I. Induction

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1 News Flash From Zuming!

- Remind Po to take all the markers from CBA 337
- Tonight's study session for Red/Blue is in **Bessey 104**
- Future Red lectures are in **NM B-7**, the "Naval Military Base #7"
- Future Red tests and study sessions are in **Bessey 104**
- All test reviews will be in **Bessey 104**

2 Warm-Up

1. A coil has an inductance of 2mH, and a current through it changes from 0.2A to 1.5A in a time of 0.2s. Find the magnitude of the average induced emf in the coil during this time.

   Solution: Just kidding.

2. Prove that the number of binary sequences of length n with an even number of 1's is equal to the number of binary sequences of length n with an odd number of 1's.

   Solution: Easy induction: Let E_n, O_n the numbers of even/odd sequences. Induction hypothesis is $E_n = O_n = 2^{n-1}$. Now break E_{n+1} into two groups: those with first coordinate 0 and with first coordinate 1. Number in first group is E_n, and number in second group is O_n. So $E_{n+1} = E_n + O_n$, which by induction is 2^n. So $O_{n+1} = 2^{n+1} - E_{n+1}$, done.

3 Equality

1. Let F_n be the Fibonacci sequence. Prove that $F_n^2 = F_{n-1}F_{n+1} \pm 1$. Determine when it's $+1$ and when it's -1.

   Solution: It's actually $F_n^2 = F_{n-1}F_{n+1} - (-1)^n$, under the convention $F_0 = 0$, $F_1 = 1$. Inductive step is just evaluating F_nF_{n+2} and expanding out $F_{n+2} = F_n + F_{n+1}$.

$$\begin{aligned}
F_nF_{n+2} &= F_n(F_n + F_{n+1}) \\
&= F_n^2 + F_nF_{n+1} \\
&= F_{n-1}F_{n+1} - (-1)^n + F_nF_{n+1} \\
&= (F_{n+1})F_{n+1} - (-1)^n \\
F_nF_{n+2} - (-1)^{n+1} &= F_{n+1}^2.
\end{aligned}$$

2. (Titu98) Let a be a real number such that $\sin a + \cos a$ is a rational number. Prove that for all $n \in \mathbb{N}$, $\sin^n a + \cos^n a$ is rational.

   Solution: Clearly true for $n = 2$, since $\sin^2 + \cos^2 = 1$; then to go from $n \mapsto n+1$, just multiply the 1-case by the n case; use the fact that $n = 1, 2$ show that $\sin\cos \in \mathbb{Q}$.

   Specifically,
$$(\sin^n + \cos^n)(\sin + \cos) = \sin^{n+1} + \cos^{n+1} + \sin\cos(\sin^{n-1} + \cos^{n-1}).$$

   Since we only need this for $n \geq 2$, we have everything we need already rational, except for $\sin\cos$. But that is $\frac{1}{2}[(\sin + \cos)^2 - \sin^2 - \cos^2]$, already known to be rational by base case.

3. You've seen maps in geography books. Did you know that they could all be colored with just 4 colors? (That is, "colored" in the sense that no two adjacent countries are the same color. Note that if two countries share a corner, they do not count as being "adjacent".) Prove it!

   Solution: Well, this is actually the 4-color theorem, which was proved via computer. So don't try too hard.

4. (Ricky03 from Internet) I'm playing the color-country game against Bob. We take turns; on my turn, I draw in a country. On Bob's turn, he chooses any color for the country, but he must make sure that no adjacent countries share the same color. Is it possible for me to force Bob to use more than 4 colors?

   Solution: Yes it is. Prove the following statement by induction: for any N, I can force Bob to use N colors, and furthermore, every time I draw a new country, there is some part of the new country's border that can see the point at infinity (topologically connected, not straight-line vision).

   This is clearly true for $N = 2$. For the inductive step, suppose we have it for N, and we are trying to get $N + 1$. Well, then start a blob of countries, and force Bob to use 1 color in that blob.

   Next, start another new blob in a remote area far away, and use the previous algorithm until Bob uses a second color (different from the 1 already used); note that this is still exposed to the point at infinity.

   Proceed until we have N blobs, spread far apart, each with a different one of the N colors exposed to infinity. Now engulf everything in a huge (not simply-connected) region; that region must have a color different from the N colors. So we are done.

5. (MOP98) Let S be the set of nonnegative integers. Let $h : S \to S$ by a bijective function. Prove that there do not exist functions f, g from S to itself, f injective and g surjective, such that $f(n)g(n) = h(n)$ for all $n \in S$.

   Solution: Use contradiction; assume that f and g exist, and define $F = f \circ h^{-1}$, $G = g \circ h^{-1}$; now F is injective and G is surjective, and $f(n)g(n) = h(n) \Leftrightarrow F(n)G(n) = n$. Prove by strong induction that $F(n) = n$: at $n = 1$, we have $F(1)G(1) = 1 \Rightarrow F(1) = 1$ since we are in nonnegative integers. But then if true up to N, then $F(N+1)G(N+1) = N+1 \Rightarrow F(N+1) \in [1, N+1]$ and by injectivity, $F(N+1) = N+1$. Hence G can only take on two values, 1 and something else, so not surjective. Contradiction.

6. Show that every $2^n \times 2^n$ board with one square removed can be covered by Triominoes.

   Solution: First we inductively prove that we can tile any $2^n \times 2^n$ board such that we only miss one of the corners. Very easy, since we can cut into quarters, tile each one such that NW, NE, and SW are missing the corners in the center of the big square, and SE is missing the SE corner. One more triomino fills in the gap in the middle of the big square.

   Then look at a general $2^n \times 2^n$ board and split it into 4 equal squares. One of the squares contains the missing block, and we can use the previous result to tile the other 3 major squares, in such a way that the missing corners form a triomino in the center of the big square. Recursively descend.

4 Inequality

1. Prove the AM-GM inequality by induction.

   Solution: Easy for $n = 1, 2$; use n to show $2n$, and then use n to show $n-1$. For $n \mapsto 2n$, plug in $(a_k + a_{k+1})/2$; for the other one, use $a_n = (a_1 + \cdots + a_{n-1})/(n-1)$.

2. (Zuming97) Let $a_1 = 2$ and $a_{n+1} = a_n/2 + 1/a_n$ for $n = 1, 2, \ldots$. Prove that $\sqrt{2} < a_n < \sqrt{2} + 1/n$.

   Solution: Draw a picture to see why it is always greater than $\sqrt{2}$. Also use AM-GM to prove that we must be beyond $\sqrt{2}$. For the other side, induct and bound

$$a_{n+1} < (\sqrt{2} + 1/n)/2 + 1/\sqrt{2} = \sqrt{2} + 1/(2n)$$

3. (Zuming97) For the positive sequence $\{a_n\}$ with $a_n^2 \le a_n - a_{n+1}$, prove that $a_n < 1/(n+2)$.

   Solution: Positive sequence, so $a_1 \le 1$, and $a_2 \le 1/4$ by AM-GM, or function theory. Now

$$a_{n+1} \le a_n(1 - a_n) \le \frac{1}{n+2}\left(1 - \frac{1}{n+2}\right) = \frac{1}{n+2}\frac{n+1}{n+2} \le \frac{1}{n+2}\frac{n+2}{n+3} = \frac{1}{n+3}$$

4. (Zuming97) For $a > 0$, prove that:

$$\sqrt{a + \sqrt{2a + \sqrt{3a + \sqrt{4a + \sqrt{5a}}}}} < \sqrt{a} + 1$$

   Solution: True for $n = 1$. Let $f_k(a)$ be the k-th iteration evaluated at a. Then if it's true for some k:

$$f_{k+1}(a)^2 < f_k(2a) + a < \sqrt{2a} + 1 + a < a + 2\sqrt{a} + 1 = (\sqrt{a} + 1)^2$$

5. Prove that

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$$

   Solution: Bunch the terms in packs of 1, 2, 4, 8, 16, etc. Each pack will exceed 1, and there are infinitely many of them.

6. Prove that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2$$

   Solution: Similar bunching, but now the bunches are bounded by $1/2^k$, geometric series converges to 2.

5 Additional Problems

1. (Titu98) Prove that for every $n \in \mathbb{N}$, there exists a finite set of points in the plane such that for every point of the set there exist exactly n other points of the set at distance equal to 1 from that point.

   Solution: n-dimensional Hypercube, squashed into the plane. Basically, at each step translate the given point set by a unit distance. Just need to show there is a valid angle for this unit translation. But there are only a finite number of conflicts, (number of points $\times$ 2 because pair of circles has 2 intersections), and an infinite number of angles.

2. (Titu98) Let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n+1) > f(f(n))$ for all $n \in \mathbb{N}$. Prove that $f(n) = n$ for all n.

   Solution: We will prove by strong induction the statement P_n: all $f(a) = a$ for $a < n$, and the n-th smallest value in the set $\{f(i)\}$ is uniquely $f(n)$. That is, the unique index which attains that mark is $i = n$. For $n = 0$, there is nothing to prove.

   For $n = 1$, consider the smallest value, and suppose it is attained (possibly not uniquely) by $f(a)$. Then the recursion says $f(a) > f(f(a-1))$. So if $a \neq 1$, then the RHS is defined, and we have another $f(i)$ smaller than the MIN. Thus $a = 1$, and that is unique. This establishes P_1.

   For P_2, consider the 2nd smallest value, say $f(a)$. Recall P_1 says the smallest value is *uniquely* attained by $f(1)$. Again, $f(a) > f(f(a-1))$. We know $a \neq 1$ since $f(1)$ is the smallest. So, $f(a-1)$ is defined; let it be b. Now we have $f(a) > f(b)$. Thus $f(b)$ must be the smallest, hence $b = 1$. So $f(a-1) = b = 1$. But 1 is the minimum of the range, so we must actually have $a - 1 = 1$, so we now know that $f(1) = 1$ and $f(2)$ is *uniquely* the 2nd smallest, giving P_2.

   Now proceed inductively. Suppose P_n is true. Consider the $(n+1)$-st smallest value in the range, suppose it is attained by $f(a)$. Again, $f(a) > f(f(a-1))$, and since we know $a \neq 1$ (or else it would be the very smallest), we have $f(a-1) = b$. So $f(b)$ is one of the n smallest values. If it were some $i \in \{1, \ldots, n-1\}$, then we would also know that $b = i$ by P_n, further giving $f(a-1) = b = i$, forcing $a - 1 = i$ as well, and hence $a = i + 1$. But then again by P_n, we know $f(i+1)$ is one of the n smallest values, contradiction.

   Therefore, $f(b)$ is precisely the n-th smallest value, so $b = n$. This gives $f(a-1) = b = n$, which (since the $n-1$ smallest values are known to be $\{1, \ldots, n-1\}$) implies that n is the n-th smallest value, i.e., $a - 1 = n$. This shows that $f(n) = n$ and $a = n + 1$, i.e., $f(n+1)$ uniquely achieves the $(n+1)$-st smallest value.

3. (MOP97) Let F_k be the Fibonacci sequence, where $F_0 = F_1 = 1$ and $F_{n+2} = F_n + F_{n+1}$. Prove that for every $n, k \in \mathbb{N}$:

   $$F_n \leq F_k F_{n-k} + F_{k+1} F_{n-k-1} \leq F_{n+1}$$

   Solution: Write-up for 1997 MOP test 11, problem 3.

4. (MOP97) Prove that for $n, k \in \mathbb{Z}, n > 0, k \geq 0$:

   $$F_{n+2} - F_k F_{n-k} - F_{k+1} F_{n-k-1} = F_{k+1} F_{n-k}$$

   Solution: Write-up for 1997 MOP test 11, problem 3.

5. (MOP97) Given a sequence of numbers $\{a_1, \ldots, a_n\}$, define the derived sequence $\{a'_1, \ldots, a'_{n+1}\}$ by $a'_k = S - a_{k-1} - a_k$, where

   $$S = \min_{1 \leq k \leq n+1} (a_{k-1} + a_k) + \max_{1 \leq k \leq n+1} (a_{k-1} + a_k)$$

and $a_0 = a_{n+1} = 0$. Thus, if we start with the sequence $\{1\}$ of length 1 and apply the derived sequence operation again and again, we get the family of sequences:

$$\{1\}, \{1,1\}, \{2,1,2\}, \{3,2,2,3\}, \{5,3,4,3,5\}, \ldots$$

Show that when we apply the operation $2n$ times in succession to the initial sequence $\{1\}$ (with $n \geq 1$), we get a sequence whose middle (i.e. $(n+1)$-st) term is a perfect square.

Solution: Write-up for 1997 MOP test 11, problem 3.

6. (MOP97) Suppose that each positive integer not greater than $n(n^2 - 2n + 3)/2$ is colored one of two colors (red or blue). Show that there must be an n-term monochromatic sequence $a_1 < a_2 < \cdots a_n$ satisfying

$$a_2 - a_1 \leq a_3 - a_2 \leq \cdots \leq a_n - a_{n-1}.$$

Solution: Write-up for 1997 MOP IMO test 3, problem 2.