}/\Lambda(\tau)$ is an elliptic curve with j-invariant $j(\tau)$. It is known that for $\tau_1, \tau_2 \in \mathbb{H}$ the elliptic curves E_{τ_1} and E_{τ_2} are isomorphic if and only if $\tau_2 = g\tau_1$ for some $g \in \mathrm{SL}_2(\mathbb{Z})$. This happens if and only if $j(\tau_1) = j(\tau_2)$.

In fact, the j-invariant of an elliptic curve can be defined in terms of the coefficients of its algebraic equation. Indeed, every elliptic curve can be embedded into the projective plane as an algebraic curve, defined by a cubic equation of the form

$$y^2 z = 4x^3 - axz^2 - bz^3.$$

Then its j-invariant is defined as $1728\frac{a^3}{\Delta}$ where $\Delta := a^3 - 27b^2 \neq 0$ is its discriminant. This can be used to give a definition of the j-function.

Let $\mathrm{GL}_2^+(\mathbb{R})$ be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of matrices with positive determinant[1]. Let $\mathrm{GL}_2^+(\mathbb{Q})$ be its subgroup of matrices with rational entries. For $g \in \mathrm{GL}_2^+(\mathbb{Q})$ we let $N(g)$ be the determinant of g scaled so that it has relatively prime integral entries. For each positive integer N there is an irreducible polynomial $\Phi_N(X,Y) \in \mathbb{Z}[X,Y]$ such that whenever $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$, the function $\Phi_N(j(z), j(gz))$ is identically zero. Conversely, if $\Phi_N(j(x), j(y)) = 0$ for some $x, y \in \mathbb{H}$ then $y = gx$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$ with $N = N(g)$. The polynomials Φ_N are called *modular polynomials*. It is well known that $\Phi_1(X,Y) = X - Y$ and all the other modular polynomials are symmetric. For $w = j(z)$ the image of the $\mathrm{GL}_2^+(\mathbb{Q})$-orbit of z under j is called the *Hecke orbit* of w. It obviously consists of the union of solutions of the equations $\Phi_N(w, X) = 0,\ N \geq 1$. Two elements $w_1, w_2 \in \mathbb{C}$ are called *modularly independent* if they have different Hecke orbits, i.e. do not satisfy any modular relation $\Phi_N(w_1, w_2) = 0$. This definition makes sense for arbitrary fields (of characteristic zero) as the modular polynomials have integer coefficients.

The j-function satisfies an order 3 algebraic differential equation over $\mathbb{Q}$, and none of lower order (i.e. its differential rank over $\mathbb{C}$ is 3). Namely, $F(j, j', j'', j''') = 0$ where

$$F(y_0, y_1, y_2, y_3) = \frac{y_3}{y_1} - \frac{3}{2}\left(\frac{y_2}{y_1}\right)^2 + \frac{y_0^2 - 1968y_0 + 2654208}{2y_0^2(y_0 - 1728)^2} \cdot y_1^2.$$

Thus

$$F(y, y', y'', y''') = Sy + R(y)(y')^2,$$

where S denotes the *Schwarzian derivative* defined by $Sy = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ and $R(y) = \frac{y^2 - 1968y + 2654208}{2y^2(y-1728)^2}$.

The following result is well known (see, for example, [FS15], Lemma 4.2).

[1] This group acts on the upper half-plane and in fact it is the biggest subgroup of $\mathrm{GL}_2(\mathbb{C})$ with this property. In fact, $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$ is the group of automorphisms of $\mathbb{H}$ as a complex manifold.

Lemma 6.1.1. *All functions $j(gz)$ with $g \in \mathrm{SL}_2(\mathbb{R})$ satisfy the differential equation $F(y, y', y'', y''') = 0$ and all solutions (defined on $\mathbb{H}$) are of that form. If we allow functions not necessarily defined on $\mathbb{H}$, then all solutions will be of the form $j(gz)$ where $g \in \mathrm{SL}_2(\mathbb{C})$.*

6.2 Ax-Schanuel and weak modular Zilber-Pink

Theorem 6.2.1 (Ax-Schanuel for j, [PT16])**.** *Let $(K; +, \cdot, ', 0, 1)$ be a differential field and let $z_i, j_i \in K \setminus C,\ j_i^{(1)}, j_i^{(2)}, j_i^{(3)} \in K^{\times},\ i = 1, \ldots, n$, be such that*

$$F\left(j_i, j_i^{(1)}, j_i^{(2)}, j_i^{(3)}\right) = 0 \wedge j_i' = j_i^{(1)} z_i' \wedge \left(j_i^{(1)}\right)' = j_i^{(2)} z_i' \wedge \left(j_i^{(2)}\right)' = j_i^{(3)} z_i'.$$

If the j_i's are pairwise modularly independent then

$$\mathrm{td}_C\, C\left(\bar{z}, \bar{j}, \bar{j}^{(1)}, \bar{j}^{(2)}\right) \geq 3n + 1. \tag{2.1}$$

Corollary 6.2.2 (Ax-Schanuel without derivatives)**.** *If z_i, j_i are non-constant elements in a differential field K with $F(j_i, \partial_{z_i} j_i, \partial_{z_i}^2 j_i, \partial_{z_i}^3 j_i) = 0$, then*

$$\mathrm{td}_C\, C(\bar{z}, \bar{j}) \geq n + 1,$$

unless for some N, i, k we have $\Phi_N(j_i, j_k) = 0$.

This theorem implies in particular that the only algebraic relation between the functions $j(z)$ and $j(gz)$ for $g \in \mathrm{SL}_2^+(\mathbb{R})$ are the modular relations (corresponding to $g \in \mathrm{GL}_2^+(\mathbb{Q})$).

A consequence of the Ax-Schanuel theorem is a weak form of the modular Zilber-Pink conjecture. Below K is an algebraically closed field.

Definition 6.2.3. A *special variety* is an irreducible component of a Zariski-closed set defined by some modular equations. Note that we allow a modular equation of the form $\Phi_N(x_i, x_i) = 0$ which is equivalent to allowing equations of the form $x_i = c$ where c is a special point (the image of a quadratic number under j).

Definition 6.2.4. Let $V \subseteq K^n$ be an algebraic variety. An *atypical subvariety* of V is an irreducible component W of some $V \cap S$, where S is a special subvariety, such that

$$\dim W > \dim V + \dim S - n.$$

An atypical subvariety W of V is said to be *strongly atypical* if it is not contained in any hyperplane of the form $x_i = a$ for some $a \in K$ (i.e. no coordinate is constant on W).

The following is an analogue of Zilber's conjecture on intersection with tori (see [KZ14, Zil02]).

Conjecture 6.2.5 (Modular Zilber-Pink)**.** *Every algebraic variety contains only finitely many maximal atypical subvarieties.*

Definition 6.2.6. When $V \subseteq K^{n+m}$ is a variety defined over $\mathbb{Q}$, and $A \subseteq K^m$ is its projection onto the last m coordinates (A is a constructible set), for each $\bar{a} \in A$ we let $V(\bar{a})$ (or $V_{\bar{a}}$) be the fibre of the projection above $\bar{a}$. The family $(V(\bar{a}))_{\bar{a}\in A}$ is called a *parametric family of varieties.*

The following theorem is a weak version of Zilber-Pink and follows from Ax-Schanuel with a compactness argument. Pila and Tsimerman [PT16] give an o-minimality proof, again using Ax-Schanuel.

Theorem 6.2.7 (Weak modular Zilber-Pink)**.** *Given a parametric family of algebraic varieties $(V_{\bar{a}})_{\bar{a}\in A}$ in K^n, there is a finite collection of proper special varieties $(S_i)_{i\leq N}$ in K^n such that for every $\bar{a} \in A$, every strongly atypical subvariety of $V_{\bar{a}}$ is contained in one of S_i.*

For simplicity, we are going to work with Ax-Schanuel without derivatives. However, most of our results remain true in the general setting as well, and in the last section we will formulate definitions and main results in that generality, pointing out an issue related to weak modular Zilber-Pink "with derivatives".

6.3 The universal theory and predimension

Let $(K; +, \cdot, ', 0, 1)$ be a differential field and let $F(y, y', y'', y''') = 0$ be the differential equation of the j-function. Consider its two-variable version[2]

$$f(x, y) := F\left(y, \partial_x y, \partial_x^2 y, \partial_x^3 y\right) = 0. \tag{3.2}$$

We prove several lemmas about this differential equation. Below C denotes the field of constants of K.

Lemma 6.3.1. *Given $a_1, a_2, b \in K \setminus C$, if $f(a_1, b) = 0$ then $f(a_2, b) = 0$ iff $a_2 = ga_1$ for some $g \in \mathrm{SL}_2(C)$.*

Proof. We can assume without loss of generality that $a_1 = t$ (recall that t satisfies $t' = 1$). For simplicity denote $a_2 = a$. Let also S_{∂_a} be the Schwarzian derivative with respect to ∂_a. Then we know that $Sb + R(b)(b')^2 = 0$, and so $S_{\partial_a} b + R(b)(\partial_a b)^2 = 0$ if and only if $(a')^2 \cdot S_{\partial_a} b = Sb$. However, straightforward calculations show that $(a')^2 \cdot S_{\partial_a} b = Sb - Sa$. Hence, $f(a, b) = 0$ iff $Sa = 0$ iff $a = gt$ for some $g \in \mathrm{SL}_2(C)$. □

Lemma 6.3.2. *If $f(z, j_1) = 0$ for some non-constants z, j_1, and j_2 satisfies $\Phi_N(j_1, j_2) = 0$ for some modular polynomial Φ_N then $f(z, j_2) = 0$.*

Proof. Embedding our differential field into the field of germs of meromorphic functions, we can assume $C = \mathbb{C}$ and j_1, j_2 are complex meromorphic functions of variable z. But then by Lemma 6.1.1 $j_1 = j(g_1 z)$ for some $g_1 \in \mathrm{SL}_2(\mathbb{C})$ where $j : \mathbb{H} \to \mathbb{C}$ is the j-invariant. Now the identity $\Phi_N(j_1(z), j_2(z)) = 0$ implies $j_2(z) = j(g_2 z)$ where $g_2 = gg_1$ for some $g \in \mathrm{GL}_2^+(\mathbb{Q})$. Applying Lemma 6.1.1 again we see that $j(g_2 z)$ satisfies the differential equation of $j(z)$. □

[2] Recall that for a non-constant x we define $\partial_x : y \mapsto \frac{y'}{x'}$.

We consider a binary predicate $E_j^*(x,y)$ which will be interpreted in a differential field as

$$\exists y_1, y_2, y_3 \left(y_1^2 y^2 (y-1728)^2 F(y, y_1, y_2, y_3) = 0 \wedge y' = y_1 x' \wedge y_1' = y_2 x' \wedge y_2' = y_3 x'\right).$$

Here we multiplied F by $y_1^2 y^2 (y-1728)^2$ in order to make it a differential polynomial. Observe that any pair (a,c), where c is a constant, is in E_j^*. In order to simplify our arguments, we remove all points (a,c) with $a \notin C, c \in C$, and define $E_j(x,y)$ by

$$E_j(x,y) \longleftrightarrow \left[E_j^*(x,y) \wedge \neg(x' \neq 0 \wedge y' = 0)\right].$$

Actually, E_j can be defined from E_j^* (without using the derivation) as C is definable by $E_j^*(0,y)$. The formula $E_j(0,y)$ defines the field of constants as well. One can also notice that for non-constant x and y the relation $E_j(x,y)$ is equivalent to $f(x,y)=0$.

Definition 6.3.3. The theory T_j^0 consists of the following first-order statements about a structure K in the language $\mathfrak{L}_j := \{+, \cdot, E_j, 0, 1\}$.

A1 K is an algebraically closed field.

A2 $C := C_K = \{c \in K : E_j(0,c)\}$ is an algebraically closed subfield. Further, $C^2 \subseteq E_j(K)$ and if $(z,j) \in E_j(K)$ and one of z, j is constant then both of them are constants.

A3 If $(z,j) \in E_j$ then for any $g \in \mathrm{SL}_2(C)$, $(gz, j) \in E_j$. Conversely, if for some j we have $(z_1, j), (z_2, j) \in E_j$ then $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$.

A4 If $(z, j_1) \in E_j$ and $\Phi_N(j_1, j_2) = 0$ for some j_2 and some modular polynomial $\Phi_N(X,Y)$ then $(z, j_2) \in E_j$.

AS If $(z_i, j_i) \in E_j,\ i = 1, \ldots, n$, with

$$\mathrm{td}_C\, C\,(\bar{z}, \bar{j}) \leq n,$$

then $\Phi_N(j_i, j_k) = 0$ for some N and some $1 \leq i < k \leq n$, or $j_i \in C$ for some i.

Remark 6.3.4. A3 and A4 (the functional equations) imply that if $E_j(z_i, j_i),\ i = 1, 2$, and j_1, j_2 are modularly dependent then z_1 and z_2 have the same $\mathrm{SL}_2(C)$-orbit. However, the converse is not true: if $z_2 = gz_1$ for some g then this does not impose a relation on j_1, j_2 (they can be algebraically independent).

A compactness argument shows that AS can be written as a first-order axiom scheme. Indeed, AS holds in all differential fields K. The compactness theorem can be applied to deduce that, given a parametric family of varieties $(W_{\bar{c}})_{\bar{c} \in C} \subseteq K^{2n}$, there is a natural number $N(W)$ such that if $\bar{c} \in C$ satisfies $\dim W_{\bar{c}} \leq n$, and if $(\bar{z}, \bar{j}) \in E_j(K) \cap W_{\bar{c}}(K)$ and $j_i \notin C$ for all i, then $\Phi_N(j_i, j_k) = 0$ for some $N \leq N(W)$ and some $1 \leq i < k \leq n$. This can clearly be written as a first-order axiom scheme. Thus, AS should be understood as the uniform version of Ax-Schanuel.

Definition 6.3.5. An E_j-*field* is a model of T_j^0. If K is an E_j-field, then a tuple $(\bar{z}, \bar{j}) \in K^{2n}$ is called an E_j-*point* if $(z_i, j_i) \in E_j(K)$ for each $i = 1, \ldots, n$. By abuse of notation, we let $E_j(K)$ denote the set of all E_j-points in K^{2n} for any natural number n (which will be obvious from the context). The subfield C_K is called the field of constants of K.

The above lemmas show that reducts of differential fields to the language $\mathfrak{L}_j$ are E_j-fields.

Let C be an algebraically closed field with $\mathrm{td}(C/\mathbb{Q}) = \aleph_0$ and let $\mathfrak{C}$ consist of all E_j-fields K with $C_K = C$. Note that C is an E_j-field with $E_j(C) = C^2$ and it is the smallest structure in $\mathfrak{C}$. From now on, by an E_j-field we understand a member of $\mathfrak{C}$. Note that for some $X \subseteq A \in \mathfrak{C}$ we have $\langle X \rangle_A = C(X)^{\mathrm{alg}}$ (with the induced structure from A) and $\mathfrak{C}_{f.g.}$ consists of those E_j-fields that have finite transcendence degree over C.

Definition 6.3.6. For $A \subseteq B \in \mathfrak{C}_{f.g.}$ an E_j-*basis* of B over A is an E_j-point $\bar{b} = (\bar{z}, \bar{j})$ from B of maximal length satisfying the following conditions:

- j_i and j_k are modularly independent for all $i \neq k$,
- $(z_i, j_i) \notin A^2$ for each i.

We let $\sigma(B/A)$ be the length of $\bar{j}$ in an E_j-basis of B over A (equivalently, $2\sigma(B/A) = |\bar{b}|$). When $A = C$ we write $\sigma(B)$ for $\sigma(B/C)$. Further, for $A \in \mathfrak{C}_{f.g.}$ define the predimension by

$$\delta(A) := \mathrm{td}_C(A) - \sigma(A).$$

Note that the Ax-Schanuel inequality for j implies that σ is finite for finitely generated structures. It is easy to see that for $A \subseteq B \in \mathfrak{C}_{f.g.}$ one has $\sigma(B/A) = \sigma(B) - \sigma(A)$. Moreover, for $A, B \subseteq D \in \mathfrak{C}_{f.g.}$ the inequality

$$\sigma(AB) \geq \sigma(A) + \sigma(B) - \sigma(A \cap B)$$

holds. Hence δ is submodular (so it is a predimension) and the Pila-Tsimerman inequality states exactly that $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$ with equality holding if and only if $A = C$. The dimension associated with δ will be denoted by d_j or simply d. We will add a superscript if we want to emphasise the model that we work in.

Observe also that for $A \subseteq B \in \mathfrak{C}_{f.g.}$

$$\delta(B/A) = \delta(B) - \delta(A) = \mathrm{td}(B/A) - \sigma(B/A).$$

6.4 Amalgamation

Definition 6.4.1. A structure $A \in \mathfrak{C}$ is said to be *full* if for every $j \in A$ there is $z \in A$ such that $A \models E_j(z, j)$. The subclass $\hat{\mathfrak{C}}$ consists of all full E_j-fields.

Lemma 6.4.2. *Every $A \in \mathfrak{C}$ has a unique (up to isomorphism over A) strong full extension $\hat{A} \in \hat{\mathfrak{C}}$ which is generated by A as a full structure. In particular, if $A \in \mathfrak{C}_{f.g.}$ then $\hat{A} \in \hat{\mathfrak{C}}_{f.g.}$. Furthermore, if $f : A \hookrightarrow B$ is a strong embedding then f extends to a strong embedding $\hat{f} : \hat{A} \hookrightarrow \hat{B}$.*

Proof. Let $A \in \mathfrak{C}$. Choose an element $j \in A$ for which $A \models \neg\exists x E_j(x, j)$ (if there is such). Pick z transcendental over A (in a big algebraically closed field). Let $A_1 := A(z)^{\text{alg}}$. Extend the relation E_j to A_1 by adding the tuple (z, j) to E_j and closing the latter under the functional equations given by axioms A3 and A4. It is easy to see that $A \leq A_1$. Repeating this construction we will get a strong chain $A \leq A_1 \leq A_2 \leq \ldots$ the union of which, $A^1 := \bigcup_i A_i$, contains a solution of the formula $E_j(x, j)$ for each $j \in A$. Now we can iterate this construction and get another strong chain $A \leq A^1 \leq A^2 \leq \ldots$ such that for every $j \in A^i$ the formula $E_j(x, j)$ has a solution in A^{i+1}. The union $\hat{A} := \bigcup_i A^i$ will be the desired strong and full extension of A. It is also clear that $\hat{A}$ is generated by A as a full E_j-field.

Now we show that if $\hat{B} \in \hat{\mathfrak{C}}$ is a strong extension of A then there is a strong embedding $\hat{A} \hookrightarrow \hat{B}$ over A. Let $j \in A$ be such that $A \models \neg\exists x E_j(x, j)$ and let $w \in \hat{B}$ satisfy $E_j(w, j)$. Since $w \notin A$, it must be transcendental over A. We claim that A_1 (as constructed above) is isomorphic to $B_1 := A(w)^{\text{alg}} \subseteq \hat{B}$ (with the induced structure). Indeed, $A \leq \hat{B}$ implies that (w, j) is an E_j-basis of B_1 over A. Similarly, (z, j) is an E_j-basis of A_1 over A. Hence, any isomorphism between the algebraically closed fields A_1 and B_1 that fixes A pointwise and sends z to w is actually an isomorphism of E_j-fields A_1 and B_1. Moreover, $B_1 \leq \hat{B}$ since $\delta(B_1/A) = 0$. We can inductively construct similar partial isomorphisms from $\hat{A}$ into $\hat{B}$ the union of which will give a strong embedding $\hat{A} \hookrightarrow \hat{B}$. Furthermore, if $\hat{B}$ is generated by A as a full E_j-field then we get an isomorphism $\hat{A} \cong \hat{B}$. $\square$

Proposition 6.4.3. *The class $\hat{\mathfrak{C}}$ has the asymmetric amalgamation property.*

Proof. Let $A, B_1, B_2 \in \hat{\mathfrak{C}}$ with an embedding $A \hookrightarrow B_1$ and a strong embedding $A \hookrightarrow B_2$. Let B be the free amalgam of B_1 and B_2 over A as algebraically closed fields. More precisely, B is the algebraic closure of the extension of A by the disjoint union of the transcendence bases of B_1 and B_2 over A. Identifying A, B_1, B_2 with their images in B we have $B_1 \cap B_2 = A$. Define E_j on B as the union $E_j(B_1) \cup E_j(B_2)$.

We show[3] that $B_1 \leq B$. By our definition of $E_j(B)$, a non-constant element $b \in B$ satisfies $B \models \exists x E_j(x, b)$ if and only if $b \in B_1 \cup B_2$. For a finitely generated $X \subseteq_{f.g.} B$ denote $X_1 := X \cap B_1, X_2 := X \cap B_2, X_0 := X \cap A$. From the above observation it follows that $\sigma(X) = \sigma(X \cap B_1) + \sigma(X \cap B_2) - \sigma(X \cap A)$. Further, X_1 and X_2 are algebraically independent over X_0 and so

$$\text{td}(X/C) \geq \text{td}(X_1/C) + \text{td}(X_2/C) - \text{td}(X_0/C).$$

Therefore

$$\delta(X) = \text{td}(X/C) - \sigma(X) \geq \delta(X \cap B_1) + \delta(X \cap B_2) - \delta(X \cap A) \geq \delta(X \cap B_1),$$

[3]Evidently, B satisfies A1-A4 but we are still to prove that AS holds in B too. So we do not know yet that B is an E_j-field. However, δ is well defined on B and it makes sense to say that $B_1 \leq B$.

where the last inequality holds as $A \leq B_2$. Thus, $B_1 \leq B$.

This shows in particular that $\delta(X) \geq 0$. If $\delta(X) = 0$ then $\delta(X \cap B_1) = 0$ and so $X \cap B_1 \subseteq C$. But then $X \cap A \subseteq C$ which implies $\delta(X \cap A) = 0$. Therefore $\delta(X \cap B_2) = 0$ and $X \cap B_2 \subseteq C$. So $X \setminus C$ is disjoint from $B_1 \cup B_2$. But then $\delta(X) > 0$ unless $X \subseteq C$.

So, B satisfies the AS axiom scheme. Hence we can extend it strongly to a full E_j-field. The symmetric argument shows that if $A \leq B_1$ then $B_2 \leq B$. □

Lemma 6.4.4. *Let $A \in \mathfrak{C}$ and let B be a strong extension of A finitely generated over A. Then B is determined up to isomorphism by the locus $\mathrm{Loc}_A(\bar{b})$ for an E_j-basis $\bar{b}$ of B over A and the number $n = \mathrm{td}(B/A(\bar{b}))$. Hence for a given A there are at most countably many strong finitely generated extensions of A, up to isomorphism.*

Proof. Let B_1 and B_2 be two strong extensions of A, finitely generated over A. Let also $\bar{b}_i := (\bar{z}_i, \bar{j}_i)$ be an E_j-basis of B_i over A, and denote $A_i := A(\bar{b}_i)$ $(i = 1, 2)$. Assume $\mathrm{Loc}_A(\bar{b}_1) = \mathrm{Loc}_A(\bar{b}_2)$ and $\mathrm{td}(B/A_1) = \mathrm{td}(B/A_2)$. The map that fixes A and sends $\bar{b}_1$ to $\bar{b}_2$ extends uniquely to a field isomorphism between A_1 and A_2, which respects the E_j-field structure. Any extension of this field isomorphism to A_1^{alg} and A_2^{alg} is actually an isomorphism of E_j-fields. Since $\bar{b}_i$ is an E_j-basis of B_i over A, $E_j(B_i) = E_j(A_i^{\mathrm{alg}})$ for $i = 1, 2$. Therefore any extension of the above map to a field isomorphism of B_1 and B_2 (which exists as $\mathrm{td}(B/A_1) = \mathrm{td}(B/A_2)$) is an E_j-field isomorphism over A.

For the second part of the lemma we just notice that there are countably many choices for $\mathrm{Loc}_A(\bar{b})$ and the number n. □

Theorem 6.4.5. *The classes $\mathfrak{C}$ and $\hat{\mathfrak{C}}$ are strong amalgamation classes with the same strong Fraïssé limit U.*

Proof. Proposition 3.2.6 shows that $\mathfrak{C}$ has the strong amalgamation property. Then $\mathfrak{C}$ and $\hat{\mathfrak{C}}$ are strong amalgamation classes and have the same strong Fraïssé limit by Proposition 3.2.7. □

Note that $\mathfrak{C}_{f.g.}$ does not have the asymmetric amalgamation property.

6.5 Normal and free varieties

Definition 6.5.1. Let n be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $\bar{i} = (i_1, \ldots, i_k)$ and define the projection map $\mathrm{pr}_{\bar{i}} : K^n \to K^k$ by

$$\mathrm{pr}_{\bar{i}} : (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_n}).$$

Further, define (by abuse of notation) $\mathrm{pr}_{\bar{i}} : K^{2n} \to K^{2k}$ by

$$\mathrm{pr}_{\bar{i}} : (\bar{x}, \bar{y}) \mapsto (\mathrm{pr}_{\bar{i}}\, \bar{x}, \mathrm{pr}_{\bar{i}}\, \bar{y}).$$

It will be clear from the context in which sense $\mathrm{pr}_{\bar{i}}$ should be understood (mostly in the second sense).

Definition 6.5.2. Let K be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{2n}$ is *normal* if and only if for any $0 < k \leq n$ and any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \operatorname{pr}_{\bar{i}} V \geq k$. We say V is *strongly normal* if the strict inequality $\dim \operatorname{pr}_{\bar{i}} V > k$ holds.

For a subfield $A \subseteq K$ we say a variety V defined over $B \subseteq K$ is (strongly) normal over A (regardless of whether V is defined over A) if, for a generic (over $A \cup B$) point $\bar{v}$ of V, the locus $\operatorname{Loc}_A(\bar{v})$ is (strongly) normal. In other words, if a projection of V is defined over A then it is strongly normal over A.

Normality is an analogue of rotundity. However, the term "rotundity" is not suitable in the context of this chapter since it refers to the group structure of exponentiation. Note that normality was the original term used by Zilber (rotundity was coined by Kirby).

Remark 6.5.3. It may seem strange that, in contrast to the exponential case, the functional equations of the j-function are not reflected in the definition of normality. The reason is that those functional equations are of "trivial" type. Indeed, one would expect the following additional condition to be present: if $\bar{v} := (\bar{z}, \bar{j}) \in V(K)$ is a generic point of V then we have not only $\operatorname{td}(\operatorname{pr}_{\bar{i}} \bar{v}) \geq k$ but also if we replace z's by arbitrary elements in their $\mathrm{SL}_2(C)$-orbits and j's by arbitrary elements in their Hecke orbits, then the transcendence degree of images of all those tuples under $\operatorname{pr}_{\bar{i}}$ must be at least k. However, it is obvious that the first condition already implies this because when we change the tuple in this manner, we do not change the transcendence degree (over C).

Remark 6.5.4. Normality is a first-order definable property. This follows from the facts that irreducibility and algebraic dimension are definable in algebraically closed fields. More generally, Morley rank is definable in strongly minimal theories.

Rotundity (in the exponential case) is first-order definable as well but it is not obvious since in its definition there are infinitely many conditions.

Definition 6.5.5. An algebraic variety $V \subseteq K^{2n}$ (with coordinates $(\bar{x}, \bar{y})$) is *free* if it is not contained in any variety defined by an equation $\Phi_N(y_i, y_k) = 0$ for some modular polynomial Φ_N and some indices i, k.

This definition makes sense for an arbitrary field K. However, when K is an E_j-field and $A \subseteq K$ is an E_j-subfield, we say $V \subseteq K^{2n}$ is *free over* A if it is free and it is not contained in a hyperplane defined by an equation of the form $y_i = a$ (for some i) where $a \in A$ with $A \models \exists z E_j(z, a)$.

We could require in the definition of freeness that V is not contained in any variety defined by an equation of the form $y_i = b$ for some $b \in K$. This would be more standard definition and in fact it would be a definable property of the variety due to weak modular Zilber-Pink (but we will not need this result). Nevertheless, we find it more convenient to work with the notion of freeness (over A) defined above since it allows us to simplify some arguments slightly.

Lemma 6.5.6. *If $A \leq B \in \mathfrak{C}_{f.g.}$ and $\bar{b} \in B^{2n}$ is an E_j-basis of B over A then the locus $\operatorname{Loc}_A(\bar{b})$ is normal and free over A, and strongly normal over C.*

Proof. Follows obviously from definitions. $\square$

Lemma 6.5.7. *Let $A = C(\bar{a})^{\text{alg}}$ be an E_j-field and V be a normal irreducible variety defined over A. Then there is a strong extension B of A which contains an E_j-point of V generic over $\bar{a}$. Furthermore, if V is normal, free over A and strongly normal over C then we can choose B so that $V(B) \cap E_j(B)$ contains a point generic in V over A.*

Proof. First, we prove the "furthermore" clause. Take a generic point of V over A, say, $\bar{b} := (\bar{z}, \bar{j})$ and let $B := \langle A\bar{b} \rangle = A(\bar{b})^{\text{alg}}$. Extend E_j by declaring (z_i, j_i) an E_j-point for each i and close it under functional equations (axioms A3 and A4). The given properties of V make sure that B is a model of T_j^0 and is a strong extension of A.

Now we prove the first part of the lemma. If for some $i_1 < i_2 < \ldots < i_k$ the projection $W := \text{pr}_{\bar{i}} V$ is defined over C and has dimension k then we pick constant elements $z_{i_s}, j_{i_s}, s = 1, \ldots, k$, such that $(\bar{z}, \bar{j})$ is generic in W over $\bar{a}$. Doing this for all projections defined over C, we consider the variety V_1 obtained from V by setting $x_{i_s} = z_{i_s}, y_{i_s} = j_{i_s}$ for all indices i_s considered above. All of those pairs of constants will be in E_j.

Further, if V_1 is contained in a hyperplane $y_i = a$ for a non-constant $a \in A$ with $A \models E_j(z, a)$ for some $z \in A$, then we intersect it with the hyperplane $x_i = gz$ where we choose the entries of g to be generic constants over $\bar{a}$. Doing this for all such a, we get a variety V_2, in a lower number of variables, which is still normal.

If V_2 is free then we proceed as above. Otherwise we argue as follows. Suppose for some $i_1 \neq i_2$ the projection $\text{pr}_{i_1,i_2} V_2$ satisfies the equation $\Phi_N(y_{i_1}, y_{i_2}) = 0$ (we can assume i_1 and i_2 are different from all indices i_s considered above). Let us assume for now that this is the only modular relation between the y-coordinates satisfied by V_2. Then we take algebraically independent elements $a, b, c \in C$ over $\bar{a}$ and over all elements from A chosen above, and denote $d := (1 + bc)/a$. Let V_3 be the subvariety of V_2 defined by the equation $x_{i_2} = \frac{ax_{i_1}+b}{cx_{i_1}+d}$. It is easy to see that $\dim V_3 = \dim V_2 - 1$ (here $V_3 \neq \emptyset$ as, by normality, $\dim \text{pr}_{i_1,i_2} V_2 \geq 2$). Now we take a generic point of V_3 over $\bar{a}abc$ and all constants taken above, and proceed as in the free case. Note that this generic point will be generic in V over A.

When there are more modular relations between the y-coordinates of V_2, we apply the above procedure for all of those modular relations, that is, we introduce new generic $\text{SL}_2(C)$-relations between the pairs of the appropriate x-coordinates (the corresponding y-coordinates of which satisfy a modular relation), and proceed as above. $\square$

6.6 Existential closedness

Consider the following statements for an E_j-field K.

EC For each normal variety $V \subseteq K^{2n}$ the intersection $E_j(K) \cap V(K)$ is non-empty.

SEC For each normal variety $V \subseteq K^{2n}$ defined over a finite tuple $\bar{a} \subseteq K$, the intersection $E_j(K) \cap V(K)$ contains a point generic in V over $\bar{a}$.

GSEC For each irreducible variety $V \subseteq K^{2n}$ of dimension n defined over a finitely generated strong E_j-subfield $A \leq K$, if V is normal and free over A and strongly normal over C, then the intersection $E_j(K) \cap V(K)$ contains a point generic in V over A.

NT $K \supsetneq C$.

ID K has infinite d_j-dimension.

EC, SEC, GSEC, NT and ID stand for existential closedness, strong existential closedness, generic strong existential closedness, non-triviality and infinite dimensionality respectively. Clearly, NT and EC are first-order axiomatisable. Notice that if an E_j-field K satisfies AS+NT+EC then $\mathrm{td}(K/C)$ is infinite. In fact, all full E_j-fields with a non-constant point have the same property (we need to apply AS repeatedly).

Lemma 6.6.1. *Let V be an irreducible algebraic variety such that for every finitely generated (over $\mathbb{Q}$) field of definition $A \subseteq K$ there is a C-point generic in V over A. Then V is defined over C.*

Proof. Let A be a field of definition of V and $\bar{a}$ be a transcendence basis of A over C (if $\bar{a}$ is empty then V is defined over C). Then V is defined over $\mathbb{Q}(\bar{a}, \bar{c})^{\mathrm{alg}}$ for some finite tuple $\bar{c} \in C$. Denote $A' := \mathbb{Q}(\bar{a}, \bar{c})^{\mathrm{alg}}$. Let $\bar{d} \in C$ be a generic point of V over A. Then $\mathrm{td}(\bar{d}/A') = \dim V$. Since $\bar{a}$ is algebraically independent over C, we have $\mathrm{td}(\bar{d}/A') = \mathrm{td}(\bar{d}/\bar{c})$. Let $W := \mathrm{Loc}(\bar{d}/C_0)$ where $C_0 = \mathbb{Q}(\bar{c})^{\mathrm{alg}} \subseteq C$. Evidently, $W \supseteq V = \mathrm{Loc}(\bar{d}/A')$ and $\dim W = \mathrm{td}(\bar{d}/C_0) = \dim V$. Since both V and W are irreducible, $V = W$ and therefore V is defined over C_0. $\square$

Proposition 6.6.2. *For E_j-fields* SEC $\Rightarrow$ GSEC.

Proof. Let V and A be as in the statement of GSEC. Choose $\bar{a} \subseteq A$ such that $\bar{a}$ contains an E_j-basis of A, V is defined over $\bar{a}$ and $A = C(\bar{a})^{\mathrm{alg}}$.

Note that it suffices to prove that V contains an E_j-point $\bar{v} = (\bar{z}, \bar{j})$ none of the coordinates of which is constant and which is generic over $\bar{a}$. Indeed, we claim that $\bar{v}$ will be generic over A. If it is not the case then $\mathrm{td}(\bar{v}/A) < \dim V = n$. However, j_i and j_k are modularly independent for $i \neq k$ as V is free and $\bar{v}$ is generic in V over $\mathbb{Q}(\bar{a})$ and hence over $\mathbb{Q}$ (and modular polynomials are defined over $\mathbb{Q}$). Since V is free over A and $\bar{a}$ contains an E_j-basis of A, $(z_i, j_i) \notin A^2$ for each i. Then we would have $\delta(\bar{v}/A) < 0$ which contradicts strongness of A in K.

We claim that $V(K)$ contains an E_j-point generic over $\bar{a}$ which is not a C-point. If this is not the case then by SEC and Lemma 6.6.1 V is defined over C. Since it is strongly normal over C, we have $\dim V > n+1$ which contradicts our assumption that $\dim V = n$.

Now we prove that V contains an E_j-point none of the coordinates of which is constant. We proceed to the proof by induction on n. The case $n = 1$ is covered by the above argument (if $(z, j) \in E_j$ and one of z, j is in C then both of them must be in C). If $n > 1$ take a point $\bar{v} = (\bar{z}, \bar{j}) \in V(K) \cap E_j(K)$ generic over $\bar{a}$. If $\bar{v}$ has some constant coordinates then we can assume $(z_i, j_i) \subseteq C$ for $i = 1, \ldots, k$ with $k < n$

(again, if one of z_i, j_i is constant then both of them must be constants) and these are the only constant coordinates. If these constants have transcendence degree at least $k+1$ over $\bar{a}$ then the transcendence degree of all elements z_i, j_i with $i > k$ over $C(\bar{a})$ will be strictly less than $n-k$ which contradicts $A \leq K$ as above.

Therefore $\operatorname{td}(\{z_i, j_i : i \leq k\}/\mathbb{Q}(\bar{a})) = k$. By the induction hypothesis we can find an E_j-point $\bar{b}$ of $\operatorname{pr}_{\bar{i}} V$ (where $\bar{i} = (1, \ldots, k)$) none of the coordinates of which is constant and which is generic in V over $\bar{a}$. Clearly, $\delta(\bar{b}/A) = 0$ and so denoting $B := A(\bar{b})^{\text{alg}}$ we have $A \leq B \leq K$. Now let $V(\bar{b})$ be the variety obtained from V by letting the corresponding k coordinates of V be equal to the corresponding coordinates of $\bar{b}$. Using the induction hypothesis we get an E_j-point $\bar{u}$ of $V(\bar{b})$ which is generic over $\bar{a}, \bar{b}$ and whose coordinates are all non-constant. It is easy to see that $(\bar{b}, \bar{u}) \in V(K) \cap E_j(K)$ is as required. $\square$

Proposition 6.6.3. *The strong Fraïssé limit U satisfies* SEC *and* ID*, and hence* GSEC*.*

Proof. Let V be a normal irreducible variety defined over a finite tuple $\bar{a}$. Let also $A := \lceil \bar{a} \rceil_U$ (we can assume $A = C(\bar{a})^{\text{alg}}$ by extending $\bar{a}$ if necessary). By Lemma 6.5.7 there is a strong extension B of A which contains an E_j-point $\bar{v}$ generic in V over $\bar{a}$. Since U is saturated for strong extensions, there is an embedding of B into U over A. The image of $\bar{v}$ under this embedding is the required generic E_j-point of V.

For $n \in \mathbb{N}$, let A_n be an algebraically closed field of transcendence degree n over C. Defining $E_j(A_n) = C^2$ we make A_n into a finitely generated E_j-field with d_j-dimension n. By universality of U, A_n can be strongly embedded into U which shows U has infinite d_j-dimension because strong extensions preserve dimension. $\square$

One can directly prove in the same manner that U satisfies GSEC (without using Proposition 6.6.2).

Lemma 6.6.4. *Let K be an infinite d_j-dimensional E_j-field and $A \subseteq K$ be a finitely generated E_j-subfield. Assume $V \subseteq K^{2n}$ is a normal irreducible variety defined over A with $\dim_A V > n$. Then we can find a strong extension $A \leq A' \leq K$, generated over A by finitely many d_j^K-independent (over A) elements, and a normal subvariety V' of V, defined over A', with $\dim_{A'} V' = n$.*

This can be proven exactly as in the exponential case by intersecting V with generic hyperplanes (see [Kir09], Proposition 2.33 and Theorem 2.35). We give full details for completeness.

For $\bar{p} := (p_1, \ldots, p_N) \in K^N \setminus \{0\}$ let the hyperplane Π_p be defined by the equation $\sum_{i=1}^N p_i x_i = 1$. It is obvious that $\bar{a} \in \Pi_{\bar{b}}(K)$ iff $\bar{b} \in \Pi_{\bar{a}}(K)$.

We will need the following result from [Kir09] which has been adapted from [Zil04a].

Lemma 6.6.5. *Let $\bar{v} \in K^N$ and let $\bar{p} \in \Pi_{\bar{v}}$ be generic over A. Then for any tuple $\bar{w} \in A(\bar{v})^{\text{alg}}$ either $\bar{v} \in A(\bar{w})^{\text{alg}}$ or $\operatorname{td}(\bar{w}/A\bar{p}) = \operatorname{td}(\bar{w}/A)$ (i.e. $\bar{w} \perp_A \bar{p}$).*

Proof. Let $P := \mathrm{Loc}(\bar{p}/A(\bar{w})^{\mathrm{alg}})$. Then

$$\dim P = \mathrm{td}(\bar{p}/A\bar{w}) \geq \mathrm{td}(\bar{p}/A\bar{v}) = N - 1,$$

the inequality following from the fact that $\bar{w} \in A(\bar{v})^{\mathrm{alg}}$. On the other hand $P \subseteq \Pi_{\bar{v}}$ and $\dim \Pi_{\bar{v}} = N-1$. Since both P and $\Pi_{\bar{v}}$ are irreducible, they must be equal. Hence $\Pi_{\bar{v}}$ is defined over $A(\bar{w})^{\mathrm{alg}}$ and so the formula $\forall \bar{y} \in \Pi_{\bar{v}}(\bar{x} \in \Pi_{\bar{y}})$ defines $\bar{v}$ over $A(\bar{w})^{\mathrm{alg}}$. Thus, $\bar{v} \in A(\bar{w})^{\mathrm{alg}}$. □

Proof of Lemma 6.6.4. Let $\dim_A V > n$. It will be enough to find A' and V' with $\dim_{A'} V' = \dim V - 1$. Pick a generic point $\bar{v} \in V(K)$. Denote $N = 2n$ and choose $p_1, \ldots, p_{N-1} \in K$ to be d_j^K-independent over A. Pick $p_N \in K$ such that $\sum_{i=1}^N p_i v_i = 1$.

Let $A' := A(\bar{p})^{\mathrm{alg}}$ and $V' := V \cap \Pi_{\bar{p}} = \mathrm{Loc}_{A'}(\bar{v})$. Obviously, V' is irreducible and $\dim V' = \dim V - 1$. We claim that V' is normal. Let $\bar{w} := \mathrm{pr}_{\bar{i}}\, \bar{v}$ for some projection map $\mathrm{pr}_{\bar{i}}$ with $|\bar{i}| = k \leq n$. Then obviously $\bar{w} \in A(\bar{v})^{\mathrm{alg}}$. Therefore by Lemma 6.6.5 either $\bar{v} \in A(\bar{w})^{\mathrm{alg}}$ or $\mathrm{td}(\bar{w}/A') = \mathrm{td}(\bar{w}/A)$. In the former case

$$\dim \mathrm{pr}_{\bar{i}}\, V' = \mathrm{td}(\bar{w}/A') = \mathrm{td}(\bar{v}/A') = \dim V' = \dim V - 1 \geq n \geq k.$$

In the latter case

$$\dim \mathrm{pr}_{\bar{i}}\, V' = \mathrm{td}(\bar{w}/A') = \mathrm{td}(\bar{w}/A) \geq k,$$

where the last inequality follows from normality of V. □

Proposition 6.6.6. *The strong Fraïssé limit U is the unique countable E_j-field satisfying* GSEC *and* ID *and having* $\mathrm{td}(C/\mathbb{Q}) = \aleph_0$.

Proof. Let K be such an E_j-field. We will show it is saturated with respect to strong embeddings. Let $A \leq B$ be finitely generated E_j-fields and let $\bar{b}$ be a basis of B over A. If $\mathrm{td}(B/A(\bar{b})) > 0$ then let $\bar{b}'$ be a transcendence basis of B over $A(\bar{b})$. We can find a strong extension $B \leq B' = B(\bar{a}')^{\mathrm{alg}}$ such that $B' \models E_j(a_i', b_i')$ for each i. Replacing B by B' we may assume that $\mathrm{td}(B/A(\bar{b})) = 0$ and hence $B = A(\bar{b})^{\mathrm{alg}}$.

Let $V := \mathrm{Loc}_A(\bar{b})$ be the Zariski closure of $\bar{b}$ over A. It is irreducible, normal and free over A and strongly normal over C. By Lemma 6.6.4 we can find a strong extension A' of A, generated by independent elements over A, and a normal irreducible subvariety V' of V over A' such that $\dim_{A'} V' = \sigma(B/A)$. Obviously, V' is also free over A' and strongly normal over C (because V is).

By GSEC there is a point $\bar{v} \in V' \cap E_j$ in K, generic in V' over A'. Then $\bar{v}$ is also generic in V over A. Let $B'' := A'(\bar{v})^{\mathrm{alg}}$ with the induced structure from K. Then $\delta(A') = \delta(B'')$ and so $B'' \leq K$. Now $B' := A(\bar{v})^{\mathrm{alg}}$ with the induced structure is isomorphic to B over A. Moreover, B'' is generated by d_j-independent elements over B' and so $B' \leq B''$ and $B' \leq K$. Therefore, K is saturated for strong extensions. □

6.7 The complete theory

Definition 6.7.1. Let T_j be the theory axiomatised by $T_j^0 + \mathrm{EC} + \mathrm{NT}$.

Note that T_j is an $\forall\exists$-theory.

Proposition 6.7.2. *All $\aleph_0$-saturated models of T_j satisfy* SEC, *and hence* GSEC.

Proof. It suffices to show that in an arbitrary model K of T_j every Zariski-open subset of an irreducible normal variety contains an E_j-point. Let $(\bar{x}, \bar{y})$ be the coordinates of K^{2n} and let $V \subseteq K^{2n}$ be a normal irreducible variety. It is enough to show that for every proper subvariety W of V, defined by a single equation, $V \setminus W$ contains an E_j-point. Suppose W (as a subvariety of V) is defined by an equation $f(z_1, \ldots, z_k) = 0$ where each z_i is one of the coordinates $\{x_i, y_i : i = 1, \ldots, n\}$. The assumption that $W \subsetneq V$ means that f does not vanish on V.

We use Rabinovich's trick to replace $V \setminus W$ by a normal irreducible variety in a higher number of variables. Consider the variety $V' \subseteq K^{2(n+1)}$ (with coordinates $(\bar{x}, x_{n+1}, \bar{y}, y_{n+1})$) defined by the equations of V and one additional equation $x_{n+1} f(\bar{z}) = 1$. It is clear that V' is normal and irreducible. By EC, V' contains an E_j-point. Its projection onto the coordinates $(\bar{x}, \bar{y})$ will be an E_j-point in $V \setminus W$. $\square$

Proposition 6.7.3. *All $\aleph_0$-saturated models of T_j satisfy* ID. *In particular, a countable saturated model of T_j (if it exists) is isomorphic to U.*

Proof. Let $K \models T_j$ be $\aleph_0$-saturated. A priori, we do not have a type whose realisations would be d_j-independent, but we can write d_j-independence by an $\mathfrak{L}_{\omega_1,\omega}$-sentence. The idea is to use weak Zilber-Pink to reduce this $\mathfrak{L}_{\omega_1,\omega}$-sentence to a type and show that it is finitely satisfiable in K.

ID means that for each n there is a $2n$-tuple $\bar{a}$ of algebraically independent (over C) elements with $\bar{a} \in E_j(K)$ (which is equivalent to $\delta(\bar{a}) = n$) such that for all tuples $\bar{x}$ one has $\delta(\bar{x}/\bar{a}) \geq 0$. Here we can assume as well that $\bar{x}$ is a $2l$-tuple for some l and is an E_j-point. The fact that $\bar{a}$ is algebraically independent over C is given by a type consisting of formulae $\varphi_i(\bar{a}) = \forall \bar{c}(\bar{a} \notin V_i(\bar{c}))$, $i < \omega$, stating that $\bar{a}$ is not in any hypersurface (defined over C) from a parametric family of hypersurfaces $(V_i(\bar{c}))_{\bar{c} \in C}$ (to be more precise, we could say that $(V_i(\bar{c}))_{\bar{c} \in C}$ is the parametric family of hypersurfaces over C with degree i).

The statement $\forall x_1, \ldots, x_{2l} \delta(\bar{x}/\bar{a}) \geq 0$ can be written as an $\mathfrak{L}_{\omega_1,\omega}$-sentence as follows. Given an algebraic variety $W \subseteq K^{2l+2n+m}$ defined over $\mathbb{Q}$, for any $\bar{c} \in C^m$ with $\dim W(\bar{a}, \bar{c}) < l$ and for any $\bar{x} \in W(\bar{a}, \bar{c}) \cap E_j$, the j-coordinates of $\bar{x}$ (i.e. $x_{l+1}, \ldots, x_{2l}$) must satisfy a modular relation $\Phi_N(x_{l+i}, x_{l+k}) = 0$ for some N and some $1 \leq i < k \leq l$, or a modular relation with $\bar{a}$, i.e. $\Phi_N(a_{n+i}, x_{l+k}) = 0$ for some $1 \leq i \leq n$, $1 \leq k \leq l$, or we must have $x_{l+i} \in C$ for some $1 \leq i \leq l$.

Now suppose, for contradiction, that ID does not hold in K. It means that for some n, for all $2n$-tuples $\bar{a}$ satisfying $\bar{a} \in E_j$ and $\bigwedge_i \varphi_i(\bar{a})$, there are a variety $W \subseteq K^{2l+2n+m}$ (for some l, m) defined over $\mathbb{Q}$, a constant point $\bar{c} \in C^m$ with $\dim W(\bar{a}, \bar{c}) < l$, and a tuple $\bar{x} \in W(\bar{a}, \bar{c}) \cap E_j$, such that $\Phi_N(x_{l+i}, x_{l+k}) \neq 0$ for all N and all $1 \leq i < k \leq l$, and $\Phi_N(a_{n+i}, x_{l+k}) \neq 0$ for all $1 \leq i \leq n$, $1 \leq k \leq l$, and $x_{l+i} \notin C$ for all $1 \leq i \leq l$.

If $(\bar{u}, \bar{v}, \bar{w})$ are the coordinates of $K^{2l} \times K^{2n} \times K^m$ then let W' be the projection of W onto the coordinates $(u_{l+1}, \ldots, u_{2l}, v_{n+1}, \ldots, v_{2n}, w_1, \ldots, w_m)$. Consider the parametric family of varieties $W'(\bar{c})_{\bar{c} \in C^m}$ in K^{l+n}. Let $N(W)$ be the maximal number N

such that Φ_N appears in the defining equations of the finitely many special varieties given by the weak modular Zilber-Pink for this parametric family. Then the following holds[4] in K:

$$\forall \bar{a}\left[\bar{a} \in E_j \wedge \bigwedge_{i<\omega} \varphi_i(\bar{a}) \longrightarrow \bigvee_{\substack{l,m\in\mathbb{N}\\ W\subseteq K^{2l+2n+m}}} \exists \bar{c} \in C^m \ \exists \bar{x} \in W(\bar{a},\bar{c}) \cap E_j \left(\dim W(\bar{a},\bar{c}) < l \right.\right.$$

$$\left.\left. \wedge \bigwedge_{\substack{p\leq N(W)\\ 1\leq i<k\leq l}} \Phi_p(x_{l+i}, x_{l+k}) \neq 0 \wedge \bigwedge_{\substack{p\leq N(W)\\ 1\leq i\leq n\\ 1\leq k\leq l}} \Phi_p(a_{n+i}, x_{l+k}) \neq 0 \wedge \bigwedge_{1\leq i\leq l} x_{l+i} \notin C \right)\right].$$

Here the disjunction (in the first line) is over all positive integers l, m and all algebraic varieties $W \subseteq K^{2l+2n+m}$ defined over $\mathbb{Q}$ (there are countably many such triples (l, m, W)).

By $\aleph_0$-saturation of K and compactness we deduce that there are a finite collection of varieties $W_s \subseteq K^{2l_s+2n+m_s}$, $s = 1, \ldots, t$, and a finite number r such that

$$\forall \bar{a}\left[\bar{a} \in E_j \wedge \bigwedge_{i\leq r} \varphi_i(\bar{a}) \longrightarrow \bigvee_{s\leq t} \exists \bar{c} \in C^{m_s} \ \exists \bar{x} \in W_s(\bar{a},\bar{c}) \cap E_j \left(\dim W_s(\bar{a},\bar{c}) < l_s \right.\right.$$

$$\left.\left. \wedge \bigwedge_{\substack{p\leq N(W_s)\\ 1\leq i<k\leq l_s}} \Phi_p(x_{l_s+i}, x_{l_s+k}) \neq 0 \wedge \bigwedge_{\substack{p\leq N(W_s)\\ 1\leq i\leq n\\ 1\leq k\leq l_s}} \Phi_p(a_{n+i}, x_{l_s+k}) \neq 0 \wedge \bigwedge_{1\leq i\leq l_s} x_{l_s+i} \notin C \right)\right].$$

The formulas $\varphi_i(\bar{u})$ state that $\bar{u}$ is not in a given parametric family of hypersurfaces $V_i(\bar{c})$. It is easy to see that we can find a strongly normal and free variety P in K^{2n} defined over C, of dimension $n+1$, which is not contained in any of the varieties $V_i(\bar{c})$ for any $\bar{c}$ and any $i \leq r$. We can also make sure that the projection of P onto the last n coordinates is the whole affine space K^n.

Now by the GSEC property we can find a non-constant E_j-point $\bar{b} \in K^{2n}$ which is generic in P over C. Indeed, we need to intersect P with a generic hyperplane as in Lemma 6.6.4, with algebraically independent coefficients (instead of d_j-independent), and get a normal and free variety over (the strong closure of) the field generated by those coefficients. Then we apply GSEC.

Then $\operatorname{td}(\bar{b}/C) = n + 1$ and $\delta(\bar{b}) = 1$ and $\varphi_i(\bar{b})$ holds for $i \leq r$. Moreover, $b_{n+1}, \ldots, b_{2n}$ are algebraically independent over C. Now by the above statement, for some $W := W_s \subseteq K^{2l+2n+m}$ (where $m = m_s, l = l_s$) there are $\bar{c} \in C^m, \bar{d} \in W(\bar{b},\bar{c})(K) \cap E_j(K)$ such that $\dim W(\bar{b},\bar{c}) < l$ and $b_{n+1}, \ldots, b_{2n}, d_{l+1}, \ldots, d_{2l}$ are non-constant and do not satisfy any modular equation $\Phi_p = 0$ for $p \leq N(W)$.

Suppose, for a moment, that $b_{n+1}, \ldots, b_{2n}, d_{l+1}, \ldots, d_{2l}$ are pairwise modularly independent. Then evidently $\delta(\bar{d}/\bar{b}) < 0$ which contradicts AS.

[4]It holds for any N instead of $N(W)$. Our choice of $N(W)$ was made so that it will lead to a contradiction.

However those elements may satisfy some modular relations $\Phi_p = 0$ with $p > N(W)$. Let S be the special variety defined by all those modular relations (more precisely, S is a component of the variety defined by those relations, which contains the point $(b_{n+1}, \dots, b_{2n}, d_{l+1}, \dots, d_{2l})$). We claim that S intersects $W'(\bar{c})$ typically. Indeed, by our choice of $N(W)$, the intersection cannot be strongly atypical. On the other hand, no coordinate is constant on the intersection since $b_i, d_k \notin C$, so the intersection is not atypical. It means that if $b_{n+1}, \dots, b_{2n}, d_{l+1}, \dots, d_{2l}$ satisfy h independent modular relations ($h = \operatorname{codim} S = n + l - \dim S$), then $\operatorname{td}(\bar{b}', \bar{d}') \leq \dim W'(\bar{c}) - h$ where $\bar{b}' := (b_{n+1}, \dots, b_{2n})$, $\bar{d}' := (d_{n+1}, \dots, d_{2n})$. Counting transcendence degrees we see that $\operatorname{td}(\bar{b}, \bar{d}/C) \leq n + l - h$ while $\sigma(\bar{b}, \bar{d}) = n + l - h$. So $\delta(\bar{b}, \bar{d}) = 0$ which contradicts AS. $\square$

Proposition 6.7.4. *The theory T_j is complete and the Fraïssé limit U is $\aleph_0$-saturated.*

Proof. Let T_j^1 be an arbitrary completion of T_j and let M be a (possibly uncountable) $\aleph_0$-saturated model of T_j^1. Let also $C := C_M$ be the field of constants (which may be uncountable as well).

Claim. For all finitely generated (i.e. of finite transcendence degree over C) strong E_j-subfields $A, B \leq M$ with an isomorphism $f : A \cong B$, and for any $a' \in M$, there are $A \leq A' \leq M$ and $B \leq B' \leq M$ with $a' \in A'$ such that f extends to an isomorphism $A' \cong B'$.

Proof of the claim. We can assume $a' \notin A$ and hence it is transcendental over A. We consider two cases.

Case 1: $d_j^M(a'/A) = 0$.

Let $A' := \lceil Aa' \rceil_M$ and let $\bar{v}$ be an E_j-basis of A' over A. Since $\delta(A'/A) = 0$, A' must be $\mathfrak{C}$-generated by $\bar{v}$ over A, i.e. $A' = \langle A\bar{v} \rangle$. Now if $V := \operatorname{Loc}_A(\bar{v})$, then V is normal and free over A and strongly normal over C, and $\dim V = n$ (since $\delta(\bar{v}/A) = 0$). Let W be the image of V under the isomorphism $f : A \to B$ (i.e. we just replace the coefficients of equations of V by their images under f). Then W is normal and free over B and strongly normal over C, and so by the GSEC property the intersection $W(M) \cap E_j(M)$ contains a point $\bar{w}$ generic in W over B. Setting $B' := B(\bar{w})^{\mathrm{alg}}$ (with the induced structure from M), we see that $\delta(B'/B) = 0$ and so $B \leq B' \leq M$. Clearly f extends to an isomorphism from A' to B'.

Case 2: $d_j^M(a'/A) = 1$.

In this case we pick an element $b' \in M$ which is d_j^M-independent from B (which exists by ID) and set $A' = \langle Aa' \rangle = A(a')^{\mathrm{alg}}$ and $B' = \langle Bb' \rangle = B(b')^{\mathrm{alg}}$. Obviously $A' \leq M$, $B' \leq M$ and $A' \cong B'$. $\square$

Thus, given two tuples $\bar{a}, \bar{b} \in M$ (of the same length) with an isomorphism $f : \lceil \bar{a} \rceil_M \cong \lceil \bar{b} \rceil_M$ sending $\bar{a}$ to $\bar{b}$, we can start with f and construct a back-and-forth system of partial isomorphisms from M to itself showing that $\operatorname{tp}^M(\bar{a}) = \operatorname{tp}^M(\bar{b})$. Combining this with Lemma 6.4.4 we see that if $A := \lceil \bar{a} \rceil_M$ and $\bar{a}'$ is an E_j-basis of A then the type of $\bar{a}$ in M is determined uniquely by $\operatorname{Loc}(\bar{a}'/C)$, $\operatorname{Loc}(\bar{a}/C(\bar{a}'))$ and the number $\operatorname{td}(A/C(\bar{a}'))$. Indeed, if for $\bar{a}, \bar{b} \in M$ these data coincide then there is an isomorphism $f : \lceil \bar{a} \rceil_M \cong \lceil \bar{b} \rceil_M$ sending $\bar{a}$ to $\bar{b}$. Moreover, if $\operatorname{Loc}(\bar{a}'/C)$ and $\operatorname{Loc}(\bar{a}, \bar{a}'/C)$

are defined over a finite set of constants $\bar{c}$, then the proof of Lemma 6.4.4 shows actually that $\mathrm{tp}^M(\bar{a})$ is determined by the algebraic varieties $\mathrm{Loc}(\bar{c}/\mathbb{Q})$, $\mathrm{Loc}(\bar{a}', \bar{c}/\mathbb{Q})$ and $\mathrm{Loc}(\bar{a}, \bar{a}', \bar{c}/\mathbb{Q})$ (in fact, the first two varieties are also uniquely determined by the third one) and the number $\mathrm{td}(A/C(\bar{a}'))$. There are countably many choices for those varieties and the transcendence degree, hence T_j^1 is small, i.e. there are countably many pure types (types over $\emptyset$).[5] This implies that T_j^1 has a countable saturated model which must be isomorphic to U by Proposition 6.7.3. Thus, U is saturated and $T_j^1 = \mathrm{Th}(U)$. Since T_j^1 was an arbitrary completion of T_j, the latter has a unique completion and so it is complete. □

We combine the results of this section in the following theorems.

Theorem 6.7.5. *The theory T_j is consistent and complete. It is the first-order theory of the strong Fraïssé limit U, which is saturated.*

Theorem 6.7.6. *The following are equivalent.*

- *The Ax-Schanuel inequality for j is adequate.*
- *The Ax-Schanuel inequality for j is strongly adequate.*
- *$\mathfrak{L}_j$-reducts of differentially closed fields are models of T_j.*
- *$\mathfrak{L}_j$-reducts of differentially closed fields satisfy* EC.
- *$\mathfrak{L}_j$-reducts of $\aleph_0$-saturated differentially closed fields satisfy* SEC.

Thus adequacy of the Ax-Schanuel inequality for j would give a complete axiomatisation of the first-order theory of the differential equation of j and show that it is nearly model complete. It will also give a criterion for a system of differential equations in terms of the equation of j to have a solution. Nevertheless, it seems to be a difficult problem and we are not able to tackle it now. Most probably Kirby's technique of proving adequacy of the exponential Ax-Schanuel will not work for j, as it is based on the theory of differential forms and the simple form of the exponential differential equation, while the equation of j is quite complicated.

6.8 The general case

In this section we study the predimension given by the "full" Ax-Schanuel inequality (with derivatives). We consider a predicate $E_j'(x, y, y_1, y_2)$ which will be interpreted in a differential field as

$$\exists y_3 \left(y_1^2 y^2 (y - 1728)^2 F(y, y_1, y_2, y_3) = 0 \wedge y' = y_1 x' \wedge y_1' = y_2 x' \wedge y_2' = y_3 x' \right).$$

Note that all quadruples of constants $(z, j, j^{(1)}, j^{(2)})$ satisfy E_j' unless $j^{(1)} = 0,\ j^{(2)} \neq 0$. For convenience we extend E_j' so that it contains all quadruples of constants.

[5] We can in fact show by a similar argument that T_j is $\aleph_0$-stable.

Also, if z is constant then $j, j^{(1)}, j^{(2)}$ must be constants as well. Moreover, if $\bar{a} = \left(z_i, j_i, j_i^{(1)}, j_i^{(2)}\right) \in E_j'(K^\times)$ and one of the coordinates of $\bar{a}$ is constant then all of them are. One can also notice that for non-constant x and y the relation E_j' is equivalent to

$$f(x,y) = 0 \wedge y_1 = \partial_x y \wedge y_2 = \partial_x^2 y.$$

Lemma 6.8.1. *In a differential field if $x_2 = gx_1$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C)$ then for any non-constant y we have*

$$\begin{aligned} \partial_{x_2} y &= \partial_{x_1} y \cdot (cx_1 + d)^2, \\ \partial_{x_2}^2 y &= \partial_{x_1}^2 y \cdot (cx_1 + d)^2 - 2c \cdot \partial_{x_1} y \cdot (cx_1 + d)^3. \end{aligned}$$

Proof. Easy calculations. □

Definition 6.8.2. The theory $(T_j^0)'$ consists of the following first-order statements about a structure K in the language $\mathfrak{L}_j := \{+, \cdot, E_j', 0, 1\}$.

A1 K is an algebraically closed field with an algebraically closed subfield $C := C_K$, which is defined by $E_j'(0, y, 0, 0)$. Further, $C^4 \subseteq E_j'(K)$ and if $\bar{a} = \left(z, j, j^{(1)}, j^{(2)}\right) \in E_j'(K^\times)$ and one of the coordinates of $\bar{a}$ is in C then $\bar{a} \subseteq C$.

A2 For any $z, j \in K \setminus C$ there is at most one pair $\left(j^{(1)}, j^{(2)}\right)$ in K with $E_j'\left(z, j, j^{(1)}, j^{(2)}\right)$.

A3 If $\left(z, j, j^{(1)}, j^{(2)}\right) \in E_j'$ then for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(C)$

$$\left(gz, j, j^{(1)} \cdot (cz + d)^2, j^{(2)} \cdot (cz + d)^2 - 2c \cdot j^{(1)} \cdot (cz + d)^3\right) \in E_j'.$$

Conversely, if for some j we have $\left(z_1, j, j^{(1)}, j^{(2)}\right), \left(z_2, j, w^{(1)}, w^{(2)}\right) \in E_j'$ then $z_2 = gz_1$ for some $g \in \mathrm{SL}_2(C)$.

A4 If $\left(z, j_1, j_1^{(1)}, j_1^{(2)}\right) \in E_j'$ and $\Phi(j_1, j_2) = 0$ for some modular polynomial $\Phi(X, Y)$ then $\left(z, j_2, j_2^{(1)}, j_2^{(2)}\right) \in E_j'$ where $j_2^{(1)}$, $j_2^{(2)}$ are determined from the following system of equations:

$$\begin{aligned} &\frac{\partial \Phi}{\partial X}(j_1, j_2) \cdot j_1^{(1)} + \frac{\partial \Phi}{\partial Y}(j_1, j_2) \cdot j_2^{(1)} = 0, \\ &\frac{\partial^2 \Phi}{\partial X^2}(j_1, j_2) \cdot \left(j_1^{(1)}\right)^2 + \frac{\partial^2 \Phi}{\partial Y^2}(j_1, j_2) \cdot \left(j_2^{(1)}\right)^2 + 2\frac{\partial^2 \Phi}{\partial X \partial Y}(j_1, j_2) \cdot j_1^{(1)} \cdot j_2^{(1)} + \\ &\frac{\partial \Phi}{\partial X}(j_1, j_2) \cdot j_1^{(2)} + \frac{\partial \Phi}{\partial Y}(j_1, j_2) \cdot j_2^{(2)} = 0. \end{aligned}$$

AS If $\left(z_i, j_i, j_i^{(1)}, j_i^{(2)}\right) \in E_j'$, $i = 1, \ldots, n$, with

$$\mathrm{td}_C C\left(\bar{z}, \bar{j}, \bar{j}^{(1)}, \bar{j}^{(2)}\right) \leq 3n$$

then $\Phi_N(j_i, j_k) = 0$ for some N and $1 \leq i < k \leq n$ or $j_i \in C$ for some i.

A4 is obtained by differentiating the equality $\Phi(j_1, j_2) = 0$. A compactness argument shows that AS can be written as a first-order axiom scheme exactly as before.

Definition 6.8.3. An E_j'*-field* is a model of $(T_j^0)'$. If K is an E_j'-field, then a tuple $\left(\bar{z}, \bar{j}, \bar{j}^{(1)}, \bar{j}^{(2)}\right) \in K^{4n}$ is called an E_j'*-point* if $(z_i, j_i) \in E_j'(K)$ for each $i = 1, \ldots, n$. By abuse of notation, we let $E_j'(K)$ denote the set of all E_j'-points in K^{4n} for any natural number n.

Let C be an algebraically closed field with $\operatorname{td}(C/\mathbb{Q}) = \aleph_0$ and let $\mathfrak{C}$ consist of all E_j'-fields K with $C_K = C$. Note that C is an E_j'-field with $E_j'(C) = C^4$ and it is the smallest structure in $\mathfrak{C}$. From now on, by an E_j'-field we understand a member of $\mathfrak{C}$.

Definition 6.8.4. For $A \subseteq B \in \mathfrak{C}_{f.g.}$ an E_j'*-basis* of B over A is an E_j'-point $\bar{b} = \left(\bar{z}, \bar{j}, \bar{j}^{(1)}, \bar{j}^{(2)}\right)$ from B of maximal length satisfying the following conditions:

- j_i and j_k are modularly independent for all $i \neq k$,
- $\left(z_i, j_i, j^{(1)}, j_i^{(2)}\right) \notin A^4$ for each i.

We let $\sigma(B/A)$ be the length of $\bar{j}$ in an E_j'-basis of B over A (equivalently, $4\sigma(B/A) = |\bar{b}|$). When $A = C$ we write $\sigma(B)$ for $\sigma(B/C)$. Further, for $A \in \mathfrak{C}_{f.g.}$ define the predimension by

$$\delta(A) := \operatorname{td}_C(A) - 3 \cdot \sigma(A).$$

As before, δ is submodular (so it is a predimension) and the Pila-Tsimerman inequality states exactly that $\delta(A) \geq 0$ for all $A \in \mathfrak{C}_{f.g.}$ with equality holding if and only if $A = C$. The dimension associated with δ will be denoted by d_j.

Definition 6.8.5. A structure $A \in \mathfrak{C}$ is said to be *full* if for every $j \in A$ there are $z, j^{(1)}, j^{(2)} \in A$ such that $A \models E_j'\left(z, j, j^{(1)}, j^{(2)}\right)$. The subclass $\hat{\mathfrak{C}}$ consists of all full E_j'-fields.

The obvious analogues of all results from Sections 6.3 and 6.4 hold in this setting as well (with obvious adaptations of the proofs). So we get a strong Fraïssé limit U.

Definition 6.8.6. Let n be a positive integer, $k \leq n$ and $1 \leq i_1 < \ldots < i_k \leq n$. Denote $i = (i_1, \ldots, i_k)$ and define the projection map $\operatorname{pr}_{\bar{i}} : K^n \to K^k$ by

$$\operatorname{pr}_{\bar{i}} : (x_1, \ldots, x_n) \mapsto (x_{i_1}, \ldots, x_{i_n}).$$

Further, define $\operatorname{pr}_{\bar{i}} : K^{4n} \to K^{4k}$ by

$$\operatorname{pr}_{\bar{i}} : (\bar{x}, \bar{y}, \bar{z}, \bar{w}) \mapsto (\operatorname{pr}_{\bar{i}} \bar{x}, \operatorname{pr}_{\bar{i}} \bar{y}, \operatorname{pr}_{\bar{i}} \bar{z}, \operatorname{pr}_{\bar{i}} \bar{w}).$$

Below $\operatorname{pr}_{\bar{i}}$ should always be understood in the second sense.

Definition 6.8.7. Let K be an algebraically closed field. An irreducible algebraic variety $V \subseteq K^{4n}$ is *normal* if and only if for any $1 \leq i_1 < \ldots < i_k \leq n$ we have $\dim \operatorname{pr}_{\bar{i}} V \geq 3k$. We say V is *strongly normal* if the strict inequality $\dim \operatorname{pr}_{\bar{i}} V > 3k$ holds.

Definition 6.8.8. An algebraic variety $V \subseteq K^{4n}$ (with coordinates $\left(\bar{x}, \bar{y}, \bar{y}^{(1)}, \bar{y}^{(2)}\right)$) is *free* if it is not contained in any variety defined by an equation $\Phi_N(y_i, y_k) = 0$ for some modular polynomial Φ_N and some indices i, k.

When K is an E_j'-field and $A \subseteq K$ is an E_j'-subfield, we say $V \subseteq K^{4n}$ is *free over* A if it is free and it is not contained in a hyperplane defined by an equation $y_i = a$ (for some i) where $a \in A$ with $A \models \exists z, u, v E_j'(z, a, u, v)$.

Consider the following statements for an E_j'-field K.

EC For each normal variety $V \subseteq K^{4n}$ the intersection $E_j'(K) \cap V(K)$ is non-empty.

SEC For each normal variety $V \subseteq K^{4n}$ defined over a finite tuple $\bar{a} \subseteq K$, the intersection $E_j'(K) \cap V(K)$ contains a point generic in V over $\bar{a}$.

GSEC For each irreducible variety $V \subseteq K^{4n}$ of dimension $3n$ defined over a finitely generated strong E_j'-subfield $A \leq K$, if V is normal and free over A and strongly normal over C, then the intersection $E_j'(K) \cap V(K)$ contains a point generic in V over A.

NT $K \supsetneq C$.

ID K has infinite d_j-dimension.

Again, the analogues of all facts established in Sections 6.5 and 6.6 are true with more or less the same proofs. Therefore U is a model of the theory T_j' axiomatised by A1-A4,AS,NT,EC.

Proposition 6.7.2 holds as well. However, the proof of Proposition 6.7.3 does not go through. The weak version of the modular Zilber-Pink conjecture that we used in that proof follows from uniform Ax-Schanuel without derivatives. As we saw, it helped us to deduce ID from the other axioms. Since the predimension is now defined as $\delta(A) = \mathrm{td}(A/C) - 3\sigma(A)$, the same weak Zilber-Pink does not work here. So one needs a weak Zilber-Pink "with derivatives", which would probably follow from the uniform version of Ax-Schanuel with derivatives. But the functional equations given by axioms A3 and A4 are quite complicated, since a modular relation on j's imposes a $\mathrm{SL}_2(C)$ relation on z's and then those relations impose some algebraic relations between $j^{(1)}$'s and $j^{(2)}$'s which depend on j's and z's. In other words, the functional equations "mix" all variables. This complicates things and it seems that even formulating a possible weak Zilber-Pink with derivatives is not an easy task. In fact, we can formulate a statement which would be enough to prove ID, but then it will not really be an analogue of Zilber-Pink and we do not have a proof for that statement (it seems Ax-Schanuel does not help). So we conclude that at the moment we cannot prove the completeness of T_j'.

It is in fact possible that T_j' is not complete. One can try to replace the $3n$ in the Ax-Schanuel theorem with something which reflects the "mixed" functional equations, which would in some cases be stronger than the current version of Ax-Schanuel. If this is possible, we would need to consider a slightly different predimension, and it might lead to a complete axiomatisation of the Fraïssé limit.

These are only some speculations and we are unable to say anything precise regarding this issue. So we finish this chapter here...

Chapter 7

Ax-Schanuel Type Theorems and Geometry of Strongly Minimal Sets in DCF_0

In this chapter we explore the connection between Ax-Schanuel type theorems (predimension inequalities) for a differential equation $E(x, y)$ and geometry of fibres of E. More precisely, given a predimension inequality (not necessarily adequate) for solutions of E of a certain type (which is of the form "$\mathrm{td} - \dim$" where dim is a dimension of trivial type) we show that the fibres of E are strongly minimal and geometrically trivial (after removing constant points). Moreover, the induced structure on the Cartesian powers of those fibres is given by *special* subvarieties.

In particular, since an Ax-Schanuel theorem (of the required form) for the (differential equation of the) j-function is known (due to Pila and Tsimerman, see Chapter 6), our results will give another proof for a theorem of Freitag and Scanlon [FS15] stating that the differential equation of j defines a strongly minimal set with trivial geometry (which is not $\aleph_0$-categorical though). In fact, the Pila-Tsimerman inequality is the main motivation for this chapter.

Thus we get a necessary condition for E to satisfy an Ax-Schanuel inequality of the given form. This is a step towards the solution of the main problem of this thesis. In particular it gives rise to an inverse problem: given a one-variable differential equation which is strongly minimal and geometrically trivial, can we say anything about the Ax-Schanuel properties of its two-variable analogue? See Section 7.4 for more details.

On the other hand, understanding the structure of strongly minimal sets in a given theory is one of the most important problems in model theory. In DCF_0 strongly minimal sets have a very nice classification, namely, they satisfy the Zilber trichotomy (Hrushovski-Sokolović [HuS93]). Hrushovski [Hru95] also gave a full characterisation of strongly minimal sets of order 1 proving that such a set is either non-orthogonal to the constants or it is trivial and $\aleph_0$-categorical. However there is no general classification of trivial strongly minimal sets of higher order and therefore we do not fully understand the nature of those sets. From this point of view the set J defined by the differential equation of j is quite intriguing since it is the first example of a trivial strongly minimal set in DCF_0 which is not $\aleph_0$-categorical. Before Freitag and Scanlon established those properties of J in [FS15], it was mainly believed that trivial strongly minimal sets in DCF_0 must be $\aleph_0$-categorical. The reason for this speculation was Hrushovski's aforementioned theorem on order 1 strongly minimal sets.

Thus, the classification of strongly minimal sets in DCF_0 can be seen as another source of motivation for the work in this chapter, where we show that these two problems (Ax-Schanuel type theorems and geometry of strongly minimal sets) are in fact closely related.

After defining the appropriate notions we formulate the main results of this chapter in Section 7.1 and prove them in Section 7.2. Then we apply our results to the differential equation of the j-function in Section 7.3. Section 7.4 is devoted to some concluding remarks and inverse questions. We have gathered definitions of several properties of strongly minimal sets that we need in Section 2.5.

The results of this chapter are combined in the preprint [Asl16b].

7.1 Setup and main results

Recall that $\mathcal{K} = (K; +, \cdot, ', 0, 1)$ is a differentially closed field with field of constants C. We may assume (without loss of generality) $\mathcal{K}$ is sufficiently saturated if necessary. Fix an element t with $t' = 1$. Let $E(x, y)$ be (the set of solutions of) a differential equation $f(x, y) = 0$ with rational (or, more generally, constant) coefficients.

Now we give several definitions and then state the main results of this chapter.

Definition 7.1.1. Let $\mathcal{P}$ be a non-empty collection of algebraic polynomials $P(X, Y) \in C[X, Y]$. We say two elements $a, b \in K$ are *$\mathcal{P}$-independent* if $P(a, b) \neq 0$ and $P(b, a) \neq 0$ for all $P \in \mathcal{P}$. The *$\mathcal{P}$-orbit* of an element $a \in K$ is the set $\{b \in K : P(a, b) = 0 \text{ or } P(b, a) = 0 \text{ for some } P \in \mathcal{P}\}$ (in analogy with Hecke orbit, see Section 6.1). Also, $\mathcal{P}$ is said to be *trivial* if it consists only of the polynomial $X - Y$.

Definition 7.1.2. For a non-constant $x \in K$ the *differentiation with respect to x* is the derivation $\partial_x : K \to K$ defined by $\partial_x : y \mapsto \frac{y'}{x'}$.

Recall that $f(x, y) = 0$ is the differential equation defining E and denote $m := \mathrm{ord}_Y f(X, Y)$ (the order of f with respect to Y).

Definition 7.1.3. We say the differential equation $E(x, y)$ has the $\mathcal{P}$-AS (Ax-Schanuel with respect to $\mathcal{P}$) property if the following condition is satisfied:
Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be non-constant elements of K with $f(x_i, y_i) = 0$. If the y_i's are pairwise $\mathcal{P}$-independent then

$$\mathrm{td}_C C\left(x_1, y_1, \partial_{x_1} y_1, \ldots, \partial_{x_1}^{m-1} y_1, \ldots, x_n, y_n, \partial_{x_n} y_n, \ldots, \partial_{x_n}^{m-1} y_n\right) \geq mn + 1. \quad (1.1)$$

The $\mathcal{P}$-AS property can be reformulated as follows: for any non-constant solutions (x_i, y_i) of E the transcendence degree in (1.1) is strictly bigger than m times the number of different $\mathcal{P}$-orbits of y_i's. Note that (1.1) is motivated by the known examples of Ax-Schanuel inequalities that we have seen in previous chapters.

Remark 7.1.4. Having the $\mathcal{P}$-AS property for a given equation E may force $\mathcal{P}$ to be "closed" in some sense. Firstly, $X-Y$ must be in $\mathcal{P}$. Secondly, if $P_1, P_2 \in \mathcal{P}$ then $P_1(x,y)=0,\ P_2(y,z)=0$ impose a relation on x and z given by $Q(x,y)=0$ for some polynomial Q. Then the $\mathcal{P}$-AS property may fail if one allows a relation $Q(y_i,y_j)=0$ between y_i and y_j (although one requires $P_l(y_i,y_j)\neq 0,\ l=1,2$). In that case one has to add Q to $\mathcal{P}$ in order to allow the possibility of an Ax-Shcanuel property with respect to $\mathcal{P}$.

Similar conditions on $\mathcal{P}$ are required in order for $\mathcal{P}$-independence to define a dimension function (number of distinct $\mathcal{P}$-orbits) of a pregeometry (of trivial type), which would imply that the $\mathcal{P}$-AS property is a predimension inequality. Note that the collection of modular polynomials has all those properties. However, the shape of $\mathcal{P}$ is not important for our results since we assume that a given equation E has the $\mathcal{P}$-AS property.

Definition 7.1.5. A $\mathcal{P}$-*special variety* (in K^n for some n) is an irreducible (over C) component of a Zariski closed set in K^n defined by a finite collection of equations of the form $P_{ik}(y_i,y_k)=0$ for some $P_{ik}\in\mathcal{P}$. For a differential subfield $L\subseteq K$ a $\mathcal{P}$-*special variety over* L is an irreducible (over L^{alg}) component of a Zariski closed set in K^n defined by a finite collection of equations of the form $P_{ik}(y_i,y_k)=0$ and $y_i=a$ for some $P_{ik}\in\mathcal{P}$ and $a\in L^{\mathrm{alg}}$.

For a definable set V, a $\mathcal{P}$-*special subvariety* (over L) of V is an intersection of V with a $\mathcal{P}$-special variety (over L).

Remark 7.1.6. If the polynomials from $\mathcal{P}$ have rational coefficients then $\mathcal{P}$-special varieties are defined over $\mathbb{Q}^{\mathrm{alg}}$. Furthermore, if E satisfies the $\mathcal{P}$-AS property then for the set $U:=\{y: f(t,y)=0\wedge y'\neq 0\}$ we have $U\cap C(t)^{\mathrm{alg}}=\emptyset$ and so $\mathcal{P}$-special subvarieties of U over $C(t)$ are merely $\mathcal{P}$-special subvarities.

Recall that for differential fields $L\subseteq K$ and a subset $A\subseteq K$ the differential subfield of K generated by L and A will be denoted by $L\langle A\rangle$. Let $C_0\subseteq C$ be the subfield of C generated by the coefficients of f and let $K_0=C_0\langle t\rangle=C_0(t)$ be the (differential) subfield of K generated by C_0 and t (clearly U is defined over K_0). We fix K_0 and work over it (in other words we expand our language with new constant symbols for elements of K_0).

Now we can formulate our main result (see Section 2.5 for definitions of geometric triviality and strict disintegratedness).

Theorem 7.1.7. *Assume $E(x,y)$ satisfies the $\mathcal{P}$-AS property for some $\mathcal{P}$. Assume further that the differential polynomial $g(Y):=f(t,Y)$ is absolutely irreducible. Then*

- *$U:=\{y: f(t,y)=0\wedge y'\neq 0\}$ is strongly minimal with trivial geometry.*
- *If, in addition, $\mathcal{P}$ is trivial then U is strictly disintegrated and hence it has $\aleph_0$-categorical induced structure.*
- *All definable subsets of U^n over a differential field $L\supseteq K_0$ are Boolean combinations of $\mathcal{P}$-special subvarieties over L.*

As the reader may guess and as we will see in the proof, this theorem holds under weaker assumptions on E. Namely, it is enough to require that (1.1) hold for $x_1 = \ldots = x_n = t$ (which can be thought of as a weak form of the "Ax-Lindemann-Weierstrass" property). However, we prefer the given formulation of Theorem 7.1.7 since the main object of our interest is the Ax-Schanuel inequality (for E).

Further, we deduce from Theorem 7.1.7 that if E has some special form, then all the fibres $E(s, y)$ for a non-constant $s \in K$ have the above properties (over $C_0\langle s\rangle$).

Corollary 7.1.8. *Let $E(x, y)$ be defined by $P(x, y, \partial_x y, \ldots, \partial_x^m y) = 0$ where $P(X, \bar{Y})$ is an irreducible algebraic polynomials over C. Assume $E(x, y)$ satisfies the $\mathcal{P}$-AS property for some $\mathcal{P}$ and let $s \in K$ be a non-constant element. Then*

- *$U_s := \{y : f(s, y) = 0 \wedge y' \neq 0\}$ is strongly minimal with trivial geometry.*
- *If in addition $\mathcal{P}$ is trivial then any distinct non-algebraic (over $C_0\langle s\rangle$) elements are independent and U_s is $\aleph_0$-categorical.*
- *All definable subsets of U_s^n over a differential field $L \supseteq C_0\langle s\rangle$ are Boolean combinations of $\mathcal{P}$-special subvarieties over L.*

Remark 7.1.9. Since $U_s \cap C = \emptyset$, in Theorem 7.1.7 and Corollary 7.1.8 the induced structure on U_s^n is actually given by *strongly special* subvarieties (over L), which means that we do not allow any equation of the form $y_i = c$ for c a constant. In particular we also need to exclude equations of the form $P(y_i, y_i) = 0$ for $P \in \mathcal{P}$.

We also prove a generalisation of Theorem 7.1.7.

Theorem 7.1.10. *Assume $E(x, y)$ satisfies the $\mathcal{P}$-AS property and let $p(Y) \in C(t)[Y] \setminus C$, $q(Y) \in C[Y] \setminus C$ be such that the differential polynomial $f(p(Y), q(Y))$ is absolutely irreducible. Then the set*

$$U_{p,q} := \{y : E(p(y), q(y)) \wedge y \notin C\}$$

is strongly minimal and geometrically trivial.

As an application of Theorem 7.1.7 we obtain a result on the differential equation of the j-function which was established by Freitag and Scanlon in [FS15]. To be more precise, let $F(j, j', j'', j''') = 0$ be the algebraic differential equation satisfied by the modular j-function (see Section 6.1).

Theorem 7.1.11 ([FS15])**.** *The set $J \subseteq K$ defined by $F(y, y', y'', y''') = 0$ is strongly minimal with trivial geometry. Furthermore, J is not $\aleph_0$-categorical.*

Strong minimality and geometric triviality of J follow directly from Theorem 7.1.7 combined with the Ax-Schanuel theorem for j (see Section 6.1). Of course the "furthermore" clause does not follow from Theorem 7.1.7 but it is not difficult to prove. Theorem 7.1.7 also gives a characterisation of the induced structure on the Cartesian powers of J. Again, that result can be found in [FS15] in a more general form.

The proof of Theorem 7.1.11 by Freitag and Scanlon is based on Pila's modular Ax-Lindemann-Weierstrass with derivatives theorem along with Seidenberg's embedding theorem and Nishioka's theorem on differential equations satisfied by automorphic functions ([Nis89]). They also make use of some tools of stability theory such as the "Shelah reflection principle". However, as one may guess, we cannot use Nishioka's theorem (or some analogue of that) in the proof of 7.1.7 since we do not know anything about the analytic properties of the solutions of our differential equation. Thus, we show in particular that Theorem 7.1.11 can be deduced from Pila's result abstractly. The key point that makes this possible is stable embedding, which means that if $\mathcal{M}$ is a model of a stable theory and $X \subseteq M$ is a definable set over some $A \subseteq M$ then every definable subset of X^n can in fact be defined with parameters from $X \cup A$ (see Section 2.5).

Let us recall once more that the set J is notable for being the first example of a strongly minimal set (definable in DCF_0) with trivial geometry that is not $\aleph_0$-categorical. Indeed the aforementioned result of Hrushovski on strongly minimal sets of order 1 led people to believe that all geometrically trivial strongly minimal sets must be $\aleph_0$-categorical. Nevertheless, it is not true as the set J illustrates.

7.2 Proofs of the main results

Proof of Theorem 7.1.7

Taking $x_1 = \ldots = x_n = t$ in the $\mathcal{P}$-AS property we get the following weak version of Ax-Lindemann-Weierstrass[1] property for U which in fact is enough to prove Theorem 7.1.7.

Lemma 7.2.1. *$\mathcal{P}$-AS implies that for any pairwise $\mathcal{P}$-independent elements $u_1, \ldots, u_n \in U$ the elements $\bar{u}, \bar{u}', \ldots, \bar{u}^{(m-1)}$ are algebraically independent over $C(t)$ and hence over K_0.*

We show that every definable (possibly with parameters) subset V of U is either finite or co-finite. Since U is defined over K_0, by stable embedding there is a finite subset $A = \{a_1, \ldots, a_n\} \subseteq U$ such that V is defined over $K_0 \cup A$. It suffices to show that U realises a unique non-algebraic type over $K_0 \cup A$, i.e. for any $u_1, u_2 \in U \setminus \mathrm{acl}(K_0 \cup A)$ we have $\mathrm{tp}(u_1/K_0 \cup A) = \mathrm{tp}(u_2/K_0 \cup A)$. Let $u \in U \setminus \mathrm{acl}(K_0 \cup A)$. We know that $\mathrm{acl}(K_0 \cup A) = (K_0\langle A \rangle)^{\mathrm{alg}} = (K_0(\bar{a}, \bar{a}', \ldots, \bar{a}^{(m-1)}))^{\mathrm{alg}}$. Since $u \notin (K_0\langle A \rangle)^{\mathrm{alg}}$, u is transcendental over $K_0(A)$ and hence it is $\mathcal{P}$-independent from each a_i. We may assume without loss of generality that a_i's are pairwise $\mathcal{P}$-independent (otherwise we could replace A by a maximal pairwise $\mathcal{P}$-independent subset). Applying Lemma 7.2.1 to $\bar{a}, u$, we deduce that $u, u', \ldots, u^{(m-1)}$ are algebraically independent over $K_0\langle A \rangle$. Hence $\mathrm{tp}(u/K_0 \cup A)$ is determined uniquely (axiomatised) by the set of formulae

$$\{g(y) = 0\} \cup \{h(y) \neq 0 : h(Y) \in K_0\langle A \rangle\{Y\},\ \mathrm{ord}(h) < m\}$$

[1] To be precise, ALW is $\mathcal{P}$-AS under an additional assumption $\mathrm{td}(\bar{x}/C) = 1$.

(Recall that g is absolutely irreducible and hence it is irreducible over any field). In other words $g(Y)$ is the minimal differential polynomial of u over $K_0\langle A\rangle$.

Thus U is strongly minimal. A similar argument shows also that if $A \subseteq U$ is a (finite) subset and $u \in U \cap \operatorname{acl}(K_0A)$ then there is $a \in A$ such that $u \in \operatorname{acl}(K_0a)$. This proves that U is geometrically trivial.

If $\mathcal{P}$ is trivial then distinct elements of U are independent, hence U is strictly disintegrated.

The last part of Theorem 7.1.7 follows from the following lemma.

Lemma 7.2.2. *Every irreducible (relatively) Kolchin closed (over $C(t)$) subset of U^n is a $\mathcal{P}$-special subvariety of U^n.*

Proof. Let $V \subseteq U^n$ be an irreducible relatively closed subset (i.e. it is the intersection of U^n with an irreducible Kolchin closed set in K^n). Pick a generic point $\bar{v} = (v_1, \ldots, v_n) \in V$ and let $W \subseteq K^n$ be the Zariski closure of $\bar{v}$ over C. Let $d := \dim W$ and assume $v_1, \ldots, v_d$ are algebraically independent over C. Then $v_i \in (C(v_1, \ldots, v_d))^{\mathrm{alg}}$ for each $i = d+1, \ldots, n$. By Lemma 7.2.1 each v_i with $i > d$ must be in a $\mathcal{P}$-relation with some v_{k_i} with $k_i \leq d$. Let $P_i(v_i, v_{k_i}) = 0$ for $i > d$. The algebraic variety defined by the equations $P_i(y_i, y_{k_i}) = 0,\ i = d+1, \ldots, n$, has dimension d and contains W. Therefore W is a component of that variety and so it is a $\mathcal{P}$-special variety.

We claim that $W \cap U^n = V$. Since $v_1, \ldots, v_d \in U$ are algebraically independent over C, by Lemma 7.2.1 $\bar{v}, \bar{v}', \ldots, \bar{v}^{(m-1)}$ are algebraically independent over $C(t)$. Moreover, the (differential) type of each $v_i,\ i > d$, over $v_1, \ldots, v_d$ is determined uniquely by an irreducible algebraic equation. Therefore $\operatorname{tp}(\bar{v}/C(t))$ is axiomatised by formulas stating that $\bar{v}$ is Zariski generic in W and belongs to U^n. In other words $\bar{v}$ is Kolchin generic in $W \cap U^n$. Now V and $W \cap U^n$ are both equal to the Kolchin closure of $\bar{v}$ inside U^n and hence they are equal. $\square$

Thus definable (over $C(t)$) subsets of U^n are Boolean combinations of special subvarieties. Now let $L \subseteq K$ be an arbitrary differential subfield over which U is defined. Then definable subsets of U^n over L can be defined with parameters from $\tilde{L} = K_0 \cup (U \cap L^{\mathrm{alg}})$ (see Section 2.5). Then Lemma 7.2.2 implies that irreducible Kolchin closed subsets of U^n defined over $\tilde{L}$ are $\mathcal{P}$-special subvarities of U^n over L.

Finally, note that since U does not contain any algebraic elements over $C(t)$, the type of any element $u \in U$ over $C(t)$ is isolated by the formula $f(t, y) = 0 \wedge y' \neq 0$.

Proof of Theorem 7.1.10

We argue as above and show that for a finite set $A = \{a_1, \ldots, a_n\} \subseteq U_{p,q}$ there is a unique non-algebraic type over $K_0\langle A\rangle$ realised in $U_{p,q}$. Here we will use full Ax-Lindemann-Weierstrass.

If $u \in U_{p,q} \setminus (K_0\langle A\rangle)^{\mathrm{alg}}$ then $q(u)$ is transcendental over $K_0(A)$ and so $q(u)$ is $\mathcal{P}$-independent from each $q(a_i)$. Moreover, we can assume $\{q(a_1), \ldots, q(a_n)\}$ is $\mathcal{P}$-independent. Then by the $\mathcal{P}$-AS property

$$\operatorname{td}_C C\left(p(u), q(u), \ldots, \partial_{p(u)}^{m-1} q(u), p(a_i), q(a_i), \ldots, \partial_{p(a_i)}^{m-1} q(a_i)\right)_{i=1,\ldots,n} \geq m(n+1)+1.$$

But then

$$\operatorname{td}_C C\left(t, u, u', \ldots, u^{(m-1)}, a_i, a_i', \ldots, a_i^{(m-1)}\right)_{i=1,\ldots,n} \geq m(n+1)+1,$$

and hence $u, u', \ldots, u^{(m-1)}$ are algebraically independent over $K_0\langle A\rangle$. This determines the type $\operatorname{tp}(u/K_0A)$ uniquely as required. It also shows triviality of the geometry.

Proof of Corollary 7.1.8

Consider the differentially closed field $\mathcal{K}_s = (K; +, \cdot, \partial_s, 0, 1)$. The given form of differential equation E implies that U_s is defined over $C_0(s)$ in $\mathcal{K}_s$. However, in general it may not be defined over $C_0(s)$ in $\mathcal{K}$, it is defined over $C_0\langle s\rangle = C_0(s, s', s'', \ldots)$. Since $\partial_s s = 1$, we know by Theorem 7.1.7 that U_s is strongly minimal in $\mathcal{K}_s$. On the other hand the derivations ∂_s and $'$ are inter-definable (with parameters) and so a set is definable in $\mathcal{K}$ if and only if it is definable in $\mathcal{K}_s$ (possibly with different parameters). This implies that every definable subset of U_s in $\mathcal{K}$ is either finite or co-finite, hence it is strongly minimal.

Further, Theorem 7.1.7 implies that U_s is geometrically trivial over $C_0(s)$ in $\mathcal{K}_s$. By Theorem 2.5.2, U_s is also geometrically trivial over $C_0\langle s\rangle$ in $\mathcal{K}_s$. On the other hand for any subset $A \subseteq U_s$ the algebraic closure of $C_0\langle s\rangle \cup A$ is the same in $\mathcal{K}$ and $\mathcal{K}_s$. This implies geometric triviality of U_s in $\mathcal{K}_s$.

The same argument (along with the remark after Theorem 2.5.2) shows that the second and the third parts of Corollary 7.1.8 hold as well.

7.3 The modular j-function

We recall some basic properties of the j-function presented in the previous chapter.

The j-invariant satisfies the following order three algebraic differential equation:

$$F(y, y', y'', y''') = Sy + R(y)(y')^2 = 0, \tag{3.2}$$

where S denotes the Schwarzian derivative defined by $Sy = \frac{y'''}{y'} - \frac{3}{2}\left(\frac{y''}{y'}\right)^2$ and $R(y) = \frac{y^2-1968y+2654208}{2y^2(y-1728)^2}$. Let J be the set defined by (3.2). Note that F is not a polynomial but a rational function. In particular constant elements do not satisfy (3.2), for Sy is not defined for a constant y. We can multiply our equation through by a common denominator and make it into a polynomial equation

$$F^*(y, y', y'', y''') = q(y)y'y''' - \frac{3}{2}q(y)(y'')^2 + p(y)(y')^4 = 0, \tag{3.3}$$

where p and q are respectively the numerator and the denominator of R. Let J^* be the set defined by (3.3). It is not strongly minimal since C is a definable subset. However $J = J^* \setminus C$ is strongly minimal and $\operatorname{MR}(J^*) = 1$, $\operatorname{MD}(J^*) = 2$. Thus whenever we speak of the formula $F(y, y', y'', y''') = 0$ (which, strictly speaking, is not a formula in the language of differential rings) we mean the formula $F^*(y, y', y'', y''') = 0 \wedge y' \neq 0$.

Let $\mathcal{P} = \Phi := \{\Phi_N(X,Y) : N > 0\}$ be the collection of modular polynomials (see Section 6.1). Two elements are modularly independent iff they are Φ-independent. For an element $a \in K$ its Hecke orbit is the same as its Φ-orbit.

Let us form the two-variable analogue of the equation (3.2):

$$f(x,y) := F(y, \partial_x y, \partial_x^2 y, \partial_x^3 y) = 0. \tag{3.4}$$

Now we reformulate the Ax-Schanuel theorem for j (see Section 6.2) in the terminology of this chapter.

Theorem 7.3.1. *The equation* (3.4) *has the Φ-AS property.*

As a consequence of Theorems 7.1.7 and 7.3.1 we get strong minimality and geometric triviality of J (note that $F^*(Y_0, Y_1, Y_2, Y_3)$ is obviously absolutely irreducible as it depends linearly on Y_3).

Lemma 7.2.1 for j is of course a special case of the Ax-Schanuel theorem for j. Nevertheless it can also be deduced from Pila's modular Ax-Lindemann-Weierstrass with derivatives theorem ([Pil13]) by employing Seidenberg's embedding theorem. Therefore only Pila's theorem is enough to prove strong minimality and geometric triviality of J. Moreover, Corollary 7.1.8 shows that all the non-constant fibres of (3.4) are strongly minimal and geometrically trivial (after removing constant points) and the induced structure on the Cartesian powers of those fibres is given by special subvarieties. Note that it is proven in [FS15] that the sets $F(y, y', y'', y''') = a$ have the same properties for any a.

Remark 7.3.2. To complete the proof of Theorem 7.1.11, that is, to show that J is not $\aleph_0$-categorical, one argues as follows (see [FS15]). The Hecke orbit of an element $j \in J$ is contained in J. Therefore J realises infinitely many algebraic types over j and thus is not $\aleph_0$-categorical.

7.4 Concluding remarks

The $\mathcal{P}$-AS property states positivity of a predimension of the form "td $-m \cdot d$", where d is the number of distinct $\mathcal{P}$-orbits.[2] Transcendence degree, being the algebraic dimension, is non-locally modular. On the other hand d is a dimension of trivial type. And it is this fact that is responsible for triviality of the geometry of U.

For the exponential differential equation the predimension is of the form "td $-$ ldim". The linear dimension is locally modular non-trivial and accordingly the strongly minimal set $y' = y$ is not trivial, indeed it is non-orthogonal to C. Here strong minimality is obvious as the equation has order one. However one may ask whether strong minimality can be deduced from this type of predimension inequalities in general. The answer is no. For example, consider a linear differential equation with constant coefficients $\partial_x^2 y - y = 0$. We showed in Chapter 5 that an Ax-Schanuel statement holds for it. However U is the set defined by $y'' - y = 0$ and the set $y' = y$ is a definable infinite

[2]Strictly speaking, we do not know this but can assume it is the case. See Remark 7.1.4.

and co-infinite subset (the differential polynomial $Y'' - Y$ is absolutely irreducible though).

An interesting question is whether there are differential equations with the $\mathcal{P}$-AS property with trivial $\mathcal{P}$. As we showed here, if $E(x, y)$ has such a property then the corresponding U must be strongly minimal and strictly disintegrated. There are quite a few examples of this kind of strongly minimal sets in DCF_0. The two-variable versions of those equations will be natural candidates of equations with the required $\mathcal{P}$-AS property.

For example, the geometry of sets of the form $y' = f(y)$, where f is a rational function over C, is well understood. The nature of the geometry is determined by the partial fraction decomposition of $1/f$. As an example consider the equation

$$y' = \frac{y}{1+y}. \tag{4.5}$$

One can show that it defines a strictly disintegrated strongly minimal set ([Mar05b]). The two variable analogue of this equation is

$$\partial_x y = \frac{y}{1+y}. \tag{4.6}$$

But this is equivalent to the equation $\frac{y'}{y} = (x - y)'$. Denoting $z = x - y$ we get the exponential differential equation $y' = yz'$. It is easy to deduce from this that (4.6) does not satisfy the $\mathcal{P}$-AS property with any $\mathcal{P}$ (it satisfies a version of the original exponential Ax-Schanuel inequality though and therefore has a predimension inequality which is of the form $\mathrm{td} - \mathrm{ldim}$). Indeed, the fibre of (4.6) above $x = t$ is of trivial type but the section by $x = t + y$ is non-orthogonal to C. So according to Theorem 7.1.10 the equation (4.6) does not satisfy any $\mathcal{P}$-AS property. Of course, all the sets $y' = f(y)$ can be treated in the same manner and hence they are not appropriate for our purpose. Thus, one needs to look at the behaviour of all the sets $E(p(y), q(y))$, and if they happen to be trivial strongly minimal sets then one can hope for a $\mathcal{P}$-AS inequality.

The classical Painlevé equations define strongly minimal and strictly disintegrated sets as well. For example, let us consider the first Painlevé equation $y'' = 6y^2 + t$. Strong minimality and algebraic independence of solutions of this equation were shown by Nishioka in [Nis04]. We consider its two-variable version

$$\partial_x^2 y = 6y^2 + x. \tag{4.7}$$

The goal is to find an Ax-Schanuel inequality for this equation. Note that (4.7) does not satisfy the $\mathcal{P}$-AS property with trivial $\mathcal{P}$. Indeed, if ζ is a fifth root of unity then the transformation $x \mapsto \zeta^2 x,\ y \mapsto \zeta y$ sends a solution of (4.7) to another solution. If one believes these are the only relations between solutions of the above equation, then one can conjecture the following.

Conjecture 7.4.1 (Ax-Schanuel for the first Painlevé equation)**.** *If* $(x_i, y_i),\ i = 1, \ldots, n$, *are solutions to the equation* (4.7) *and* $(y_i/y_j)^5 \neq 1$ *for* $i \neq j$ *then*

$$\mathrm{td}(\bar{x}, \bar{y}, \partial_{\bar{x}} \bar{y}) \geq 2n + 1.$$

One could in fact replace y's with x's in the condition $(y_i/y_j)^5 \neq 1$ as those are equivalent. Though we do not have a proof of this conjecture at the moment, it seems a weaker version, namely, the Ax-Lindemann-Weiertsrass part of Ax-Schanuel, can be proven by elaborating Nishioka's method. It asserts that the above conjecture is true under the additional assumption $\operatorname{td}(\bar{x}) = 1$.

Nagloo and Pillay showed in [NP14] that the other generic Painlevé equations define strictly disintegrated strongly minimal sets as well. So we can analyse relations between solutions of their two-variable analogues and ask similar questions for them too.

In general, proving that a certain equation has the required transcendence properties may be much more difficult than proving that there are equations with those properties. In this regard we believe that "generic" (in a suitable sense) equations must satisfy the $\mathcal{P}$-AS property with trivial $\mathcal{P}$. However we do not go into details here and finish with a final remark.

Zilber constructed "a theory of a generic function" where the function has transcendence properties analogous to trivial AS property described here ([Zil05]). He also conjectured that it has an analytic model, i.e. there is an analytic function that satisfies Zilber's axioms and, in particular, the given transcendence properties. Wilkie constructed *Liouville functions* in [Wil05] and showed that they indeed satisfy the transcendence properties formulated by Zilber. Later Koiran [Koi03] proved that Liouville functions satisfy Zilber's existential closedness axiom scheme. However, those functions do not satisfy any algebraic differential equation, so we cannot translate the result into differential algebraic language. If there is a differentially algebraic function with similar properties then it may give rise to a differential equation with the desired properties.

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