Combinatorics of sets

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1 Definitions

- An r-set is a set of size r.
- The set $\{1, 2, \ldots, n\}$ is often denoted $[n]$.
- Given a set S, its power set 2^S is the collection of all subsets of S.
- A hypergraph is a set of vertices V, together with a collection $E \subset 2^V$ of subsets of V. The members of E are called hyperedges, or just edges for short.
- An r-uniform hypergraph is a hypergraph in which all edges have are r-subsets of V. Note that a 2-uniform hypergraph is just a graph.
- If $U \subset V$, the sub-hypergraph $H[U]$ of H induced by U is the hypergraph with vertex set U, together with all edges of the original H which were completely contained within U.
- An independent set in a hypergraph is a collection of vertices which induces no edges.

2 Problems and famous results

1. Let $\mathcal{F}$ be a collection of subsets $A_1, A_2, \ldots$ of $\{1, \ldots, n\}$, such that for each $i \neq j$, $A_i \cap A_j \neq \emptyset$. Prove that $\mathcal{F}$ has size at most 2^{n-1}.

   Solution: For each set $S \in 2^{[n]}$, observe that at most one of S and $\overline{S}$ is contained in $\mathcal{F}$.

2. Suppose that $\mathcal{F}$ above has size exactly 2^{n-1}. Must there be a common element $x \in \{1, \ldots, n\}$ which is contained by every A_i?

   Solution: No. First, observe that in [3], one can create an intersecting family with the four sets $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, and $\{1, 2, 3\}$. Then for every $n \geq 3$, one can blow up this construction by taking all sets S which are obtained by taking the union of one of these 4 sets together with an arbitrary subset of $\{4, \ldots, n\}$.

3. (Sauer-Shelah.) A family $\mathcal{F}$ shatters a set A if for every $B \subset A$, there is $F \in \mathcal{F}$ such that $F \cap A = B$. Prove that if $\mathcal{F} \subset 2^{[n]}$ and $|\mathcal{F}| > \binom{n}{0} + \cdots + \binom{n}{k}$, then there is a set $A \subset [n]$ of size $k + 1$ such that $\mathcal{F}$ shatters A.

   Solution: Compressions: show that if a set family has no shattered subset of size t, then after compressing, it still has no shattered subset of size t. To visualize this, a shattered subset S of size t is specified by a collection of t elementary basis vectors in the hypercube, and the plane P determined by the 2^t vectors in their span. Now, consider the orthogonal complement of P, and imagine partitioning the entire hypercube into $|P|$ fibers, each of which is running orthogonally into P. The shattering means that each of the fibers contains some point in the set family. Now a compression is a wind which blows in one dimension. If the wind blows orthogonally to P, and S is shattered, then it is clear that

S was also shattered before. Also, if the wind blows parallel to P, and S is now shattered, then it is also clear that S was shattered before. Therefore, we may assume that all compressions have been completed, and now if a set is in the family, all of its subsets are as well. This forces there to be a set of size $k+1$.

4. (Erdős-Ko-Rado.) Let $\mathcal{F}$ be a collection of distinct r-subsets of $[n]$, with the property that every pair of subsets $A, B \in \mathcal{F}$ intersects. Prove that $|\mathcal{F}| \le \binom{n-1}{r-1}$ whenever $n \ge 2r$.

5. (Putnam 1956/A7.) Call an integer $0 \le r \le 1000$ *strange* if the number of r-subsets of $\{1, \ldots, 1000\}$ is odd. Prove that the number of strange integers is a power of 2. (To make this an AIME problem: how many strange integers are there?)

   Solution: Lucas's theorem: it is going to be 2 raised to the power which is equal to the number of 1's in the binary expansion of 1000.

6. (From Peter Winkler.) The 53 MOPpers were divided into 7 teams for Team Contest 1. They were then divided into 6 teams for Team Contest 2. Prove that there must be a MOPper for whom the size of her team in Contest 2 was strictly larger than the size of her team in Contest 1.

   Solution: In Contest 1, suppose the team breakdown was $s_1 + \cdots + s_7 = 60$. Then in the i-th team, with s_i people, say that each person did $\frac{1}{s_i}$ of the work. Similarly, in Contest 2, account equally for the work within each team, giving scores of $\frac{1}{s_i'}$.

   However, the total amount of work done by all people in Contest 1 was then exactly 7, and the total amount of work done by all people in Contest 2 was exactly 6. So somebody must have done strictly less work in Contest 2. That person saw
$$\frac{1}{s_i'} < \frac{1}{s_i},$$
   i.e., the size of that person's team on Contest 2 was strictly larger than her team size on Contest 1.

7. Let $\mathcal{F}$ be a collection of r-subsets of $[n]$, and let $t = |\mathcal{F}|/n$. Then there is always a set $S \subset [n]$ of size at least $n/(4t^{\frac{1}{r-1}})$, which does not completely contain any member of $\mathcal{F}$.

8. (Turán.) Let G be an n-vertex graph with average degree d. Then it contains an independent set of size at least $\frac{n}{d+1}$, and this is tight.

9. (Oddtown.) Let $\mathcal{F}$ be a collection of distinct subsets of $2^{[n]}$ such that every $A \in \mathcal{F}$ has size which is nonzero modulo 2, but every pair of distinct $A, B \in \mathcal{F}$ has intersection size which is zero modulo 2. Prove that $|\mathcal{F}| \le n$.

10. (Open.) What if 2 is replaced by 6?

11. (Folklore.) The chromatic number χ of a hypergraph H is the minimum integer k such that it is possible to assign a color from $[k]$ to each vertex of H, with no edge having all of its vertices in the same color. Prove that if $\chi > 2$, then H must have two edges which intersect in exactly one vertex.

12. (L.) An r-tree is an r-uniform hypergraph created in the following way: starting with a single hyperedge of size r, repeatedly add new hyperedges by selecting one existing vertex v, and adding $r-1$ new vertices, together with a new hyperedge through v and the $r-1$ new vertices. Let T be an arbitrary r-tree with t edges. Observe that T will always have exactly $1+(r-1)t$ vertices. Prove that every r-uniform hypergraph H with chromatic number $\chi > t$ must contain T as a sub-hypergraph.

13. (TST 2005/1.) Let n be an integer greater than 1. For a positive integer m, let $X_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a family $\mathcal{F}$ of $2n$ subsets of X_m such that:

    (a) each member of $\mathcal{F}$ is an m-element subset of X_m;

(b) each pair of members of $\mathcal{F}$ shares at most one common element;

(c) each element of X_m is contained in exactly 2 elements of $\mathcal{F}$.

Determine the maximum possible value of m in terms of n.

Solution: Count in two ways:

$$\sum_v \binom{d_v}{2} = \#\text{ set pairs intersect} \le \binom{2n}{2}.$$

But all $d_v = 2$, so we have $mn \le n(2n-1)$, i.e., $m \le 2n-1$. Equality is possible: take $2n$ lines in general position in $\mathbb{R}^2$, and let their $\binom{2n}{2} = mn$ intersection points be the points.

14. (USAMO 2011/6.) Let X be a set with $|X| = 225$. Suppose further that there are eleven subsets $A_1, \dots, A_{11}$ of X such that $|A_i| = 45$ for $1 \le i \le 11$ and $|A_i \cap A_j| = 9$ for $1 \le i < j \le 11$. Prove that $|A_1 \cup \dots \cup A_{11}| \ge 165$, and give an example for which equality holds.

Solution: The $|X| = 225$ condition is unnecessary. Count in two ways:

$$\sum_{v \in X} \binom{d_v}{2} = \binom{11}{2} \cdot 9 = 495.$$

But also $\sum_{v \in X} d_v = 11 \cdot 45 = 495$. This suggests that equality occurs when all $d_v = \binom{d_v}{2}$, which is precisely at $d_v = 3$.

Indeed, by Cauchy-Schwarz, if we let Let $n = |X|$, we find

$$\left(\sum_{v \in X} 1 d_v\right)^2 \le \left(\sum_{v \in X} 1\right)\left(\sum_{v \in X} d_v^2\right)$$
$$495^2 \le n \cdot 3 \cdot 495,$$

where we deduced $\sum d_v^2 = 3 \cdot 495$ from $\sum \binom{d_v}{2}$ and $\sum d_v$. We conclude that $n \ge 165$.

For the construction, take 11 planes in general position in $\mathbb{R}^3$, and let their $\binom{11}{3} = 165$ points of intersection be the points in the set.