# B-V. Graph Theory

Po-Shen Loh

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## Warm-Ups

1. (Russia 1990) There are 30 senators in a senate. For each pair of senators, they are either friends or enemies. Every senator has exactly 6 enemies. Find $A+B$, where $A$ is the number of 3-subsets of senators that are all enemies, and $B$ is the number of 3-subsets of senators that are all friends.

   **Solution:** Make graph with black if enemies and blue if friends; count the number of formations of blue/black meeting at one vertex. Answer is 1990.

2. (Zuming 7.3) There are 12 students in Mr. Fat's combinatorics class. At the beginning of each week, Mr. Fat assigns a project to his students. The students pair up into 6 groups. Each pair works on the project independently and submits the work at the end of the week. Each week, the students can pair up as they wish. Prove that, regardless of the way the students choose their partners, there are always 2 students such that $|A \cup B| \geq 5$, where $A$ is the set of students who have worked with both of them and $B$ is the set that has worked with neither.

3. (1J) Let $G$ be a graph on $n > 3$ vertices with no vertex of degree $n-1$. Suppose that for any two vertices of $G$, there is a unique vertex joined to both of them. Prove that $G$ is regular.

4. (4.3) Let $G$ be a graph on $n > 2$ vertices with all degrees at least $n/2$. Prove that $G$ contains a Hamiltonian circuit.

5. (5.5) Let $A$ be a square matrix with nonnegative integer entries, such that all row and column sums are the same. Prove that $A$ is the sum of some number of *permutation matrices*, where a permutation matrix is a square matrix containing only 0's and 1's, where every row and column has exactly one 1.

   **Solution:** Hall's marriage Lemma

## Problems

1. (2F) Suppose $G$ is a graph with exactly one vertex of degree $i$ for $2 \leq i \leq m$ and $k$ other vertices, all of degree 1. For each $m$, give a construction with $k = \lfloor \frac{m+3}{2} \rfloor$.

2. (4C) If a graph on $n$ vertices has $e$ edges, then it has at least $\frac{e}{3n}(4e - n^2)$ triangles.

   **Solution:** Number of triangles on a given edge $\{v_i, v_j\}$ is at least $d_i + d_j - n$. Sum over all edges and divide by three, and then use RMS-AM.

3. (3.3) **Ramsey's Theorem.** Let $r \geq 1$ and $q_i \geq r$, $i = 1, 2, \ldots, s$ be given. There exists a minimal positive integer $N(q_1, q_2, \ldots, q_s; r)$ with the following property. Let $S$ be a set with $n$ elements. Suppose that all $\binom{n}{r}$ $r$-subsets of $S$ are divided into $s$ mutually exclusive families $T_1, \ldots, T_s$ ("colors"). Then if $n \geq N(q_1, q_2, \ldots, q_s; r)$ there is an $i$, $1 \leq i \leq s$, and some $q_i$-subset of $S$ for which every $r$-subset is in $T_i$.

   **Solution:** Induction on $r$, for case $s = 2$:

   (a) Trivial for $r = 1$: $N(p, q; 1) = p + q - 1$.

(b) For any $r$ and $p \geq r$, clear that $N(p, q; r) = p$ and symmetrically for the second argument as well.

(c) Now induct on $r$. Assume true for $r - 1$, and induct on $p + q$.

$$N(p, q; r) \leq N\left[N(p-1, q; r), N(p, q-1; r); r-1\right] + 1$$

4. (3G) Let $m$ be given. Show that if $n$ is large enough, every $n \times n$ $(0, 1)$-matrix has a principal submatrix of size $m$, in which all the elements below the diagonal are the same, and all the elements above the diagonal are the same.

   **Solution:** On $K_n$, color edge $\{i, j\}$ with $i < j$ in one of 4 colors; color being the ordered pair $(a_{ij}, a_{ji})$. A $K_m$ corresponds to the principal submatrix on the rows/cols indexed by the $m$ indices picked.

5. (21D) A *strongly regular* graph with parameters $(v, k, \lambda, \mu)$ is a graph with $v$ vertices, regular with degree $k$, and such that for any pair of adjacent vertices $x$ and $y$, there are exactly $\lambda$ vertices adjacent to both $x$ and $y$, and for any pair of non-adjacent vertices $x$ and $y$, there are exactly $\mu$ vertices adjacent to both $x$ and $y$. Prove that no strongly regular graph with parameters $(28, 9, 0, 4)$ exists.

6. (cit.t3.7.1) A connected graph is said to be $k$-edge-connected if one must delete at least $k$ edges in order to disconnect the graph. Prove that $G$ is $k$-edge-connected if and only if there are at least $k$ edge-disjoint paths linking any pair of distinct vertices.

7. (cit.t3) A connected graph is said to be $k$-vertex-connected if one must delete at least $k$ vertices in order to disconnect the graph. Prove that $G$ is $k$-vertex-connected if and only if there are at least $k$ totally-internally-disjoint paths linking any pair of distinct non-adjacent vertices.

8. (cit.t3.7.2) Let $G$ be a $k$-vertex-connected graph. Prove that for any set of $k$ vertices of $G$, there exists a cycle that contains all of them. (A cycle is a closed path that never visits a vertex more than once.)

9. (cit.t3) Let $G$ be a bipartite graph. Then $\chi'(G) = \Delta(G)$, where $\chi'(G)$ is the *edge chromatic number* of $G$, which is the minimum number of different colors needed to color the edges of $G$ in such a way that no incident edges receive the same color, and $\Delta(G)$ is the maximum degree of a vertex.

   **Solution:** Induction on number of edges, alternating path

10. (cit.t3) **Vizing's Theorem.** Let $G$ be a graph. Then $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

    **Solution:** Suppose we have a colouring of all but one edge $xy \in E(G)$ using colours $\{1, 2, \cdots, \Delta(G) + 1\}$. Then we wish to recolour so all the edges are coloured.

    Note that one colour is unused ("missing") at every vertex.

    Let $xy_0$ be the uncoloured edge. We construct a sequence of edges $xy_0, xy_1, \cdots$ and a sequence of colours $c_0, c_1, \cdots$ as follows.

    Pick $c_i$ to be a colour missing at $y_i$. Let $xy_{i+1}$ be an edge with colour $c_i$. We stop with $k = i$ when either $c_k$ is a colour unused at $x$, or $c_k$ is already used on $xy_j$ for $j < k$.

    If $c_k$ was a colour unused at $x$ then we recolour $xy_i$ with $c_i$ for $0 \leq i \leq k$. This finishes the easy case where we can recolour the edges touching $x$ to give a a colouring for $G$.

    Otherwise we recolour $xy_i$ with $c_i$ for $0 \leq i < j$ and uncolour $xy_j$. Notice that $c_k$ (red) is missing at both $y_j$ and $y_k$. Let blue be a colour unused at $x$.

    (a) If red is missing at $x$, we colour $xy_j$ red.

    (b) If blue is missing at $y_j$ we colour $xy_j$ blue.

    (c) If blue is missing at $y_k$ we colour $xy_i$ with $c_i$ for $j \leq i < k$ and colour $xy_k$ blue. (None of the $xy_i$, $j \leq i < k$ are red or blue.)

If none of the above hold, then we consider the subgraph of red and blue edges. The components of this subgraph are paths or cycles. The vertices $x, y_i, y_k$ are the end vertices of paths. Therefore they cannot all belong to the same component.

Select a component that contains exactly one of these vertices. Now swap over red and blue in this component. Now one of the conditions above must apply.

11. (cit.t3.8.4) Let $G$ be a graph. We say that $G$ is *$k$-list-colorable* if $G$ satisfies the following property: Let $X$ be a set of colors, and for each vertex $v \in G$, let $S_v \subset X$ be a set of $k$ of the colors. Then under any choice of $X$ and $\{S_v\}$, there exists a coloring of each vertex $v \in G$ with a color from its associated list, in such a way that adjacent vertices receive different colors.

    On the other hand, we say that $G$ is *$k$-vertex-colorable* if there exists a way to color the vertices of $G$ with $k$ colors in such a way that adjacent vertices receive different colors.

    For every integer $k$, construct a graph $G_k$ that is 2-vertex-colorable but is not $k$-list-colorable.

12. (6A) Let $a_1, a_2, \ldots, a_{n^2+1}$ be a permutation of the integers $1, 2, \ldots, n^2 + 1$. Show that there is a subsequence of length $n + 1$ that is monotone.

    **Solution:** Dilworth's theorem on poset of $\{(i, a_i)\}$. Define $(x, y) \leq (x', y')$ iff $x \leq x'$ and $y \geq y'$.

13. (6.1) **Dilworth's Theorem.** Let $P$ be a finite poset. The minimum number $m$ of disjoint chains which together contain all elements of $P$ is equal to the maximum number $M$ of elements in an antichain of $P$.

# References

[1] J.H. van Lint and R. Wilson, *A Course in Combinatorics*, Cambridge University Press, 2001.