Extremal Combinatorics

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1 Warm-up

1. (Mantel.) Prove that every graph with strictly more than $\frac{n^2}{4}$ edges contains a triangle.

   Solution: Assume no triangles. Then for every edge uv, must have $d(u) + d(v) \leq n$. Sum over all edges. Then

   $$nE \geq \sum d(v)^2 \geq n \cdot \left(\frac{2E}{n}\right)^2 = \frac{4E^2}{n},$$

   i.e., $E \leq n^2/4$, contradiction.

2 Turán-type problems

1. (Katona.) Let $v_1, \ldots, v_n$ be given vectors in $\mathbb{R}^d$. Let W_1 and W_2 be random vectors chosen independently and uniformly from the $\{v_i\}$. Prove that

   $$\mathbb{P}\left[\|W_1 + W_2\| \geq 1\right] \;\geq\; \frac{1}{2}\mathbb{P}\left[\|W_1\| \geq 1\right]^2.$$

   Solution: LEM: suppose m of the vectors are ≥ 1. Make graph on m vertices, with edge if $\|v_i + v_j\| < 1$. This is triangle-free.

   PF: Take 3 vectors v_1, v_2, v_3 of norm ≥ 1. Show two sum to ≥ 1. WLOG one has norm 1, or scale everything down. Change axes: WLOG $v_1 = (1, 0, \ldots, 0)$. Assume all pairs sum to norm < 1. Then v_2, v_3 are in open ball of radius 1 about $(-1, 0, \ldots, 0)$. Also they are outside open ball of radius 1 about 0. Let $v_2 = (x_2, y_2)$ and $v_3 = (x_3, y_3)$. Then:

   $$\begin{aligned} (x_i + 1)^2 + \|y_i\|_2^2 &< 1 \\ x_i^2 + \|y_i\|_2^2 &\geq 1 \\ -x_i^2 - \|y_i\|_2^2 &\leq -1 \\ 2x_i + 1 &< 0 \\ x_i < -1/2. \end{aligned}$$

   So $v_2 + v_3$ already has first coordinate strictly less than -1, giving norm > 1. Contradiction.

   THUS: number of edges is at most $m^2/4$. We sample 2 vectors with replacement. Looking only on the m, we already have at least $m^2 - 2 \cdot m^2/4 = m^2/2$ ways to pick 2 vectors with sum ≥ 1 (since even if we pick the same vector twice, its length will double to ≥ 2 since it was from the m). The factor 2 is from converting unordered pairs (edges) to ordered pairs.

2. (Turán.) Let $r \geq 3$ be given, and let $T_r(n)$ be the complete r-partite graph with its n vertices distributed among its r parts as evenly as possible (because rounding errors may occur). Then the

Turán graph $T_r(n)$ is the unique n-vertex graph with the maximum number of edges subject to having no K_{r+1} subgraphs.

Solution: Zykov symmetrization. First show any non-adjacent vertices have same degree. Indeed, if one has more degree, then delete the smaller and clone the larger, and get strictly more edges without making any bigger cliques.

Next show non-adjacency is an equivalence relation, to get a complete multipartite graph (and then use convexity). So suppose we had x, y_1, y_2 such that x is not adjacent to either y_i, but the y_i are adjacent to each other. By previous, $d(y_1) = d(x) = d(y_2)$. But note that the degree of y_i includes $+1$ for their internal edge (and this is double-counted when adding $d(y_1) + d(y_2)$, so if we delete both y_i and clone x twice, then we get at least $+1$ in total edges.

3. (Application of Kővári-Sós-Turán.) Given n points in the plane, prove that the number of pairs of points which are distance 1 apart is at most $n^{3/2}$.

   Extra credit: Improve the bound to $100n^{4/3}$.

   Mega bonus: Improve the bound to $100^{100}n^{1.333333333333333333333333333333333333333333333333333333332}$.

   Solution: Zarankiewicz counting for $K_{2,3}$. Assume we have average degree at least $2n^{1/2}$ and no $K_{2,3}$. For each pair of vertices u, v, let the codegree $d(u, v)$ be their number of common neighbors. By assumption, all of these are at most 2, so their sum is at most $\binom{n}{2} \cdot 2 \leq n^2$.

   Now consider an arbitrary vertex x. It contributes to exactly $\binom{d(x)}{2}$ codegrees. Since $\binom{t}{2}$ is quadratic and convex, the total contribution to codegrees is

   $$\sum_x \binom{d(x)}{2} \geq n \cdot \binom{\overline{d}}{2}.$$

   Since we assumed $\overline{d} \geq 2n^{1/2}$, we get a total contribution of at least roughly $2n^2$, contradiction.

3 More problems

1. (Classical.) Let G be a graph. It is possible to partition the vertices into two groups such that for each vertex, at least half of its neighbors ended up in the other group.

   Solution: Take a max-cut: the bipartition which maximizes the number of crossing edges.

2. (Andrásfai, Erdős, Sós.) In general, graphs without triangles are not necessarily bipartite. Show that if we also know that all degrees of the graph are strictly greater than $\frac{2}{5}n$, where n is the number of vertices, then the graph is bipartite. Also show that this is tight.

   Solution: For contradiction, suppose that $\delta > \frac{2}{5}n$ and the graph is triangle-free, but not bipartite. Take a shortest odd cycle C, say with $t \geq 5$ vertices. Consider the vertices B outside the cycle C. Each vertex $v \in B$ can only have at most 2 neighbors in C, or else it would make a triangle or a shorter odd cycle. Indeed, suppose there were 3 entry points on the cycle. Since the cycle is odd, one of the distances between the points, say between w, x is odd. So adding v will make another odd cycle. To see it is shorter, note that the third entry point, z, will be cut out of the cycle, as well as the neighbors of z (we cannot have w or x adjacent to z or else we will have made a triangle). So the new odd cycle is strictly shorter. Therefore, the number of edges between B and C is at most $2(n - t)$.

   On the other hand, the minimum degree condition lets us lower bound this number. First note that there are no edges between vertices of C. If there were, then the two endpoints would cut C into two parts, and since it was odd, one of the parts would have odd length, and length at least 3. So short-circuiting that odd side with the new chord will give a strictly shorter odd cycle. Therefore, the number of edges from C to B is strictly greater than $t \cdot \left(\frac{2}{5}n - 2\right)$, so strictly greater than $2n - 2t$, contradiction.

   Tightness comes from the blow-up of C_5.

3. (Moscow, 1964, from GDC 2008.) King Arthur summoned $2n$ knights to his court. Each knight has at most $n-1$ enemies among the other knights present. Prove that the knights can sit at the Round Table so that no two enemies sit next to each other. (The "enemy" relation is symmetric.)

   Solution: This is Dirac's theorem in disguise. Suppose the longest path has t vertices $x_1, \ldots, x_t$. We will show there is a cycle of t vertices as well. Suppose not. All neighbors of x_1 and x_t must lie on the path or else it is not longest. Minimum degree condition implies that both have degree $\geq t/2$. But if $x_1 \sim x_k$, then $x_t \not\sim x_{k-1}$ or else we can re-route to get a cycle. So, each of x_1's $t/2$ neighbors on the path prohibit a potential neighbor of x_t. Yet x_t's neighbors come from indices $1 \ldots t-1$, so there is not enough space for x_t to have $t/2$ neighbors there, avoiding the prohibited ones.

   Now if this longest path is not the full n vertices, then we get a cycle C missing some vertex x. But min-degree $n/2$ implies that the graph is connected (smallest connected component is $n/2+1$), so there is a shortest path from x to C, and adding this to the cycle gives a longer path than t, contradiction.

4. (China, 1986, from GDC 2008.) In a chess tournament with n players, every pair of players plays once and there are no draws. Show that there must be some player A such that for every other player B, either A beat B or A beat some other player C who in turn beat B.

   Solution: (From Havet and Thomassé, 2000.) A vertex v in a directed graph is a *king* if every other vertex can be reached from it via directed paths of length at most 2. That is, $V = \{v\} \cup N^+(v) \cup N^{++}(v)$. Show that every tournament has a king.

   Take the first vertex v_1 in a median order, and consider v_k. If already $\overrightarrow{v_1 v_k}$, then done. Otherwise, by the feedback property, at least half of the edges from v_1 to $v_2 \ldots v_k$ point forward. These give at least $\frac{k-1}{2}$ landing points in $v_2 \ldots v_{k-1}$, because we assumed that v_1 does not go to v_k. Similarly, there are at least $\frac{k-1}{2}$ vertices in $v_2 \ldots v_{k-1}$ which are origination points for edges directed to v_k. Yet there are only $k-2$ vertices in this window, so by pigeonhole some vertex is both an origination point and a landing point, giving a directed path of length 2.

5. (Classical.) Prove that every tournament (complete graph with all edges directed) has a Hamiltonian path, i.e., a directed path which visits every vertex exactly once.

   Solution: Take a median order.

6. (Romania.) Given n points in the plane, prove that there exists a set of $\sqrt{n}$ points such that no 3 points in the set form an equilateral triangle.

   Solution: First we prove the Erdős-Szekeres theorem, that every sequence of n^2 distinct numbers contains a subsequence of length n which is monotone (i.e. either always increasing or always decreasing). Indeed, for each of the n^2 indices in the sequence, associate the ordered pair (x, y) where x is the length of the longest increasing subsequence ending at x, and y is the length of the longest decreasing one. All ordered pairs must obviously be distinct. But if they only take values with $x, y \in \{1, \ldots, n-1\}$, then there are not enough for the total n^2 ordered pairs. Thus n appears somewhere, and we are done.

   Next, we find an x-axis direction such that all points have distinct projection on both x and y (orthogonal) axes. Order the points according to this direction, and let $y_1, \ldots, y_n$ be their corresponding y-coordinates (distinct). By Erdős-Szekeres, there is a monotone subsequence of the desired length. But this cannot contain an equilateral triangle.

7. (Sylvester.) Prove that for every finite set of at least 3 distinct points in the plane, not all collinear, there is a line which passes through exactly 2 of the points.

   Solution: (Wikipedia.) Suppose for contradiction that we have a finite set of points not all collinear but with at least three points on each line. Call it S. Define a connecting line to be a line which contains at least two points in the collection. Let (P, l) be the point and connecting line that are the smallest positive distance apart among all point-line pairs.

By the supposition, the connecting line l goes through at least three points of S, so dropping a perpendicular from P to l there must be at least two points on one side of the perpendicular (one might be exactly on the intersection of the perpendicular with l). Call the point closer to the perpendicular B, and the farther point C. Draw the line m connecting P to C. Then the distance from B to m is smaller than the distance from P to l, contradicting the original definition of P and l. One way to see this is to notice that the right triangle with hypotenuse BC is similar to and contained in the right triangle with hypotenuse PC.

Thus there cannot be a smallest positive distance between point-line pairs—every point must be distance 0 from every line. In other words, every point must lie on the same line if each connecting line has at least three points.

8. (Matoušek.) A country has n signal towers, no three of which are collinear. The towers communicate via straight-line wireless links between each other. The enemy's spy agency wishes to block all pairs of towers from communicating with each other, by placing jammers on the ground between the towers. (Two towers are prevented from communicating if a jammer is located on the line segment between them.) Prove that at least $2n - 3$ jammers are required.

   Bonus: Can you prove that at least $n \cdot \log \log \log \log n$ are required?

   Solution: We first show that it is possible to triangulate the point set using at least $2n - 3$ line segments. One way to see this is to choose an x-axis direction such that all projections onto it have different coordinates. Then scan in the positive x direction, adding one point at a time. Each time, the new point is not in the convex hull of the previous, so we can draw a triangle from it to two adjacent points of the previous convex hull. Then possibly add a new edge to make the new body convex, and repeat. Each time we add at least 2 edges, and we start with 3 points with 3 edges, giving the base case.

   Now note that each of these line segments needs a jammer, and they are all disjoint, so we need at least $2n - 3$ jammers.

9. (Problem of Nowakowski and Winkler; result of Frankl.) Let G be an n-vertex graph. Consider the following game of Cops vs. Robber, in which some number of cops attempt to catch a single robber. It is a perfect information game, so both players know each others' positions at all times. First, the cops choose their starting vertices. Then, the robber chooses his starting vertex (knowing the cops' positions). The players now alternate moves, with the cops moving first. On a move, each cop can either move to an adjacent vertex, or stay still. Similarly, on his turn, the robber can either move to an adjacent vertex or remain stationary. The cops win if eventually one of them moves on top of the robber.

   Prove that given any graph, there is always a strategy for $n \cdot \frac{\log \log n}{\log n}$ cops to catch the robber.

   Bonus: It is known that there are graphs for which $\sqrt{n}$ cops are required to catch the robber. Can you prove there is always a strategy for $n^{0.999999}$ cops to catch a robber? (That is a lot of cops.)

   Solution: 1 cop can guard neighborhood, so can bound maximum degree. Also 1 cop can guard geodesic, by maintaining the invariant that for every intermediate vertex on the geodesic, the cop is at least as close to the vertex as the robber is. So can bound diameter. Yet we have n vertices, so by vertex exploration we get the bound.