Solutions to Homework Set 1

1) Let R be a ring and $e \in R$ such that $e^2 = e$. Show that $(xe - exe)^2 = (ex - exe)^2 = 0$ for all $x \in R$.

Solution:

$$(xe - exe)^2 = (xe - exe)(xe - exe) = xexe - xeexe - exexe + exeexe =$$

$$= xexe - xexe - exexe + exexe = 0$$

$$= exex - exexe - exex + exexe = exex - exexe - exeex + exeexe =$$

$$= (ex - exe)(ex - exe) = (ex - exe)^2.$$

2) Let D be an integral domain with unit of characteristic $p \neq 0$. Prove that p is prime. Extra credit: What if D has no unit?

Solution: Suppose that $p = rs$ with $r, s > 1$. Then $0 = p(1) = (rs)(1) = r(s1) = (r1)(s1)$. Since D is a domain, either $r1 = 0$ or $s1 = 0$, assume the former. Then for any $x \in R$, we have $rx = r(1x) = (r1)x = 0x = 0$. Since $r < p$, this is a contradiction, hence p is prime.

3) Let D be a commutative ring. Prove that D is an integral domain if and only if the following holds: for every $a, b, c \in D$ with $a \neq 0$, if $ab = ac$ then $b = c$.

Solution: If $ab = ac$, then $a(b - c) = 0$ and since D is a domain and $a \neq 0$, we conclude that $b - c = 0$ or $b = c$. For the converse, suppose that $ab = 0$, then $ab = a0$ so by the cancellation property $a = 0$ or $b = 0$.

4) Let R be a ring in which $x^3 = x$ for every $x \in R$. Prove that R is commutative.

Solution: By 1), we know that $(xe - exe)^2 = 0$. Multiplying by another factor of $xe - exe$ and using $a^3 = a$ gives $xe = exe$. The same argument applied to the equation $(ex - exe)^2 = 0$ yields $ex = exe$. Putting these together gives $xe = ex$ whenever e is a square. Now

$$ab = ab^3 = (ab)b^2 = b^2(ab) = b^2(a^3b) = b^2a(a^2b) =$$

$$= b^2a(ba^2) = b((ba)^2a) = b(a(ba)^2) = (ba)^3 = ba.$$