Homework Set 3

1) Let R be a commutative ring with unit element. Prove that $a_0 + a_1x + \cdots + a_mx^m \in R[x]$ is a unit in $R[x]$ if and only if a_0 is a unit in R and $a_1, \ldots, a_n$ are nilpotent elements in R (an element is nilpotent if some power of it is zero).

2) Let V be a finite dimensional vector space over the reals and $W = \{w_1, \ldots, w_m\}$ be an orthonormal set in V such that

$$\sum_{i=1}^{m} |\langle w_i, v\rangle|^2 = ||v||^2$$

for every $v \in V$. Prove that W is a basis of V.

3) Let V be the set of real functions $y = f(x)$ satisfying

$$\frac{d^2y}{dx^2} + 9y = 0.$$

a) Prove that V is a two-dimensional real vector space.
b) In V, define

$$\langle u, v\rangle = \int_0^{\pi} uv\ dx.$$

Show that this defines an inner product on V and find an orthonormal basis for V.

4) Let W be a subspace of V and $v \in V$ satisfy $2\langle v, w\rangle \leq \langle w, w\rangle$ for every $w \in W$. Prove that v lies in the orthogonal complement of W.