B-II. Inequalities

Po-Shen Loh

15 June 2004

1 Warm-Up

(Po98) Prove that for all ordered triples (a, b, c) of prime numbers:

$$a^2b + a^2 + ac^2 + 115a + b^2c + b^2 + c^2 + 27c + 176 > 6ab + 22ac + 14bc + 5b.$$

Solution: Complete the square

2 The p-Norm

Here's the easiest way to think of this: let's define the "p-norm" of a sequence as follows: take a sequence $\{a_1, a_2, \ldots, a_n\}$ and call it a. Then write $\|a\|_p$ to denote $\sqrt[p]{|a_1|^p + \cdots + |a_n|^p}$. (By the way, p doesn't have to be an integer; the p-th root is then defined as the $1/p$ power.)

This is kind of intuitive; if $p = 2$, then if we have n-dimensional space and a is a coordinate vector, $\|a\|_2$ is the length of it. And different p's give different notions of length; for instance, $p = 1$ corresponds to the Manhattan (taxicab) distance. For completeness, definte $\|a\|_\infty$ to be the maximum of the $|a_k|$. Amazingly enough, this notion of p-norm appears in many areas of mathematics. You'll see it again and again as you learn more math.

Also, we should define "addition" and "multiplication" on sequences. That is, given $a = \{a_1, a_2, \ldots, a_n\}$ and $b = \{b_1, b_2, \ldots, b_n\}$, define $a + b$ to be $\{a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n\}$ and ab to be $\{a_1b_1, a_2b_2, \ldots, a_nb_n\}$.

3 Cute Inequalities (inspired by Kiran97)

AM-GM-HM For positive sequences a:

$$\frac{a_1 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 \cdots a_n} \geq \frac{n}{\frac{1}{a_1} + \cdots + \frac{1}{a_n}}$$

with equality when all of the a_k are equal.

Cauchy-Schwarz $(a_1b_1 + \cdots + a_nb_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2)$. Note that this is a special case of Hölder. Can you see why?

Minkowski Given the same sequences as above, and $p \geq 1$, $\|a + b\|_p \leq \|a\|_p + \|b\|_p$.

Hölder Given the same sequences as above, and $p, q \geq 1$ such that $1/p + 1/q = 1$,

$$\|ab\|_1 \leq \|a\|_p \cdot \|b\|_q$$

Note: we can take $p = 1$ and $q = \infty$; their reciprocals "add up to 1."

Chebyshev If a and b are increasing positive sequences, then:

$$\frac{a_1b_1 + \cdots + a_nb_n}{n} \geq \frac{a_1 + \cdots + a_n}{n}\frac{b_1 + \cdots + b_n}{n}$$

If one sequence is increasing but the other is decreasing, then the inequality flips.

Rearrangement If we have the series $a_1b_1 + \cdots + a_nb_n$, it is maximized when the a's and b's are both sorted in the same direction, and minimized when they are sorted in opposite directions.

Bernoulli If $x > -1$ and $r \geq 1$, then $(1+x)^r \geq 1 + rx$.

1. (IMO95) Let a, b, and c be positive real numbers such that $abc = 1$. Prove:

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

2. (ELMO 1999/2) Prove that for $a, b, c, d > 0$:

$$2^6\frac{abcd+1}{(a+b+c+d)^2} \leq a^2+b^2+c^2+d^2+\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}.$$

Solution: Split by the + in the numerator of LHS. The first inequality uses RMS-GM: $\sqrt[4]{abcd}/\sqrt{a^2+b^2+c^2+d^2} \leq 1/2$, so that we bound

$$2\sqrt{\frac{abcd}{a^2+b^2+c^2+d^2}} \leq \sqrt[4]{abcd} \leq \frac{a+b+c+d}{4}.$$

For the second part, we use HM2-AM:

$$\sqrt{\frac{4}{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}}} \leq \frac{a+b+c+d}{4}.$$

3. (Ireland 1998/7a) Prove that if a, b, c are positive real numbers, then

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

Solution: AM-HM or C-S by rewriting:

$$\frac{9}{(a+b)+(b+c)+(c+a)} \leq \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}.$$

4. (Po 2004) Suppose that $0 < a, b, c < 4$ and $abc = 1$. Prove that:

$$\frac{1}{1+a+b}+\frac{1}{1+b+c}+\frac{1}{1+c+a} \leq \frac{1}{4-a}+\frac{1}{4-b}+\frac{1}{4-c}.$$

Solution: Define the unordered lists:

$$\begin{aligned} S &= \{4-a,\ 4-b,\ 4-c\}, \\ T &= \{1+b+c,\ 1+c+a,\ 1+a+b\}. \end{aligned}$$

It suffices to show that HM(S) $\leq$ HM(T), where HM denotes the harmonic mean. Yet the sets are just translates of each other on the number line:

$$T = S + (a + b + c - 3),$$

so the following consequence of the AM-GM inequality completes our proof:

$$\frac{a+b+c}{3} \geq (abc)^{1/3} = 1.$$

5. (Romania 1996/11) Let x_1, x_2, $\ldots$, x_n be positive reals, and let S be their sum. Prove that:

$$\sum_{i=1}^{n} \sqrt{x_i(S - x_i)} \leq \sqrt{\sum_{i=1}^{n} S(S - x_i)}.$$

Solution: RHS simplifies to $S\sqrt{n-1}$. Divide both sides by that; now STS

$$\sum_{i=1}^{n} \sqrt{\frac{x_i}{S} \cdot \frac{1}{n-1}\left(1 - \frac{x_i}{S}\right)} \leq 1.$$

AM-GM wrt the dot, sum both parts, get $1/2 + 1/2 = 1$.

6. (Austrian-Polish Math Competition 1996/4) The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that:

$$-1 \leq xy + yz + zt + tx.$$

Solution: RHS groups into $(x+z)(y+t)$. First equality tells us this is both $-(x+z)^2$ and $-(y+t)^2$. Yet RMS-ineq yields that these are bounded by $-2(x^2+z^2)$ and $-2(y^2+t^2)$, so their average is bounded by -1, as desired.

7. (Kiran97) Let a, b, c be positive. Prove:

$$\frac{1}{a(1+b)} + \frac{1}{b(1+c)} + \frac{1}{c(1+a)} \geq \frac{3}{1+abc}$$

with equality iff $a = b = c = 1$.

8. (Iran 1998/9) Let $x, y, z > 1$ and $1/x + 1/y + 1/z = 2$. Prove that

$$\sqrt{x+y+z} \geq \sqrt{x-1} + \sqrt{y-1} + \sqrt{z-1}.$$

Solution: C-S LHS against $\sqrt{(x-1)/x + (y-1)/y + (z-1)/z}$.

9. (Asian Pacific Math Olympiad 1998/3) Let a, b, c be positive real numbers. Prove that

$$\left(1 + \frac{a}{b}\right)\left(1 + \frac{b}{c}\right)\left(1 + \frac{c}{a}\right) \geq 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right).$$

Solution: Multiply out:

$$\begin{aligned} & (a/b + a/c + a/a) + (b/c + b/a + b/b) + (c/a + c/b + c/c) - 1 \\ \geq\ & 2\left(\frac{a+b+c}{\sqrt[3]{abc}}\right) + \left(\frac{a+b+c}{\sqrt[3]{abc}}\right) - 1 \\ \geq\ & 2\left(1 + \frac{a+b+c}{\sqrt[3]{abc}}\right). \end{aligned}$$

10. (Vietnam 1998/4) Let $x_1, \ldots, x_n$ $(n \geq 2)$ be positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

Solution: Let $y_i = 1998/(x_i + 1998)$; then $y_i \geq 0$ and $1 - y_i = \sum_{j \neq i} y_j$. AM-GM:

$$1 - y_i \geq (n-1) \sqrt[n-1]{\prod_{j \neq i} y_j}.$$

Multiply over all i and get:

$$\prod_1^n (1 - y_i) \geq (n-1)^n \prod_1^n y_i.$$

11. (IMO 2001 Shortlist) Prove that for all positive reals a, b, c:

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

Solution: Prove that each term exceeds

$$\frac{a^{(4/3)}}{a^{(4/3)} + b^{(4/3)} + c^{(4/3)}}$$

Cross multiply and square. Then factor the following difference of squares

$$(a^{(4/3)} + b^{(4/3)} + c^{(4/3)})^2 - (a^{(4/3)})^2$$

and apply AM-GM on the product. We get $8a^{2/3}bc$.

4 Convexity and Smoothing

Jensen A *convex* function is a function $f(t)$ for which the second derivative is nonnegative. This is equivalent to having the property that for any a, b in the domain, $f((a+b)/2) \leq (f(a) + f(b))/2$. Then given weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive numbers $a_1, a_2, \ldots, a_n$:

$$f\left(\frac{\lambda_1 a_1 + \cdots + \lambda_n a_n}{\lambda_1 + \cdots + \lambda_n}\right) \leq \frac{\lambda_1 f(a_1) + \cdots + \lambda_n f(a_n)}{\lambda_1 + \cdots + \lambda_n}$$

If f is concave, then the inequality flips.

Jensen (Probabilistic interpretation) Let $f(t)$ be a convex function, let X be a random variable, and let $E[Y]$ denote the expected value of random variable Y. Then

$$f(E[X]) \leq E[f(X)].$$

If f is concave, then the inequality flips.

Smoothing Given an inequality, show that it becomes less true when you "squish" the values of the variables together. Then if it's still true after you're done squishing, hey, it must have been true in the first place!

1. (Ireland 1998/7b) Prove that if a, b, c are positive real numbers, then

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \le \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

   Solution: Jensen with $f(t) = 1/t$:

$$\frac{f(a)+f(b)}{2} \ge f\left(\frac{a+b}{2}\right).$$

2. (Zvezda98) Prove for all nonnegative number a, b, c:

$$\frac{(a+b+c)^2}{3} \ge a\sqrt{bc} + b\sqrt{ca} + c\sqrt{ab}.$$

3. (IMO84) For $x, y, z > 0$ and $x + y + z = 1$, prove that $xy + yz + xz - 2xyz \le 7/27$.

   Solution: Smooth with the following expression: $x(y+z) + yz(1-2x)$. Now, if $x \le 1/2$, then we can push y and z together. The mushing algorithm is as follows: first, if there is one of them that is greater than $1/2$, pick any other one and mush the other two until all are within $1/2$. Next we will be allowed to mush with any variable taking the place of x. Pick the middle term to be x; then by contradiction, the other two terms must be on opposite sides of $1/3$. Hence we can mush to get one of them to be $1/3$. Finally, use the $1/3$ for x and mush the other two into $1/3$. Plugging in, we get $7/27$.

4. (Po, 2004) Prove that if $X \ge 1$ is a random variable taking integer values in $\{1, 2, \ldots, N\}$, then

$$E[X] = \sum_{k=1}^{N} P(X \ge k).$$

5. (Po, 2004) Let $1 = a_1 \ge \cdots \ge a_n \ge a_{n+1} = 0$ be a sequence of real numbers. Prove that:

$$\sqrt{\sum_{k=1}^{n} a_k} \ge \sum_{k=1}^{n} \sqrt{k}(a_k - a_{k+1}).$$

   Solution: Define the random variable X such that $P(X \ge k) = a_k$. Then Jensen

$$\sqrt{E[X]} \ge E[\sqrt{X}].$$

6. (MOP98/5/5) Let $a_1 \ge \cdots \ge a_n \ge a_{n+1} = 0$ be a sequence of real numbers. Prove that:

$$\sqrt{\sum_{k=1}^{n} a_k} \le \sum_{k=1}^{n} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}).$$

   Solution: Since the inequality is homogeneous, we can normalize the a_k so that $a_1 = 1$. (If they are all zero, it is trivial anyway.) Now define the random variable X such that $P(X \ge k) = \sqrt{a_k}$. Then STS

$$\sqrt{E[\min\{X_1, X_2\}]} \le E[\sqrt{X}],$$

where X, X_1, X_2 are i.i.d. Prove by induction on n. Base case is if $n = 1$, trivial. Now if you go to $n+1$ by shifting q amount of probability from $P(X = n)$ to $P(X = n+1)$, RHS will increase by exactly $q(\sqrt{n+1} - \sqrt{n})$. Yet LHS increases by exactly q^2 under the square root. Now since the probability shifted from $P(X = n)$, the square root was originally at least $q^2 n$. In the worst case, the LHS increases by $\sqrt{q^2 n + q^2} - \sqrt{q^2 n}$, which equals the RHS increase.

7. (MMO63) For $a, b, c > 0$, prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$$

Solution: Same idea as previous, except first you multiply it out and normalize $a + b + c = 1$. Then you get:

$$(a+b)(a^2 + b^2 + c(a+b)) + c(c+a)(c+b) \geq \frac{3}{2}(a+b)(b+c)(c+a)$$

Show that the difference is always at least 0, and if you mush together a and b, it gets better as long as $(3/2)(a+b) - c \geq 0$, which happens as long as $c \leq 3/5$. Hence the same algorithm as previous works.

8. (88 Friendship Competition) For $a, b, c > 0$:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$$

9. (USAMO98) Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $(0, \pi/2)$ such that

$$\tan(a_0 - \pi/4) + \tan(a_1 - \pi/4) + \ldots + \tan(a_n - \pi/4) \geq n - 1$$

Prove that $\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}$.

Solution: Let $t_k = \tan(x_k - \pi/4)$. Then $\tan x_k = (1 + t_k)/(1 - t_k)$, and we want this product to be at least n^{n+1}. Next the given inequality is equivalent to $1 + t_k \geq \sum_{j \neq k}(1 - t_j)$, and by AM-GM, it is at least $n \sqrt[n]{\prod_{j \neq k}(1 - t_j)}$. Finally, take the product over all possible LHS and the result falls out.

5 Brute Force (stolen from Kiran98)

Weighted Power Mean Given weights $\lambda_1, \lambda_2, \ldots, \lambda_n$ and positive numbers $a_1, a_2, \ldots, a_n$, and powers p and q such that $p \leq q$:

$$\left(\frac{\lambda_1 a_1^p + \cdots + \lambda_n a_n^p}{\lambda_1 + \cdots + \lambda_n}\right)^{1/p} \leq \left(\frac{\lambda_1 a_1^q + \cdots + \lambda_n a_n^q}{\lambda_1 + \cdots + \lambda_n}\right)^{1/q}$$

with equality when all of the a_k are equal.

Schur's Inequality For x, y, z positive and r real:

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \geq 0$$

with equality when $x = y = z$.

Now in all of these problems, all variables should be assumed positive.

1. $4(a^3 + b^3) \geq (a+b)^3$

Solution: Expand; to get the $ab(a+b) \leq a^3 + b^3$, take it as a product of two guys and use Weighted Power Mean for each.

2. $9(a^3+b^3+c^3) \geq (a+b+c)^3$

   Solution: Expand and get:

$$8\sum_{\text{sym}} a^3 \geq 3\sum_{\text{sym}} a^2b + 6abc$$

   (count terms; it works)

   Next by AM-GM, get rid of 6abc; cancels 2 of the LHS. Divide through by 3 and write out the rest (6 terms per side, split cyclically) then use rearrangement.

3. If $abc = 1$ then

$$\frac{1}{a+b+1} + \frac{1}{b+c+1} + \frac{1}{c+a+1} \leq 1$$

4. (MOP98) Prove that for $x, y, z > 0$,

$$\frac{x}{(x+y)(x+z)} + \frac{y}{(y+z)(y+x)} + \frac{z}{(z+x)(z+y)} \leq \frac{9}{4(x+y+z)}$$

5. If $abc = 1$ then

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

6. If $abc = 1$ then

$$\frac{c}{a+b+1} + \frac{a}{b+c+1} + \frac{b}{c+a+1} \geq 1$$

7. If $abc = 1$ then

$$\frac{1}{a+ab} + \frac{1}{b+bc} + \frac{1}{c+ca} \geq \frac{3}{2}$$

8. Prove:

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}$$