Pidgeonhole principal

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1 Warm-up

1. (Gelca-Andreescu 44.) Inside a circle of radius 4 are chosen 61 points. Show that among them there are two at distance at most $\sqrt{2}$ from each other.

   Solution: Put down unit graph paper. The unit disk hits $15 \times 4 = 60$ unit squares.

2. (Gelca-Andreescu 46, Moscow Math Olympiad.) Show that any convex polyhedron has two faces with the same number of edges.

   Solution: Consider the dual graph, where faces are vertices and adjacent faces give edges. Every graph has two vertices of equal degree.

2 Problems and famous results

1. (Erdős-Szekeres upper bound.) The *Ramsey Number* $R(s,t)$ is the minimum integer n for which **every** red-blue coloring of the edges of K_n contains a completely red K_s or a completely blue K_t. Prove that

$$R(s,t) \le \binom{s+t-2}{s-1}$$

   Solution: Observe that $R(s,t) \le R(s-1,t) + R(s,t-1)$, because if we have that many vertices, then if we select one vertex, then it cannot simultaneously have $< R(s-1,t)$ red neighbors and $< R(s,t-1)$ blue neighbors, so we can inductively build either a red K_s or a blue K_t. But

$$\binom{(s-1)+t-2}{(s-2)} + \binom{s+(t-1)-2}{s-1} = \binom{s+t-2}{s-1},$$

   because in Pascal's Triangle the sum of two adjacent guys in a row equals the guy directly below them in the next row.

2. (USAMO 1990/1.) A license plate has six digits from 0 to 9 and may have leading zeros. If two plates must always differ in at least two places, what is the largest number of plates that is possible?

   Solution: Answer: 10^5. Take the plates whose sum of digits modulo 10 is, say, 0. then changing one digit forces you to also change another digit too. Now suppose we have more than 10^5. Then look on "holes" which are the last 5 digits. There are 10^5 holes. "Pigeons" are the license plates. So some hole gets at least 2 pigeons.

3. (Sweden 2010.) In a mathematical competition, the number of competitors is greater than k times the number of problems. If all competitors solved at least one problem, prove that there exists one among them such that any problem he/she solved was solved by at least k more competitors.

   Solution: Assume false. Then for every competitor, one can designate a low-degree problem that it is adjacent to: one which has degree $\le k$ in the incidence graph. Suppose there are n problems.

Then there are $> kn$ competitors. So we have $> kn$ vertices each linked with a low-degree problem. The total number of low-degree problems is $\leq n$, but the total number of edges they can absorb by definition of low-degree is $\leq kn$, contradiction.

4. (Ireland 2010/12.) The numbers $1, 2, \ldots, 4n^2$ are written in the unit squares of a $2n \times 2n$ array, $n \geq 3$. Prove that there exist $n+1$ columns in the array such that in each of them any number is less than the sum of the remaining $2n-1$ numbers in that column.

   Solution: Suppose for contradiction that the first n columns all have that the largest number is at least the sum of all other numbers in that column. Let B be the sum by taking the largest number in each of those columns, and let A be the sum by taking all but the largest number in each of those columns. Then we have $B \geq A$. However, the smallest A can be is if it is the first $n(2n-1)$ numbers, and the largest B can be is if it is the last n numbers. Hence

$$\begin{aligned} 1 + \cdots + n(2n-1) &\leq (4n^2 - n + 1) + \cdots + 4n^2 \\ n(2n-1)(2n^2 - n + 1) &\leq n \cdot (8n^2 - n + 1) \\ (2n-1)(2n^2 - n + 1) &\leq (8n^2 - n + 1) \\ 4n^3 - 4n^2 + 3n - 1 &\leq 8n^2 - n + 1 \\ 4n^3 - 12n^2 + 4n - 2 &\leq 0 \\ 2n^3 - 6n^2 + 2n - 1 &\leq 0 \\ 2n^3 &< 6n^2 \\ n &< 3\,, \end{aligned}$$

   contradiction.

5. (Sweden 2010.) Let $x_1, \ldots, x_m$ be integers such that $1 \leq x_1 < x_2 < \cdots < x_m \leq n$, and assume that $m > \frac{n+1}{2}$. Prove that there exist indices i, j, k such that $1 \leq i \leq j < k \leq m$, for which $x_i + x_j = x_k$. What if we only assume that $m \geq \frac{n+1}{2}$?

   Solution: If $n = 2k+1$ is odd, and we only take the integers $\{k+1, k+2, \ldots, 2k+1\}$, then this is clearly sum-free, but there are $k + 1 = \frac{n+1}{2}$ integers. So the result is tight.

   To prove it, let $X = \{x_1, \ldots, x_m\}$. Let $X - X$ denote the set $\{a - b : a, b \in X\}$. The key observation is that $(X - X) \cap \mathbb{Z}^+ \geq |X| - 1$. This can be seen by considering all differences of the form $x_i - x_1$, $i \geq 2$. However, we cannot have $X - X$ and X intersect, or else we will have an equality of the form $x_k - x_j = x_i$. Yet X covers $m > \frac{n+1}{2}$ elements of $[n]$, and $X - X$ must cover at least $m - 1 > \frac{n-1}{2}$ elements of $[n]$. Thus $|X| + |X - X| > n$, contradiction.

6. (Erdős-Szekeres.) Prove that every sequence of n^2 distinct numbers contains a subsequence of length n which is monotone (i.e. either always increasing or always decreasing).

   Solution: For each of the n^2 indices in the sequence, associate the ordered pair (x, y) where x is the length of the longest increasing subsequence ending at x, and y is the length of the longest decreasing one. All ordered pairs must obviously be distinct. But if they only take values with $x, y \in \{1, \ldots, n-1\}$, then there are not enough for the total n^2 ordered pairs. Thus n appears somewhere, and we are done.

7. (Sweden 2010.) The numbers $1, 2, \ldots, n^2$ are placed randomly in an $n \times n$ table. Prove that there are two adjacent cells (in a row or a column) such that the numbers in them differ by at least n.

   Solution: Isoperimetric inequality. Consider where the numbers $1, 2, \ldots, t$ have been placed. Look on all squares that are adjacent to these (the "vertex" expansion). If the number of such squares is at least n, then immediately we know that by the time $t+1, \ldots, t+n$ are placed, in particular the $t+n$ one must be close to a square with a number $\leq t$ (like a pigeonhole type of idea). The key transition point is when $t = \frac{n(n-1)}{2}$. At this point, the boundary must have at least n squares.

Some ideas on how to show this: you can compress the hit squares toward the left, and only reduce the boundary. You can then compress all hit squares down, and only reduce the boundary. Now we have a staircase pattern. Let the nonzero rows have $a_1 \geq a_2 \geq \cdots \geq a_m$ hit squares. Any a_i which is not the full n can contribute $+1$ to the boundary (at its right end). Also, if $a_i > a_{i+1}$, then we contribute $a_i - a_{i+1}$. These can be reconciled by saying that the boundary is the sum of all $a_i - a_{i+1}$, adding another $+1$ for every pair with $a_i = a_{i+1}$.

3 Bonus problems

1. (Hamming code.) A license plate has seven binary digits (0 or 1), and may have leading zeros. If two plates must always differ in at least three places, what is the largest number of plates that is possible?
2. (Erdős.) Every set A of n nonzero integers contains a sum-free subset (one with no $x + y = z$, with $x, y, z \in A$, not necessarily distinct) of size $|A| > \frac{n}{3}$.
3. Show that the previous result also holds for nonzero real numbers.